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to the classic two-stage scheme, provided that the SNR is sufficiently high. For example, given a target BER of 10^{-5}, the three-stage receiver using SD ($N_{\rm cand} = 32$) is capable of achieving a performance gain of 2.5 dB over its two-stage counterpart in an uplink (8×4) SDMA/OFDM 4-QAM system. Furthermore, an additional 2-dB performance gain can be attained with the aid of the novel center-shifting-based SD amalgamated with an IrCC.

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On the Optimality of the Null Subcarrier Placement for Blind Carrier Offset Estimation in OFDM Systems

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***Abstract*—Liu and Tureli proposed a blind carrier frequency offset (CFO) estimation method for orthogonal frequency-division multiplexing (OFDM) systems, making use of null subcarriers. The optimal subcarrier placement that minimizes the Cramer–Rao bound (CRB) of the CFO estimation was reported by Ghogho *et al.* In this paper, we study the optimality of the null subcarrier placement from another perspective. We first show that the SNR of the CFO estimation using null subcarriers is a function of the null subcarrier placement. We then formulate the CFO-SNR optimization for the null subcarrier placement as a convex optimization problem for small CFO values and derive the optimal placement when the number of subcarriers is a multiple of the number of null subcarriers. In addition, we show that the SNR-optimal null subcarrier placement also minimizes the theoretical mean square error in the high SNR region. When the number of subcarriers is not a multiple of the number of null subcarriers, we propose a heuristic method for the null subcarrier placement that still achieves good performance in the CFO estimation. We also discuss the optimality of the null subcarrier placement in practical OFDM systems, where guard bands are required at both ends of the spectrum.**

***Index Terms*—Blind carrier offset estimation, convex optimization, orthogonal frequency-division multiplexing (OFDM).**

I. INTRODUCTION

Orthogonal frequency-division multiplexing (OFDM) is known to be more sensitive to carrier frequency offset (CFO). For an OFDM system with CFO, we can write the received time-domain signal in the following form [1]:

$$\mathbf{y}^m = \mathbf{E}\mathbf{W}_P\mathbf{H}^m\mathbf{s}^m e^{j2\pi\phi_0(m-1)(1+N_g/N)} + \mathbf{n}^m. \qquad (1)$$

Here, we use superscript m to indicate the OFDM symbol index. $\mathbf{E} = \text{diag}(1, e^{j2\pi\phi_0/N}, \ldots, e^{j2\pi(N-1)\phi_0/N})$ is a diagonal matrix containing CFO ϕ_0, which we assume to be normalized with respect to subcarrier spacing $2\pi/N$. In a practical OFDM system, there are some subcarriers that do not carry any data. They are called null subcarriers, whereas the data-carrying subcarriers are simply called data subcarriers. Let P out of N subcarriers be the data subcarriers. Then, $\mathbf{W}_P$ is an $N \times P$ submatrix that is obtained from the $N \times N$ inverse discrete Fourier transform (DFT) matrix $\mathbf{W}_N$. $\mathbf{H}^m$ is a diagonal matrix containing the channel frequency response, $\mathbf{s}^m$ is a $P \times 1$ vector containing the transmitted data in the mth OFDM symbol, N_g denotes the length of the cyclic prefix, and $\mathbf{n}^m$ is an additive white Gaussian noise (AWGN) vector.

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Liu and Tureli [1] presented a blind CFO estimation method based on the received signal on the null subcarriers. Let us define $\mathbf{l}^m = [l_1^m, l_2^m, \ldots, l_L^m]$ as the null subcarrier indexes in OFDM symbol m and $L = N - P$ as the number of null subcarriers. It was shown that the CFO estimate can be obtained from the minimization of the cost function, which is given as

$$\mathcal{J}(z) = \sum_{m=1}^{M} \sum_{l \in \mathbf{l}^m} \left\| \mathbf{w}_l^H \mathbf{Z}^{-1} \mathbf{y}^m \right\|^2 \tag{2}$$

where M is the total number of OFDM symbols that are used for the CFO estimation, $\mathbf{w}_l^H$ is the lth row of the DFT matrix, and $\mathbf{Z} = \mathrm{diag}(1, z, z^2, \ldots, z^{(N-1)})$. The optimal CFO estimate is normally obtained through a search over the range of possible CFO values.

The null subcarrier placement that guarantees the identifiability of the CFO estimation method in [1] was studied in [2] and [3]. In [3], it is reported that the null subcarrier placement that minimizes the Cramer–Rao bound (CRB) is achieved by placing them with even spacing across the whole OFDM symbol. In this paper, we study the optimality of the null subcarrier placement using a very different approach compared to [3]. We found that the SNR of the CFO estimation using the method in [1] is a function of the null subcarrier placement. Therefore, we want to find the optimal placement of null subcarriers such that the SNR of the CFO estimation is maximized. We formulate the SNR maximization problem of the null subcarrier placement and derive the optimal solution from a convex optimization procedure for small CFO values. We found that when the number of subcarriers is divisible by the number of null subcarriers, the exact optimal null subcarrier placement can be found. We also prove that the SNR-optimal null subcarrier placement is also optimal in minimizing the theoretical MSE, which is given in [4], of the CFO estimation. Interestingly, this is the same null subcarrier placement that minimizes the CRB in [3]. Therefore, the main contribution of this paper does not lie in the finding of a new optimal null subcarrier placement. This paper has rather contributed additional theoretical insights on why the null subcarriers should be placed evenly. We show that the evenly spaced null subcarrier placement not only minimizes the CRB but also maximizes the SNR and minimizes the theoretical MSE of the CFO estimation.

When the number of subcarriers is not divisible by the number of null subcarriers, it is difficult to prove the optimality of the null subcarrier placement due to the integer constraint on the optimization variables. However, we will show a heuristic procedure on how to place the null subcarriers where good performance can still be achieved. We extend the optimization problem to a practical OFDM system where guard bands are required at both ends of the spectrum. In this case, if given a few more null subcarriers that can be freely inserted in the OFDM symbol, we show how to place them to guarantee the SNR optimality in the CFO estimation. We show that for practical OFDM systems with guard bands, the introduction of a few extra null subcarriers leads to much better performance of the blind CFO estimation.

II. Placement of Null Subcarriers Based on the CFO-SNR Maximization

Given a CFO value of ϕ_0, the received signal on null subcarrier l_i of OFDM symbol m can be written as

$$r_{l_i}^m = \sum_{n=0, n \neq l_i}^{N-1} h_n^m s_n^m C_{n-l_i}^m(\phi_0) + n_{l_i}^m = \mathrm{ICI}_{l_i}^m(\phi_0) + n_{l_i}^m \tag{3}$$

where $h_{l_i}^m$ and $s_{l_i}^m$ are the channel response and the transmitted data on subcarrier l_i of OFDM symbol m, respectively. $\mathrm{ICI}_{l_i}^m(\phi_0)$ is the Inter-Carrier Interference (ICI) due to the CFO of ϕ_0, and $n_{l_i}^m$ is an AWGN noise. The value of $C_k^m(\phi_0)$ is given by [5]

$$C_k^m(\phi_0) = \frac{\sin[\pi(k+\phi_0)]}{N \sin\left[\frac{\pi}{N}(k+\phi_0)\right]} \exp\left(j\pi(k+\phi_0)\left(1 - \frac{1}{N}\right)\right) \times \exp\left(j2\pi\phi_0(m-1)(1+N_g/N)\right). \tag{4}$$

Using (3), the cost function in (2), which is the summation of the received signal power over all the null subcarriers, can be equivalently rewritten as

$$\mathcal{J}(\phi) = \sum_{m=1}^{M} \sum_{i=1}^{L} \left| \sum_{n=0, n \notin \mathbf{l}^m}^{N-1} h_n^m s_n^m C_{n-l_i}^m(\phi_0 - \phi) + n_{l_i}^m \right|^2. \tag{5}$$

Correspondingly, the estimate of the CFO is given by

$$\hat{\phi} = \arg\min_{\phi} \mathcal{J}(\phi). \tag{6}$$

Note that the received signal on a null subcarrier l_i in (3) is the sum of $\mathrm{ICI}_{l_i}^m$ and $n_{l_i}^m$. $\mathrm{ICI}_{l_i}^m$ is the useful signal term that we can use for the estimation of CFO ϕ_0, and $n_{l_i}^m$ is the noise term, which is uncorrelated with $\mathrm{ICI}_{l_i}^m$. Therefore, using (3), we can define an objective function, so called $\mathrm{SNR}_{\mathrm{CFO}}$, as follows:

$$\mathrm{SNR}_{\mathrm{CFO}} = \frac{\mathrm{E}\left(\sum_{m=1}^{M} \sum_{i=1}^{L} \left| \mathrm{ICI}_{l_i}^m(\phi_0) \right|^2 \right)}{\mathrm{E}\left(\sum_{m=1}^{M} \sum_{i=1}^{L} \left| n_{l_i}^m \right|^2 \right)} \tag{7}$$

where E denotes statistical expectation. Note that the objective function can be interpreted as the SNR of the CFO estimation. The power of the ICI on subcarrier l_i in OFDM symbol m can be written as

$$\mathrm{E}\left|\mathrm{ICI}_{l_i}^m(\phi_0)\right|^2 = \left\{ \sum_{n=0, n \notin \mathbf{l}}^{N-1} \mathrm{E}\left|h_n^m s_n^m\right|^2 \frac{\sin^2[\pi(n - l_i + \phi_0)]}{N^2 \sin^2\left[\frac{\pi}{N}(n - l_i + \phi_0)\right]} \right\}.$$

Notice that the ICI power for the mth OFDM symbol depends only on the signals in OFDM symbol m and is not affected by other OFDM symbols. As the noise in OFDM symbol m is also independent from the noise in other OFDM symbols, the $\mathrm{SNR}_{\mathrm{CFO}}$ optimization for M OFDM symbols can be independently performed on each OFDM symbol. Therefore, the optimization only needs to be performed for one OFDM symbol. From now on, for ease of notation, we will drop OFDM symbol index m. In this case, null subcarrier placement $\mathbf{l}$ that maximizes estimation $\mathrm{SNR}_{\mathrm{CFO}}$ in (7) can be found by

$$\begin{aligned} \mathbf{l} &= \arg\max_{\mathbf{l}} (\mathrm{SNR}_{\mathrm{CFO}}) = \arg\max_{\mathbf{l}} \mathrm{E}\left(\sum_{i=1}^{L} \left|\mathrm{ICI}_{l_i}(\phi_0)\right|^2 \right) \\ &= \arg\max_{\mathbf{l}} \sum_{i=1}^{L} \left\{ \sum_{n=0, n \notin \mathbf{l}}^{N-1} \mathrm{E}|h_n s_n|^2 \frac{\sin^2[\pi(n - l_i + \phi_0)]}{N^2 \sin^2\left[\frac{\pi}{N}(n - l_i + \phi_0)\right]} \right\} \\ &= \arg\max_{\mathbf{l}} \sum_{i=1}^{L} \left\{ \sum_{n=0, n \notin \mathbf{l}}^{N-1} \frac{1}{\sin^2\left[\frac{\pi}{N}(n - l_i + \phi_0)\right]} \right\} \end{aligned} \tag{8}$$

as $\mathrm{E}|h_n s_n|^2 = \mathrm{E}\{|h_n|^2\}\mathrm{E}\{|s_n|^2\}$ is independent of the null subcarrier placement. The numerator $\sin^2[\pi(n - l_i + \phi_0)]$ is equal to $\sin^2(\pi\phi_0)$ and is also independent of the null subcarrier placement. In practice, ϕ_0 is normally modeled as a random variable with a uniform distribution between $[-\theta, \theta)$. In this case, the cost function can be rewritten as

$$\begin{aligned}\mathbf{l} &= \arg\max_{\mathbf{l}} \frac{1}{2\theta} \int_{-\theta}^{+\theta} \sum_{i=1}^{L} \left[\sum_{n=0, n\notin \mathbf{l}}^{N-1} \frac{1}{\sin^2\left[\frac{\pi}{N}(n-l_i+\phi_0)\right]} \right] d\phi_0 \\ &= \arg\max_{\mathbf{l}} \left\{ \sum_{i=1}^{L} \sum_{n=0, n\notin \mathbf{l}}^{N-1} \frac{1}{2\theta}\frac{N}{\pi} f(n-l_i) \right\} \qquad (9)\end{aligned}$$

where $f(k)$ is given by

$$f(k) = \left[\cot\left(\frac{\pi}{N}(k-\theta)\right) - \cot\left(\frac{\pi}{N}(k+\theta)\right)\right] \text{ for } k = -(N-1), \ldots, -1, 1, \ldots, N-1. \qquad (10)$$

Note that $k = n - l_i \neq 0$ for $n \notin \mathbf{l}$. It could be easily shown that function $f(k)$ is periodic with period N, i.e., $f(k) = f(k+N)$. Therefore, for the subsequent optimization, we only need to consider function $f(k)$ over one period, i.e., $k = 1, 2, \ldots, N-1$. Another property of $f(k)$ is that it is an even function of k, i.e., $f(k) = f(-k)$ for any integer k.

Discarding the constants, we can rewrite (9) in the following form:

$$\mathbf{l} = \arg\max_{\mathbf{l}} \sum_{i=1}^{L} \left\{ \sum_{n=0}^{N-1} f(n-l_i) - \sum_{n\in\mathbf{l}, n\neq l_i} f(n-l_i) - f(0) \right\}. \qquad (11)$$

The third term in (11), i.e., $f(0)$, is independent of $\mathbf{l}$ and, hence, can be dropped. Using the periodicity of $f(k)$, it can be easily shown that the first term in the summation $\sum_{n=0}^{N-1} f(n-l_i)$ in (11) is also independent of l_i. Therefore, the cost function in (11) can be simplified to

$$\mathbf{l} = \arg\min_{\mathbf{l}} \left\{ \sum_{i=1}^{L} \sum_{j=1, j\neq i}^{L} f(l_i - l_j) \right\}. \qquad (12)$$

Notice that the new cost function in (12) depends only on the spacing, not the absolute positions, of the null subcarriers. Let us define the spacing between the ith and $(i+1)$th null subcarriers as $k_i = l_{i+1} - l_i$ for $i = 1, 2 \ldots, L-1$ and $k_L = N + l_1 - l_L$. We further define $p_{i,m} = \sum_{j=0}^{m-1} k_{[(i+j-1) \bmod L]+1}$ for $i = 1, 2, \ldots, L$ and $m = 1, 2, \ldots, L-1$. Here, we use $[i \bmod L]$ for integers i and L to denote the integer remainder of i/L. The subscript i indicates the k index of the first term in the summation because $[(i+0-1) \bmod L] + 1 = i$. The subscript m indicates the total number of terms in the summation. Therefore, $p_{i,m}$ is actually the spacing between the ith null subcarrier and its mth neighboring null subcarrier to the right in the cyclic sense. Therefore, the contribution to the total cost function due to a particular null subcarrier i is the summation of $f(l_i - l_j)$ from all its $L-1$ neighboring null subcarriers, i.e., $\sum_{j=1, j\neq i}^{L} f(l_i - l_j)$. As $p_{i,m}$ is the spacing between the ith null subcarrier and its mth neighboring null subcarrier, we can write $\sum_{j=1, j\neq i}^{L} f(l_i - l_j) = \sum_{m=1}^{L-1} f(p_{i,m})$. Summing this over all the L null subcarriers, i.e., L possible values of i, the new cost function can be written as

$$\mathcal{J}(k_1, k_2, \ldots, k_L) = \sum_{i=1}^{L} \sum_{m=1}^{L-1} f(p_{i,m}) = \sum_{m=1}^{L-1} \left\{ \sum_{i=1}^{L} f(p_{i,m}) \right\} \qquad (13)$$

with $\sum_{i=1}^{L} k_i = N$.

The optimization problem in (13) has all variables being integers. Such integer programming problems are difficult to solve analytically. Therefore, we first relax the constraints on all k_i's being integers and assume them to be real positive numbers. This approach has been commonly used in finding the optimal bit allocations for multiuser or multicarrier systems (see, for example, [6]). For ease of analysis, we also assume that $\theta < 1$ so that $k - \theta > 0$ and $k + \theta < N$ are satisfied for all possible values of k. It can be easily shown that if the above condition is satisfied, $(d^2/dk^2) f(k) > 0$ for $1 < k < N-1$. Therefore, $f(k)$ is a convex function for $1 < k < N-1$, and $\theta < 1$. According to Jensen's inequality [7], if $f(k)$ is convex for $k_1, k_2, \ldots, k_L$, and given $\lambda_1, \lambda_2, \ldots, \lambda_L$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_L = 1$, then

$$f(\lambda_1 k_1 + \cdots + \lambda_L k_L) \leq \lambda_1 f(k_1) + \cdots + \lambda_L f(k_L). \qquad (14)$$

By setting $\lambda_1 = \lambda_2 = \cdots = \lambda_L = (1/L)$, we have

$$\frac{1}{L} \sum_{i=1}^{L} f(p_{i,m}) \geq f\left(\frac{1}{L}\sum_{i=1}^{L} p_{i,m}\right) = f\left(\frac{mN}{L}\right). \qquad (15)$$

Here, we make use of $\sum_{i=1}^{L} p_{i,m} = mN$ because $\sum_{i=1}^{L} k_i = N$. The equality in (15) holds for a given m when all the $p_{i,m}$'s for $i = 1, 2, \ldots L$ are equal. When $k_1 = k_2 = \cdots = k_L = N/L$, the equality in (15) holds for all values of m. Therefore, we obtain

$$\mathcal{J}(k_1, k_2, \ldots, k_L) \geq \sum_{m=1}^{L-1} L \left\{ f\left(\frac{mN}{L}\right) \right\}. \qquad (16)$$

The right-hand side of (16) is independent of k_i's and is the lower bound of the cost function. Therefore, this cost function is minimized when $k_1 = k_2 = \cdots = k_L = N/L$. This means that the null subcarriers should be placed evenly spaced across the whole OFDM symbol.

If N/L is an integer, the null subcarriers should be placed N/L apart to maximize $\mathrm{SNR}_{\mathrm{CFO}}$. Therefore, in system design, when we can freely choose the number of null subcarriers L, we should always choose L such that N/L is an integer to ensure the optimality of the null subcarrier placement. However, for systems where N is not divisible by L, it turns out to be difficult to prove the optimality of a particular null subcarrier placement because of the integer constraints on the values of k_i's. For real number k_i's, we know that, to maximize $\mathrm{SNR}_{\mathrm{CFO}}$, the spacing between the null subcarriers should be the same. In the following, we propose a heuristic method in placing the null subcarriers as evenly as possible for integer k_i's.

Let $k_l = \lfloor N/L \rfloor$ and $k_u = \lceil N/L \rceil$, where k_l and k_u are both integers. Here, we use $\lfloor x \rfloor$ to denote the largest integer that is smaller than or equal to x, whereas we use $\lceil x \rceil$ to denote the smallest integer that is larger than or equal to x. We know that to achieve close to even spacing between the null subcarriers, all the k_i values should be chosen as either k_l or k_u. Next, we determine how many k_i's should take the value k_l and how many k_i's should take the value k_u, and we use n_l and n_u to denote the two numbers, respectively. The values of n_l and n_u can be obtained by solving

$$\begin{cases} n_l + n_u = L \\ n_l \times k_l + n_u \times k_u = N. \end{cases} \qquad (17)$$

Now, the problem of placing the null subcarriers is equivalent to placing these $n_l k_l$'s and $n_u k_u$'s as evenly as possible. It is obvious that if we place all the k_l's consecutively and all the k_u's consecutively, the spacing between the null subcarriers is not going to be very even. They should be alternatively placed in some way. Without loss of generality, let us assume that $n_l \geq n_u$. If $(n_l/n_u) = q$ is an integer, we should

TABLE I
HEURISTIC NULL SUBCARRIER PLACEMENT WHEN N IS NOT DIVISIBLE BY L $(n_l > n_u)$

• Find k_l, k_u and solve for the corresponding n_l n_u.
• **if** n_l is divisible by n_u, i.e. $\frac{n_l}{n_u} = q\{$
  • The spacing between null subcarriers should be

$$[\overbrace{\underbrace{k_l, k_l, \cdots, k_l}_{q}, \underbrace{k_u}_{1}, \cdots, \underbrace{k_l, k_l, \cdots, k_l}_{q}, \underbrace{k_u}_{1}}^{n_u \text{groups}}]$$

$\}$
else$\{$
  • calculate $q_l = \lfloor \frac{n_l}{n_u} \rfloor$ and g_l and g_u from (18),
  • The spacing between the null subcarriers should be

$$[\overbrace{\underbrace{k_l, k_l, \cdots, k_l}_{q_l}, \underbrace{k_u}_{1}}^{\text{type 1 group}}, \overbrace{\underbrace{k_l, k_l, \cdots, k_l}_{q_l+1}, \underbrace{k_u}_{1}}^{\text{type 2 group}}, \overbrace{\underbrace{k_l, k_l, \cdots, k_l}_{q_l}, \underbrace{k_u}_{1}}^{\text{type 1 group}} \cdots\cdots]$$

$\}$

TABLE II
HEURISTIC NULL SUBCARRIER PLACEMENT FOR $L = 4$ TO 11 FOR $N = 64$ OFDM SYSTEMS

L	k_i	index of null subcarriers
5	[13 13 13 13 12]	[1 14 27 40 53]
6	[11 11 10 11 11 10]	[1 12 23 33 44 55]
7	[9 9 9 9 9 9 10]	[1 10 19 28 37 46 55]
8	[8 8 8 8 8 8 8 8]	[1 9 17 25 33 41 49 57]
9	[7 7 7 7 7 7 7 7 8]	[1 8 15 22 29 36 43 50 57]
10	[6 7 6 6 7 6 7 6 6 7]	[1 7 14 20 26 33 39 46 52 58]
11	[6 6 6 6 5 6 6 6 6 6 5]	[1 7 13 19 25 30 36 42 48 54 60]

group q k_l's followed by one k_u into one group and place n_u of such groups as illustrated in Table I. Otherwise, we let $q_l = \lfloor n_l/n_u \rfloor$. In this case, we should have two kinds of placing groups. The type 1 group consists q_l k_l's followed by one k_u, and the type 2 group consists $q_l + 1$ k_l's followed by one k_u. The number of type 1 groups g_l and the number of type 2 groups g_u can be obtained by solving

$$\begin{cases} g_l + g_u = n_u \\ g_l \times q_l + g_u \times (q_l + 1) = n_l. \end{cases} \tag{18}$$

These two types of groups should be placed alternatively. A summary of this heuristic placement method is given in Table I.

The null subcarrier placement for $L = 4$ to $L = 11$ null subcarriers for an OFDM system with $N = 64$ subcarriers using the proposed heuristic method is listed in Table II. For the case of $L = 5$ and 6, we have verified that the null subcarrier placement using the heuristic method is the same as the optimal placement that is obtained through an exhaustive computer search.

Note that our previous derivation is based on the assumption that $\theta < 1$ to ensure that $f(k)$ is convex for $k = 1, 2, \ldots, N-1$. This is a valid assumption for most indoor communication systems that are operating at the 2.4- and 5-GHz bands, such as wireless local area network (LAN) systems [8]. According to the IEEE 802.11a standard [8], the tolerance of the transmit and receive center frequency should be ± 20 ppm. Therefore, the worst case frequency offset is 40 ppm, which is about 200 kHz for a 5.2-GHz center frequency. This worst case frequency offset corresponds to the value of $\theta = 0.66$. Moreover, for indoor applications, due to low mobility and high carrier frequency (5 GHz for the IEEE 802.11a system), the CFO due to the Doppler shift is negligible. Therefore, this is a valid assumption in practice, particularly for indoor wireless LAN-based applications due to the high-quality oscillators that are currently used.

III. PLACEMENT OF NULL SUBCARRIERS BASED ON THE THEORETICAL MSE MINIMIZATION

In this section, we prove that the SNR-optimal null subcarrier placement is also optimal in minimizing the MSE of the CFO estimation in the high SNR region. Let us set $\Delta\phi = \phi_0 - \hat{\phi}$ as the CFO estimation error. The linear approximation of $\Delta\phi$ can be obtained as [4]

$$\Delta\phi = \frac{\left.\frac{\partial \mathcal{J}(\phi)}{\partial \phi}\right|_{\phi=\phi_0}}{\left.\frac{\partial^2 \mathcal{J}(\phi)}{\partial \phi^2}\right|_{\phi=\phi_0}}. \tag{19}$$

After some algebraic manipulations, the estimation error can be expressed as in (20), shown at the bottom of the page. Assuming the noise on different subcarriers to be independent and identically distributed (i.i.d.), with zero mean and variance σ_n^2, we can show that $\mathrm{E}_n(\Delta\phi) = 0$. Therefore, the linearized estimator is unbiased. This also means that the MSE of the CFO estimation is equal to the variance of $\Delta\phi$. Let us also assume that transmitted signal s_k^m is also i.i.d., with zero mean and unit variance, and the channel is appropriately normalized such that the channel component on each subcarrier has zero mean with unit variance, i.e., $\mathrm{E}(h_n^m) = 0$ and $\mathrm{E}(|h_n^m|^2) = 1$ for $n = 0, 1, \ldots, N-1$ and $m = 1, \ldots, M$. We obtain that the MSE of CFO estimation in the high SNR region is given by

$$\mathrm{E}_{n,s,h}\left[(\Delta\phi)^2\right] = \frac{N^2\sigma_n^2}{2\pi^2 \sum\limits_{m=1}^{M} \sum\limits_{i=1}^{L} \sum\limits_{n=0, n\notin \mathbf{l}^m}^{N-1} \frac{1}{\sin^2\left[\frac{\pi}{N}\left(n - l_i^m\right)\right]}}. \tag{21}$$

Now, let us look at the null subcarrier optimization problem again. The null subcarrier placement that minimizes the MSE can be formulated as

$$\begin{aligned} \mathbf{l} &= \arg\min_{\mathbf{l}} \left(\mathrm{E}_{n,s,h}\left[(\Delta\phi)^2\right]\right) \\ &= \arg\max_{\mathbf{l}} \left(\sum_{m=1}^{M} \sum_{i=1}^{L} \sum_{n=0, n\notin \mathbf{l}^m}^{N-1} \frac{1}{\sin^2\left[\frac{\pi}{N}\left(n - l_i^m\right)\right]}\right). \end{aligned} \tag{22}$$

If $\sum_{i=1}^{L} \sum_{n=0, n\notin \mathbf{l}^m}^{N-1} (1/(\sin^2[(\pi/N)(n - l_i^m)]))$ is maximized for every value of m, i.e., for each OFDM symbol, then the cost function

$$\Delta\phi = -\frac{\Re\left\{\sum\limits_{m=1}^{M} \sum\limits_{i=1}^{L} \sum\limits_{n=0, n\notin \mathbf{l}}^{N-1} h_n^m s_n^m \left(n_{l_i}^m\right)^* \frac{\exp\left[-j\pi \frac{n - l_i^m}{N}\right]}{\sin\left[\frac{\pi}{N}\left(n - l_i^m\right)\right]}\right\}}{\sum\limits_{m=1}^{M} \sum\limits_{i=1}^{L} \sum\limits_{k=0, k\notin \mathbf{l}^m}^{N-1} \sum\limits_{n=0, n\notin \mathbf{l}^m}^{N-1} h_k^m \left(h_n^m\right)^* s_k^m \left(s_n^m\right)^* \frac{\pi \exp\left(-j\pi \frac{k-n}{N}\right)}{N \sin\left(\frac{\pi}{N}\left(k - l_i^m\right)\right) \sin\left(\frac{\pi}{N}\left(n - l_i^m\right)\right)}} \tag{20}$$

in (22) is maximized. Therefore, the optimization problem over M OFDM symbols is equivalent to the optimization in one OFDM symbol, which is given by

$$\mathbf{l} = \arg\max_{\mathbf{l}} \left(\sum_{i=1}^{L} \sum_{n=0, n \notin \mathbf{l}}^{N-1} g(n - l_i) \right) \tag{23}$$

where $g(k) = (1/(\sin^2[(\pi/N)(n - l_i)]))$. It is straightforward to show that $g(k)$ is also periodic with period N. Using a similar approach as we have done in Section II, the optimization problem in (23) can be simplified to

$$\mathbf{l} = \arg\min_{\mathbf{l}} \left\{ \sum_{i=1}^{L} \sum_{j=1, j \neq i}^{L} g(l_i - l_j) \right\}. \tag{24}$$

It can also be shown that $g(x)$ is a convex function of real number x for $1 < x < N - 1$. Therefore, the optimization problem in (24) is essentially the same as the optimization problem in (12), as $g(x)$ and $f(x)$ are both convex. This means that the optimal solutions to the two optimization problems are the same. Thus, we have proven the following.

Proposition: The null subcarrier placement that maximizes the SNR of the CFO estimation defined in (7) also minimizes the MSE of the CFO estimation in the high SNR region given in (21).

IV. Placement of Null Subcarriers Based on Practical Considerations

For practical OFDM systems, it is usually necessary to consecutively place some null subcarriers at both ends of the spectrum as guard bands. These are called the guard null subcarriers. In this section, we show that, given the fixed positions of the guard null subcarriers, if there are a few null subcarriers to freely place in the OFDM symbol for the purpose of CFO estimation (we call these null subcarriers as free null subcarriers), how should we place them to achieve optimality in the SNR sense?

Let us study an OFDM system having two guard bands with L_1 and L_2 null subcarriers, respectively, at both ends of the spectrum. Suppose that we have L_n free null subcarriers that we can freely place between subcarrier $-N/2 + L_1$ and $N/2 - L_2 + 1$. The whole set of all the null subcarriers becomes

$$\mathbf{l} = [l_1, l_2, \ldots, l_{L_1}, l_{L_1+1}, \ldots, l_{L_1+L_n}, l_{L_1+L_n+1}, \ldots, l_L]$$

where $L = L_1 + L_2 + L_n$ is the total number of null subcarriers. Again, we define $k_i = l_{i+1} - l_i$ as the spacing between the l_{i+1}th null subcarrier and the lth null subcarrier, and $p_{i,m} = \sum_{j=0}^{m-1} k_{[(i+j-1) \bmod L]+1}$ as the spacing between the ith null subcarrier and its mth neighboring null subcarrier to the right in the cyclic sense. Following the similar procedures as in Section II, we could obtain the cost function for the OFDM system with the guard band as

$$\mathcal{J}(k_{L_1}, k_{L_1+1}, \ldots, k_{L_1+L_n}) = \sum_{i=1}^{L} \sum_{m=1}^{L-1} f(p_{i,m}) \tag{25}$$

subject to $\sum_{i=1}^{L} k_i = N$. Comparing (25) with (13), we can see that the summation is still taking over all the L null subcarriers, including the guard and free null subcarriers. However, for this problem, we would not be able to reach the same optimal solution as in Section II because that solution requires the k_i's to be equal for all $i = 1, 2, \ldots, L$, which means that the null subcarriers should be evenly placed across the whole OFDM symbol. This is impossible for our problem, as we do not have the freedom to freely place all the null subcarriers due to the fixed positions of the guard null subcarriers. As a result, the closed-form optimal solution for (25) is difficult to find. However, in practice, the number of free null subcarriers L_n must be kept small to minimize the loss of the transmission data rate, as they occupy the useful spectrum of the data subcarriers. Therefore, it is usually possible to resort to a computer search to find the optimal placement of these subcarriers offline. Table III shows the optimal placement of L_n free null subcarriers with different L_n values for an IEEE 802.11a compliant system, which is obtained by the computer search. In such a system, there are a total of $N = 64$ subcarriers. Subcarriers $[-31{:}-27, 27{:}32]$ are used as guard bands, i.e., $L_1 = 5$ and $L_2 = 6$. Here, the θ value used is 0.5. We can see that the placement is close to the evenly spaced placement.

TABLE III
SNR-Optimal Free Null Subcarrier Placement for IEEE 802.11a Systems

L_n	Free Null Subcarrier Index
2	[-7, 7]
3	[-12, 0, 12]
4	[-15, -5, 5, 15]

V. Simulation Results

Computer simulations were performed for an OFDM system with 64 subcarriers and a length-16 cyclic prefix. According to the specifications that are given in IEEE 802.11a, there are 11 null subcarriers that are consecutively placed from subcarriers 27 to 37 [8]. To achieve a fair comparison, we are also using 11 null subcarriers in our simulations. We only use one OFDM symbol for the CFO estimation, i.e., $M = 1$. We use channel model A of the HiperLan II channel models [9] in all the simulations. It is a multipath Rayleigh fading channel with an exponential power delay profile and a root-mean-square delay spread that is equal to one modulation symbol interval. To assess the performance of the proposed null subcarrier placement, we define the estimation MSE as

$$\text{MSE} = \frac{1}{N_s} \sum_{i=1}^{N_s} (\phi_0 - \hat{\phi})^2 \tag{26}$$

where $\hat{\phi}$ and ϕ_0 represent the estimated and true CFOs, respectively, and N_s denotes the total number of Monte Carlo trials.

A comparison between the MSE that is obtained through simulations and the theoretical MSE that is obtained from (21) is depicted in Fig. 1. The SNR on the x-axis is the SNR of the received signal and not the CFO estimation SNR that we are trying to optimize. The CFO value that we use in the simulation is uniformly distributed between -0.5 and $+0.5$. We compare the theoretical MSE and the MSE that is obtained from simulations for both the consecutive null subcarriers that are placed from 27 to 37 according to IEEE 802.11a, and the proposed null subcarrier placement according to Table II. From the comparison, we can see that the theoretical MSE approximates the actual MSE very closely for the proposed scheme for an SNR that is larger than 10 dB.

Fig. 2 shows the performance of the blind carrier offset estimation using the method in [1] with null subcarriers that are placed with different spacings. The proposed scheme places the null subcarriers according to Table II. We can see that with the proposed null subcarrier

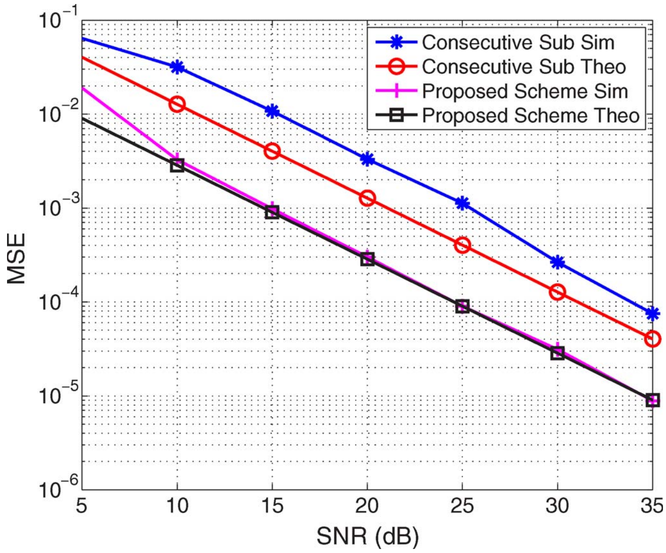

Fig. 1. Comparison between the theoretical MSE and the MSE that is obtained from simulations.

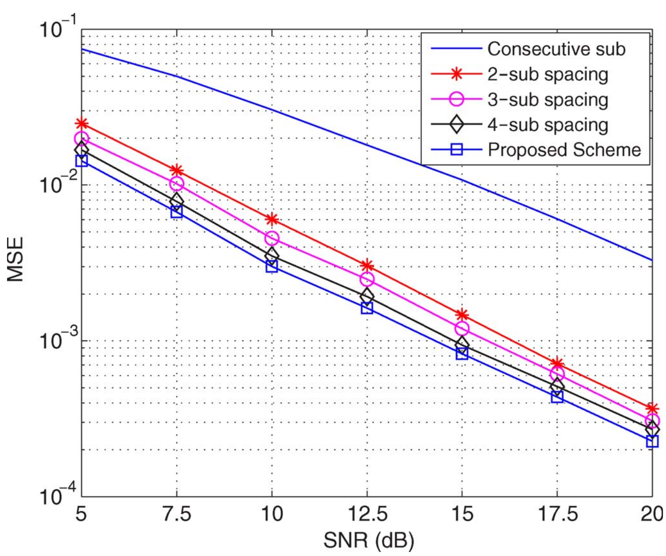

Fig. 2. MSE performance of the CFO estimation using different null subcarrier placements.

placement, the CFO estimation accuracy is improved significantly. The performance gain, compared to the consecutive null subcarrier placement, is as large as 10 dB. We can also see that the further apart the null subcarriers are placed, the better the MSE performance will become. Although we could not prove the optimality of the null subcarrier placement that is obtained from the heuristic method in Table I, from the results, we can see that it still leads to very good performance in the CFO estimation. The symbol error rate (SER) performance is shown in Fig. 3 for quaternary phase-shift keying modulations. From the SER performance, we can see a performance gain of 3.5 dB compared to the consecutive null subcarrier placement.

Fig. 4 shows the improvement in the CFO estimation that is achieved by introducing a few optimally placed free null subcarriers besides the guard null subcarriers. The system follows the IEEE 802.11a specifications with 11 guard null subcarriers. The CFO value that we used in the simulation is, again, uniformly distributed between -0.5 and $+0.5$. We can see that, by introducing two extra free null subcarriers,

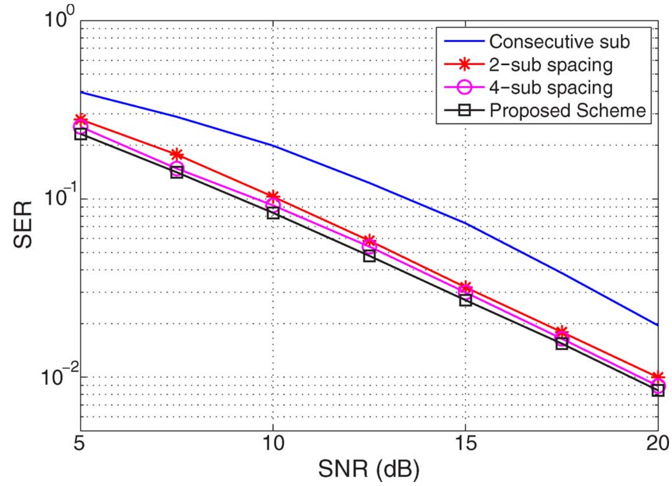

Fig. 3. SER performance of systems using CFO estimation with different null subcarrier placements.

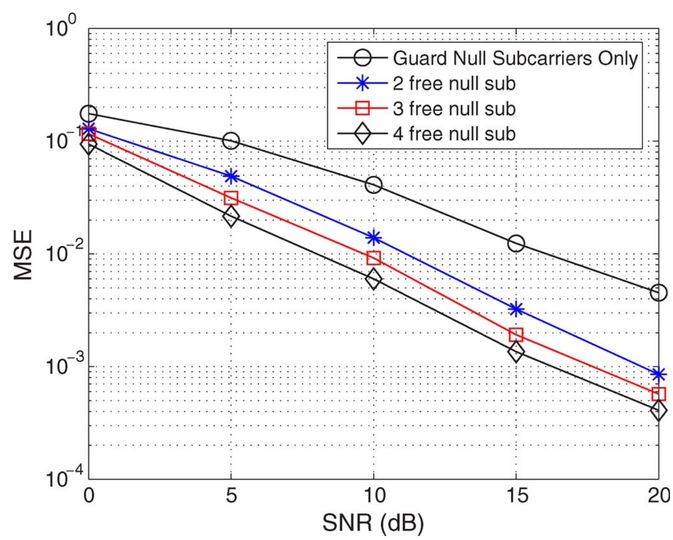

Fig. 4. MSE performance of the CFO estimation for OFDM systems with guard bands and a different number of optimally placed free null subcarriers.

the performance of the CFO estimation could be improved by 5 dB compared to using guard null subcarriers alone. The performance can be further improved by introducing more free null subcarriers. The gain, on the other hand, becomes smaller as the number of free null subcarriers increases.

VI. Conclusion

In this paper, we have formulated the null subcarrier placement problem for blind CFO estimation in an OFDM system using the $\mathrm{SNR}_{\mathrm{CFO}}$ maximization criterion. We have showed that for small CFO values, this leads to a convex optimization problem, and the optimal placement is achieved by placing the null subcarriers evenly across the OFDM symbol. We have proved that this optimal null subcarrier placement also minimizes the theoretical MSE, which is an accurate approximation of the MSE of the CFO estimation in the high SNR region. For systems where the number of subcarriers is divisible by the number of null subcarriers, this optimal placement can be achieved. Otherwise, based on a heuristic procedure, we have showed how to place the null subcarriers such that good performance in the CFO estimation can still be achieved. We have also studied the optimal free null subcarrier placement problem for practical OFDM systems with guard bands. We have demonstrated that the proposed null subcarrier placement significantly improves the performance of the CFO estimation.

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Performance of Variable-Power Adaptive Modulation With Space–Time Coding and Imperfect CSI in MIMO Systems

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***Abstract*—The performance analysis of multi-input–multi-output (MIMO) systems with M-ary quadrature amplitude modulation (MQAM) and a space–time block code (STBC) over flat Rayleigh fading channels for imperfect channel state information (CSI) is presented. In this paper, the optimum fading gain switching thresholds for attaining maximum spectrum efficiency (SE) subject to a target bit-error rate (BER) and an average power constraint are derived. It is shown that the Lagrange multiplier in the constrained SE optimization does exist and is unique for imperfect CSI and for single-input–single-output (SISO) systems under perfect CSI. On the other hand, the Lagrange multiplier will be unique if the existence condition for MIMO under perfect CSI is satisfied. Numerical evaluation shows that the variable-power (VP) adaptive modulation (AM) with STBC provides better SE than its constant-power (CP) counterpart.**

***Index Terms*—Adaptive modulation (AM), multi-input–multi-output (MIMO) system, space–time coding, spectrum efficiency (SE), variable power (VP).**

I. INTRODUCTION

Adaptive modulation (AM) is a powerful technique for improving the spectrum efficiency (SE), which can take advantage of the time-varying nature of wireless channels to transmit data at higher rates under favorable channel conditions and to maintain the bit error rate (BER) by varying the transmit power and symbol rate under poor channel conditions [1]–[3]. The multiple-antenna approach is another well-known SE technique with diversity and/or coding gain. In particular, space–time coding in a multiantenna system provides effective transmit diversity for combating fading effects [4], [5]. Therefore, the effective combination of AM and multiple-antenna technique has received much attention in the literature [6]. Most of the above systems, however, employ constant-power (CP) AM schemes. This CP approach restricts the systems performance because the freedom of a variable power (VP) has been ignored.

VP in AM has been considered in [7]–[9], which are mostly single-input–single-output (SISO) systems. For the optimization of the SE for most of the aforementioned schemes, the Lagrange multiplier technique is used to integrate constraints to the optimization problem. However, the existence and uniqueness of the Lagrange multiplier have yet to be studied in the literature. Furthermore, no practical algorithm for computing the Lagrange multiplier has been developed.

The notations we use throughout this paper are as follows. Bold uppercase and lowercase letters denote matrices and column vectors, respectively. The superscripts $(\cdot)^H$, $(\cdot)^T$, and $(\cdot)^*$ denote the

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