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Synchronization of delay-coupled nonlinear oscillators: An approach based on the stability analysis of synchronized equilibria

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We consider the synchronization problem of an arbitrary number of coupled nonlinear oscillators with delays in the interconnections. The network topology is described by a directed graph. Unlike the conventional approach of deriving directly sufficient synchronization conditions, the approach of the paper starts from an exact stability analysis in a (gain, delay) parameter space of a synchronized equilibrium and extracts insights from an analysis of its bifurcations and from the corresponding emerging behavior. Instrumental to this analysis a factorization of the characteristic equation is employed that not only facilitates the analysis and reduces computational cost but also allows to determine the precise role of the individual agents and the topology of the network in the (in)stability mechanisms. The study provides an algorithm to perform a stability and bifurcation analysis of synchronized equilibria. Furthermore, it reveals fundamental limitations to synchronization and it explains under which conditions on the topology of the network and on the characteristics of the coupling the systems are expected to synchronize. In the second part of the paper the results are applied to coupled Lorenz systems. The main results show that for sufficiently large coupling gains, delay-coupled Lorenz systems exhibit a generic behavior that does not depend on the number of systems and the topology of the network, as long as some basic assumptions are satisfied, including the strong connectivity of the graph. Here the linearized stability analysis is strengthened by a nonlinear stability analysis which confirms the predictions based on the linearized stability and bifurcation analysis. This illustrates the usefulness of the exact linearized analysis in a situation where a direct nonlinear stability analysis is not possible or where it yields conservative conditions from which it is hard to get qualitative insights in the synchronization mechanisms and their scaling properties. In the examples several network topologies are considered. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3187792]

Synchronization is an important problem in the study of coupled dynamical systems, motivated by applications in science and engineering. This article considers the synchronization problem in networks of identical nonlinear oscillators, where the signal exchanges are affected by time delays. The main interest lies in gaining insight in the occurrence of synchronized behavior and in studying its dependence on parameters of the coupling (coupling strength, delay) and on the topology of the network. Because synchronization is a notion of relative stability, the existing methods for the stability analysis of nonlinear time-delay systems can be directly applied or adapted to the synchronization problem, possibly combined with a boundedness argument of the solutions. Such methods are almost exclusively time-domain based and heavily rely on an appropriately chosen Lyapunov function(al). Although they are applicable to the study of complex synchronized behavior (e.g., synchronized chaotic behavior), the resulting conditions for synchronization are typically in the form of sufficient, yet not necessary conditions, from which it may be hard to extract qualitative insights in the synchronization mechanisms. Inspired by this observation a distinct approach is taken in this paper which builds on the stability and bifurcation analysis of a special type of synchronized solutions, for which necessary and sufficient stability conditions can be obtained, namely, synchronized equilibria.

NOMENCLATURE

$\mathbb{C}$, $\mathbb{R}$	=	Set of complex numbers, set of real numbers
$\mathbb{N}$	=	Set of natural numbers, includes zero
j	=	Imaginary identity
$\lambda_1(G), E_1, \lambda_2(G), E_2 \ldots$	=	Eigenvalues and corresponding eigenvectors of matrix G
$\Re(\lambda)$, $\Im(\lambda)$, $\lvert\lambda\rvert$, $\lambda \in \mathbb{C}$	=	Real part, imaginary part, and modulus of λ

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$\angle\lambda,\ \lambda\in\mathbb{C}$ = Argument of λ, following the convention $\angle\lambda \in [0,2\pi)$
$r_\sigma(A)$ = Spectral radius of matrix A
$\sigma(A)$ = Spectrum of matrix A
$\sigma_1(A)\geq\sigma_2(A)\geq\cdots$ = Singular values of matrix A
$\|a\|,\ a=(a_1,\ldots,a_m)\in\mathbb{C}^m$ = Euclidean norm of a, $\|a\| = \sqrt{\sum_{k=1}^m a_k^2}$
$\bar{E},\ \partial E,\ E\subseteq\mathbb{C}$ = Closure of E, boundary of E
$A\otimes B$ = Kronecker product of matrices A and B (see, e.g., Ref. 7)
$A\oplus B,\ A\in\mathbb{C}^{n\times n},\ B\in\mathbb{C}^{m\times m}$ = Kronecker sum of A and B, $A\oplus B=(A\otimes I_m)+(I_n\otimes B)$.

I. INTRODUCTION

We consider p identical nonlinear oscillators described by

$$\dot{x}_i(t)=f(x_i)+Bu_i(t),\quad y_i(t)=Cx_i(t),\quad i=1,\ldots,p, \tag{1}$$

where $x_i\in\mathbb{R}^n$, $i=1,\ldots,p$, $B,C\in\mathbb{R}^{n\times 1}$, and $f:\mathbb{R}^n\mapsto\mathbb{R}^n$ is twice continuously differentiable. We further assume that for $u_i=0$ the system (1) has at least one unstable equilibrium of focus type, which we denote by x^* in what follows.

In order to describe the coupling between the oscillators we define a directed graph

$$\mathcal{G}(\mathcal{V},G), \tag{2}$$

characterized by the node set $\mathcal{V}=\{1,\ldots,p\}$ and a weighted adjacency matrix G with zero diagonal entries and nondiagonal entries equal to $\alpha_{i,l}\geq 0$. The corresponding edge set $\mathcal{E}$ satisfies $(i,l)\in\mathcal{E}$ if and only if $\alpha_{i,l}\neq 0$, Next, we couple the systems (1) by means of the "control" law

$$u_i(t)=k\left(\sum_{(i,l)\in\mathcal{E}}\alpha_{i,l}(y_l(t-\tau)-y_i(t))\right),\quad i=1,\ldots,p, \tag{3}$$

where $k>0$ represents the gain parameter and τ the transmission delay. We assume the transmission delay to be fixed and independent of the nodes. It is important to point out that we do not assume that G is symmetric.

The aim of the paper is to study the effect of the coupling (3), with k and τ as parameters, on the synchronization of the systems (1) and to reveal synchronization mechanisms and conditions. Hereto a complete characterization of the stability-instability regions of the synchronized equilibrium $(x^*,\ldots,x^*)$ of Eqs. (1) and (3) in the (k,τ) parameter space will be made. Note that achieving stability can be seen as an extension of the use of Pyragas-type feedback[16] to stabilize an unstable equilibrium as in Ref. 8 though, in contrast to the original idea of the Pyragas feedback, the time delay τ is not linked to any possible periodicity in the system. It can also be interpreted as a situation where so-called oscillator death is achieved.[1] The proposed approach to compute stability regions in the (k,τ) parameter space is inspired by Chap. 4 of Ref. 12 where an overview of methods to compute stability regions in parameter spaces of linear control systems with delays is presented, yet not in the context of networks of interconnected systems. Beyond the stability analysis of (synchronized) equilibria, the goal is to gain insights in and reveal explanations for the occurrence of more complex synchronized behavior.

A motivation and overview of synchronization problems and results can be found in Refs. 18 and 20. An overview of the available results on synchronization problems with coupling delays is presented in Sec. II of Ref. 14. As it is apparent from this overview some of the existing results assume a coupling similar to Eq. (3) but where also the output $y_i(t)$ is delayed over an interval of length τ. A control law of the form (3) is employed in Refs. 10, 14, and 21. The motivation for this form is that each agent has immediate access to its own state, while the information about the state of the other agents needs to be communicated over the network, which may lead to delays. In Ref. 10 the anticipative synchronization problem of two systems is considered. In Ref. 21 a symmetric network of four systems is analyzed. In Ref. 14 the synchronization problem in general networks with delays is considered Necessary conditions on the network topology and delays for the existence of synchronized solutions are presented, and sufficient conditions for asymptotic synchronization are stated. However, unlike this paper, no qualitative analysis is performed and a different methodology is taken, relying on time-domain methods and, in particular, Lyapunov functionals. Synchronization in general networks is also addressed in Refs. 2 and 15 and the references therein but without taking into account delay effects. The latter effects are, however, crucial in this paper. Finally, in Ref. 3 a complete bifurcation analysis of a model for three interacting neurons with delay in the interconnections is presented under the assumption of a ring configuration and bidirectional coupling.

The structure of the paper is as follows. In Sec. II some preliminary results are presented, which include necessary conditions for the occurrence of synchronized solutions of Eqs. (1) and (3). In Sec. III a linearized stability analysis of synchronized equilibria is performed. In Sec. IV the results are applied to networks of coupled Lorenz systems and complemented with a nonlinear stability analysis. Particular attention will be paid to the asymptotic behavior of stability properties for large values of the gain parameter and on the derivation of generic results that do not depend on the network topology and the number of oscillators. The conclusions are formulated in Sec. V.

II. PRELIMINARIES

A. Assumptions on the graph

The following assumptions are made throughout the paper.

Assumption 2.1: The graph $\mathcal{G}$ is strongly connected.

Assumption 2.2: The adjacency matrix G satisfies

$$\sum_{l=1}^{p}\alpha_{i,l}=1,\quad i=1,\ldots,p.$$

The first assumption is natural in the context of synchronization with delayed coupling; the second assumption will be motivated in Sec. II B. The following results are direct corollaries.

Corollary 2.3: G has a simple eigenvalue equal to 1 *and* $[1\cdots1]^T$ *is the corresponding eigenvector. Furthermore, there exits a vector* $\gamma=[\gamma_1\cdots\gamma_p]^T$ *such that*

$$\gamma_l > 0, \quad 1 \le l \le p, \quad \sum_{l=1}^{p} \gamma_l = 1, \quad \gamma^T(G-I) = 0.$$

Proof: The matrix $G-I$ is a Metzler matrix with zero row sums by Assumption 2.2, and it is irreducible by Assumption 2.1. The assertion follows. □

Corollary 2.4: All eigenvalues of G have modulus smaller than or equal to 1.

Proof: For all $x \in \mathbb{C}^p$ we have $\|Gx\|_\infty \le \|x\|_2$, as the premultiplication of G implies that every element is replaced with a weighted average of the other elements. This implies that $\|G\|_\infty \le 1$ and the assertion follows. □

In what follows we denote the eigenvalues of G as $\lambda_i(G)$, $1 \le i \le p$, where we take the following convention.

Convention 2.5: $\lambda_1(G)=1$.

B. A coordinate transformation

Define the matrix $\tilde{G}$:

$$\tilde{G} = \begin{bmatrix} 0 & \alpha_{2,3} & \alpha_{2,4} & \cdots & \alpha_{2,p} \\ \alpha_{3,2} & 0 & \alpha_{3,4} & & \alpha_{3,p} \\ & & & \ddots & \\ \alpha_{p,2} & \alpha_{p,3} & \cdots & \alpha_{p,p-1} & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [\alpha_{1,2} \quad \alpha_{1,3} \quad \cdots \quad \alpha_{1,p}], \tag{4}$$

which satisfies the following property.

Property 2.6: $\sigma(\tilde{G}) = \sigma(G) \setminus \{1\}$.

Proof: By Assumption 2.2 and Corollary 2.3 the matrix

$$\bar{G} := G - \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} [0 \quad \alpha_{1,2} \quad \cdots \quad \alpha_{1,n}]$$

satisfies

$$\sigma(\bar{G}) = \{0, \lambda_2(G), \ldots, \lambda_p(G)\}.$$

Furthermore, we have by construction

$$\bar{G} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \alpha_{2,1} & & & \\ \vdots & & \tilde{G} & \\ \alpha_{p,1} & & & \end{bmatrix}.$$

It follows that $\sigma(\tilde{G}) = \{\lambda_2(G), \ldots, \lambda_p(G)\}$. □

By means of the new variables

$$e_2(t) = x_2(t) - x_1(t),$$
$$\vdots,$$
$$e_p(t) = x_p(t) - x_1(t),$$

we can bring Eqs. (1) and (3) in the form

$$\dot{x}_1(t) = f(x_1(t)) + kBC\left(\sum_l \alpha_{1,l}\right)(x_1(t-\tau) - x_1(t)) + kBC\sum_l \alpha_{1,l} e_l(t-\tau),$$

$$\begin{bmatrix} \dot{e}_2(t) \\ \vdots \\ \dot{e}_p(t) \end{bmatrix} = \begin{bmatrix} f(x_1+e_2) - f(x_1) \\ \vdots \\ f(x_1+e_p) - f(x_1) \end{bmatrix} - k\left(\begin{bmatrix} \sum_l \alpha_{2,l} & & \\ & \ddots & \\ & & \sum_l \alpha_{p,l} \end{bmatrix} \otimes BC\right) \begin{bmatrix} e_2(t) \\ \vdots \\ e_p(t) \end{bmatrix} + k\tilde{G} \otimes BC^T \begin{bmatrix} e_2(t-\tau) \\ \vdots \\ e_p(t-\tau) \end{bmatrix} + \left(k \begin{bmatrix} \sum_l \alpha_{1,l} - \sum_l \alpha_{2,l} \\ \sum_l \alpha_{1,l} - \sum_l \alpha_{3,l} \\ \vdots \\ \sum_l \alpha_{1,l} - \sum_l \alpha_{p,l} \end{bmatrix} \otimes BC\right)(x_1(t) - x_1(t-\tau)). \tag{5}$$

From this equation it can be seen that a synchronized solution, characterized by

$$e_2 \equiv 0, \ldots, e_p \equiv 0,$$

can only exist in three cases:

(1) the delay is equal to zero,
(2) the overall motion is τ-periodic, and
(3) Assumption 2.2 holds.

Because we are primarily interested in explaining synchronized complex behavior in the presence of delays in the coupling, we can take Assumption 2.2 without losing generality and the equations in Eq. (5) simplify to

$$\dot{x}_1 = f(x_1(t)) + kBC(x_1(t-\tau) - x_1(t)) + kBC\sum_{l=1}^{p} \alpha_{1,l} e_l(t-\tau), \tag{6}$$

$$\begin{bmatrix} \dot{e}_2 \\ \vdots \\ \dot{e}_p \end{bmatrix} = \begin{bmatrix} f(x_1+e_2)-f(x_1)-kBCe_2 \\ \vdots \\ f(x_1+e_p)-f(x_1)-kBCe_p \end{bmatrix} + k\tilde{G}\otimes BC \begin{bmatrix} e_2(t-\tau) \\ \vdots \\ e_p(t-\tau) \end{bmatrix}. \tag{7}$$

The solutions on the synchronization manifold are characterized by

$$\dot{x}_1(t) = f(x_1(t)) + kBC(x_1(t-\tau)-x_1(t)). \tag{8}$$

If all the solutions of Eqs. (6) and (7) converge to a bounded forward invariant set, then the synchronization between the agents is achieved locally if the linearization of Eq. (7),

$$\begin{bmatrix} \dot{e}_2(t) \\ \vdots \\ \dot{e}_p(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{\partial f}{\partial x}(x_1(t))-kBC\right)e_2(t) \\ \vdots \\ \left(\frac{\partial f}{\partial x}(x_1(t))-kBC\right)e_p(t) \end{bmatrix} + k\tilde{G}\otimes BC \begin{bmatrix} e_2(t-\tau) \\ \vdots \\ e_p(t-\tau) \end{bmatrix}, \tag{9}$$

is uniformly asymptotically stable. In order to simplify the analysis, we let R and I be defined as

$$R = \{i \in \{2,\ldots,p\} : \Im(\lambda_i(G)) = 0\},$$

$$I = \{i \in \{2,\ldots,p\} : \Im(\lambda_i(G)) > 0\}$$

and we let T_r be a matrix satisfying

$$T_r^{-1}\tilde{G}T_r = D,$$

where D is a block triangular matrix whose diagonal blocks are given by

$$\{\lambda_i(G) : i \in R\} \cup \left\{ \begin{bmatrix} \Re(\lambda_i(G)) & \Im(\lambda_i(G)) \\ -\Im(\lambda_i(G)) & \Re(\lambda_i(G)) \end{bmatrix} : i \in I \right\}.$$

The matrices T_r and D always exist by the identity $\sigma(D)=\sigma(\tilde{G})$ and Property 2.6.

From the state transformation induced by the matrix $(T_r \otimes I)$ it follows that its zero solution of Eq. (9) is uniformly asymptotically stable if the following equations are uniformly asymptotically stable:

$$\dot{\xi}_i(t) = \left(\frac{\partial f}{\partial x}(x_1(t))-kBC\right)\xi_i(t) + k\lambda_i(G)BC\xi_i(t-\tau), \quad \forall\, i \in I, \tag{10}$$

$$\begin{bmatrix} \dot{\xi}_i \\ \dot{\eta}_i \end{bmatrix} = I \otimes \left(\frac{\partial f}{\partial x}(x_1(t))-kBC\right)\begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix} + k\begin{bmatrix} \Re(\lambda_i(G)) & \Im(\lambda_i(G)) \\ -\Im(\lambda_i(G)) & \Re(\lambda_i(G)) \end{bmatrix} \otimes BC \begin{bmatrix} \xi_i(t-\tau) \\ \eta_i(t-\tau) \end{bmatrix}, \quad \forall\, i \in J. \tag{11}$$

Equivalently, a full triangularization of $\tilde{G}$ results in the analysis of

$$\dot{\xi}_i(t) = \left(\frac{\partial f}{\partial x}(x_1(t))-kBC\right)\xi_i(t) + k\lambda_i(G)BC\xi_i(t-\tau), \quad \forall\, i = 2,\ldots,p, \tag{12}$$

at the price that some of the equations in Eq. (12) are complex valued if $\tilde{G}$ has complex eigenvalues.

Remark 2.7: The analysis of networks using the master stability function[4,15] *is based on a similar decomposition of the error dynamics. For* $\tau=0$, this function maps $z \in \mathbb{C}$, $\Re(z) \le 0$ *to the largest Lyapunov exponent of*

$$\dot{\xi}_i = \frac{\partial f}{\partial x}(x_1(t))\xi_i(t) + zBC\xi_i(t). \tag{13}$$

Equation (13) is related to Eq. (12) with $\tau=0$ *via*

$$z = k\lambda_i(G-I).$$

Unlike the undelayed case considered in the literature, the dynamics on the synchronization manifold, governed by Eq. (8), depend on both k *and* τ, *since the coupling is invasive if* $\tau \neq 0$. *Furthermore, in the presence of delay both parameters* k *and* $\lambda_i(G)$ *can no longer be simultaneously absorbed in the variable* z.

III. STABILITY ANALYSIS OF SYNCHRONIZED EQUILIBRIA

When we linearize the system (1) and (3) around the synchronized equilibrium $(x^*,\ldots,x^*)$, we obtain

$$\begin{bmatrix} \dot{\nu}_1(t) \\ \vdots \\ \dot{\nu}_p(t) \end{bmatrix} = I \otimes (A-kBC)\begin{bmatrix} \nu_1(t) \\ \vdots \\ \nu_p(t) \end{bmatrix} + kG \otimes BC \begin{bmatrix} \nu_1(t-\tau) \\ \vdots \\ \nu_p(t-\tau) \end{bmatrix}, \tag{14}$$

where

$$A = \frac{\partial f}{\partial x}(x^*).$$

A. The characteristic equation

1. Factorization

The characteristic function of Eq. (14) is given by

$$f(\lambda;k,\tau) := \det F(\lambda;k,\tau),$$

where the characteristic matrix F is defined as

$$F(\lambda;k,\tau) = I \otimes (\lambda I - A + kBC) - G \otimes kBCe^{-\lambda\tau}. \tag{15}$$

If we factorize $G=T\Lambda T_c^{-1}$, where $\Lambda \in \mathbb{C}^{p\times p}$ is triangular and $T_c \in \mathbb{C}^{p\times p}$, the characteristic function becomes

$$f(\lambda;k,\tau) = \det(I \otimes (\lambda I - A + kBC_c) - T_c \Lambda T_c^{-1} \otimes kBC_c e^{-\lambda\tau})$$
$$= \det(T_c^{-1} \otimes I)\det(I \otimes (\lambda I - A + kBC) - T_c \Lambda T_c^{-1} \otimes kBCe^{-\lambda\tau})\det(T_c \otimes I)$$
$$= \det(I \otimes (\lambda I - A + kBC - kBC\lambda_i(G)e^{-\lambda\tau}))$$
$$= \Pi_{i=1}^{p} f_i(\lambda;k,\tau), \tag{16}$$

where

$$f_i(\lambda;k,\tau) := \det F_i(\lambda;k,\tau),$$

$$F_i(\lambda;k,\tau) := \lambda I - A + kBC - kBC\lambda_i(G)e^{-\lambda\tau}, \quad i = 1,\ldots,p.$$

Remark 3.1: This factorization of the characteristic function can also be obtained from the factorization of Eqs. (6) and (7) into Eqs. (6) and (12) when taking into account that $x_1(t) \equiv x^*$. It follows from this observation that the zeros of

$$f_1(\lambda;k,\tau) = \det(\lambda I - A + kBC - kBCe^{-\lambda\tau})$$

describe the dynamics of the linearization of the "nominal" system Eq. (8), while the zeros of

$$f_2(\lambda;k,\tau),\ldots,f_p(\lambda;k,\tau)$$

describe the behavior of the synchronization error dynamics.

2. Eigenspaces and behavior on the onset of instability

We investigate the eigenspace of the characteristic matrix (15), corresponding to a characteristic root. For reasons of simplicity we restrict ourselves to the generic case where all the eigenvalues of the adjacency matrix G are simple. Let E_i be the eigenvector of G corresponding to the eigenvalue $\lambda_i(G)$, $i=1,\ldots,p$. By Corollary 2.3, we have $E_1=[1\ \ 1\cdots1]^T$.

If for some $l \in \{1,\ldots,p\}$, the equation

$$f_l(\lambda;k,\tau) = 0$$

has a simple root at $\lambda=\hat{\lambda}$ such that

$$F_l(\hat{\lambda};k,\tau)V = 0, \quad V \in C^{m\times 1}, \tag{17}$$

then it can be verified that

$$F(\hat{\lambda};k,\tau)(E_l \otimes V) = 0. \tag{18}$$

This implies that the linearized system (14) has an exponential solution

$$\begin{bmatrix} \nu_1(t) \\ \vdots \\ \nu_p(t) \end{bmatrix} = c(E_l \otimes V)e^{\hat{\lambda}t} = c\begin{bmatrix} e_{l,1}V \\ \vdots \\ e_{l,p}V \end{bmatrix} e^{\hat{\lambda}t}, \tag{19}$$

with the constant c depending on the initial conditions.

The above analysis can be generalized to the case where the zero $\hat{\lambda}$ of f_l has multiplicity larger than 1—for the theory of multiple eigenvalues of nonlinear eigenvalue problems we refer to Ref. 9 If the vectors $(V_r,\ldots,V_1)$ form a Jordan chain of length r of F_l corresponding to the eigenvalue $\hat{\lambda}$, that is,

$$V_1 \neq 0,$$

$$F_l(\hat{\lambda})V_1 = 0,$$

$$F_l(\hat{\lambda})V_2 + \frac{1}{1!}\frac{dF_l}{d\lambda}(\hat{\lambda})V_1 = 0,$$

$$\vdots,$$

$$F_l(\hat{\lambda})V_3 + \frac{1}{1!}\frac{dF_l}{d\lambda}(\hat{\lambda})V_2 + \frac{1}{2!}\frac{d^2F_l}{d\lambda^2}(\hat{\lambda})V_1 = 0,$$

$$F_l(\hat{\lambda})V_r + \frac{1}{1!}\frac{dF_l}{d\lambda}(\hat{\lambda})V_{r-1} + \frac{1}{2!}\frac{d^2F_l}{d\lambda^2}(\hat{\lambda})V_{r-2} + \cdots + \frac{1}{(r-1)!}\frac{d^{r-1}F_l}{d\lambda^{r-1}}(\hat{\lambda})V_1, = 0,$$

then the vectors

$$(E_l \otimes V_r,\ldots,E_l \otimes V_1)$$

form a Jordan chain of F. Moreover, the corresponding solution of Eq. (14) takes the form

$$\begin{bmatrix} \nu_1(t) \\ \vdots \\ \nu_p(t) \end{bmatrix} = \sum_{i=1}^{r} c_i \left(\sum_{k=1}^{i} \frac{t^{i-k}}{(i-k)!}(E_l \otimes V_k) \right) e^{\hat{\lambda}t}, \tag{20}$$

where the constants $(c_1,\ldots,c_r)$ depend on the initial condition. The following can be concluded:

- In an exponential solution of Eq. (14) corresponding to a zero of $f_l(\lambda;k,\tau)$, the relation between the state variables of an individual subsystem is determined by the Jordan system of F_l, while the relation between the corresponding state variables of the different subsystems is *solely* determined by the eigenvector E_l, corresponding to the lth eigenvalue of the adjacency matrix G. This implies that all modes can be classified in at most p types based on the relations between the behavior of the different subsystems. The bifurcations of the synchronized equilibria of the original nonlinear system can be classified in the same way.
- The modes induced by the zeros of $f_1(\lambda;k,\tau)$ all correspond to synchronized behavior of the different subsystems because $E_1=[1\cdots1]^T$. This is expected because they describe the dynamics on the synchronization manifold. By Corollary 2.3, the occurrence of these modes is independent of the topology of the network. The modes induced by $E_2,\ldots,E_p$ correspond to the synchronization error dynamics.

B. Computation of stability regions in the delay parameter

For a fixed value of k the following propositions allow to characterize delay values corresponding to characteristic roots of Eq. (14) on the imaginary axis.

Proposition 3.2: For every $i \in \{1,\ldots,p\}$ we have

$$f_i(0;k,0) = 0 \Leftrightarrow f_i(0;k,\tau) = 0, \quad \forall\ \tau \geq 0. \tag{21}$$

Proof: The relation (21) is implied by the identity

$f_i(0;k,\tau)=f_i(0;k,0)$. □

Proposition 3.3: The following assertions hold:

(1) *The equation*

$$f_i(\lambda;k,\tau)=0, \quad i\in\{1,\ldots,p\}, \tag{22}$$

has a root $j\omega$, $\omega>0$, *for some value of* $\tau\geq 0$ *if and only if there exists a complex number* z *such that*

$$|z|=1, \quad j\omega\in\sigma(A-kBC+kBC|\lambda_i(G)|z). \tag{23}$$

(2) *The corresponding delay values are given by*

$$\mathcal{T}_\omega := \cup\{\mathcal{T}_\omega^z : z \text{ satisfies Eq. (23)}\}, \tag{24}$$

with

$$\mathcal{T}_\omega^z := \left\{\frac{\angle(\lambda_i(G)\bar{z})+2\pi\ell}{\omega} : \ell\in\mathbb{N}\right\}. \tag{25}$$

(3) *If* $j\omega$ *is a simple root of Eq. (22) for the delay value* $\tau=\tau_0$ *and* $\tau_0\in\mathcal{T}_\omega^z$, *then by sweeping the delay through* τ_0 *the root crosses the imaginary axis toward instability (stability) if*

$$\Re\left(\frac{u^*BCvj\omega z}{u^*v}\right)<0(>0), \tag{26}$$

where u *and* v *are left and right null vectors of*

$$j\omega I-A+kBC-kBC|\lambda_i(G)|z.$$

Proof: This proposition is an adaptation of Proposition 4.5 of Ref. 12. □

Proposition 3.4: The conditions (23) imply

$$|z|=1,$$

$$\det\left(\begin{bmatrix} 0 & -I \\ I\otimes k|\lambda_i(G)|(BC)^T & (A-kBC)\oplus(A-kBC)^T \end{bmatrix} + z\begin{bmatrix} I & 0 \\ 0 & k|\lambda_i(G)|(BC)\otimes I \end{bmatrix}\right)=0. \tag{27}$$

Proof: Similar to the proof of Proposition 4.5 of Ref. 12. □

By combining the above results with a continuation argument we obtain the following algorithm for computing stability/instability regions of Eq. (14) in the delay parameter space:

Algorithm 3.5: [Stability regions of Eq. (14) in the delay parameter space]

(1) *Repeat for* $i=1,\ldots,p$:
Compute the zeros of the polynomial $f_i(\lambda;k,0)$ *in the closed right half-plane. Free the delay parameter. Compute all delay values for which* $f_i(\lambda;k,\tau)$ *has a zero on the positive imaginary axis and the corresponding crossing direction from Propositions 3.3 and 3.4, in the following way:*
   (a) *Compute all solutions of the eigenvalue problem (27).*
   (b) *For every* z *satisfying Eq. (27), compute the eigenvalues on the positive imaginary axis of the matrix* $A-kBC+kBC|\lambda_i(G)|z$.
   (c) *Use the results of steps (a) and (b) to determine all pairs* (ω,z) *satisfying Eq. (23).*
   (d) *Determine all critical delay values as well as the corresponding crossing direction of the corresponding zeros of* f_i *on the imaginary axis using Eqs. (24)–(26) (in the generic case of simple zeros).*

(2) *Combine the obtained results for all* $i\in\{1,\ldots,p\}$. *This yields a full characterization of the stability regions of Eq. (14) in the delay parameter, because all critical delay values and stability switches are covered.*

Remark 3.6: Step 1(a) is facilitated by the following symmetry property of the generalized eigenvalue problem: a number $z\in\mathbb{C}\backslash\{0\}$ *satisfies the second condition of Eq. (27) if and only if* $\bar{z}^{-1}$ *satisfies this condition.*

Remark 3.7: The argument $\angle\lambda_i(G)$ *does not affect the solutions of Eqs. (23) and (26). From Eq. (24) a change in the argument only leads to a shift in the critical delay values. This property will be apparent in the examples presented in Sec. IV.*

As an alternative to Algorithm 3.5 the curves separating stability-instability regions in the (k,τ) parameter space can be computed as Hopf bifurcation curves by numerical continuation (see, e.g., Ref. 11). At the one hand, the advantage of numerical continuation is that curves in the two-parameter space (k,τ) are directly computed in a computationally efficient way (whereas Algorithm 3.5 only sweeps the delay parameter for a fixed value of the gain parameter and needs to be repeated for a set of gain values chosen on a grid). On the other hand, isolated curves may not be automatically detected since starting values are required in a continuation procedure. The latter problem does not occur with Algorithm 3.5 as it is based on a complete description of critical delay values.

The computations for the numerical examples presented Sec. IV B are based on numerical continuation using the package DDE-BIFTOOL,[6] where Algorithm 3.5 is used to generate starting values for the curves. The asymptotic analysis of coupled Lorenz systems presented in Sec. IV A is based on Propositions 3.2–3.4, on which Algorithm 3.5 relies.

IV. APPLICATION TO COUPLED LORENZ SYSTEMS

In this section the nonlinear oscillators (1) are specified as Lorenz systems:

$$\begin{aligned}
\dot{x}_{i,1}(t)&=\sigma(x_{i,2}(t)-x_{i,1}(t)),\\
\dot{x}_{i,2}(t)&=rx_{i,1}(t)-x_{i,2}(t)-x_{i,1}(t)x_{i,3}(t)+u_{i,1}(t)\\
\dot{x}_{i,3}(t)&=-bx_{i,3}(t)+x_{i,1}(t)x_{i,2}(t)+u_{i,2}(t),\\
y_{i,1}(t)&=x_{i,2}(t),
\end{aligned} \tag{28}$$

$$y_{i,2}(t) = x_{i,3}(t) - r, i = 1, \ldots, p,$$

where

$$u_i = [u_{i,1} u_{i,2}]^T, \quad y_i = [y_{i,1} y_{i,2}]^T.$$

The parameter values are given by

$$\sigma = 10, \quad r = 28, \quad b = 8/3. \tag{29}$$

Note that for $u_i \equiv 0$ each Lorenz system has three equilibria given by

$$(0,0,0), (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1), \tag{30}$$

the latter two corresponding to unstable foci. Furthermore, with the parameter values (29) it exhibits a chaotic attractor.[17]

If we linearize the coupled system (28) and (3) around the synchronized equilibrium

$$(x^*, \ldots, x^*), \quad x^* = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1), \tag{31}$$

then we obtain the linear system (14), where the matrices are specified as

$$A = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{bmatrix}, \tag{32}$$

$$B = C^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

It is easy to show that the stability of the linearized system does not depend on which equilibrium x^* in Eq. (31) is considered, Therefore, we will restrict ourselves to the one in the positive octant.

In what follows we analyze the stability properties of the synchronized equilibria (31) in the (k, τ) parameter space. First we study the asymptotic behavior for large values of the gain parameters in Sec. IV A. For the standard parameters (29) this will allow us to make assertions about stability regions, stability switches, and emerging behavior, which do *not* depend on the network topology. Next we present several numerical examples in Sec. IV B. Finally we perform a nonlinear stability analysis of Eqs. (28) and (3) in Sec. IV C.

A. Asymptotic behavior for large gain values

We first state some technical lemmas.

Lemma 4.1: For large values of k the zeros of the functions $f_i(\lambda;k,0)$, $i=2,\ldots,p$, *are in the open left half-plane. Furthermore, the system (14) and (32) has exactly two characteristic roots in the closed right half-plane for* $\tau=0$, *which are equal to the unstable eigenvalues of A .*

Proof: Let $i \in \{2, \ldots, p\}$. As $k \to \infty$ the function

$$\frac{f_i(\lambda;k,0)}{k^2} = \det\left(\begin{bmatrix} \lambda+\sigma & -\sigma & 0 \\ -\frac{1}{k} & \frac{\lambda+1}{k} - (\lambda_i(G)-1) & \frac{\sqrt{b(r-1)}}{k} \\ -\frac{\sqrt{b(r-1)}}{k} & -\frac{\sqrt{b(r-1)}}{k} & \frac{\lambda+b}{k} - (\lambda_i(G)-1) \end{bmatrix}\right)$$

uniformly converges on compact subsets of $\mathbb{C}$ to the Hurwitz polynomial

$$\det\left(\begin{bmatrix} \lambda+\sigma & -\sigma & 0 \\ 0 & -(\lambda_i(G)-1) & 0 \\ 0 & 0 & -(\lambda_i(G)-1) \end{bmatrix}\right) = (1-\lambda_i(G))^2(\lambda+\sigma). \tag{33}$$

From Rouché's theorem it follows that for sufficiently large k the function $f_i(\lambda;k,0)$ has at least one zero in the left half-plane which converges to $-\sigma$ as $k \to \infty$. Next, from the normalized function

$$\frac{f_i(\lambda;k,0)}{k^3} = \det\left((\lambda/k)I - \begin{bmatrix} -\frac{\sigma}{k} & \frac{\sigma}{k} & 0 \\ \frac{1}{k} & -\frac{1}{k} + (\lambda_i(G)-1) & -\frac{\sqrt{b(r-1)}}{k} \\ \frac{\sqrt{b(r-1)}}{k} & \frac{\sqrt{b(r-1)}}{k} & -\frac{b}{k} + (\lambda_i(G)-1) \end{bmatrix}\right)$$

it follows that for sufficiently large k the function $f_i(\lambda;k.0)$ has two zeros equal to $k\tilde{\lambda}_j(l)$, $l=1,2$, where

$$\lim_{k\to\infty} \tilde{\lambda}_l(k) = \lambda_i(G) - 1, \quad l = 1,2.$$

Because $\Re(\lambda_i) - 1 < 0$ one concludes that, as $k \to \infty$, two zeros of f_i move off to infinity without leaving the open left half-plane, while the other zero converges to $-\sigma$.

The second assertion follows from the identity

$$f_1(\lambda;k,\tau) = \det(\lambda I - A).$$

□

Lemma 4.2: Assume that $|\lambda_i| < 1$. *Then for large values of* k *the zeros of the function*

$$f_i(\lambda;k,\tau)$$

are in the open left half-plane for all values of the delay parameter.

Proof: The equation

$$f_i(j\omega;k,\tau) = 0$$

is equivalent to

$$\det(I - (j\omega I - A + kBC)^{-1} kBC\lambda_i(G) e^{-j\omega\tau}) = 0.$$

A necessary solvability condition is given by

$$r_\sigma((j\omega I - A + kBC)^{-1} kBC\lambda_i(G)) = 1.$$

This condition is always violated for large k. Indeed, in the complex plane the nonzero eigenvalues of the matrix

$$(j\omega I - A + kBC)^{-1} kBC\lambda_i(G), \tag{34}$$

which can be written as

$$\left(j\frac{\omega}{k} I - \frac{A}{k} + BC\right)^{-1} BC\lambda_i(G),$$

converge to the curve

$$\Omega \geq 0 \mapsto \frac{1}{j\Omega + 1}\lambda_i(G)$$

as $k \to \infty$, uniformly in the parameter $\omega \geq 0$. Furthermore, we have

$$\left|\frac{1}{1 + j\Omega}\lambda_i(G)\right| \leq |\lambda_i(G)| < 1, \quad \forall\, \Omega \geq 0.$$

It follows that zeros on the imaginary axis are not possible for large values of k. Combining this result with Lemma 4.1 leads to the assertion to be proven. □

Lemma 4.3: Assume that $|\lambda_i| = 1$. *If* $(z(k), \omega(k))$ *satisfies Eq. (23) for all* $k > 0$, *then*

$$\lim_{k\to\infty} z(k) = 1.$$

Proof: The assertion follows from the same arguments as spelled out in the proof of Lemma 4.2. □

A combination of the above results leads to the main result of this paragraph.

Theorem 4.4: *Consider a network of coupled Lorenz systems (28) with parameters (29) and coupling (3). Assume that the network satisfies Assumptions 2.1 and 2.2. Then there exists a number* $\hat{k} > 0$ *and a function*

$$\tau^*:[\hat{k},\infty] \to \mathbb{R}_+, \quad k \mapsto \tau^*(k), \tag{35}$$

satisfying the following properties:

(1) there is a constant $\tilde{k} > \hat{k}$ *such that for every* $k > \tilde{k}$, *the synchronized equilibrium has two characteristic roots in the open right half-plane for all* $\tau \in [0, \tau^*)$, *while it is asymptotically stable for* $\tau \in (\tau^*, \tau^* + \epsilon)$, *with* ϵ *sufficiently small;*

(2) at $\tau = \tau^*$ *a synchronization preserving Hopf bifurcation occurs;*

(3) for all $k \in [\hat{k}, \infty]$ *we can factor*

$$\tau^*(k) = \frac{\upsilon(k)}{k}, \tag{36}$$

where

$$\lim_{k\to\infty} \upsilon(k) = 0.586\,004. \tag{37}$$

Furthermore, the number $\hat{k}$ *and the function (35) are independent of the number of subsystems and of the network topology.*

Proof: By Lemmas 4.1 and 4.2 the functions $f_i(\lambda, k, \tau)$, where $|\lambda_i(G)| < 1$, have their zeros in the left half-plane for all values of τ if k is sufficiently large. So for large k all stability switches in the delay parameter space are due to the functions $f_i(\lambda;k,\tau)$ such that $|\lambda_i| = 1$, and Eq. (23) simplifies to

$$j\omega \in \sigma(A - kBC + kBCz), \quad |z| = 1. \tag{38}$$

If we set $z = 1 + j\rho/k$ then this expression becomes

$$j\omega \in \sigma(A + j\rho BC) \tag{39}$$

under the constraint

$$\left|1 + j\frac{\rho}{k}\right| = 1. \tag{40}$$

We analyze the solutions of Eqs. (39) and (40) as $k \to \infty$. From Lemma 4.3 and the constraint (40) it follows that ρ/k must converge to zero along the *real axis* as $k \to \infty$. Hence the asymptotic behavior for $k \to \infty$ is determined by the solutions (ω, ρ) of Eq. (39), where ρ is restricted to be real. To find these solutions matrix pencil techniques can be used, similar to Proposition 3.4. Equation (39) implies

$$-j\omega \in \sigma(A^T - j\rho(BC)^T) \tag{41}$$

and under the condition $\rho \in \mathbb{R}$, Eqs. (39) and (41) imply on their turn

$$\det(A \oplus A^T + \rho j[(BC) \otimes I - I \otimes (BC)^T]) = 0. \tag{42}$$

Thus all solutions of Eq. (39) under the constraint $\rho \in \mathbb{R}$ can be directly computed by calculating the real solutions of the eigenvalue problem (42) in the first step in order to obtain a finite number of candidate values for ρ and next, solving Eq. (39) for ω. With the parameter values (29) and with matrices (32) these solutions are given by $(\hat{\rho}_1, \hat{\omega}_1)$ and $(\hat{\rho}_2, \hat{\omega}_2)$, where

$$\hat{\rho}_1 = -3.995\,906\,4, \quad \hat{\omega}_1 = 6.818\,903\,4,$$
$$\hat{\rho}_2 = 5.223\,604\,5, \quad \hat{\omega}_1 = 14.811\,554. \tag{43}$$

As a consequence, for large values of k the solutions of Eq. (38) are

$$(\omega, z) = (\omega_l(k), z_l(k)), \quad l = 1,2,$$

where

$$z_l(k) = 1 + j\frac{\rho_l(k)}{k}, \quad l = 1,2$$

and

$$\lim_{k\to\infty} \rho_l(k) = \hat{\rho}_l, \quad \lim_{k\to\infty} \omega_l(k) = \hat{\omega}_l, \quad l = 1,2.$$

Next, from Eqs. (24) and (43) it follows that for sufficiently large k, the *first* critical delay value, as the delay is increased from zero, is given by

$$\tau^*(k) := \frac{\angle(\bar{z}_1(k))}{\omega_1(k)} = \frac{\nu(k)}{k},$$

where

$$\begin{aligned}\lim_{k\to\infty} \nu(k) &= \frac{1}{\hat{\omega}_1}\lim_{k\to\infty} k \angle(\bar{z}_1(k)) \\ &= -\frac{1}{\hat{\omega}_1}\lim_{k\to\infty} k \arctan(\rho_1(k)/k) \\ &= -\frac{\hat{\rho}_1}{\hat{\omega}_1} \\ &= 0.586\,004.\end{aligned}$$

As this switch is due to a zero of $f_1(\lambda; k, \tau)$ it is independent of the network topology and the emanating solutions have the form

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_p(t) \end{bmatrix} = \begin{bmatrix} V \\ \vdots \\ V \end{bmatrix} e^{j\omega t},$$

where $F_1(j\omega; k, \tau)V = 0$, i.e., synchronization is preserved in the emanating solutions.

Finally, we consider the crossing direction of the characteristic roots on the imaginary for

$$(k, \tau) = (k, \tau^*(k))$$

when the delay is varied. According to Eq. (26) the crossing direction is determined by the sign of

$$s(k) := \Re\left(\frac{u(k)^* BC v(k) j\omega_1(k) z_1(k)}{u(k)^* v(k)}\right),$$

where $u(k)$ and $v(k)$ are left and right null vectors of

$$j\omega_1(k)I - A + kBC - BCz_1(k) = j\omega_1(k)I - A - BCj\rho_1(k).$$

It follows that

$$\lim_{k\to\infty} s(k) = \Re\left(\frac{\hat{u}^* BC\hat{v} j\hat{\omega}_1}{\hat{u}^*\hat{v}}\right),$$

with $\hat{u}$ and $\hat{v}$ left and right null vectors of $j\hat{\omega}_1 I - A - BCj\hat{\rho}_1$. For the parameters (29) and with matrices (32) we arrive at

$$\lim_{k\to\infty} s(k) = 3.498\,024\,1 > 0.$$

Thus for large k the first stability switch, which occurs at $\tau = \tau^*(k)$, is toward stability, and it results in asymptotic stability by Lemma 4.1. When putting together the above results the statements of the theorem follow. □

B. Examples

We illustrate the obtained results with several examples with different network topologies. The computations of stability regions are done as described in Sec. III B.

1. Ring topology, unidirectional coupling

We consider a ring topology with unidirectional coupling, described by the adjacency matrix

$$G = \begin{bmatrix} 0 & \cdots & & 0 & 1 \\ 1 & 0 & & & \\ & \ddots & \ddots & & \\ & & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{p\times p}, \tag{44}$$

which has the following properties:

$$\lambda_l(G) = e^{-j[2\pi(l-1)/p]}, \quad E_l = \begin{bmatrix} 1 \\ e^{-j[2\pi(l-1)/p]} \\ \vdots \\ e^{-j[2\pi(p-1)(l-1)/p]} \end{bmatrix}$$

for $l = 1, \ldots, p$.

If Eq. (17) is satisfied for $\hat{\lambda} = j\omega$, $\omega > 0$, then the emanating solution (19) becomes

$$\begin{bmatrix} \nu_1(t) \\ \vdots \\ \nu_p(t) \end{bmatrix} = c\begin{bmatrix} Ve^{j\omega t} \\ Ve^{j\omega t[-2\pi(l-1)/p]} \\ Ve^{j\omega t[-4\pi(l-1)/p]} \\ \vdots \\ Ve^{j\omega t - [2(p-1)\pi(l-1)/p]} \end{bmatrix}. \tag{45}$$

It can be interpreted as a traveling wave solution, where the agents follow each other with a phase shift of $360(l-1)/p$ (in degrees). Therefore, if the characteristic root $\hat{\lambda}$ on the imaginary axis corresponds to a Hopf bifurcation of the original nonlinear system (1) and (3) for a critical value of some free parameter, we refer to this bifurcation as a "Hopf $360(l-1)/p$" bifurcation. In a sense, this type of traveling wave solution strongly reminds that of the gait of an animal, be it that the underlying oscillator is different from a Lorenz oscillator.

With the individual agents taken as Lorenz systems (28) with parameters (29) and with $p=4$ and $p=12$ we display the stability regions in the delay parameter space of the synchronized equilibria (31) in Fig. 1. The Hopf 0 bifurcation curves are independent of the number of subsystems, because they

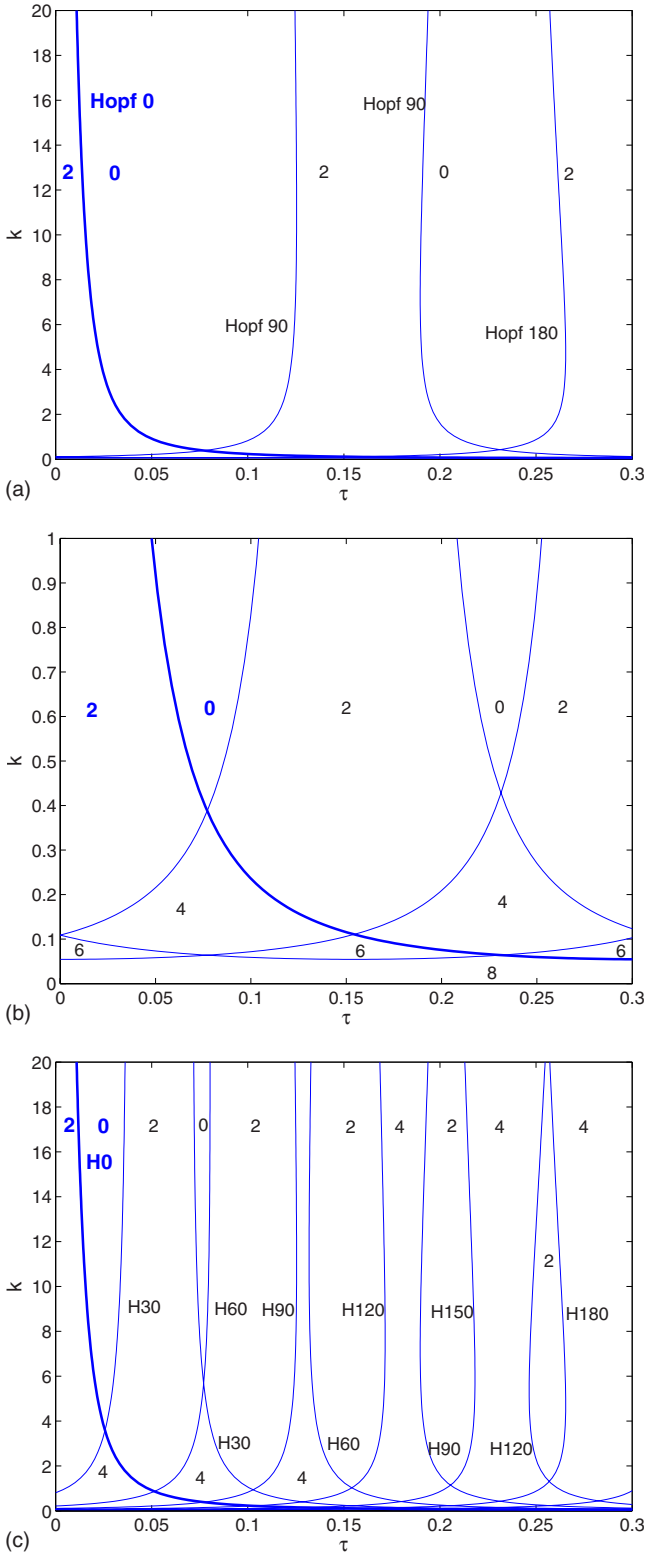

FIG. 1. (Color online) Stability regions of the synchronized equilibrium (31) of Lorenz systems (28) and (29) coupled in a ring configuration described by Eq. (44) for $p=4$ (top frame and middle frame, on two different scales) and $p=12$ (lower frame). The numbers refer to the number of characteristic roots in the closed right half-plane. The quantities indicated in boldface are independent of the network topology and the number of subsystems.

are induced by the zeros of $f_1(\lambda;k,\tau)$. The first one corresponds to the function (35). By Theorem 4.4 the quantities indicated in boldface on the figure are independent of the number of agents and of the network topology.

2. Ring topology, bidirectional coupling

A ring topology with bidirectional coupling between the agents is described by the matrix

$$G=\frac{1}{2}\begin{bmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 1 & & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{p\times p}, \tag{46}$$

satisfying

$$\lambda_l(G)=\cos\left(\frac{2\pi}{p}(l-1)\right), \quad l=1,\ldots,q,$$

where $q=(p+2)/2$ if p is even and $q=(p+1)/2$ if p is odd. All eigenvalues have multiplicity 2, excepting $\lambda_1(G)=1$ and, if p is even, $\lambda_{(p+2)/2}(G)$. The corresponding eigenvectors are

$$\left[\cos\left(\frac{2\pi(l-1)\cdot(p-1)}{p}\right)\cdots\cos\left(\frac{2\pi(l-1)\cdot 1}{p}\right)1\right]^T$$

and

$$\left[\sin\left(\frac{2\pi(l-1)\cdot(p-1)}{p}\right)\cdots\sin\left(\frac{2\pi(l-1)\cdot 1}{p}\right)0\right]^T.$$

Note that if all subsystems are Lorenz systems described by Eqs. (28) and (29) then for large values of k the stability switches are only associated with the eigenvalues ± 1 and corresponding eigenvectors $[1\ 1\cdots 1]^T$ and $[1(-1)(-1)^2\cdots(-1)^{p-1}]^T$ (see Lemma 4.2). They result in either synchronized motion or standing waves. This is due to the bidirectional coupling and is in contrast to the case of unidirectional coupling addressed above, where traveling wave solutions naturally appear.

For $p=4$ the stability regions in the (k,τ) parameter space are shown in Fig. 2. Note from the lower pane that the number of characteristic roots in the right half-plane changes from 2 to 6 when crossing the horizontal curve. This is due to the double eigenvalue of G at zero.

3. Cross topology

The topology induced by the matrix

$$G=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{47}$$

is displayed in Fig. 3.

The eigenvalues and corresponding eigenvectors of G are

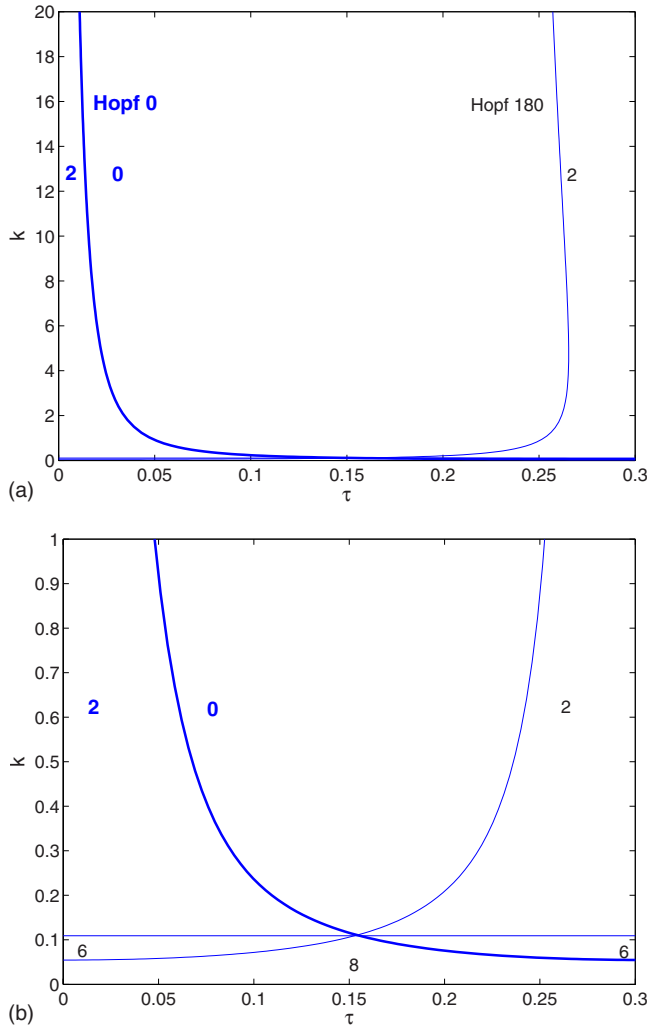

FIG. 2. (Color online) Stability regions of the synchronized equilibrium (30) of Lorenz systems (28) and (29) coupled in a ring configuration described by Eq. (46) on two different scales.

$$[\lambda_1(G)\cdots\lambda_7(G)] = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & -1 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

and

$$[E_1\cdots E_7] = \begin{bmatrix} 1 & 2 & 2 & 1 & \sqrt{2} & \sqrt{2} & 0 \\ 1 & 1 & -1 & -1 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & -1 & 1 & 0 \\ 1 & 2 & 2 & 1 & \sqrt{2} & -\sqrt{2} & 0 \\ 1 & -2 & 2 & -1 & 0 & 0 & -1 \\ 1 & -2 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

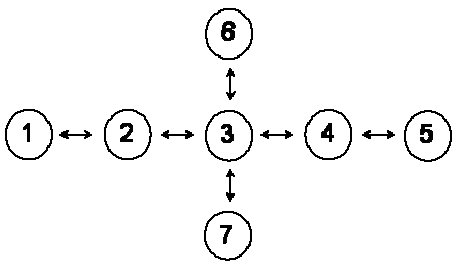

FIG. 3. Network topology with adjacency matrix (47).

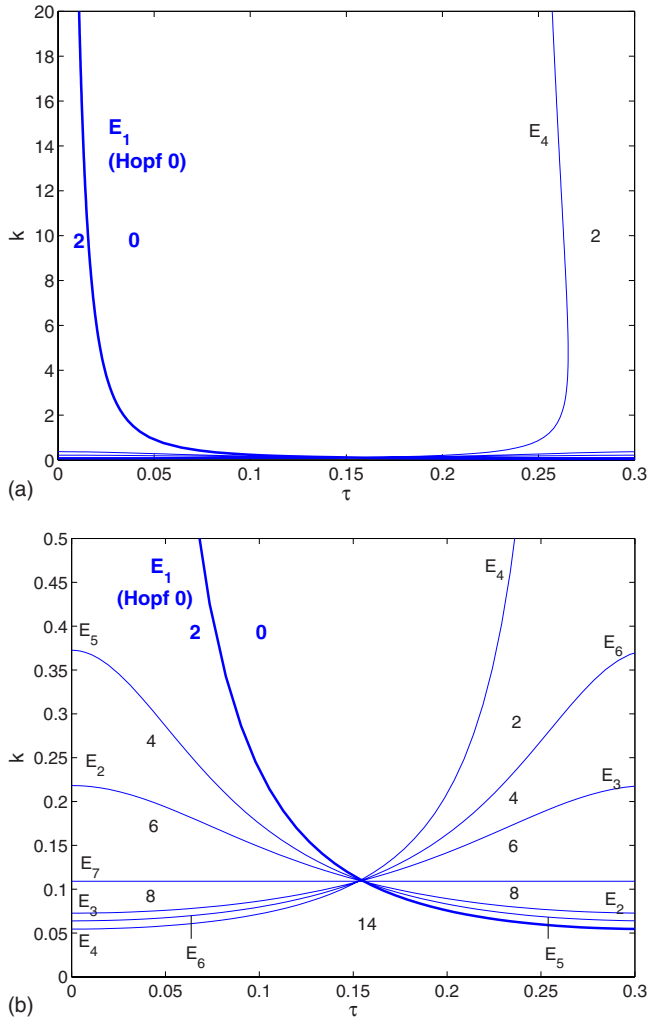

FIG. 4. (Color online) Stability regions in the (k,τ) parameter space of the synchronized equilibrium of coupled Lorenz systems on two different scales. The network topology is described by Eq. (47).

With all agents described by Eqs. (28) and (29) we display in Fig. 4 the stability regions in the (k,τ) parameter space of the synchronized equilibria (31). The type of Hopf bifurcation is displayed by indicating the corresponding eigenvector of G. Following from Lemma 4.2 only the Hopf curves corresponding to eigenvalues on the unit circle of G persist for large k, namely, $\lambda_1(G)=1$ and $\lambda_4(G)=-1$. The bold curve once again corresponds to the function (35).

C. Beyond the linearized stability analysis

The exact linearized stability analysis in the previous paragraphs illustrates that for large values of the coupling gain and small values of the delay-coupled Lorenz systems exhibit some generic behavior independent of the network topology. In what follows the results are strengthened by a nonlinear stability analysis.

We reconsider Theorem 4.4 and make some observations. The presence of the synchronization preserving Hopf bifurcation at $\tau=\tau^*(k)$, the fact that for large values of k the functions $f_2(\lambda;k,\tau),\ldots,f_p(\lambda;k,\tau)$, that describe the synchronization error around the synchronized equilibrium, have all zeros in the open left half-plane for all $\tau\in[0,\tau^*(k)]$, and

the observed synchronized behavior in our experiments for all $\tau \in [0, \tau^*(k)]$ suggest that asymptotic synchronization can also be achieved for all $\tau \in [0, \tau^*(k))$, though the dynamics on the synchronization manifold are no longer characterized by the presence of stable equilibria. Furthermore, the asymptotic behavior of the curve $k \mapsto \tau^*(k)$, described by Eqs. (36) and (37), suggests that the natural parameters in the analysis are rather the gain parameter k and the normalized delay parameter $k\tau$. These observations do hold and are apparent from the following theorem.

Theorem 4.5: *Consider a network of coupled Lorenz systems (28) with parameters (29) and coupling (3). Assume that the network satisfies Assumptions 2.1 and 2.2. Let*

$$y = (y_1, \ldots, y_p)$$

and define the functions

$$V_s(y) := \sum_{i=1}^{p} \gamma_i V(y_i), \quad H_s(y) := \sum_{i=1}^{p} \gamma_i H(y_i),$$

where γ is defined as in Corollary 2.3,

$$V(y_i) := \tfrac{1}{2} y_i^T y_i$$

and

$$H(y_i) := y_{i,1}^2 + b y_{i,2}^2 + b r y_{i,2}. \tag{48}$$

The following results hold.

(1) All solutions of Eqs. (28) and (3) are bounded and converge to the set Ω, defined as

$$\Omega := \left\{ x \in \mathbb{R}^{3p} : V_s(y) < v_m \text{ and } |x_{i,1}| \le \sqrt{\frac{2v_m}{\gamma_i}}, \ i = 1, \ldots, p \right\}, \tag{49}$$

where the constant $v_m > 0$ is such that

$$V_s(y) \ge v_m \Rightarrow H_s(y) \ge 0. \tag{50}$$

(2) The set Ω is a forward invariant set of Eqs. (28) and (3).

(3) For all $C > 0$, there exists a number $\hat{k} > 0$, such that all synchronized solutions in Ω exhibit asymptotically stable error dynamics whenever $k > \hat{k}$ and $k\tau < C$.

Sketch of proof: Because the coupling affects the dynamics on the synchronization manifold for $\tau \neq 0$, proving boundedness properties of the solutions is a necessary step in the analysis (see also the discussion in Ref. 21 in this context). The first and second statements are due to a semipassivitylike property of the individual oscillators, more precisely the fact that the derivative of the function $V(y_i)$ along the solutions of Eq. (28) satisfies

$$\dot{V} = -H(y_i) + y_i^T u_i,$$

with $H(y_i) > 0$ for large values of $\|y_i\|$. The proofs rely on a composed Lyapunov–Krasovskii functional and a Lyapunov–Razumikhin function for the output y, where, inspired by Ref. 5 the components are weighted by the left eigenvector γ of the adjacency matrix G. The third statement of the theorem follows from the uniform stability of the null solution of Eq. (12) for large k when x_1 is confined to a compact set. This is proved using techniques from $\mathcal{L}_2$ gain analysis,[19] where $x_1(t)$ in Eq. (12) is interpreted as a time-varying perturbation. For a detailed proof we refer to the Appendix. □

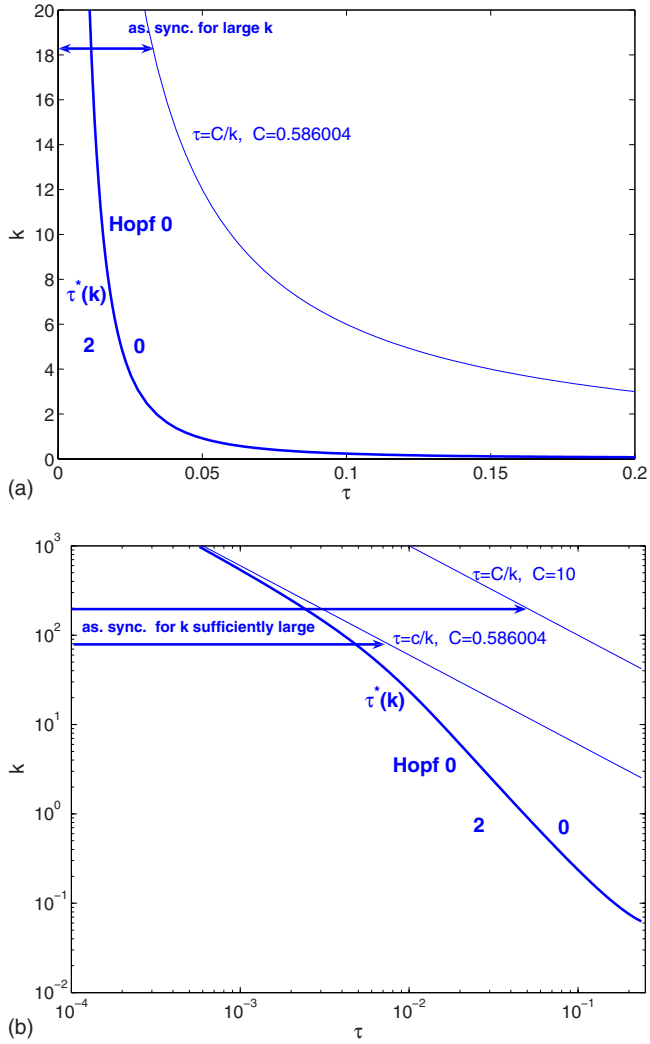

FIG. 5. (Color online) Graphical illustration of the results of Theorems 4.4 and 4.5 for large values of the coupling gain k on a linear scale (left) and on a logarithmic scale (right). The quantities indicated in boldface are independent of the number of agents and network topology.

The main results of the section, Theorems 4.4 and 4.5, are graphically displayed in Fig. 5. Whereas Theorem 4.5 only makes assertions about preservation of synchronized behavior in the emanating solutions in the Hopf 0 bifurcation of the synchronized equilibrium, Theorem 4.5 states that for an arbitrary value of C asymptotic synchronization can be achieved for all $\tau \in [0, C/k)$. If k is chosen such that asymptotic synchronization is guaranteed for $\tau \in [0, \tau^*(k) + \epsilon]$ with $\epsilon > 0$ some small number and if the system is initialized close to the synchronized equilibrium and the delay parameter slowly swept from $\tau^*(k) + \epsilon$ to zero, then the attractor of the solution evolves from the stable synchronized equilibrium to synchronized chaotic behavior for $\tau = 0$ because the synchronization between the agents is maintained throughout *every* bifurcation. Recall that the dynamics on the

synchronization manifold are described by Eq. (8), which reduces to $\dot{x}_1 = f(x_1)$ for $\tau = 0$.

V. CONCLUSIONS

We studied the synchronization of coupled nonlinear oscillators with delay in the coupling, Eqs. (1) and (3), with the emphasis on coupled Lorenz systems. First, the state transformation to Eq. (5) led us to necessary conditions on the network topology for the existence of synchronized solutions. Next we performed a stability analysis of synchronized equilibria in a (gain, delay) parameter space. Instrumental to this study we employed a factorization of the characteristic equation, which separates the nominal behavior and the synchronization error dynamics, and we revealed the precise role of the eigenvalues and the eigenvectors of the adjacency matrix of the graph on the behavior of the solutions. The latter allowed us to classify the modes of the system, as well as the Hopf bifurcation curves and the emerging behavior on the onset of instability. As a result of this analysis for the case of coupled Lorenz systems we proved that for sufficiently large gain values, there always exists a stability interval in the delay parameter space that does not contain the zero delay value. Furthermore, this behavior is generic because both the critical delay value, $\tau^*(k)$, and the type of corresponding bifurcation (a synchronization preserving Hopf bifurcation in the sense that if the delay is reduced beyond the critical value the equilibrium becomes unstable without losing the synchronization between the agents) do not depend on the network topology and the number of agents. Finally, these results were complemented with a nonlinear stability analysis, which among others showed that by choosing the gain parameter sufficiently large asymptotic synchronization can actually be achieved over any finite interval in the normalized delay $k\tau$, again independently of the network.

Instead of directly deriving synchronization conditions for the nonlinear system (1) and (3) the methodology of the paper was based on considering first the linearized stability problem around a synchronized equilibrium, which can be exactly solved. Such an approach directly leads to insights in the problem, because not only the stability regions in the (k, τ) parameter space can be characterized but also the solutions on the onset of instability by considering the structure of the eigenspaces in the bifurcations. In addition, the gained qualitative insights and observations may lead to a better targeted and less conservative nonlinear stability analysis. This was illustrated in this paper with coupled Lorenz systems. Indeed, the formulation and proof of Theorem 4.5 were based on the following properties suggested by the linear stability analysis: (i) behavior independent of the network for large coupling gains and small delays, (ii) natural parameters $(k, k\tau)$ rather than (k, τ), and (iii) instead of analyzing stability of the full error dynamics Eq. (9) directly, it is preferred to analyze the decoupled systems (12), where the magnitudes of the eigenvalues of the adjacency matrix suggest the natural type of criterion to be used.

It should be noted that the results of the article and, in particular, Algorithm 3.5 can be directly extended to the case where self-feedback is also considered, though the qualitative results described in Sec. IV will be different. If the agents are not completely identical, then in general (perfectly) synchronized solutions do not exist. This can be seen from Eq. (5) where terms related to the deviations would appear in the right-hand side. Though the analysis in the paper has been performed step by step using a particular decomposition or factorization, holding for identical agents and uniform delays only, the final results for the coupled system (presence of a synchronized steady state solution, its stability regions and Hopf bifurcation curves in the (k, τ) plane, the structure of the eigenfunctions corresponding to the Hopf bifurcations) will be slightly perturbed only if the differences between the agents and the delay parameters are sufficiently small. This means that Theorem 4.4 remains approximately valid in the sense that for large k and particular values of τ there exists an almost synchronized equilibrium, which is stable but loses stability beyond $\tau \approx \tau^*$ while maintaining the solutions close to being synchronized. This indicates that for τ sufficiently small, the synchronization error dynamics exhibits an attractor whose size can be made arbitrarily small by reducing the difference between the agents. The effect of a small time variation of delays and other system parameters around a nominal value can be taken into account using the ideas of Ref. 13, where the time-varying parameters are essentially treated as perturbations of time-invariant parameters. However, to analyze the effect of large variations or the effects of a time-varying network topology, time-domain methods become necessary at the cost of introducing conservatism in the analysis.

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APPENDIX: PROOF OF THEOREM 4.5

We split the proof in two parts.

1. Part I: Boundedness properties (1) and (2)

From the first equation of Eq. (28),

$$x_{i,1}(t) = -\sigma x_{i,1}(t) + \sigma x_{i,2}(t),$$

it follows that $x_{i,1}$ can be interpreted as the result of a first order low-pass filter applied to $x_{i,2}$. Therefore it is sufficient to show that the outputs $y=(y_1, \ldots, y_p)$ converge to the set

$$\Omega_r = \{y \in \mathbb{R}^{2p} : V_s(y) < v_m\}$$

and to show that this set is a forward invariant set.

The derivative of the function V_s along the solutions of Eqs. (28) and (3) satisfies

$$\dot{V}_s(t) = -\sum_{i=1}^{p} \gamma_i H(y_i(t)) + \beta(t),$$

where

$$\begin{aligned}
\beta(t) &= k\sum_{i=1}^{p} \gamma_i y_i(t)^T \left(\sum_{l=1}^{p} \alpha_{i,l}(y_l(t-\tau) - y_i(t)) \right) \\
&= -k\sum_{i=1}^{p} \gamma_i y_i(t)^T y_i(t) + k\sum_{i=1}^{p} \gamma_i \sum_{\nu=1}^{2} \sum_{l=1}^{p} \alpha_{i,l} y_{i,\nu}(t) y_{l,\nu}(t-\tau) \\
&\le -\frac{k}{2}\sum_{i=1}^{p} \gamma_i y_i(t)^T y_i(t) + \frac{k}{2}\sum_{i=1}^{p} \gamma_i \sum_{\nu=1}^{2} \sum_{l=1}^{p} \alpha_{i,l} y_{l,\nu}(t-\tau)^2 \\
&= -\frac{k}{2}\sum_{i=1}^{p} \gamma_i y_i^T(t) y_i(t) + \frac{k}{2}\sum_{\nu=1}^{2} \gamma^T G \begin{bmatrix} y_{1,\nu}^2(t-\tau) \\ \vdots \\ y_{p,\nu}^2(t-\tau) \end{bmatrix} \\
&= -\frac{k}{2}\sum_{i=1}^{p} \gamma_i y_i^T(t) y_i(t) + \frac{k}{2}\sum_{\nu=1}^{2} \gamma^T \begin{bmatrix} y_{1,\nu}(t-\tau)^2 \\ \vdots \\ y_{p,\nu}(t-\tau)^2 \end{bmatrix} \\
&= -\frac{k}{2}\sum_{i=1}^{p} \gamma_i (y_i(t)^T y_i(t) - y_i(t-\tau)^T y_i(t-\tau)) \\
&= -\frac{k}{2}(V_s(y(t)) - V_s(y(t-\tau))).
\end{aligned}$$

We conclude

$$\dot{V}_s(t) \le -\sum_{i=1}^{p} \gamma_i H(y_i(t)) + \frac{k}{2}(V_s(y(t)) - V_s(y(t-\tau))). \tag{A1}$$

Furthermore, when defining y_t as the function segment $\theta \in [t-\tau, t] \mapsto y(\theta)$, the derivative of the functional

$$W(y_t) := V_s(y(t)) + \frac{k}{2}\int_{t-\tau}^{t} V_s(y(s))ds$$

along the solutions of Eqs. (1) and (3) satisfies

$$\dot{W}(t) \le -\sum_{i=1}^{p} \gamma_i H(y_i(t)). \tag{A2}$$

First, assume that $y_0 \subset \Omega_r$. From Eqs. (50) and (A1) we have $y_t \subset \Omega_r$ for all $t \ge 0$. The argument is by contradiction: if the solution would reach $\partial\Omega_r$ for the first time at $t=\hat{t}$, then we would have $V_s(\hat{t}) = v_f$ and $\dot{V}_s(\hat{t}) \ge 0$. However, since $V_s(y(\hat{t})) - V_s(y(\hat{t}-\tau)) > 0$, Eqs. (50) and (A1) imply $\dot{V}_s(\hat{t}) < 0$. This proves that Ω_r is an invariant set.

Next, assume that $y_0 \not\subset \Omega_r$. If $y(0) \notin \Omega_r$, then Eq. (A2) implies that W is a strictly decreasing function as long as $\Sigma_{i=1}^{p} \gamma_i H(y_i(t)) > 0$. Hence, whenever $y_0 \not\subset \Omega_r$ and $y(t) \notin \Omega_r$ for all $t \ge 0$, there exists a finite time $\hat{t}_1 > 0$ such that $y(\hat{t}_1) \in \partial\Omega_r$. Let $\zeta_1 = \sup_{t \in [\hat{t}_1 - \tau, \hat{t}_1]} V_s(y(t))$. If $y(t) \notin \Omega_r$ for some $t \in [\hat{t}_1, \hat{t}_1 + \tau]$ then Eq. (A1) implies

$$\dot{V}_s(t) \le \frac{k}{2}(\zeta_1 - V_s(t)).$$

Hence,

$$\zeta_2 := \sup_{t \in [\hat{t}_1, \hat{t}_1 + \tau]} V_s(t) \le \zeta_1(1 - e^{-(k/2)\tau}) + e^{-(k/2)\tau} v_m.$$

Repeating the same argument yields

$$\begin{aligned}
\zeta_{\ell+1} &:= \sup_{t \in [\hat{t}_1 + (\ell-1)\tau, \hat{t}_1 + \ell\tau]} V_s(t) \\
&\le \zeta_\ell(1 - e^{-(k/2)\tau}) + e^{-(k/2)\tau} v_m, \quad \forall\, \ell \ge 1.
\end{aligned}$$

Consequently, we have

$$\lim_{i \to \infty} \zeta_i = v_m.$$

This shows that, whatever the initial condition $x_0 := x(\theta), \theta \in [-\tau, 0]$, the outputs y converge to the forward invariant set Ω_r.

2. Part II: Asymptotic synchronization

Choose $C > 0$. According to the decomposition (12) we have to show that the $p-1$ systems

$$\begin{aligned}
\dot{\xi}_{i,1} &= \sigma(\xi_{i,2} - \xi_{i,1}), \\
\dot{\xi}_{i,2} &= r\xi_{i,1} - \xi_{i,2} - x_{1,3}(t)\xi_{i,1} - x_{1,1}(t)\xi_{i,3} - k\xi_{i,2} \\
&\quad + k\lambda_i(G)\xi_{i,2}(t-\tau), \\
\dot{\xi}_{i,3} &= -b\xi_{i,3} + x_{1,2}(t)\xi_{i,1} + x_{1,1}(t)\xi_{i,2} - k\xi_{i,3} \\
&\quad + k\lambda_i(G)\xi_{i,3}(t-\tau), \quad i = 2, \ldots, p,
\end{aligned} \tag{A3}$$

are asymptotically stable for $k\tau < C$ and k sufficiently large. When applying the transformation of time

$$t^{(\text{new})} = kt^{(\text{old})},$$

Eq. (A3) can be written as

$$\dot{\xi}_i(t) = A_0(k)\xi_i(t) + A_1\xi_i(t-\nu) + B_0\Delta(t,k)\xi_i(t), \tag{A4}$$

where

$$A_0(k) = \begin{bmatrix} -\frac{1}{k}\sigma & \frac{1}{k}\sigma & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_i(G) & 0 \\ 0 & 0 & \lambda_i(G) \end{bmatrix},$$

$$B_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \Delta(t,k) = \frac{1}{k}\begin{bmatrix} r - x_{1,3}(t) & -1 & -x_{1,1}(t) \\ x_{1,2}(t) & x_{1,1}(t) & -b \end{bmatrix},$$

$$\nu = k\tau.$$

Note that Eq. (A4) can be seen as an equation with two independent parameters: the gain k and the scaled delay $\nu = k\tau$.

In what follows we take a perturbation point of view and consider $\Delta(t,k)$ as a time-varying, complex uncertainty. Furthermore, we interpret Eq. (A4) as the feedback interconnection of the nominal system

$$\dot{\xi}_i(t) = A_0(k)\xi_i(t) + A_1\xi_i(t-\nu) + B_0 w(t), \tag{A5}$$

where the feedback is closed with

$$w(t) = \Delta(t,k)\xi_i(t).$$

The nominal system with $w \equiv 0$ is asymptotically stable for all values of ν. Hence, by arguments of $\mathcal{L}_2$ gain analysis (see, e.g., Ref. 19), the system (A4) is uniformly asymptotically stable if the product of the induced $\mathcal{L}_2$ gains in the feedback loop is smaller than 1, that is,

$$\max_{t\geq 0} \|\Delta(t,k)\|_2 \|G(j\omega;k,\nu)\|_{\mathcal{H}_\infty} < 1, \tag{A6}$$

where

$$G(j\omega;k,\nu) := (j\omega I - A_0(k) - A_1 e^{-j\omega\nu})^{-1} B_0$$

is the transfer function of Eq. (A5) from w to ξ_i. Here $\|\cdot\|_{\mathcal{H}_\infty}$ denotes the H-infinity norm (see Ref. 22) for the definition and an introduction on H-infinity control theory). A simple calculation yields

$$\lim_{k\to\infty} \|G(j\omega;k,\nu)\|_{\mathcal{H}_\infty} = \sup_{\omega\geq 0} \frac{1}{|j\omega + 1 - \lambda_i(G)e^{-j\omega\nu}|},$$

which is bounded for all values of ν. It follows that

$$\lim_{k\to\infty} \sup_{\nu\in[0,C]} \|G(j\omega;k,\nu)\|_{\mathcal{H}_\infty} < \infty.$$

In the light of this result we reconsider the stability condition (A6). Since $x_1(t)$ is confined to the compact set Ω for all $t \geq 0$, we have

$$\lim_{k\to\infty} \max_{t\geq 0} \|\Delta(t,k)\|_2 = 0.$$

Hence, there exists a threshold $\hat{k}$ such that for $k > \hat{k}$ and all $\nu \in (0, C)$ the stability condition (A6) is satisfied. The third assertion of Theorem 4.5 follows.

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