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July 28, 2003

Abstract

We analyse an argument of Deutsch, which purports to show that the deterministic part of classical quantum theory together with deterministic axioms of classical decision theory, together imply that a rational decision maker behaves as if the probabilistic part of quantum theory (Born's law) is true. We uncover two missing assumptions in the argument, and show that the argument also works for an instrumentalist who is prepared to accept that the outcome of a quantum measurement is random in the frequentist sense: Born's law is a consequence of functional and unitary invariance principles belonging to the deterministic part of quantum mechanics. Unfortunately, it turns out that after the necessary corrections we have done no more than give an easier proof of Gleason's theorem under stronger assumptions. However, for some special cases the proof method gives positive results while using *different* assumptions to Gleason. This leads to the conjecture that the proof could be improved to give the same conclusion as Gleason under unitary invariance together with a much *weaker* functional invariance condition.

The first draft of this paper dates back to early 1999, was posted on my web page, but never completed. It has since been partly overtaken by Barnum et al. (2000), Saunders(2002), and Wallace (2002). However there remain new points of view, new results, and most importantly, a still open conjecture.

Mathematics Subject Classification: 60B99, 62P35, 81P10, 81P15.

http://arxiv.org/abs/quant-ph/0307188

1 Introduction: are quantum probabilities fixed by quantum determinism?

Quantum mechanics has two components: a deterministic component, concerned with the time evolution of an isolated quantum system; and a stochastic component, concerned with the random jump which the state of that system makes when it comes into interaction with the outside world, sending at the same time a piece of random information into the outside world. The perceived conflict between these two behaviours is ‘the measurement problem’ as exemplified by Schrödinger’s cat.

Here we do not resolve this problem but just address the peaceful co-existence, or possibly even the harmony, between the two behaviours. We will show that a kernel of classical deterministic quantum mechanical assumptions, together with the admission that randomness is inescapable when observables are measured on a state which is not an eigenstate, makes the precise value of those probabilities inescapable too—harmony indeed. More specifically, two generally accepted invariance properties of observables and quantum systems determine the shape of the probability distribution of measured values of an observable—namely, the shape specified by Born’s law. The invariance properties are connected to unitary evolution of a quantum system, and to functional transformation of an observable, respectively.

This work was inspired by Deutsch (1999). There it is claimed that a still smaller kernel of deterministic classical quantum theory together with a small part of deterministic decision theory together force a rational decision maker to behave as if the probabilistic predictions of quantum theory are true. In our opinion there are three problems with the paper. The first is methodological: we do not accept that the behaviour of a rational decision maker should play a role in modelling physical systems. We are on the other hand happy to accept a stochastic component (with a frequentist interpretation) in physics, so we translate Deutsch’s axioms and conclusions about the behaviour of a rational decision maker into axioms and conclusions about the relative frequency with which various outcomes of a physical experiment take place. The second problem is that it appears that Deutsch has implicitly made use of a further axiom of unitary invariance alongside his truely minimalistic collection, and needs to greatly strengthen one of the existing assumptions concerning functional invariance, from one-to-one functions also to many-to-one functions. Neither addition nor strengthening is controversial from a classical deterministic quantum physics point of view, but both are very substantial from a mathematical point of view. The third problem is that the strengthening of the functional invariance assumption puts us in the position that we have assumed enough to apply Gleason's (1957) theorem. Thus at best, Deutsch's proof is an easy proof of Gleason's theorem using an extra, heavy, assumption of unitary invariance. The fact that Deutsch's proof is incomplete has been observed by Barnum et al. (2000). However these authors did not attempt to reconstruct a correct proof. In the concluding section we relate our work to theirs. Wallace (2002) has also studied Deutsch's claims in depth. I did not yet attempt to relate his work to mine. The same goes for Saunders (2002).

The paper is organised as follows. In Section 2 we put forward functional and unitary invariance assumptions, which are usually considered *consequences* of traditional quantum mechanics, but are here to be taken as *axioms* from which some of the traditional ingredients are to be derived, turning the tables so to speak. One would like to make the axioms as modest as possible, while still obtaining the same conclusions. Hence it is important to distinguish between different variants of the assumptions. In particular, we distinguish between (stronger) assumptions about the complete probability law of outcomes of measurements of observables, and (weaker) assumptions about the mean values of those probability laws. An invariance assumption concerning a class of functions, is weaker, if it only demands invariance for a smaller class of functions, and in particular we distinguish between invariance for all functions, including many-to-one functions, and invariance just for one-to-one functions.

In Section 3 we prove the required result, Born's law, for a special state (equal weight superposition of two eigenstates). This case is the central part of Deutsch (1999), who only sketches the generalization to arbitrary states. And already, it seems an impressive result. We prove the result, *for this special state*, in two forms—in law, and in mean value—the former being stronger of course; using appropriate variants of our assumptions. Deutsch's proof is incomplete, since he only appeals to unitary invariance, while it is clear that a functional invariance assumption is also required.

The strengthening of the functional invariance assumption can also be used to derive probabilities as well as mean values, and it is moreover useful from Deutsch's point of view of rational behaviour, if one wants to extend in a very natural way the class of games being played. Roughly speaking, we extend from the game of buying a lottery ticket to a game at the roulette table. In the former game the only question is, how much is one ticket worth. In the latter game one may make different kinds of bets, and the question is how much is any bet worth.

However, so far we have only been concerned with a rather special state: an equal weight superposition of two eigenstates. As mentioned before Deutsch only sketches the extension to the general case of an arbitrary, possibly mixed, state. He outlined a step-by-step argument of successive generalizations. In Section 4 we follow the same sequence of steps, strengthening the assumptions as seems to be needed.

In Section 5, we look back at the various versions of our assumptions, in the light of what can be got from them. We also evaluate the overall result of completing Deutsch's programme. From a mathematical point of view, it turns out that we have done no more, at the end of the day, than derive the same conclusion as that of Gleason's theorem, while making stronger assumptions. The payoff has just been a much easier proof. Gleason's theorem only assumes functional invariance, we have assumed unitary invariance too. We argue that unitary invariance corresponds to a natural physical intuition, while functional invariance is something which one could not have expected in advance. It *is* supported by experiment, and *is* theoretically supported in special cases (measurements of components of product systems) by locality.

We conclude with the conjecture that unitary invariance together with a weakened functional invariance assumption is sufficient to obtain the same conclusion.

2 Assumptions: degeneracy, functional invariance, unitary invariance

Recall that a quantum system in a pure state is described or represented by a unit vector $\big|\psi\big\rangle$ in a Hilbert space, which I take to be infinite-dimensional, and that an observable or physical quantity is described or represented by a self-adjoint (perhaps unbounded) operator X on that space. I shall assume that X has a discrete and nondegenerate spectrum; thus there is a countably infinite collection of real eigenvalues x and eigenstates $\big|X\!=\!x\big\rangle$, so that one can write $X = \sum_x x\big|X\!=\!x\big\rangle\big\langle X\!=\!x\big|$, while $\big|\psi\big\rangle = \sum \lambda_x\big|X\!=\!x\big\rangle$ where $\lambda_x = \big\langle X\!=\!x\big|\psi\big\rangle$. Throughout the paper we make the following background assumption:

Assumption 0. *Random outcome, in spectrum.* The outcome of measuring X is one of its eigenvalues x. Which one, is random. Its probability distribution (law) depends on X and on $\big|\psi\big\rangle$.

I write $\mathrm{meas}_\psi(X)$ for the random outcome of measuring observable X on state $\big|\psi\big\rangle$, and $\mathrm{law}\big(\mathrm{meas}_\psi(X)\big)$ for its probability distribution, i.e., the collection of probabilities $\Pr\big\{\mathrm{meas}_\psi(X) \in B\big\}$ for all Borel sets B of the real line. Deutsch's paper has the more modest aim just to compute the mean value of this probability law, $\mathrm{E}\big(\mathrm{meas}_\psi(X)\big)$, though as I shall argue before, even under his own terms (computing values of betting games) the whole probability law is of interest.

Throughout the paper I will be playing with three main assumptions, though sometimes in stronger and sometimes in weaker forms. Here are the three, in their strongest versions:

Assumption 1. *Degeneracy in eigenstates.*

$$\Pr\bigl\{\text{meas}_{|X=x\rangle}(X) \;=\; x\bigr\} \;=\; 1. \tag{1}$$

In an eigenstate of an observable, the corresponding eigenvalue is the certain outcome of measurement.

Assumption 2. *Functional invariance.*

$$\Pr\bigl\{f(\text{meas}_\psi(X)) \;=\; y\bigr\} \;=\; \Pr\bigl\{\text{meas}_\psi(f(X)) \;=\; y\bigr\}. \tag{2}$$

Measuring a function f of an observable is operationally indistinguishable from measuring the observable, and then taking the same function of the outcome. Parenthetically remark that this indistinguishability is only as far as the outcome is concerned; as far as the new state of the quantum system is concerned there will be a difference, if the function is many-to-one. Parts of Deutsch's proof only need this assumption for one-to-one functions. In fact he only explicitly used this assumption for the affine functions $f(x) = ax+b$, but implicitly other functions, including many-to-one functions, are involved too.

Assumption 3. *Unitary invariance.*

$$\Pr\bigl\{\text{meas}_{U\psi}(X) \;=\; x\bigr\} \;=\; \Pr\bigl\{\text{meas}_\psi(U^*XU) \;=\; x\bigr\}. \tag{3}$$

We will see that at first instance, we only require this assumption to hold for a special class of unitary operations U, namely those which permute eigenstates of X. There is then a one-to-one correspondence u on the eigenvalues of X with inverse u^* such that $UXU^* = u(X)$, $U^*XU = u^*(X)$, and $U\bigl|X=x\bigr\rangle = \bigl|X=u(x)\bigr\rangle$. In the special case that ψ is an eigenstate $\bigl|X=x\bigr\rangle$, Assumption 3 follows from Assumption 1 (degeneracy-in-eigenstates). Later we also need Assumption 3 for unitary operations, diagonal in the basis corresponding to X.

Since in the above assumptions, x and y are arbitrary, one could also restate the three main assumptions as:

$$\text{law}\bigl(\text{meas}_{|X=x\rangle}(X)\bigr) \;=\; \text{law}\bigl(x\bigr), \tag{1$'$}$$

$$\text{law}\bigl(f(\text{meas}_\psi(X))\bigr) \;=\; \text{law}\bigl(\text{meas}_\psi(f(X))\bigr), \tag{2$'$}$$

$$\text{law}\bigl(\text{meas}_{U\psi}(X)\bigr) \;=\; \text{law}\bigl(\text{meas}_\psi(U^*XU)\bigr), \tag{3$'$}$$

where law denotes the probability law of the random variable in question, so that in particular law(x) denotes the probability distribution degenerate at the point x. An apparently weaker still set of assumptions would only restrict the mean values of the distributions in assumptions 2 and 3:

$$\begin{aligned} \mathrm{E}\big(f(\mathrm{meas}_\psi(X))\big) &= \mathrm{E}\big(\mathrm{meas}_\psi(f(X))\big), & (2'') \\ \mathrm{E}\big(\mathrm{meas}_{U\psi}(X)\big) &= \mathrm{E}\big(\mathrm{meas}_\psi(U^*XU)\big). & (3'') \end{aligned}$$

As mentioned above, one can weaken the assumptions by restricting the class of functions f or unitaries U for which the relevant equalities are supposed to hold.

3 The first part of the proof

I return to a discussion of the assumptions after an outline of the proof of my main result:

$$\Pr\big\{\mathrm{meas}_\psi(X) \;=\; x\big\} \;=\; \big|\big\langle\psi\big|X\!=\!x\big\rangle\big|^2. \tag{4}$$

I will make use of Assumptions 1–3 in their original form, postponing discussion of how one might reach the same conclusion from weaker versions of the assumptions. In this section, following Deutsch, I only prove the result in the special case (a)

$$\big|\psi\big\rangle = \tfrac{1}{\sqrt{2}}\big(\,\big|X\!=\!x_1\big\rangle + \big|X\!=\!x_2\big\rangle\,\big), \tag{5}$$

for which I am going to obtain the probabilities 1/2 for $x = x_1$ and $x = x_2$, and zero for all other possibilities. After this, Deutsch attempts to generalize, first (b) to equal weight superpositions of a binary power of eigenstates of X, next (c) to an arbitrary number, then (d) to dyadic rational superpositions, next (e) to arbitrary real superpositions, and finally (f) to arbitrary superpositions. The proofs he gives of these steps are similarly incomplete. I will complete the proof by an alternative and rather short route in the next section, but return to Deutsch's completion in the section after that.

Suppose u, a one-to-one correspondence on the eigenvalues of X, maps x_1 to x_2 and vice-versa, and, after we have labelled the other eigenvales as x'_n, $n \in \mathbb{Z}$, maps x'_n to x'_{n+1}. Let U denote the unitary which performs the same permutation of the eigenvectors. Let u^* denote the inverse of u. Exploiting the relationship between u and U, and their relationship to X and ψ, as well as our other assumptions, we find

$$\begin{aligned}
\Pr\big\{\text{meas}_\psi(X) \;=\; x_1\big\} \;&=\; \Pr\big\{\text{meas}_{U\psi}(X) \;=\; x_1\big\} \\
&=\; \Pr\big\{\text{meas}_\psi(U^*XU) \;=\; x_1\big\} \\
&=\; \Pr\big\{\text{meas}_\psi(u(X)) \;=\; x_1\big\} \\
&=\; \Pr\big\{u(\text{meas}_\psi(X)) \;=\; x_1\big\} \\
&=\; \Pr\big\{\text{meas}_\psi(X) \;=\; u^*(x_1)\big\} \\
&=\; \Pr\big\{\text{meas}_\psi(X) \;=\; x_2\big\} \qquad (6)
\end{aligned}$$

Replacing x_1 by an eigenvalue x'_n, i.e., any other than x_1 or x_2, and running through the same derivation, we see that all other eigenvalues have equal probabilities. Since there are an infinite number of them, and since according to our background assumption the outcome of measuring X lies in its spectrum, we have obtained the required result: the probabilities of x_1 and x_2 must both equal $1/2$, all the other eigenvalues x'_n must get zero probability.

We used Assumptions 2 and 3 (functional and unitary invariance), not Assumption 1 (degeneracy in an eigenstate). However, this assumption is needed to deal with the case of ... an eigenstate. The proof method allows us to deal with an equal weight superposition of any positive finite number of eigenstates of X. We only used functional invariance for one-to-one functions.

Deutsch was only interested in mean values of the probability distributions of outcomes, since the fair value of the game: measure X on $|\psi\rangle$ and receive the value of the outcome in euro's (€), is precisely € $\mathrm{E}\big(\text{meas}_\psi(X)\big)$. (Here we are assuming that the utility of having some number of euro's is equal to that number. The reader may replace euro's by dollars, camels, or whatever else he or she prefers). In a moment I will also add a new game to the discussion: measure X on $|\psi\rangle$ and receive € 1 if the outcome x_0 is found. The value of this game should be € $|\langle x_0|\psi\rangle|^2$.

Let us assume that the spectrum of X consists of *all* the integers (negative and non-negative). Then for given x_1 and x_2 there is an affine map $u(x) = ax + b = x_1 + x_2 - x$ which defines a unitary transformation U as above. For these U, X and the same ψ as before we rewrite the argument before as

$$\begin{aligned}
\mathrm{E}\big(\text{meas}_\psi(X)\big) \;&=\; \mathrm{E}\big(\text{meas}_{U\psi}(X)\big) \\
&=\; \mathrm{E}\big(\text{meas}_\psi(U^*XU)\big) \\
&=\; \mathrm{E}\big(\text{meas}_\psi(u(X))\big) \\
&=\; \mathrm{E}\big(u(\text{meas}_\psi(X))\big) \\
&=\; x_1 + x_2 - \mathrm{E}\big(\text{meas}_\psi(X)\big) \qquad (7)
\end{aligned}$$

yielding the required

$$\mathrm{E}\big(\mathrm{meas}_\psi(X)\big) \;=\; \tfrac{1}{2}(x_1 + x_2). \tag{8}$$

Deutsch's proof was a cryptic version of the argument I have just given, except that he did not mention the unitary invariance assumption. He writes v for value, instead of E. In my opinion, without the extra (unitary invariance) assumption, his proof fails. The degeneracy Assumption 1 is not used at this stage. However one may note that Assumption 1 (degeneracy) implies that Assumption 3 (unitary invariance) holds when the state $|\psi\rangle$ is an eigenstate of the observable X. One could therefore consider Assumption 3 as a natural interpolation from Assumption 1. I return to this later.

As has been shown by de Finetti and by Savage, a rational decision maker who must make choices when outcomes are 'indeterminate' (I must avoid all terminology suggestive of probability theory, since the words 'random', 'probability' and so on, are not allowed to be in our vocabulary) behaves as if he (or she) has a prior probability distribution and indeed updates it according to Bayes' law when new information (outcomes) becomes available. Thus it seems to me that whether one starts with utilities and assumes rationality, or with probability and the frequency interpretation, is very much a matter of taste. In my opinion the latter is closer to physical experience and indeed we know that casinos and insurance companies make good money from the frequency interpretation of chance.

I consider the many repetitions in the frequency interpretation to be no more and no less than a thought experiment. When one claims that the probability of some event is some number, one is asserting that the situation in question is indistinguishable from a certain roulette game or lottery. This allows me also to talk about probabilities of outcomes of once-off experiments. For instance, a certain physical experiment might have some chance of producing a black hole which would swallow the whole universe. The probability that this would indeed happen, if the devilish experiment were actually carried out, would be computed by doing real physics in which one would imaginarily set the chain of events into motion, many many times, in which uncontrolled initial conditions would vary in all kinds of ways from repetition to repetition. How they would vary, and what possibilities could be considered equally likely, should be a matter of scientific discussion. This may appear circular reasoning or an infinite regress or just plain subjectivism, but this does not bother me: it works, and it is not subjective, since we may rationally discuss the probability modelling. When I use the mathematical model of probability I am only claiming an analogy with something familiar, like a casino, lottery, or coin toss. I think that it is the same in the rest of physics, when we talk about mass, electric charge, or magnetic field: we might think or we might hope that we are talking about real things in the real world but we can only be certain that we are talking about ingredients of mathematical models which are anchored to the real world by analogies with familiar down to earth daily experience. My frequentistic position is perhaps better labelled "Laplacian counterfactual frequentism" and though one might collapse this label to "subjectivism", I believe it is as instrumentalistic or as operationalistic as anything else in physics.

4 Completing the proof

More can be got out of the functional invariance assumption, by considering other functions f, and most crucially, certain many-to-one functions. In my opinion we must do this anyway, in order to complete the proof on the lines indicated by Deutsch (see next section). It is an open question, whether we can do without.

With the choice $f = \mathbb{1}_{\{x\}}$, and writing $[X = x]$ for the projector onto the eigenspace of X corresponding to eigenvalue x (and later also for the eigenspace itself), since $\mathbb{1}_{\{x\}}(X) = [X = x]$, we read off:

$$\Pr\bigl\{\mathrm{meas}_\psi(X) \;=\; x\bigr\} \;=\; \Pr\bigl\{\mathrm{meas}_\psi([X = x]) \;=\; 1\bigr\}. \tag{9}$$

Indeed, if we only assume the mean value form of the functional invariance assumption, we can read off the same conclusion, since the random variables $\mathbb{1}_{\{x\}}(\mathrm{meas}_\psi(X))$ and $\mathrm{meas}_\psi([X = x])$ are both zero-one valued.

Till this point we had dealt with nondegenerate observables and equal weight superpositions of eigenstates. Now we can add to this, also degenerate observables (since these can always be written as functions of nondegenerate observables). Moreover, even if we start with the assumptions in their weaker mean value form, we can still obtain the stronger conclusion about the whole probability law of the outcome.

In fact, with brute force we arrive now very quickly at the most general result (it remains, namely, to consider arbitrary states). From functional invariance (whether in terms of probability laws or whether in terms of their mean values) we have shown that a probability can be assigned to each closed subspace of our Hilbert space, countably additive over orthogonal subspaces, and equal to 1 on the whole space. Now we can invoke Gleason's theorem to conclude that the probability of any subspace is of the form $\mathrm{tr}\{\rho A\}$ for some density matrix ρ. It remains to show that $\rho = \bigl|\psi\bigr\rangle\bigl\langle\psi\bigr|$ but this follows from our first axiom that measuring an observable on an eigenstate yields with certainty the corresponding eigenvalue: consider the observable $X = \bigl|\psi\bigr\rangle\bigl\langle\psi\bigr|$

itself, and subspace $A = [\psi]$ (the one-dimensional subspace generated by $|\psi\rangle$)!

Deutsch's extension of his results to the most general case (see next section) is very hard to follow. He repeatedly invokes substitutability, whereby an outcome of one game may be replaced by a new game of the same value. He does not say *which* substitutions are being made. However he is clearly thinking of substitutions, leading to composite games with composite quantum systems, product states, and observables on each subsystem. During these constructions and substitutions, the observables being measured and the states on which they are being measured, keep changing, while the Spartan notation $v(x)$ in which the symbol x refers to an observable, an eigenvalue, and an eigenstate simultaneously, begs confusion. The mere construction of product systems implies that more is being assumed above the structure so far (so far we only spoke of observables and states on one fixed quantum system). As I will indicate below, it appears that the extra assumption of unitary invariance and the strengthened functional invariance assumption involving many-to-one functions as well as one-to-one functions, together with a natural assumption about measuring separate observables on a product system in a product state, enable one to fill the gaps. If the repair job is not too difficult, one finishes with a relatively easy proof of Gleason's theorem, under the supplementary condition of unitary invariance.

The construction of product systems will also help us extend results from infinite-dimensional quantum systems to finite dimensional, including 2-dimensional—the case not covered by Gleason.

Functional invariance assumptions on product systems, or more generally, for compatible observables, play a key role in many foundational discussions of quantum mechanics. Recall that observables X, Y commute (or are compatible with one another) if and only if both are functions of a third Z; and the third can be chosen in such a way (with a minimal set of eigenspaces) to make the mapping $Z \mapsto (X, Y)$ a one-to-one correspondence in the sense that we can write $X = f(Z)$, $Y = g(Z)$, $Z = h(X, Y)$ where h is the inverse of (f, g). In other words, two (or more) commuting observables can be thought of as components of a vector-valued observable, or equivalently as defining together one 'ordinary' observable. Whether one thinks of them together as a vector or as a scalar observable is merely a question of how the eigenspaces are labelled. One can define joint measurement of compatible observables in several equivalent ways. Assuming Lüders' projection postulate for how a state changes on measurement, the sequential measurements, in any order, of a collection of compatible observables, are operationally indistinguishable from one another. One may therefore think equally well of 'one-shot' measurement of Z, sequential measurement of X then Y, and sequential measurement of Y then X.

This leads to a further extended functional invariance assumption:

$$\mathrm{law}\big(f(\mathrm{meas}_\psi(\vec{X}))\big) \;=\; \mathrm{law}\big(\mathrm{meas}_\psi(f(\vec{X}))\big) \tag{10}$$

where $\vec{X} = (X_1, \ldots, X_k)$ is a vector of mutually compatible observables and $f : \mathbb{R}^k \to \mathbb{R}^m$. Apparently weaker is the mean value form of this:

$$\mathrm{E}\big(f(\mathrm{meas}_\psi(\vec{X}))\big) \;=\; \mathrm{E}\big(\mathrm{meas}_\psi(f(\vec{X}))\big); \tag{11}$$

though as I showed above, by playing around with indicator functions, the two are equivalent. We can recover from the assumption the fact that the probability law of a measurement of X alone is the same as the first marginal of the joint law of the two outcomes of a joint measurement of commuting X, Y. As I have argued elsewhere (see my lecture notes on hidden variables, on my web pages `www.math.uu.nl/people/gill`), these consequences of the standard theory form a crucial though often only implicit ingredient in many of the famous no-go arguments against hidden variables in the literature. Somewhat irreverently I have dubbed (11) 'the law of the unconscious quantum physicist'.

Deutsch's approach is similar to that of some probabilists, in that he would prefer to make Expectation central, and have Probability a consequence (in fact, he would prefer to do without the word Probability altogether). This is fine, and indeed many probabilists do take this approach (Whittle in his textbook on Probability argues that one should do the same for quantum probability, too). Now in our situation we want to start with hypothesizing existence of mean values, and by making some structural assumptions about them. From this we want to derive the form of the mean values. As I have noted above, since $\mathbb{1}_{\{x\}}(X)$ is a both an observable itself, and a function of the observable X, it would appear that fixing all mean values of (outcomes of measurements of) all observables, fixes all probability laws of (outcomes of measurements of) all observables. The point I want to make, is that this indeed works, provided we have the functional invariance assumption (for mean values only, if you like, but we must have if for a very large class of functions). Do we need to consider many-to-one functions? If our assumptions are *only* about expectations, I think we do need many-to-one functions. However, with modest distributional input, one need further only consider one-to-one functions, as follows. Suppose we know the mean value of $\mathrm{meas}_\psi(\exp(it \arctan X))$, and suppose we assume functional invariance, *in law*, for all one-to-one functions; in particular, the functions $f(x) = \exp(it \arctan x)$, each real t. Then we know law(meas$_\psi X$). It is possible to avoid complex-valued functions, try for instance $f(x) = s\cos(\frac{1}{2}(\arctan x + \pi/2)) + t\sin(\frac{1}{2}(\arctan x + \pi/2))$ for all real s and t.

Let me return to the contrast between Deutsch's and Gleason's argument. Deutsch's proof, on completion, seems a little simpler and more direct. His assumptions are much stronger: he needs unitary invariance. His assumptions are more representative of classical quantum mechanics—unitary evolution *has* to be considered an essential part of this. In the first stages of his argument, deriving mean values for some rather special observables and rather special states, he moreover only needed to consider functional invariance under one-to-one transformations. This assumption is close to tautological (the apparatus for measuring $a + bX$ is not going to be essentially different from that for measuring X). However, even from the point of view of deriving fair values of games, probability laws as well as mean values are equally relevant. For instance, what is the fair value of the game: measure X and receive € 1 if the outcome x_0 is obtained? The easiest way to deal with this game too, is to include functional invariance for the indicator functions too, and then one need not work any more but simply appeal to Gleason's theorem.

5 Discussion

Later in this section I will run through Deutsch's steps to complete his proof. The aim will be to see whether, with weaker versions of our main assumptions, not strong enough to give us Gleason's assumptions so easily, we could also arrive at the desired conclusion. (The answer is that at present, I do not know). But first I would like to discuss what grounds one could have for the functional and unitary invariance assumptions, against the background assumptions that measuring an observable yields an eigenvalue, and that in an eigenstate, the outcome is certain.

Functional invariance for one-to-one functions seems to me to me more or less definitional. For many-to-one it is much less definitional, also less empirical, since there will vary rarely truely exist essentially different measurement apparatuses for 'doing' X and doing $f(X)$. Just occasionally there will be empirical evidence *supporting* functional invariance: for instance when X and Y do not commute, but for some many-to-one functions, one has $f(X) = g(Y)$, there might be empirical (statistical) data supporting it, based on the quite different experiments for measuring X and for measuring Y, and finding the same statistics (or mean values) for f of the outcomes of the first experiment, g of the outcomes of the second. There is one very strong empirical fact supporting the assumption (in its form for vector observables): when we simultaneously measure observables on separate components of a product system (even if in an entangled state) we have the same marginal statistics, as if only one component was being measured. Altogether, the nature of this assumption would seem to me to be: we extend a definitional assumption concerning a smaller class of functions f—the affine functions—to a much larger class, by mathematical analogy, trusting that the world is so elegantly and mathematically put together, that the 'obvious' sweeping mathematical generalization of an indubitable fact is usually correct; we are supported in this by some empirical (statistical) evidence for some special cases.

Similarly the assumption of unitary invariance seems to be largely a leap of faith, since there will be little empirical (statistical) evidence to support it. But again, one might prefer to think of the leap of faith as a natural mathematical generalization. Our first assumption—that measuring an observable on an eigenstate produces the eigenvalue—tells us that

$$\mathrm{law}\big(\mathrm{meas}_{U\psi}(X)\big) \;=\; \mathrm{law}\big(\mathrm{meas}_{\psi}(U^*XU)\big) \tag{12}$$

whenever U permutes eigenspaces and ψ is an eigenvector! Extending this to arbitrary states can be thought of as an interpolation, in harmony with ideas of wave-particle duality. It seems to me that wave-particle duality—the very heart of quantum physics—essentially forces probability on us, since it is the only way to get a smooth interpolation between the distinct discrete behaviours at different eigenstates of an observable. We just have to live with smoothness at the statistical level, instead of at the (counterfactual) level of individual outcomes.

I would now like to discuss the remaining steps of Deutsch's proof. As we saw, functional invariance in its strongest form implies the conditions of Gleason's theorem, which makes all further conditions and further work superfluous. Now the reason functional invariance is so powerful, is that we assumed it to hold for *all* functions f, in particular, many-to-one functions. In the spirit of the first part of Deutsch's proof it would make sense to demand it only for *one-to-one* functions. It seems to me a reasonable conjecture that Deutsch's theorem is true under the three assumptions: functional invariance for one-to-one functions, unitary invariance, and the degeneracy assumption.

As was stated earlier, after (a) the two-eigenstate equal weight superposition, Deutsch extends this (b) to binary powers, (c) to arbitrary whole numbers of equal weight superpositions, (d) to rational superpositions, (e) to real and finally (f) to arbitrary. As we saw, steps (b) and (c) can also be dealt with by his own method for the two-eigenstate case. Deutsch's argument for (d) involves completely new ingredients and assumptions. He supposes that an auxiliary quantum system can be brought into interaction with the system under study, thus yielding a product space and a product state. The observable of interest X is identified with $X \otimes \mathbf{1}$, and this is considered as one of a pair $(X \otimes \mathbf{1}, \mathbf{1} \otimes Y)$ where the observable Y is cleverly chosen, so that in the product system, and with this product observable, we are back in an equal weight superposition of eigenstates. He then makes the assumption: measuring X on the original system is the same as measuring (X, Y) on the product sytem and discarding the outcome of Y. Uncontroversial though this may be, we are greatly expanding on the background assumptions. Moreover we are actually assuming functional invariance for a many-to-one function: namely, the function which delivers the x-component of a pair (x, y). By the way, Deutsch's proofs of steps (b) and (c) similarly involve such constructions. Step (e) is an approximation argument which can presumably be made rigorous, though perhaps differently to how Deutsch does it. Step (f) as presented by Deutsch involves yet another new assumption: measuring an observable can be represented as a unitary transformation on a suitable product system, so that after a new unitary transformation mapping $|x\rangle$ to $e^{-i\phi}|x\rangle$ one can remove complex phases from a superposition of eigenstates. This argument is unnecessarily complicated, it seems to me. Our unitary invariance assumption together with the unitary transformation just described, takes care of extending results from real to complex superpositions.

The work of Deutsch has been strongly criticised by Finkelstein (1999) and by Barnum et al. (1999). They also point out that the first step of Deutsch's proof is incorrect, however, do not recognise that it can be repaired by a supplementary, natural, condition. They also point out that Gleason's theorem does the same job as Deutsch purports to do, but do not see the very close connection between Gleason's and Deutsch's assumptions. They point out also that the later steps of Deutsch's proof depend on various appeals to the substitutability principle, without stating which games were to be substituted for which. I must admit that it took me a long email correspondence with David Deutsch, before I was able for myself to fill in all the gaps. Finally they also point out that the work of de Finetti and Savage implies that rational behaviour under uncertainty implies behaviour as if probability is there. It is therefore just a question of taste whether or not one adds a probability interpretation to the 'values of games' derived by Deutsch.

My conclusion is that Deutsch's proof as it stands is valid, though the author is implicitly using unitary as well as functional invariance. All his assumptions together imply the assumptions of Gleason's theorem, and much more. Consequently the proof as given does not have a great deal of mathematical interest. However the fact that distributional conclusions could already be drawn for *some* states and *some* observables, at a point at which only functional invariance for one-to-one functions had been used, and in my opinion, with a most elegant argument, justifies the conjecture I have already mentioned:

Conjecture 1. Deutsch's theorem is true under the three assumptions: functional invariance for one-to-one functions, unitary invariance, and the degeneracy assumption.

Unitary invariance alone tells us that the law of the outcome of a measurement of X only depends on the absolute innerproducts $|\langle x|\psi\rangle|$. So the task is to determine the form of the dependence.

6 Acknowledgments

I am grateful for the warm hospitality and support of the Quantum Probability group at the department of mathematics of the University of Greifswald, Germany, during my sabbatical there, Spring 2002. My research there was supported by European Commission grant HPRN-CT-2002-00279, RTN QP-Applications. This research has also been supported by project RESQ (IST-2001-37559) of the IST-FET programme of the European Commission.

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