# Mean-field behavior for the survival probability and the point-to-surface connectivity

# Mean-field behavior for the survival probability and the percolation point-to-surface connectivity

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**Abstract:** We consider the critical survival probability (up to time $t$) for oriented percolation and the contact process, and the point-to-surface (of the ball of radius $t$) connectivity for critical percolation. Let $\theta_t$ denote both quantities. We prove in a unified fashion that, if $\theta_t$ exhibits a power law and both the two-point function and its certain restricted version exhibit the same mean-field behavior, then $\theta_t \asymp t^{-1}$ for the time-oriented models with $d > 4$ and $\theta_t \asymp t^{-2}$ for percolation with $d > 7$.

*Keywords: Percolation; oriented percolation; the contact process; survival probability; point-to-surface connectivity; critical exponents; mean-field behavior.*
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# 1 Introduction

Percolation, oriented percolation and the contact process are known to exhibit a phase transition. Various interesting properties around the model-dependent critical point $p_c$ have been studied and revealed, but still there are many open problems. One of the most important problems is to investigate *critical exponents* that characterize singular behavior of observables. Some of them were identified in certain situations.

In this paper, we consider the critical survival probability up to time $t$ for oriented percolation and the contact process, and the probability of the origin $o \in \mathbb{Z}^d$ being connected to the surface of the ball of radius $t$, centered at the origin, for critical percolation. Since the survival probability is a time-oriented version of the point-to-surface connectivity, we denote both quantities by $\theta_t$. It is believed that $\theta_t$ exhibits a power law: $\theta_t \approx t^{-1/\delta_r}$ as $t \to \infty$ (in some appropriate sense). In the percolation school, $\delta_r$ is sometimes called the *one-arm exponent*. Lawler, Schramm and Werner proved $\delta_r = 48/5$ for the two-dimensional site percolation on the triangular lattice, using the estimates for the stochastic Loewner evolution with parameter 6 (see [21] for a precise statement). Except for this result, there has been no proof of existence of $\delta_r$, or identification of its values for finite-range models in mathematically rigorous manner, even in high dimensions.

In contrast, the behavior of the two-point function is well-understood in high dimensions. For percolation, the two-point function at $p_c$, denoted $\tau(x)$, is the probability of $o, x \in \mathbb{Z}^d$ being connected to each other, defined at $p_c$. It has been proved that $\tau(x) \asymp |x|^{-(d-2+\eta)}$ as $|x| \to \infty$ with $\eta = 0$ when $d > 6$ and the number $N$ of neighbors is sufficiently large [9, 10], where "$\asymp$" means that the left-hand side divided by the right-hand side is bounded away from zero and infinity. For the time-oriented models, the two-point function at $p_c$, denoted $\tau_t(x)$, is, in terms of the contact process, the probability of $x \in \mathbb{Z}^d$ being infected at time $t$ by the infected individual at $o \in \mathbb{Z}^d$ at time 0, defined at $p_c$. It has been proved that $\sup_x \tau_t(x) \asymp t^{-d/\alpha}$, $\hat{\tau}_t \equiv \sum_x \tau_t(x) \asymp t^\eta$ and $\sum_x |x|^2 \tau_t(x)/\hat{\tau}_t \asymp t^{2\nu}$ as $t \to \infty$, with $\alpha = 2$, $\eta = 0$ and $\nu = 1/2$, when the spatial dimension $d$ is above 4 and $N$ is sufficiently large [17, 19, 20, 23]. These dimension-independent values of the critical exponents are equal to the values for branching random walk (*mean-field model*). Let $\rho$ ($\risingdotseq 1/\delta_r$) be defined by $\theta_t \asymp t^{-\rho}$ as $t \to \infty$. It is not so hard to see that $\eta = 0$ implies $\rho \le 2$ for percolation and $\rho \le 1$ for the time-oriented models (see Section 3.1), where the upper bounds are the mean-field values of $\rho$.

On the other hand, the critical exponents are known to satisfy the so-called *hyperscaling inequalities*, e.g., $d - 2 + \eta \ge 2\rho$ for percolation [27] and $d\nu \ge \eta + 2\rho$ for the time-oriented models [25, (5.2) and (5.4)], where the critical exponents were defined in a wider sense. Other hyperscaling inequalities were also derived in [7, 25, 27]. By those inequalities, the mean-field values are known to be incompatible with $d < 6$ for percolation and with $d < 4$ for the time-oriented models. These threshold dimensions are called the *upper critical dimensions* for the corresponding models.

In this paper, we prove in a unified way that $\rho$ takes on the mean-field values for the time-oriented models with $d > 4$ and for percolation with $d > 7$, if $\rho$ exists and both the two-point function and its certain restricted version exhibit the same mean-field behavior (see Assumption 2.1). The assumption on the restricted two-point function is expected to hold above the upper critical dimension for each model, but is still insufficient to extend $\rho = 2$ for percolation down to $d > 6$. For sufficiently spread-out oriented percolation with $d > 4$, the asymptotic behavior of $\theta_t$ with $\rho = 1$ will be reported in [15, 16], without any assumption on the restricted two-point function. In this respect, our results are not so strong as the results in [15, 16] for oriented percolation. However, the approach reported in this paper is short and intuitive, and more importantly, gives a unified approach for both the time-oriented models and percolation. We expect that, with the help of the *random-current representation* [1], our unified approach could be applied to the single-spin expectation $\langle\sigma_o\rangle_t$ for Ising ferromagnet in the box of side length $t$ (with plus-boundary condition), and result in the mean-field behavior, i.e., $\langle\sigma_o\rangle_t \asymp t^{-1}$ as $t \to \infty$, at the critical temperature in high dimensions. This will be discussed in [26].

We organize the rest of this paper as follows. In Section 2, we define the models and state the main result. A brief explanation of the proof is given at the end of Section 2, and the detailed proof is given in Section 3.

# 2 Models and the results

## 2.1 Models

We consider the $d$-dimensional integer lattice $\mathbb{Z}^d$ as space. For $L \geq 1$, let

$$\Omega = \{x \in \mathbb{Z}^d : 0 < |x| \leq L\}, \qquad\qquad D(x) = N^{-1}\,\mathbb{1}_{\{x\in\Omega\}}, \tag{2.1}$$

where $|x|$ is the Euclidean norm of $x$, $N$ is the cardinality of $\Omega$, and $\mathbb{1}_{\{\cdots\}}$ is the indicator function. The model with $L = 1$ is the *nearest-neighbor model*, where $N = 2d$. We call the model with $L > 1$ the *spread-out model*, where $N = O(L^d)$ (see, e.g., [17] for a more general definition). Our models are defined in terms of $D$ as follows.

*Percolation.* A bond $\{x, y\}$ is an unordered pair of distinct sites in $\mathbb{Z}^d$, and is *occupied* with probability $p\,D(y - x)$ and *vacant* with probability $1 - p\,D(y - x)$, independently of the other bonds, where $p \in [0, N]$ is the expected number of occupied bonds growing out of a single site. We denote by $\mathbb{P}_p$ the probability distribution for the bond variables. We say that $x$ is *connected to* $y$, and write $x \leftrightarrow y$, if either $x = y$ or there is a path of occupied bonds between $x$ and $y$. We define $\mathcal{C}(x) = \{y \in \mathbb{Z}^d : x \leftrightarrow y\}$. For $\mathcal{Z} \subset \mathbb{Z}^d$, we write $\{x \leftrightarrow \mathcal{Z}\} = \{\mathcal{C}(x) \cap \mathcal{Z} \neq \varnothing\}$.

It is known that there is a critical value $p_c = p_c(d, L) \geq 1$ such that $\sum_x \mathbb{P}_p(o \leftrightarrow x)$ is finite if and only if $p < p_c$ and diverges as $p \uparrow p_c$. Let

$$\mathcal{B}_t = \{x \in \mathbb{Z}^d : |x| \leq t\}, \qquad \partial\mathcal{B}_t = \{x \in \mathbb{Z}^d : t \leq |x| \leq t + L\}. \tag{2.2}$$

and define the two-point function and the point-to-surface connectivity at $p_c$ as

$$\tau(x) = \mathbb{P}_{p_c}(o \leftrightarrow x), \qquad \theta_t = \mathbb{P}_{p_c}(o \leftrightarrow \partial\mathcal{B}_t). \tag{2.3}$$

We are interested in the critical exponents $\eta$ and $\rho$, defined by

$$\tau(x) \asymp |||x|||^{-(d-2+\eta)}, \qquad \theta_t \asymp |||t|||^{-\rho}, \tag{2.4}$$

where $f \asymp g$ means that $f/g$ is bounded away from zero and infinity, and where $|||\cdot||| = |\cdot| \vee 1$. Note that $|||\cdot|||$ is not a norm on $\mathbb{R}^d$, but it satisfies the following properties: for $x, y \in \mathbb{R}^d$ and $r > 0$,

$$|||x + y||| \leq |||x||| + |||y|||, \qquad |||rx||| \begin{cases} \leq r|||x|||, & \text{if } r \geq 1, \\ \geq r|||x|||, & \text{if } r < 1. \end{cases} \tag{2.5}$$

We also note that the above definition of $\rho$ is based on the assumption that $\theta_t$ decays as $t \to \infty$. This has been confirmed only when $d = 2$ or $d \geq 19$ with $L = 1$, and $d > 6$ with $L \gg 1$ (see, e.g., [8, 12]).

It has been proved that $\eta = 0$ for the nearest-neighbor model with $d \gg 6$ [9] and for the spread-out model with $d > 6$ and $L \gg 1$ [10]. The critical exponent $\eta$ is believed to be independent of the range $L$, as long as it is finite (*universality*), and thus is expected to be zero for all $d > 6$ and $L \geq 1$. This dimension-independent value of $\eta$ equals the corresponding value for the mean-field model. Various other critical exponents are also known to take on their respective mean-field values, if (see [3] and references therein)

$$\triangledown_\ell \equiv \sup_{x \notin \mathcal{B}_\ell} (\tau * D * \tau * \tau)(x) \to 0, \qquad \text{as } \ell \to \infty, \tag{2.6}$$

where "$*$" represents a convolution in $\mathbb{Z}^d$. With the help of [10, Proposition 1.7(i)], $\eta = 0$ implies $\triangledown_\ell = O(|||\ell|||^{-(d-6)})$ if $d > 6$, and thus implies the mean-field values for all the other critical exponents, except for $\rho$ until now.

*Oriented percolation and the contact process.* We begin with oriented percolation. A bond $((x, t), (y, t+1))$ is an ordered pair of sites in $\mathbb{Z}^d \times \mathbb{Z}_+$, and is *occupied* with probability $p\,D(y-x)$ and *vacant* with probability $1-p\,D(y-x)$, independently of the other bonds, where $p \in [0, N]$. We say that $(x, s)$ is *connected to* $(y, t)$, and write $(x, s) \to (y, t)$, if either $(x, s) = (y, t)$ or there is an oriented path of occupied bonds from $(x, s)$ to $(y, t)$. Let $\mathcal{C}(x, s) = \{(y, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ : (x, s) \to (y, t)\}$. For $\mathcal{Z} \subset \mathbb{Z}^d \times \mathbb{Z}_+$, we define $\{(x, s) \to \mathcal{Z}\} = \{\mathcal{C}(x, s) \cap \mathcal{Z} \neq \varnothing\}$.

The contact process is a model for the spread of an infection in $\mathbb{Z}^d$, and is regarded as continuous-time oriented percolation in $\mathbb{Z}^d \times \mathbb{R}_+$, via the following *graphical representation*. Along each time line $\{x\} \times \mathbb{R}_+$, we place points in the manner of a Poisson process with intensity 1, independently of the other time lines. For each ordered pair of distinct time lines from $\{x\} \times \mathbb{R}_+$ to $\{y\} \times \mathbb{R}_+$, we place oriented bonds $((x,t),(y,t))$, $t \geq 0$, in the manner of a Poisson process with intensity $p\,D(y-x)$, independently of the other Poisson processes, where $p \geq 0$ is the infection rate. We say that $(x,s)$ is *connected to* $(y,t)$, and write $(x,s) \to (y,t)$, if either $(x,s) = (y,t)$ or there is an oriented path in $\mathbb{Z}^d \times \mathbb{R}_+$ from $(x,s)$ to $(y,t)$ using the Poisson bonds and time-line segments traversed in the increasing-time direction without traversing the Poisson points. We define $\mathcal{C}(x,s)$ and $\{(x,s) \to \mathcal{Z}\}$ for $\mathcal{Z} \subset \mathbb{Z}^d \times \mathbb{R}_+$ similarly to oriented percolation.

We denote by $\mathbb{P}_p$ the probability distributions for these time-oriented models. It is known that there is a critical value $p_c = p_c(d,L) \geq 1$, depending on the models, such that the sum over $t \in \mathbb{Z}_+$ of $\sum_x \mathbb{P}_p((o,0) \to (x,t))$ for oriented percolation, or the integral of $\sum_x \mathbb{P}_p((o,0) \to (x,t))$ with respect to $t \in \mathbb{R}_+$ for the contact process, is finite if and only if $p < p_c$ and diverges as $p \uparrow p_c$. Let

$$\mathcal{B}_t = \mathbb{Z}^d \times [0,t], \qquad \partial\mathcal{B}_t = \mathbb{Z}^d \times \{t\}, \tag{2.7}$$

and define the two-point function and the survival probability at $p_c$ as

$$\tau_t(x) = \mathbb{P}_{p_c}((o,0) \to (x,t)), \qquad \theta_t = \mathbb{P}_{p_c}((o,0) \to \partial\mathcal{B}_t). \tag{2.8}$$

We are interested in the critical exponents $\alpha$, $\eta$, $\nu$ and $\rho$, defined by

$$\bar{\tau}_t \equiv \sup_{x \in \mathbb{Z}^d} \tau_t(x) \asymp \|\!|t|\!\|^{-d/\alpha}, \qquad \hat{\tau}_t \equiv \sum_{x \in \mathbb{Z}^d} \tau_t(x) \asymp \|\!|t|\!\|^{\eta}, \tag{2.9}$$

$$\sum_{x \in \mathbb{Z}^d} |x|^2 \frac{\tau_t(x)}{\hat{\tau}_t} \asymp \|\!|t|\!\|^{2\nu}, \qquad \theta_t \asymp \|\!|t|\!\|^{-\rho}, \tag{2.10}$$

where, by analogy, we used the same letters $\eta$ and $\rho$ for the critical exponents of the spatial sum of the two-point function and the survival probability, respectively.

It has been proved that $(\alpha,\eta,\nu) = (2,0,\frac{1}{2})$ for the time-oriented models with $d > 4$ and $L \gg 1$ [17, 20]. The same result except for $\alpha = 2$ was proved in [23] for nearest-neighbor oriented percolation with $d \gg 4$, but there have been no results on this set of exponents for the nearest-neighbor contact process. Other critical exponents for both the nearest-neighbor and spread-out time-oriented models are known to take on their respective mean-field values, if (see [4] and references therein)

$$\nabla_\ell \equiv \sup_{\substack{x:|x| \geq \ell \\ t \geq 0}} \nabla(x,t) \to 0, \qquad \text{as } \ell \to \infty, \tag{2.11}$$

where, for oriented percolation,

$$\nabla(x,t) = \sum_{\substack{s,s'\in\mathbb{Z}_+:\\ t\le s'\le s}} \sum_{y\in\mathbb{Z}^d} \tau_{s+1}(y)\ (\tau_{s'-t} * D * \tau_{s-s'})(y-x), \tag{2.12}$$

and for the contact process,

$$\nabla(x,t) = \int_t^\infty ds \int_t^s ds' \sum_{y\in\mathbb{Z}^d} \tau_s(y)\ (\tau_{s'-t} * D * \tau_{s-s'})(y-x). \tag{2.13}$$

Since the range of the set of infected sites almost surely grows at most linearly [5], $(\alpha,\eta) = (2,0)$ implies $\nabla_\ell = O(|||\ell|||^{-(d-4)/2})$ if $d > 4$, and thus implies the mean-field values for all the other critical exponents than $\rho$.

## 2.2 Results

In this paper, we prove in a unified fashion for all three models that the mean-field behavior for the two-point function implies the mean-field values of $\rho$, assuming existence of $\rho$ and the following assumption.

**Assumption 2.1.** *There are positive constants $C_1 = C_1(d,L)$ and $C_2 = C_2(d,L)$ that are independent of $t$ such that, for the time-oriented models,*

$$\sum_{(x,s)\in\mathcal{B}_{t/2}} \mathbb{P}_{p_c}((o,0)\to(x,s),\ (o,0)\not\to\partial\mathcal{B}_t) \ge C_1|||t|||, \tag{2.14}$$

*and for percolation,*

$$\sum_{x\in\mathcal{B}_{t/2+L}} \mathbb{P}_{p_c}(o\leftrightarrow x,\ o\not\leftrightarrow\partial\mathcal{B}_t) \ge C_2|||t|||^2, \tag{2.15}$$

*where $\mathcal{B}_{t/2+L} = \mathcal{B}_{t/2}\cup\partial\mathcal{B}_{t/2}$.*

The unrestricted two-point functions defined in (2.3) and (2.8), with $\eta = 0$, satisfy the above inequalities. Therefore, Assumption 2.1 states, in a weak sense, that the above restricted two-point functions exhibit the same mean-field behavior as the unrestricted two-point functions.

**Theorem 2.2.** *Suppose that $\eta = 0$ and $\alpha = 2$ (the latter is only for the time-oriented models). If $\rho$ exists and Assumption 2.1 holds, then $\rho = 1$ for the time-oriented models with $d > 4$ and $\rho = 2$ for percolation with $d > 7$.*

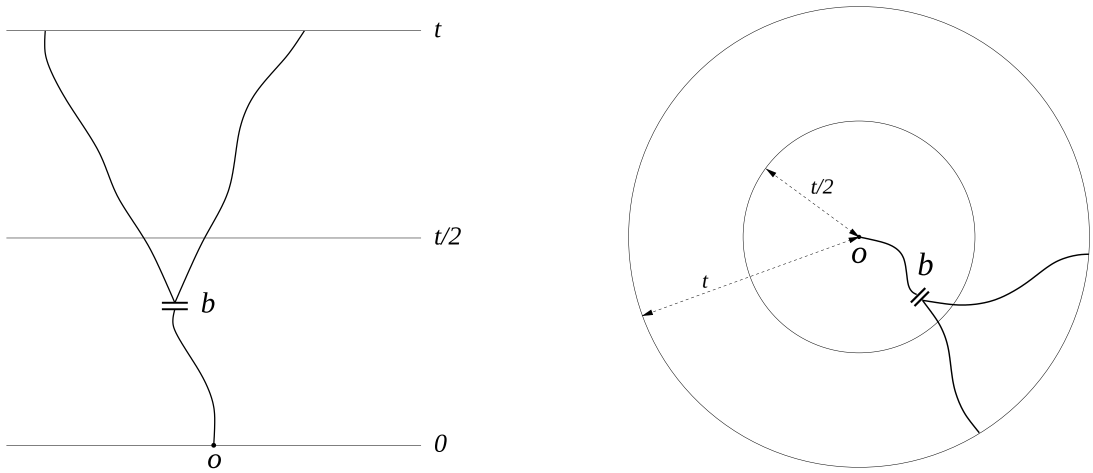


Figure 1: Typical configurations for $\theta_t$.

We briefly explain the main idea of the proof. It is easy to show that $\eta = 0$ implies $\rho \le 1$ for the time-oriented models and $\rho \le 2$ for percolation (see Section 3.1). It thus suffices to prove the opposite inequalities for $\rho$. Let us consider typical configurations for $\theta_t$. When $t \gg 1$, there may be a *pivotal bond* for the connection from the origin to the boundary $\partial\mathcal{B}_t$. We take notice of the *last* pivotal bond $b$, where we have a connection from the origin to the first endpoint of $b$ and two disjoint connections from the second endpoint of $b$ to $\partial\mathcal{B}_t$ (see Figure 1). If we could bound the probability of these configurations from below by $\theta_t^2$ times the sum of the *unrestricted* two-point function (over $b = (\underline{b}, \overline{b})$ with $\overline{b} \in \mathcal{B}_{t/2}$, as in Figure 1), then $\eta = 0$ implies

$$t^{-\rho} \ge \begin{cases} ct^{1-2\rho}, & \text{for the time-oriented models,} \\ ct^{2-2\rho}, & \text{for percolation,} \end{cases} \tag{2.16}$$

for some positive constant $c$, and thus $\rho \ge 1$ for the time-oriented models and $\rho \ge 2$ for percolation.

To realize the above idea, we have to control the correction. As we will show in Section 3.2, most error terms can be made small by letting $\nabla_\ell \ll 1$ and $t \gg 1$ in high dimensions. However, the correction due to the above approximation using the unrestricted two-point function cannot be controlled by a finite number of applications of the *BK inequality* (see, e.g., [6, 8]), and here we will use Assumption 2.1. The desired asymptotic behavior of $\theta_t$ for spread-out oriented percolation with $d > 4$ and $L \gg 1$ will be reported in [15, 16], with no assumption on the restricted two-point function. The proof in [15, 16] is based on the *lace expansion* for $\theta_t$, and the difference between the restricted and unrestricted two-point functions is efficiently taken into account along the expansion. Our proof of Theorem 2.2 does not depend on the full expansion as in [15, 16], and Assumption 2.1 is inevitable.

We remark that Assumption 2.1 is still insufficient to fully control the boundary effect and thus to obtain $\rho = 2$ for percolation with $d > 6$. To improve the result down to $d > 6$, we may also need some information on the restricted two-point function close to the boundary (see Remark at the end of Section 3.2).

# 3 Proofs

We prove Theorem 2.2 in two steps. First, in Section 3.1, we prove that $\eta = 0$ implies $\rho \leq 1$ for the time-oriented models and $\rho \leq 2$ for percolation. Then, in Section 3.2, we prove that $\eta = 0$ and $\alpha = 2$ (the latter is only for the time-oriented models) imply the opposite inequalities for $\rho$, if $d > 4$ for the time-oriented models and $d > 7$ for percolation, assuming existence of $\rho$ and Assumption 2.1.

In the rest of this paper, we omit the subscript $p_{\rm c}$ and write $\mathbb{E}$ for the expectation with respect to $\mathbb{P} = \mathbb{P}_{p_{\rm c}}$. We will use $c$ to denote a finite positive constant which may depend on $d$ and $L$, but whose exact value is unimportant and may change from line to line.

## 3.1 Proof of the upper bound

*Proof for the time-oriented models.* Let

$$I_t = \mathbb{1}_{\{(o,0)\to\partial\mathcal{B}_t\}}, \qquad\qquad X_t = \sum_{x\in\mathbb{Z}^d} \mathbb{1}_{\{(o,0)\to(x,t)\}}, \tag{3.1}$$

so that $\mathbb{E}(I_t) = \theta_t$ and $\mathbb{E}(X_t) = \hat{\tau}_t$. By the Schwarz inequality, we obtain

$$\hat{\tau}_t^2 = \mathbb{E}(I_t X_t)^2 \leq \mathbb{E}(I_t^2)\ \mathbb{E}(X_t^2) = \theta_t \sum_{x,y} \mathbb{P}_{p_{\rm c}}((o,0)\to(x,t),\ (o,0)\to(y,t)). \tag{3.2}$$

If $(o,0)\to(x,t)$ and $(o,0)\to(y,t)$ occur simultaneously, then there exists a $(z,s)\in\mathcal{B}_t$ such that $(o,0)\to(z,s)$ occurs and that $(z,s)\to(x,t)$ and $(z,s)\to(y,t)$ occur *disjointly*, i.e., on disjoint sets of bonds. Using the Markov property, the BK inequality and $\eta = 0$, we can bound the sum in (3.2) by

$$\int_0^t ds \sum_{x,y,z\in\mathbb{Z}^d} \tau_s(z)\ \tau_{t-s}(x-z)\ \tau_{t-s}(y-z) = \int_0^t ds\ \hat{\tau}_s\ \hat{\tau}_{t-s}^2 \leq c|\!|\!|t|\!|\!|. \tag{3.3}$$

(The integral is replaced by $\sum_{s=0}^t$ for oriented percolation.) Together with (3.2), we thus obtain $\rho \leq 1$, if $\rho$ exists. $\square$

*Remark.* For spread-out oriented percolation with $d > 4$ and $L \gg 1$, Theorem 4.1 and Lemma 4.2 in [14] imply that the left-hand side of (3.2) is asymptotically $A^2$, while the sum in the right-hand side of (3.2) is asymptotically $A^3Vt$, where $A$ and $V$ are constants depending only on $d$ and $L$. This leads to a lower bound on $\theta_t$ like $(AVt)^{-1}$, which is consistent with [14, Theorem 1.5], where the limit $\lim_{t\to\infty} t\,\theta_t$, if it exists, equals $2(AV)^{-1}$.

*Proof for percolation.* We follow the same strategy as above. Let

$$I_t = \mathbb{1}_{\{o\leftrightarrow\partial\mathcal{B}_t\}}, \qquad X_t = \sum_{x\in\partial\mathcal{B}_t} \mathbb{1}_{\{o\leftrightarrow x\}}. \tag{3.4}$$

Using the Schwarz inequality as in (3.2), we obtain

$$\Big[\sum_{x\in\partial\mathcal{B}_t} \tau(x)\Big]^2 = \mathbb{E}(I_tX_t)^2 \le \mathbb{E}(I_t^2)\;\mathbb{E}(X_t^2) = \theta_t \sum_{x,y\in\partial\mathcal{B}_t} \mathbb{P}_{p_c}(o\leftrightarrow x,\ o\leftrightarrow y). \tag{3.5}$$

Since $\eta = 0$, the leftmost quantity is bounded from below by $c|\!|\!|t|\!|\!|^2$. If $o\leftrightarrow x$ and $o\leftrightarrow y$ occur simultaneously, then there is a $z\in\mathbb{Z}^d$ such that $o\leftrightarrow z$, $z\leftrightarrow x$ and $z\leftrightarrow y$ occur disjointly. By the BK inequality and $\eta = 0$, the sum in the right-hand side of (3.5) is bounded by

$$\begin{aligned}\sum_{\substack{x,y\in\partial\mathcal{B}_t\\ z\in\mathbb{Z}^d}} \tau(z)\;\tau(x-z)\;\tau(y-z) &= \sum_{\substack{x,y\in\partial\mathcal{B}_t\\ z\in\mathcal{B}_{t/2}}} \tau(z)\;\tau(x-z)\;\tau(y-z) + \sum_{\substack{x,y\in\partial\mathcal{B}_t\\ z\notin\mathcal{B}_{t/2}}} \tau(z)\;\tau(x-z)\;\tau(y-z)\\ &\le c|\!|\!|t|\!|\!|^{2(2-d)+2(d-1)} \sum_{z\in\mathcal{B}_{t/2}} |\!|\!|z|\!|\!|^{2-d} + c|\!|\!|t|\!|\!|^{2-d} \sum_{\substack{x,y\in\partial\mathcal{B}_t\\ z\in\mathbb{Z}^d}} |\!|\!|x-z|\!|\!|^{2-d}\;|\!|\!|y-z|\!|\!|^{2-d},\end{aligned} \tag{3.6}$$

where we used $|x-z| \ge t/2$ and $|y-z| \ge t/2$ in the first sum, and $|z| \ge t/2$ in the second sum. By [10, Proposition 1.7(i)], the convolution of $|\!|\!|x-z|\!|\!|^{2-d}$ and $|\!|\!|y-z|\!|\!|^{2-d}$ is bounded by $c|\!|\!|x-y|\!|\!|^{4-d}$, whose sum over $x,y\in\partial\mathcal{B}_t$ is bounded by $c|\!|\!|t|\!|\!|^{2(d-1)+4-d} = c|\!|\!|t|\!|\!|^{d+2}$. Therefore, (3.1) is bounded by $c|\!|\!|t|\!|\!|^4$, and we obtain $\rho \le 2$ using (3.5). □

## 3.2 Proof of the lower bound

In this section, we will use $\epsilon = \epsilon(\rho)$ defined by

$$\epsilon(\rho)\begin{cases} > 0 \text{ (but arbitrarily small)}, & \text{if } \rho = 1,\\ = 0, & \text{if } \rho \ne 1,\end{cases} \tag{3.7}$$

for both the time-oriented models and percolation.

*Proof for the time-oriented models.* We only consider oriented percolation, since the same idea given below also applies to the time-discretized contact process in [17, 24] that weakly converges to the original contact process as the discretized-time unit tends to zero. We prove below

$$\theta_t \ge c\,[1 - O(\bar\nabla) - O(|\!|\!|t|\!|\!|^{-(d-4)/2+\epsilon})]\;|\!|\!|t|\!|\!|^{1-2\rho}, \tag{3.8}$$

and thus prove Theorem 2.2 for the time-oriented models, assuming $\bar{\nabla} \equiv \sup_x \nabla(x,0) \ll 1$. In the proof of (3.8), we will require $p_c \le 3/2$, which is a consequence of $\bar{\nabla} \ll 1$, if $d > 4$ [18, 22, 24]. We will also assume existence of a constant $a > 1$, which is independent of $d$ and $L$, such that $\sum_{s \le t/2} \hat{\tau}_s \le aC_1 \|t\|$ (cf., (2.14)) and $K \le \theta_t \|t\|^\rho \le aK$ for some $K > 0$, which may depend on $d$ and $L$. After the proof, we briefly discuss how to remove all these extra assumptions.

The survival probability $\theta_t$ is the probability of the event that there is a path of occupied bonds from $(o,0)$ to $\partial\mathcal{B}_t$. This event can be decomposed into two disjoint events depending on whether or not $(o,0)$ is *doubly connected* to $\partial\mathcal{B}_t$, denoted by $(o,0) \rightrightarrows \partial\mathcal{B}_t$, which means that there are at least two bond-disjoint occupied paths from $(o,0)$ to $\partial\mathcal{B}_t$. If $(o,0)$ is connected but not doubly connected to $\partial\mathcal{B}_t$, then there is an occupied *pivotal bond* $b = (\underline{b}, \bar{b})$ for $(o,0) \to \partial\mathcal{B}_t$ such that $(o,0) \to \underline{b}$, $\bar{b} \rightrightarrows \partial\mathcal{B}_t$ and $\mathcal{C}^b(o,0) \cap \partial\mathcal{B}_t = \varnothing$, where $\mathcal{C}^b(o,0)$ is the set of sites in $\mathbb{Z}^d \times \mathbb{Z}_+$ connected from $(o,0)$ without using $b$. Restricting the location of $\bar{b}$ in $\mathcal{B}_{t/2}$ gives

$$\theta_t \ge \sum_{b:\bar{b} \in \mathcal{B}_{t/2}} \frac{1}{N} \mathbb{P}\big((o,0) \to \underline{b},\ \bar{b} \rightrightarrows \partial\mathcal{B}_t,\ \mathcal{C}^b(o,0) \cap \partial\mathcal{B}_t = \varnothing\big), \tag{3.9}$$

where we used $p_c \ge 1$.

To investigate the right-hand side of the above inequality, we introduce the following two notions. For an event $E$ and $\mathcal{Z} \subset \mathbb{Z}^d \times \mathbb{Z}_+$, let $\{E \text{ on } \mathcal{Z}\}$ be the set of bond configurations whose restriction on bonds $b$ *touching* $\mathcal{Z}$ (i.e., $\underline{b}$ or $\bar{b}$ is in $\mathcal{Z}$) are in $E$. Similarly, we define the event $\{E \text{ in } \mathcal{Z}\}$ to be the set of bond configurations whose restriction on bonds $b$ *contained in* $\mathcal{Z}$ (i.e., both $\underline{b}$ and $\bar{b}$ are in $\mathcal{Z}$) are in $E$. Then, we can rewrite the probability in the right-hand side of (3.9) as (see [13, Lemma 2.5])

$$\mathbb{P}\big(\{(o,0) \to \underline{b},\ \mathcal{C}^b(o,0) \cap \partial\mathcal{B}_t = \varnothing\} \text{ on } \mathcal{C}^b(o,0),\ \{\bar{b} \rightrightarrows \partial\mathcal{B}_t\} \text{ in } \mathcal{C}^b(o,0)^{\mathrm{c}}\big). \tag{3.10}$$

By the "conditioning on cluster" technique [2, 12, 13], (3.10) equals

$$\begin{aligned} &\mathbb{E}\Big(\mathbb{1}_{\{(o,0) \to \underline{b},\ \mathcal{C}^b(o,0) \cap \partial\mathcal{B}_t = \varnothing\}}\, \mathbb{P}\big(\bar{b} \rightrightarrows \partial\mathcal{B}_t \text{ in } \mathcal{C}^b(o,0)^{\mathrm{c}}\big)\Big) \\ &\quad = \mathbb{P}\big((o,0) \to \underline{b},\ \mathcal{C}^b(o,0) \cap \partial\mathcal{B}_t = \varnothing\big)\, \mathbb{P}(\bar{b} \rightrightarrows \partial\mathcal{B}_t) \\ &\qquad - \mathbb{E}\Big(\mathbb{1}_{\{(o,0) \to \underline{b},\ \mathcal{C}^b(o,0) \cap \partial\mathcal{B}_t = \varnothing\}}\, \big[\mathbb{P}(\bar{b} \rightrightarrows \partial\mathcal{B}_t) - \mathbb{P}\big(\bar{b} \rightrightarrows \partial\mathcal{B}_t \text{ in } \mathcal{C}^b(o,0)^{\mathrm{c}}\big)\big]\Big). \end{aligned} \tag{3.11}$$

First, we consider the first term in (3.11). By translation invariance and monotonicity, $\mathbb{P}(\bar{b} \rightrightarrows \partial\mathcal{B}_t)$ is bounded from below by $\mathbb{P}((o,0) \rightrightarrows \partial\mathcal{B}_t)$. Since $\mathcal{C}^b(o,0) \subset \mathcal{C}(o,0)$, the contribution to (3.9) is bounded from below by

$$\mathbb{P}((o,0) \rightrightarrows \partial\mathcal{B}_t) \sum_{b:\bar{b} \in \mathcal{B}_{t/2}} \frac{1}{N} \mathbb{P}((o,0) \to \underline{b},\ (o,0) \not\to \partial\mathcal{B}_t) \ge C_1 \|t\|\, \mathbb{P}((o,0) \rightrightarrows \partial\mathcal{B}_t), \tag{3.12}$$

where we used the definition of $D$ in (2.1) and Assumption 2.1. We now prove that the right-hand side of (3.12) is bounded from below by the same formula as in the right-hand side of (3.8). By restricting the number of occupied bonds growing out of $(o,0)$ to two, $\mathbb{P}((o,0) \rightrightarrows \partial\mathcal{B}_t)$ can be bounded from below by

$$\Big(\frac{p_{\rm c}}{N}\Big)^2\Big(1-\frac{p_{\rm c}}{N}\Big)^{N-2}\sum_{\langle x,y\rangle}\mathbb{P}((x,1)\to\partial\mathcal{B}_t,\ (y,1)\to\partial\mathcal{B}_t,\ \mathcal{C}(x,1)\cap\mathcal{C}(y,1)=\varnothing\text{ in }\mathcal{B}_t),\tag{3.13}$$

where $\sum_{\langle x,y\rangle}$ is the sum over all pairs of distinct sites in $\Omega$. We note that $p_{\rm c}^2(1-\frac{p_{\rm c}}{N})^{N-2}$ is always bounded from above by an $N$-independent constant, while it is bounded from below by $e^{-1}$ using $p_{\rm c}\le 3/2$. By conditioning on $\mathcal{C}(x,1)$, (3.13) equals

$$\Big(\frac{p_{\rm c}}{N}\Big)^2\Big(1-\frac{p_{\rm c}}{N}\Big)^{N-2}\sum_{\langle x,y\rangle}\mathbb{E}\Big(\mathbb{1}_{\{(x,1)\to\partial\mathcal{B}_t\}}\ \mathbb{P}((y,1)\to\partial\mathcal{B}_t\text{ in }\mathcal{C}(x,1)^{\rm c})\Big).\tag{3.14}$$

If we ignore the condition "in $\mathcal{C}(x,1)^{\rm c}$", we obtain the main contribution $\frac{e^{-1}}{N^2}\binom{N}{2}\theta_t^2\ge\frac{K^2}{4e}|\!|\!|t|\!|\!|^{-2\rho}$. The correction is

$$\Big(\frac{p_{\rm c}}{N}\Big)^2\Big(1-\frac{p_{\rm c}}{N}\Big)^{N-2}\sum_{\langle x,y\rangle}\mathbb{E}\Big(\mathbb{1}_{\{(x,1)\to\partial\mathcal{B}_t\}}\ \mathbb{P}\big(\{(y,1)\to\partial\mathcal{B}_t\}\setminus\{(y,1)\to\partial\mathcal{B}_t\text{ in }\mathcal{C}(x,1)^{\rm c}\}\big)\Big).\tag{3.15}$$

We need an upper bound on (3.15) to obtain a lower bound on the left-hand side of (3.12). Since the event inside $\mathbb{P}$ in (3.15) is the event that all occupied paths from $(y,1)$ to $\partial\mathcal{B}_t$ go through $\mathcal{C}(x,1)$, there must be a $(z,s)\in\mathcal{C}(x,1)$ such that $(y,1)\to(z,s)\to\partial\mathcal{B}_t$. By the Markov property, the expectation in (3.15) is bounded by

$$\begin{aligned}&\mathbb{E}\Big(\mathbb{1}_{\{(x,1)\to\partial\mathcal{B}_t\}}\sum_{(z,s)\in\mathcal{C}(x,1)}\tau_{s-1}(z-y)\ \theta_{t-s}\Big)\\&\quad=\sum_{(z,s)}\mathbb{P}((x,1)\to\partial\mathcal{B}_t,\ (z,s)\in\mathcal{C}(x,1))\ \tau_{s-1}(z-y)\ \theta_{t-s}.\end{aligned}\tag{3.16}$$

We consider $\sum_{s\le t/2}$ and $\sum_{s>t/2}$ separately. For the former sum, we use the BK inequality to bound (3.16) by

$$\sum_{s=2}^{t/2}\sum_{s'=1}^{s}\sum_{z,z'\in\mathbb{Z}^d}\tau_{s'-1}(z'-x)\ \tau_{s-s'}(z-z')\ \tau_{s-1}(z-y)\ \theta_{t-s}\ \theta_{t-s'}.\tag{3.17}$$

Since $t-s'\ge t-s\ge t/2$ and $s\ge 2$ (because $x\ne y$), the contribution to (3.15) is bounded by $4^\rho(aK)^2\bar\nabla|\!|\!|t|\!|\!|^{-2\rho}$, where we used (2.5). On the other hand, we use (3.16) to bound the sum over $s>t/2$. If we ignore the condition $(x,1)\to\partial\mathcal{B}_t$, then (3.16) is bounded by

$$\sum_{s=t/2}^{t}\hat\tau_{s-1}\ \bar\tau_{s-1}\ \theta_{t-s}\le c|\!|\!|t|\!|\!|^{-d/2}\sum_{s=t/2}^{t}|\!|\!|t-s|\!|\!|^{-\rho}.\tag{3.18}$$

Since $\rho \le 1$, the right-hand side is further bounded by $c\|\!|t|\!\|^{-d/2+1-\rho+\epsilon} \le c\|\!|t|\!\|^{-2\rho-(d-4)/2+\epsilon}$. Therefore, (3.12) is bounded from below by

$$\frac{C_1K^2}{4e}\,(1-4^{\rho+1}ea^2\bar{\nabla}-c\|\!|t|\!\|^{-(d-4)/2+\epsilon})\,\|\!|t|\!\|^{1-2\rho}. \tag{3.19}$$

Next, we investigate the second term in (3.11). Note that the event $\{\bar{b} \rightrightarrows \partial\mathcal{B}_t\} \setminus \{\bar{b} \rightrightarrows \partial\mathcal{B}_t \text{ in } \mathcal{C}^b(o,0)^{\mathrm{c}}\}$ implies existence of a $(z,s) \in \mathcal{C}^b(o,0)$ such that $\bar{b} \to \partial\mathcal{B}_t$ and $\bar{b} \to (z,s) \to \partial\mathcal{B}_t$ occur disjointly. By the BK inequality and the definition (2.1), the contribution to (3.9) from the second term in (3.11) is bounded by

$$\begin{aligned}
&\sum_{\substack{(z,s),(v,s'):\\ 1\le s'<t/2}} \sum_{b:\bar{b}=(v,s')} \frac{1}{N}\,\mathbb{P}((o,0)\to \underline{b},\ (z,s)\in\mathcal{C}^b(o,0))\ \tau_{s-s'}(z-v)\ \theta_{t-s}\ \theta_{t-s'}\\
&\le \sum_{\substack{(z,s),(v,s'):\\ 1\le s'<t/2}} \sum_{\substack{(y,r),(u,s'-1)\\ 0\le r<s'}} \tau_r(y)\ \tau_{s'-1-r}(u-y)\ D(v-u)\ \tau_{s-s'}(z-v)\ \tau_{s-r}(z-y)\ \theta_{t-s}\ \theta_{t-s'}\\
&\le \frac{2^\rho aK}{\|\!|t|\!\|^\rho}\sum_{r=0}^{t/2-1}\hat{\tau}_r\sum_{s=r+1}^{t}\sum_{s'=r+1}^{(t/2)\wedge s}\sum_{x\in\mathbb{Z}^d}(\tau_{s'-1-r}*D*\tau_{s-s'})(x)\ \tau_{s-r}(x)\ \theta_{t-s},
\end{aligned} \tag{3.20}$$

where we used $s' \le t/2$ to bound $\theta_{t-s'}$. We separate the sum over $s$ into $\sum_{s\le 3t/4}$ and $\sum_{s>3t/4}$. When $s \le 3t/4$, we bound $\theta_{t-s}$ by $4^\rho aK\|\!|t|\!\|^{-\rho}$, and then bound the remaining term by $\bar{\nabla}\sum_{r=0}^{t/2-1}\hat{\tau}_r \le aC_1\bar{\nabla}\|\!|t|\!\|$. When $s > 3t/4$, we bound $\bar{\tau}_{s-r}$ by $c\|\!|t|\!\|^{-d/2}$ using $r < t/2$, and then bound the remaining term, using $\rho \le 1$, by

$$c\|\!|t|\!\|\sum_{r=0}^{t/2-1}\sum_{s=3t/4}^{t}\|\!|t-s|\!\|^{-\rho} \le c\|\!|t|\!\|^{3-\rho+\epsilon}. \tag{3.21}$$

By summarizing the above estimates, (3.20) is bounded by

$$(8^\rho a^3C_1K^2\bar{\nabla}+c\,\|\!|t|\!\|^{-(d-4)/2+\epsilon})\,\|\!|t|\!\|^{1-2\rho}. \tag{3.22}$$

The proof of (3.8) is completed by (3.19) and (3.22).

We obtain (2.16) from (3.8) if $\bar{\nabla} \ll 1$, $t \gg 1$ and $d > 4$. Together with $\rho \le 1$ proved in Section 3.1, this completes the proof of $\rho = 1$. □

*Remark.* In the above proof, we exploited the assumptions stated below (3.8). These assumptions can be removed via a *delocalization argument* [4] (or, it is also called *ultraviolet regularization* [2, 3, 12]). In fact, we can prove that there is a $c_\ell > 0$ such that

$$t^{-\rho} \ge c_\ell\,[1-O(\nabla_\ell)-O(t^{-(d-4)/2+\epsilon})]\,t^{1-2\rho}, \qquad \text{for } \ t \gg \ell. \tag{3.23}$$

Recall that $(\alpha, \eta) = (2, 0)$ implies $\lim_{\ell\to\infty} \triangledown_\ell = 0$, as explained below (2.11). Taking $\ell$ and $t$ in (3.23) sufficiently large, independently of $d$ and $L$, we obtain (2.16) for the time-oriented models. Therefore, we do not need the extra assumptions stated below (3.8).

We briefly explain the idea for the proof of (3.23). Recall (3.9), where $b$ is the last pivotal bond for $(o, 0) \to \partial\mathcal{B}_t$. The space-time rectangle $\mathcal{R}_\ell(b)$ is defined as

$$\mathcal{R}_\ell(b) = \big\{\underline{b} + (re_b, s) \in \mathbb{Z}^d \times \mathbb{Z}_+ : r \in [-\ell, \ell],\ s \in [0, \ell]\big\}, \tag{3.24}$$

where $e_b = (v - u)/|v - u|$ for $b = ((u, s), (v, s + 1))$. We may modify the occupation status of bonds contained in $\mathcal{R}_\ell(b)$, in order to thin the connection from $(o, 0)$ to $\partial\mathcal{B}_t$. Let $E_{\mathcal{R}_\ell(b)}$ be such an event that $\underline{b}$ is "minimally" connected, via $b$, to both $X_\pm \equiv \underline{b} + (\pm\ell e_b, \ell)$. Then, we obtain (cf., (3.9))

$$\theta_t \geq \sum_{b:\overline{b}\in\mathcal{B}_{t/2}} \mathbb{P}(E_{\mathcal{R}_\ell(b)})\, \mathbb{P}\big((o, 0) \to \underline{b},\ \mathcal{C}^{\mathcal{R}_\ell(b)}(o, 0) \cap \partial\mathcal{B}_t = \varnothing,\ \{X_+ \to \partial\mathcal{B}_t\} \circ \{X_- \to \partial\mathcal{B}_t\}\big), \tag{3.25}$$

where $E_1 \circ E_2$ is the event that $E_1$ and $E_2$ occur disjointly, and $\mathcal{C}^{\mathcal{R}_\ell(b)}(o, 0)$ is the set of sites connected from $(o, 0)$ without using any bonds contained in $\mathcal{R}_\ell(b)$. In (3.25), we used the fact that $E_{\mathcal{R}_\ell(b)}$ is independent of the other three events in $\mathbb{P}$. We choose $c_\ell = \inf_b \mathbb{P}(E_{\mathcal{R}_\ell(b)})$. For the remaining term, we follow the same strategy as in the proof for the case $\bar{\triangledown} \ll 1$, except that we do not need an argument around (3.13). This leads to (3.23).

It remains to determine $E_{\mathcal{R}_\ell(b)}$. This was well-explained in [4] for the time-discretized contact process. A variant of $E_{\mathcal{R}_\ell(b)}$ in [4] was chosen in such a way that $c_\ell$ is bounded away from zero uniformly in the discretized-time unit. It is not hard to adapt the idea of [4] to our settings, and we refrain from giving its details. See [4, Figure 1].

*Proof for percolation.* The strategy is the same as above. We prove below

$$\theta_t \geq c\,[1 - O(\triangledown_0) - O(|\!|\!|t|\!|\!|^{-(d-5-\rho\vee 1)+\epsilon})]\, |\!|\!|t|\!|\!|^{2-2\rho}, \tag{3.26}$$

for $t \geq 2L$ (so that $\partial\mathcal{B}_{t/2} \subset \mathcal{B}_t$), and hence Theorem 2.2 for percolation, assuming $\triangledown_0 \ll 1$. Similarly to the proof for the time-oriented models, we will also assume that $p_{\mathrm{c}} \leq 3/2$, which is indeed the case when $\triangledown_0 \ll 1$ and $d > 6$ [11, 18], and that there is a $(d, L)$-independent constant $a > 1$ such that $\sum_{x\in\mathcal{B}_{3t/2+L}} \tau(x) \leq aC_2 |\!|\!|t|\!|\!|^2$ (cf., (2.15)) and $K \leq \theta_t |\!|\!|t|\!|\!|^\rho \leq aK$ for some $K > 0$, which may depend on $d$ and $L$. These assumptions can be removed as discussed above and as in [2, 3, 12], and thus we omit its details for simplicity.

The percolation version of the joint inequality of (3.9)–(3.11) is

$$\begin{aligned}\theta_t \geq &\sum_{b:\overline{b}\in\mathcal{B}_{t/2}} \frac{1}{N}\, \mathbb{P}(o \leftrightarrow \underline{b},\ \mathcal{C}^b(o) \cap \partial\mathcal{B}_t = \varnothing)\, \mathbb{P}(\overline{b} \Leftrightarrow \partial\mathcal{B}_t)\\ &- \sum_{b:\overline{b}\in\mathcal{B}_{t/2}} \frac{1}{N}\, \mathbb{E}\Big(\mathbb{1}_{\{o\leftrightarrow\underline{b},\ \mathcal{C}^b(o)\cap\partial\mathcal{B}_t=\varnothing\}}\, \big[\mathbb{P}(\overline{b} \Leftrightarrow \partial\mathcal{B}_t) - \mathbb{P}(\overline{b} \Leftrightarrow \partial\mathcal{B}_t \text{ in } \mathcal{C}^b(o)^{\mathrm{c}})\big]\Big),\end{aligned} \tag{3.27}$$

where "⇔" represents a double connection for percolation. Similarly to the argument around (3.12), by using $\mathbb{P}(\bar{b} \Leftrightarrow \partial\mathcal{B}_t) \geq \mathbb{P}(o \Leftrightarrow \partial\mathcal{B}_{3t/2})$ and $\mathcal{C}^b(o) \subset \mathcal{C}(o)$, together with the definition (2.1) and Assumption 2.1, the first sum in (3.27) is bounded from below by

$$C_2 \|t\|^2 \, \mathbb{P}(o \Leftrightarrow \partial\mathcal{B}_{3t/2}). \tag{3.28}$$

We first prove that (3.28) is bounded from below by the same formula as in the right-hand side of (3.26). There are minor changes to investigate $\mathbb{P}(o \Leftrightarrow \partial\mathcal{B}_{3t/2})$, and now we discuss these modifications. Let $\tilde{\mathcal{C}}_{3t/2}(x) \subset \mathcal{B}_{3t/2+L}$ be the set of sites to which there is an occupied path from $x$ that includes at most one bond touching $\partial\mathcal{B}_{3t/2}$ and no bonds touching $o \in \mathbb{Z}^d$. By restricting the number of occupied bonds touching $o \in \mathbb{Z}^d$ to two, $\mathbb{P}(o \Leftrightarrow \partial\mathcal{B}_{3t/2})$ is bounded from below by (cf., (3.13))

$$\Big(\frac{p_{\rm c}}{N}\Big)^2\Big(1-\frac{p_{\rm c}}{N}\Big)^{N-2}\sum_{\langle x,y\rangle}\mathbb{P}\big(x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c},\ y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c},\ \tilde{\mathcal{C}}_{3t/2}(x)\cap\tilde{\mathcal{C}}_{3t/2}(y)=\varnothing\big). \tag{3.29}$$

By conditioning on $\tilde{\mathcal{C}}_{3t/2}(x)$, the above expression equals

$$\begin{aligned}
&\Big(\frac{p_{\rm c}}{N}\Big)^2\Big(1-\frac{p_{\rm c}}{N}\Big)^{N-2}\sum_{\langle x,y\rangle}\mathbb{E}\Big(\mathbb{1}_{\{x\leftrightarrow\partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c}\}}\,\mathbb{P}\big(y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c}\cap\tilde{\mathcal{C}}_{3t/2}(x)^{\rm c}\big)\Big)\\
&=\Big(\frac{p_{\rm c}}{N}\Big)^2\Big(1-\frac{p_{\rm c}}{N}\Big)^{N-2}\sum_{\langle x,y\rangle}\Big[\mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c})\,\mathbb{P}(y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c})\\
&\quad-\mathbb{E}\Big(\mathbb{1}_{\{x\leftrightarrow\partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c}\}}\,\mathbb{P}\big(\{y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c}\}\setminus\{y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c}\cap\tilde{\mathcal{C}}_{3t/2}(x)^{\rm c}\}\big)\Big)\Big].
\end{aligned} \tag{3.30}$$

Here, we have $\mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c})$, instead of $\mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2})$. The correction is the probability of the event that all occupied paths between $x$ and $\partial\mathcal{B}_{3t/2}$ go through the origin, and thus is bounded by the probability of the event that $x \leftrightarrow o$ and $o \leftrightarrow \partial\mathcal{B}_{3t/2}$ occur disjointly. By the BK inequality and monotonicity, we obtain

$$\mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c}) \geq \mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2}) - \tau(x)\,\theta_{3t/2} \geq \theta_{3t/2+L} - \tau(x)\,\theta_{3t/2}. \tag{3.31}$$

The contribution to (3.30) from $\theta^2_{3t/2+L}$ is bounded from below by $(4^{\rho+1}e)^{-1}K^2\|t\|^{-2\rho}$, where we used $p_{\rm c} \leq 3/2$ (cf., the argument below (3.13)) and $t \geq 2L$ together with (2.5). Since $N^{-2} = D(x)\,D(y)$ in (3.30), the contribution from the terms containing $\tau(x)\,\theta_{3t/2}$ or $\tau(y)\,\theta_{3t/2}$ is bounded by $K^2O(\triangledown_0)\|t\|^{-2\rho}$.

To complete bounding (3.28), it suffices to prove that the expectation in (3.30) is bounded by

$$(a^2K^2\triangledown_0 + c\|t\|^{-(d-5-\rho\vee1)+\epsilon})\,\|t\|^{-2\rho}. \tag{3.32}$$

Since the event inside $\mathbb{P}$ is the event that all occupied paths from $y$ to $\partial\mathcal{B}_{3t/2}$ in $\{o\}^{\mathrm{c}}$ go through $\tilde{\mathcal{C}}_{3t/2}(x) \subset \mathcal{B}_{3t/2+L}$, there must be a $z \in \tilde{\mathcal{C}}_{3t/2}(x)$ such that $y \leftrightarrow z$ and $z \leftrightarrow \partial\mathcal{B}_{3t/2}$ occur disjointly. Therefore, the expectation in (3.30) is bounded, using the BK inequality, by

$$\begin{aligned}
&\mathbb{E}\Big(\mathbb{1}_{\{x \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\mathrm{c}}\}} \sum_{z \in \tilde{\mathcal{C}}_{3t/2}(x)} \tau(z-y)\, \mathbb{P}(z \leftrightarrow \partial\mathcal{B}_{3t/2})\Big) \\
&\quad \le \sum_{z \in \mathcal{B}_{3t/2+L}} \mathbb{P}(x \leftrightarrow \partial\mathcal{B}_{3t/2},\ z \in \tilde{\mathcal{C}}_{3t/2}(x))\, \tau(z-y)\, \mathbb{P}(z \leftrightarrow \partial\mathcal{B}_{3t/2}). \qquad (3.33)
\end{aligned}$$

We separate the sum into $\sum_{z \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}}$ and $\sum_{z \in \mathcal{B}_{t/2}}$. As in (3.18), by ignoring[1] the condition $x \leftrightarrow \partial\mathcal{B}_{3t/2}$ and using $\mathbb{P}(z \leftrightarrow \partial\mathcal{B}_{3t/2}) \le \theta_{(3t/2-|z|)\vee 0}$, the former sum is bounded by

$$\begin{aligned}
\sum_{z \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}} \tau(z-x)\, \tau(z-y)\, \theta_{(3t/2-|z|)\vee 0} &\le c|\!|\!|t|\!|\!|^{(d-1)+2(2-d)} \Big(L + \sum_{s=0}^{t} |\!|\!|s|\!|\!|^{-\rho}\Big) \\
&\le c|\!|\!|t|\!|\!|^{-2\rho-(d-\rho-3-\rho\vee 1)+\epsilon}. \qquad (3.34)
\end{aligned}$$

This is further bounded by (3.32), because $\rho \le 2$. For the sum $\sum_{z \in \mathcal{B}_{t/2}}$, we first bound $\mathbb{P}(z \leftrightarrow \partial\mathcal{B}_{3t/2})$ by $aK|\!|\!|t|\!|\!|^{-\rho}$. Then, note that the event inside the former $\mathbb{P}$ in (3.33) implies existence of $w \in \mathcal{B}_{3t/2+L}$ such that $x \leftrightarrow w$, $w \leftrightarrow z$ and $w \leftrightarrow \partial\mathcal{B}_{3t/2}$ occur disjointly. Again by the BK inequality, the contribution to (3.33) from $z \in \mathcal{B}_{t/2}$ is bounded by

$$aK|\!|\!|t|\!|\!|^{-\rho} \sum_{\substack{z \in \mathcal{B}_{t/2} \\ w \in \mathcal{B}_{3t/2+L}}} \tau(x-w)\, \tau(w-z)\, \tau(z-y)\, \mathbb{P}(w \leftrightarrow \partial\mathcal{B}_{3t/2}). \qquad (3.35)$$

We further separate the sum over $w$ into $\sum_{w \in \mathcal{B}_{t/2}}$ and $\sum_{w \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}}$. For the former sum, we bound $\mathbb{P}(w \leftrightarrow \partial\mathcal{B}_{3t/2})$ by $aK|\!|\!|t|\!|\!|^{-\rho}$, and then bound the remaining term by $\triangledown_0$, using $x \ne y$. For the latter sum, we use $\mathbb{P}(w \leftrightarrow \partial\mathcal{B}_{3t/2}) \le \theta_{(3t/2-|w|)\vee 0}$ and perform the sum over $z$ using [10, Proposition 1.7(i)]. Since $x, y \in \Omega$, the expression (3.35) due to the sum over $w \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}$ is bounded by

$$\begin{aligned}
c|\!|\!|t|\!|\!|^{-\rho} \sum_{w \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}} |\!|\!|w|\!|\!|^{(2-d)+(4-d)}\, |\!|\!|\tfrac{3}{2}t - |w|\,|\!|\!|^{-\rho} &\le c|\!|\!|t|\!|\!|^{-\rho+(6-2d)+(d-1)} \Big(L + \sum_{s=0}^{t} |\!|\!|s|\!|\!|^{-\rho}\Big) \\
&\le c|\!|\!|t|\!|\!|^{-2\rho-(d-5-\rho\vee 1)+\epsilon}. \qquad (3.36)
\end{aligned}$$

[1]Some readers might wonder whether the condition $x \leftrightarrow \partial\mathcal{B}_{3t/2}$ could be used to have less power in (3.34). In fact, if we use the inequality

$$\mathbb{P}\big(x \leftrightarrow \partial\mathcal{B}_{3t/2},\ z \in \tilde{\mathcal{C}}_{3t/2}(x)\big) \le \sum_{w \in \mathcal{B}_{3t/2+L}} \tau(w-x)\, \tau(z-w)\, \theta_{(3t/2-|w|)\vee 0},$$

then the contribution due to $w \in \mathcal{B}_{t/2}$ is bounded by (3.36), while the contribution from $w \in \mathcal{B}_{3t/2+L} \setminus \mathcal{B}_{t/2}$ has a worse bound $c|\!|\!|t|\!|\!|^{-2\rho+\mu}$, where $\mu$ is negative only when $d > 9$.

Summarizing the above estimates, we conclude that (3.28) is bounded from below by the same formula as in the right-hand side of (3.26), where a multiple constant corresponding to $c$ in (3.26) is $O(C_2K^2)$. The second sum in (3.27) can be estimated similarly to (3.35), where $z$ in (3.35) corresponds to $\bar{b}$ in (3.27), and is bounded by a similar formula to (3.32), multiplied by $O(C_2)\|\!|t|\!\|^2$. This completes the proof of (3.26).

We obtain (2.16) from (3.26) if $\nabla_0 \ll 1$, $t \gg 1$ and $d > 5 + \rho \vee 1$, and thus obtain $\rho = 2$ for $d > 7$. This completes the proof. $\square$

*Remark.* The value of $\rho$ for percolation is expected to be 2 as soon as $d > 6$. The main obstacle to going down from $d > 7$ is in (3.34) and (3.36), which correspond respectively to (3.18) and (3.21) for the time-oriented models. In (3.18) and (3.21), the sum over $s$ is fully controlled using $\theta_{t-s} \asymp \|\!|t-s|\!\|^{-\rho}$. On the other hand, the point-to-surface connectivity $\theta_{(3t/2-|v|)\vee 0}$, with $v = z$ in (3.34) and $v = w$ in (3.36), is insufficient to obtain the desired bound, when $v$ is close to the boundary $\partial\mathcal{B}_{3t/2}$. This difficulty is considered to be caused by naively bounding the probability inside $\mathbb{E}$ in (3.30) as in (3.33). Since $\big\{y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c}\big\} \setminus \big\{y \leftrightarrow \partial\mathcal{B}_{3t/2} \text{ in } \{o\}^{\rm c} \cap \tilde{\mathcal{C}}_{3t/2}(x)^{\rm c}\big\}$ is the event that all occupied paths from $y$ to $\partial\mathcal{B}_{3t/2}$ (in $\{o\}^{\rm c}$) have to go through $\tilde{\mathcal{C}}_{3t/2}(x)$ *before* reaching to the boundary, the approximation by the unrestricted two-point function $\tau(z-y)$ in (3.33) could be very crude when $z$ is close to $\partial\mathcal{B}_{3t/2}$, due to the isotropic property for percolation. If we assume that there is a $\kappa \geq 1$ such that, for $|z| = \ell$,

$$\mathbb{P}(o \leftrightarrow z,\ o \not\leftrightarrow \partial\mathcal{B}_t) \leq c\|\!|\ell|\!\|^{2-d-\kappa}(\|\!|\ell|\!\| \wedge \|\!|t-\ell|\!\|)^{\kappa}, \tag{3.37}$$

then we will be able to obtain the desired inequality (2.16) down to $d > 6$. Note that (3.37) contains the factor $\|\!|t-\ell|\!\|$ that decreases as $z$ approaches the boundary $\partial\mathcal{B}_t$, that the sum of the right-hand side over $z \in \mathcal{B}_t$ is bounded by $c\|\!|t|\!\|^2$, and that the limit $t \to \infty$ of the right-hand side, while $\ell$ or $\ell/t$ is fixed, is $c\|\!|\ell|\!\|^{2-d}$. Therefore, (3.37) is a good candidate for the bound on the restricted two-point function, though we have not proved whether (3.37) really holds or does not. (For random walk, a similar inequality with $\ell = t$ and $\kappa = 1$ has been verified by our rough calculation.)

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