Stochastic processes in mechanical engineering

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Preface

Stochastic or random vibrations occur in a variety of applications of mechanical engineering. Examples are: the dynamics of a vehicle on an irregular road surface; the variation in time of thermodynamic variables in municipal waste incinerators due to fluctuations in heating value of the waste; the vibrations of an airplane flying through turbulence; the fluctuating wind loads acting on civil structures; the response of off-shore structures to random wave loading (figure 1).

Suppose we know the value of a random variable x at a certain time t_0. For future times $t > t_0$ the value of x will become more and more uncertain. Its value becomes less and less correlated to the starting value. It can only be specified by probability distributions. The theory of stochastic processes aims to describe this behavior. Although x is stochastic, by describing it in terms of probability distributions and other statistical characteristics (correlation functions, peak distributions, etc.) calculation procedures and codes can be established by which decisions on design issues can be made.

Attention will be focussed on problems of external noise. That is, we shall consider models of mechanical engineering structures where the source of random behavior comes from outside: e.g. a prescribed random force or a prescribed random displacement. The statistical behavior of the source is supposed to be known: for example, a Gaussian random process with known power density spectrum. No attention will be given to questions on the origin of the noise: how deterministic laws of physics can produce random fluctuations. A classical example is the chaotic motion of colliding molecules in otherwise empty space. Another example, closer to the applications considered here: how winds blowing over the sea surface can lead to the occurrence of random waves. Answering these questions is not an easy task; they require studies of their own and we do not address them here. We focus attention to the dynamical behavior of mechanical engineering structures subject to external noise of specified form. The structures inhibit inertia, damping and restoring, linear and non-linear. The main questions we intend to answer are: how do these structures respond to random excitation, and how can we quantify the random behavior of response variables in a manner that an engineer is able to make rational design decisions.

In chapter 1, some basic notions on probability theory are recapitulated, followed in chapter 2 by an introduction to the description of a variable which varies randomly in time. In chapter 3, methods of Fourier transform are extended to problems of random vibration. These methods are used in chapter 4, to analyze a linear spring-mass system subject to Gaussian random excitation in the frequency-domain. The description of a random signal in the time-domain is given in chapter 5; it forms the starting point for analysis in the time-domain of the spring-mass system; this is presented in chapter 6.

The tensioned beam subject to random excitation provides an excursion to partial differential equations with random right-hand sides. Formulation and analytical methods of solution are presented in chapters 7-9. The statistics of peak values of randomly and Gaussianly varying signals, and the statistics of the extreme value in a signal of a certain duration, are the subject of analysis in chapter 10.

The analysis of non-linear systems is presented in chapters 11-14. In chapter 11 we summarize numerical time-domain simulation techniques applied to non-linear systems. Quasi-static non-linear response to random excitation is treated in chapter 12, non-linearly damped resonance in chapter 13. Stability analysis of systems with parametric random excitation is presented in chapter 14.

Literature: a number of topics presented in subsequent chapters can be found in the introductory textbooks of Robson [1] and Crandall & Mark [2]. More detailed treatises of stochastic theory are presented in the books of Stratonovich [3] and Van Kampen [4]. Other references are given where relevant.

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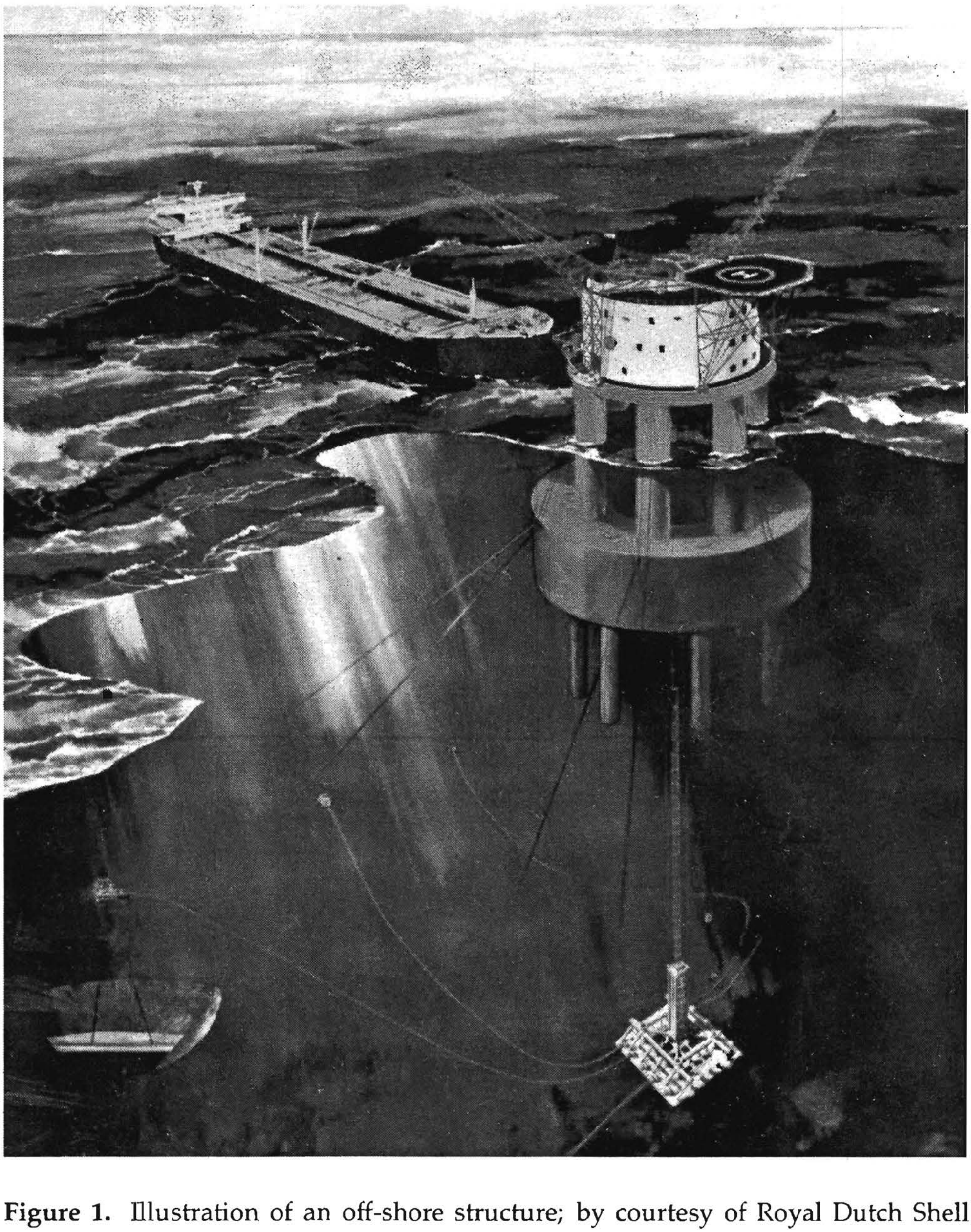


Figure 1. Illustration of an off-shore structure; by courtesy of Royal Dutch Shell Exploration & Production.

Chapter 1

Some basic properties of probability

To describe random processes, probabilistic concepts are used. In this chapter, some basic notions of probability are explained.

1.1 Probability

We define the concept of *probability* here as the probability Pr of the occurrence of a certain event n. This probability is equal to $Pr[n] = 0$ in case of an event which can not possibly occur and $Pr[n] = 1$ if the event is certain to occur. If we consider for example the case of a six-sided die, we can expect that the result N of any given throw is equally likely to be any number between 1 and 6. We can therefore write: $Pr[N = n] = p(n) = 1/6$ for where n can be any number between 1 and 6.

Probability theory is the mathematical study of probability and plays an important role in many aspects of life. As an example, imagine that a certain investment costs 1.000.000 EUR. The chance of earning 10.000.000 EUR with the investment is 50%, but it's equally probable that the investment results in no profit at all. If faced with the choice whether to invest or not, an individual with a total fortune of just over 1.000.000 EUR won't choose to go ahead with the investment. Because when the gamble has a bad outcome for him, he will lose all his money and can't play the game a second time. A big multinational, on the other hand, has the capacity of playing this game many times and the theory of probability ensures the company that when playing the game many times, 'on average' the investment will result in a large profit. It illustrates the power of capital. Another example of how to deal with uncertainty, in a rational manner, is insurance. The risk an individual is exposed to is covered by mutual insurance via an insurance company. In this way, the risk is spread over many people and over a long period of time. The premium every individual will have to pay to the company will correspond to the statistical expectation (expected or mean value) of the financial risk he is exposed to (supplemented with the profit of the insurance company!).

The above examples illustrate that uncertainty becomes a certainty once the game is played many times. It is this principle which makes probability theory a useful tool in practice. Scientific concepts based on probabilistic models are verifiable (as should be!) because we can repeat measurements and check the outcome. Equally, designers can make rational decisions in engineering practice, for example by meeting the standards required by insurance companies (such as Lloyd's insuring ships and offshore structures).

The quantities on which we will focus in the remainder of this text have a continuous range of possible values as opposed to the discrete value which the random variable N can have in the case of a die. It is convenient to define a quantity called the *cumulative distribution function* (CDF) as the probability that a random variable has a value X less than a certain specified value x:

$$P(x) = Pr[X \leq x] = \int_{-\infty}^{x} p(x)dx. \tag{1.1}$$

The previously defined quantity $p(x)$, the *probability density function* (PDF), on its turn can be defined as:

$$p(x) = \frac{dP(x)}{dx}. \tag{1.2}$$

Figure 1.1 and 1.2 show the typical shapes of both quantities in the special case of a zero-mean *Gaussian process* (which will be discussed in section 1.4). The shapes of these graphs give a nice qualitative indication of the nature of the random variable. For example, a variable whose values are closely clustered around a mean value would give a tall narrow $p(x)$ curve and a $P(x)$ curve steeply rising near the mean value. The sum of all probabilities, and therefore the area under the $p(x)$ curve is equal to one: $\int_{-\infty}^{\infty} p(x)dx = 1$.

Suppose that we measured the room temperature x of a classroom infinitely many times and that the CDF on the lower side of figure 1.2 characterizes the temperature distribution $P(x)$. The probability that the measured temperature is within the range $x_1 < x < x_2$ is equal to $P(x_2) - P(x_1)$, as illustrated in the figure.

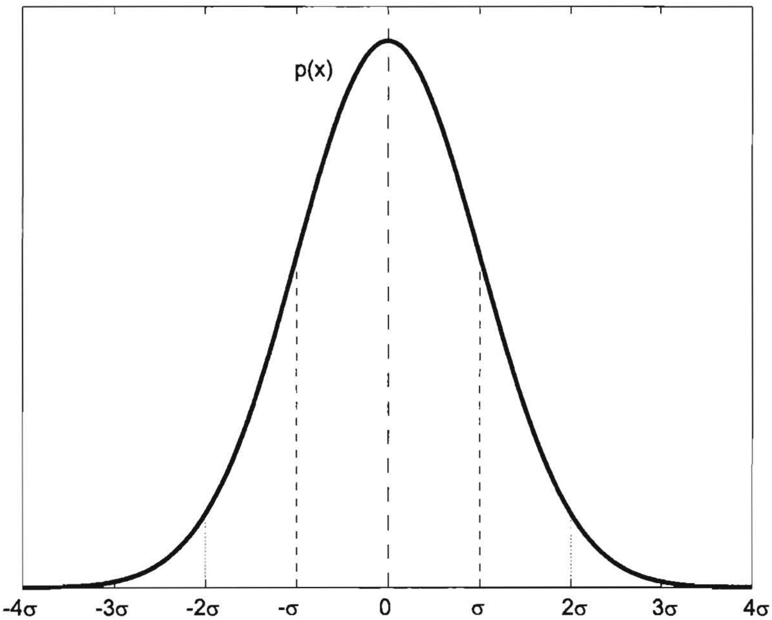


Figure 1.1. The PDF of a zero-mean Gaussian process

1.2 Expectation or mean value

An important quantity in probability theory is the *expected value* $E[x]$ or *mean value* μ of the variable x. It is also called the *first moment* of the distribution $p(x)$ and is given by

$$\mu = \langle x \rangle = \int_{-\infty}^{\infty} xp(x)dx. \tag{1.3}$$

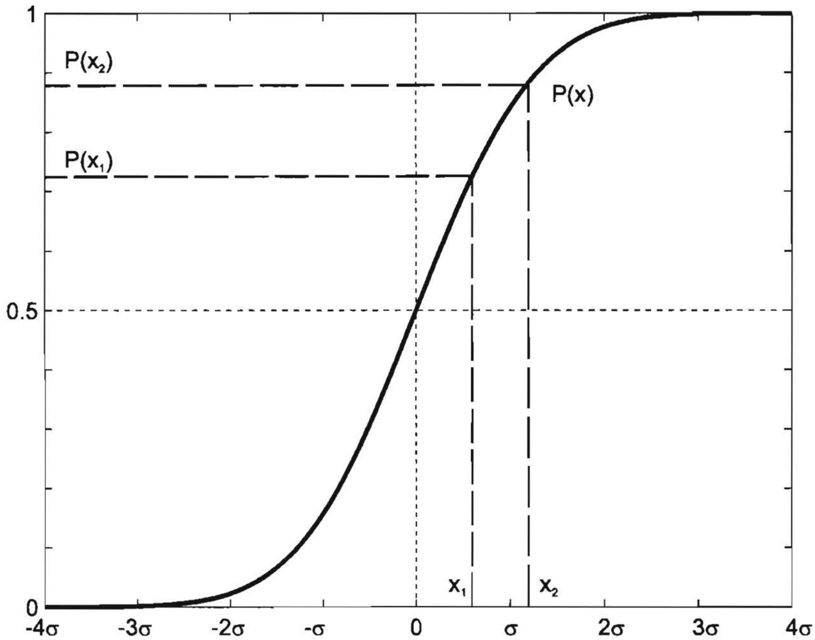


Figure 1.2. The CDF of a zero-mean Gaussian process

The mean value operator $\langle . \rangle$ is a linear operator, which means that it satisfies the following conditions:

$$\begin{aligned} \langle f + g \rangle &= \langle f \rangle + \langle g \rangle \\ \langle \alpha f \rangle &= \alpha \langle f \rangle \\ \langle \langle f \rangle g \rangle &= \langle f \rangle \langle g \rangle \end{aligned} \tag{1.4}$$

where x is random variable and α a constant.

1.3 Variance

To describe the nature of the spread of x completely would require the construction of either the $P(x)$ or $p(x)$ functions. A measure for the spread of the signal around its mean value is given by the *variance* of x and is commonly denoted by σ^2. The variance is *second moment about the mean* of the distribution $p(x)$ and is defined by:

$$\sigma^2 = \left\langle [x-\mu]^2 \right\rangle. \tag{1.5}$$

In other words, the variance is the average of the square of the distance of each data point from the mean. In terms of the probability density function, the variance can be written as:

$$\sigma^2 = \int_{-\infty}^{\infty} [x-\mu]^2 \, p(x) dx. \tag{1.6}$$

In the remainder of this text, all random processes will be considered to be *zero-mean processes* if not specified otherwise. In this case, the above equation obviously simplifies to $\sigma^2 = \int_{-\infty}^{\infty} x^2 p(x) dx$: the variance is then equal to the *mean square value* (MSV). The square root σ of the variance is known as the *root-mean-square value* (RMS) and is also called the *standard deviation* of x. It is a direct measure for the amount of variation around the mean.

1.4 Gaussian distribution

Many real-life random processes can often be considered *Gaussian* meaning that their probability density function, see figure 1.1, can be described by the following relation:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \tag{1.7}$$

with μ the mean value. A *Gaussian distribution* is also known as a *normal distribution*. Equation (1.7) shows that for *Gaussian processes*, the whole distribution is defined by its MSV and mean value. For the CDF we can write:

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(x-\mu)^2/2\sigma^2} dx. \tag{1.8}$$

With the definition of the *error function* as:

$$\text{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-t^2} dt, \tag{1.9}$$

this can be written as:

$$P(x) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right). \tag{1.10}$$

The Gaussian distribution has several important properties:

- The probability density function is symmetric about its mean value;
- 68.27% of the area under the curve is within one standard deviation of the mean;
- 95.45% of the area is within two standard deviations;
- 99.73% of the area is within three standard deviations.

A distribution of two zero-mean variables x and y is said to be Gaussian if the *joint probability density function* (see figure 1.3) is given by:

$$p(x,y) = \frac{1}{2\pi\sqrt{\sigma_x^2\sigma_y^2 - \sigma_{xy}^4}} e^{-\frac{\sigma_y^2 x^2 - 2\sigma_{xy}^2 xy + \sigma_x^2 y^2}{2(\sigma_x^2\sigma_y^2 - \sigma_{xy}^4)}}, \quad (1.11)$$

where σ_x^2 and σ_y^2 are the variances of x and y respectively and $\sigma_{xy}^2 = \langle xy \rangle$ is called the *covariance*.

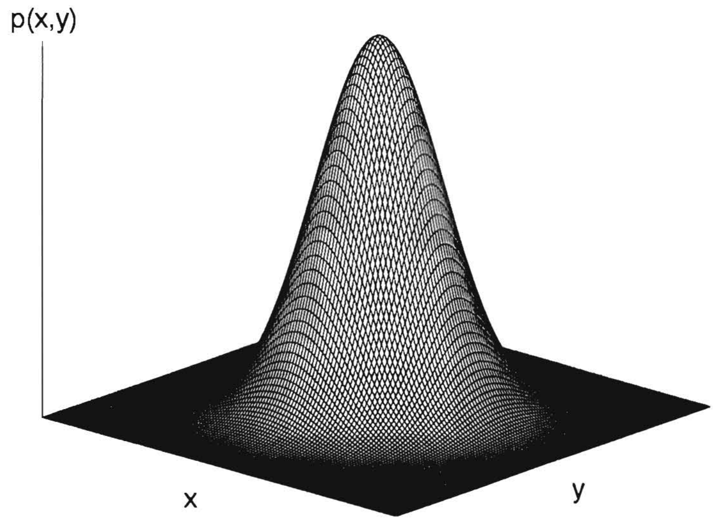


Figure 1.3. Example of a joint Gaussian PDF of two random variables x and y.

1.5 Higher order moments

In sections 1.2 and 1.3, we defined the first and second moment about the mean value. In general, the *n-th moment about the mean*, or *n-th central moment* is defined as follows:

$$\mu_n = \langle (x-\mu)^n \rangle. \tag{1.12}$$

The third central moment normalized with the second central moment, $\mu_3/\mu_2^{3/2}$, is known as *skewness*. Skewness is a measure of the degree of asymmetry of a distribution. If the left tail (tail at small end of the distribution) is more pronounced than the right tail (tail at the large end of the distribution), the function is skewed to the left and is said to have negative skewness. If the reverse is true, it has positive skewness. If the two are equal, it has zero skewness as is the case for a Gaussian distribution. Examples of skewed distributions are shown in figure 1.4.

The fourth central moment normalized with the second central moment, μ_4/μ_2^2, is known as the *kurtosis* or *flatness* of a distribution. It is a measure of the peakedness of a distribution and is equal to 3 for a Gaussian distribution. A kurtosis larger than 3 indicates a "peaked" distribution and when $\mu_4/\mu_2^2 < 3$, the distribution is described as "flat". Examples of peaked and flat distributions are shown in figure 1.5.

Instead of moments, one can express statistical properties in terms of *cumulants*. The *n-th order cumulant*, $\ll x^n \gg$ or κ_n, is defined as the moment of n-th order minus all sub-order-moments:

$$\begin{aligned}
\kappa_1 &= \langle x \rangle \\
\kappa_2 &= \langle x^2 \rangle - \langle x \rangle \langle x \rangle = \mu_2 = \sigma^2 \\
\kappa_3 &= \langle x^3 \rangle - 3\langle x \rangle \langle x^2 \rangle + 2\langle x \rangle^3 = \mu_3 \\
\kappa_4 &= \langle x^4 \rangle - 4\langle x \rangle \langle x^3 \rangle - 3\langle x^2 \rangle^2 + 12\langle x \rangle^2 \langle x^2 \rangle - 6\langle x \rangle^4 \\
&= \mu_4 - 3\mu_2^2
\end{aligned} \tag{1.13}$$

For a Gaussian distribution, all cumulants of order > 2 are equal to zero. A distribution is almost Gaussian when the cumulants of order n (with $n > 2$), normalized with the standard deviation to the power n, are small compared to unity: $\kappa_n/\sigma^n \ll 1$ for $n \geq 3$.

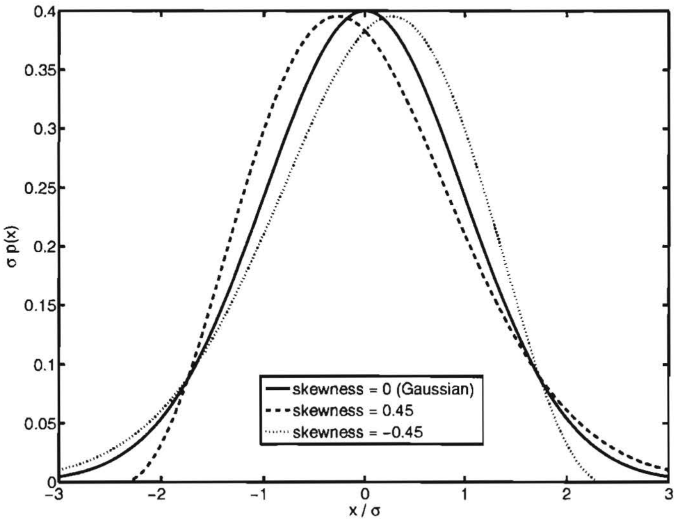


Figure 1.4. Examples of skewed distributions.

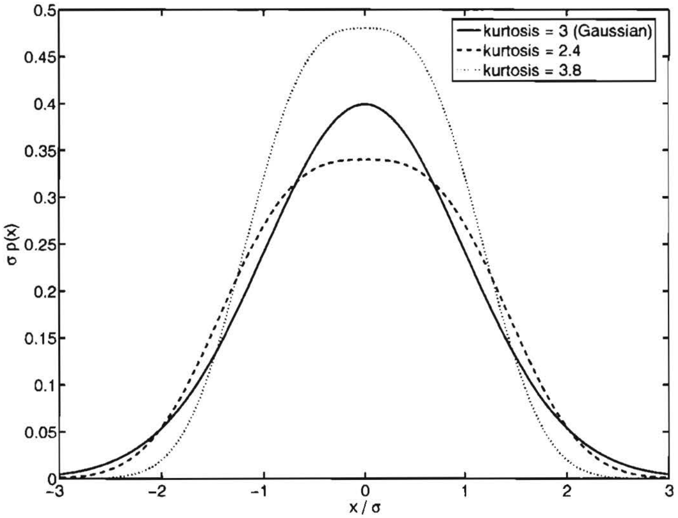


Figure 1.5. Examples of peaked and flat distributions.

Chapter 2

Random Vibrations: Introductory remarks

In the previous chapter we were concerned with the statistical description of some variable. Variation of this variable with time was not taken into consideration. This will be subject of study in this chapter.

2.1 Random events

Let us consider a quantity $x(t)$ randomly varying with an independent variable t. This independent variable will be considered to represent time. The quantity x may be any physical quantity such as velocity, pressure or position. If we carefully design an experimental setup, it is possible to record a randomly varying signal (for example: the velocity of a turbulent flow) over a time interval T. Figure 2.1 shows a typical result for such an experiment. Although it is possible to plot the measured signal $x(t)$, its random behavior makes it impossible to predict its value at a certain time t_1 outside the measured interval T. In order to develop a method to statistically describe such random signals, it is helpful to first consider some elementary statistical theories.

Let $x(t)$ represent the force acting on the wheel of a car during a trip. The forces working on the same wheel during different trips under identical

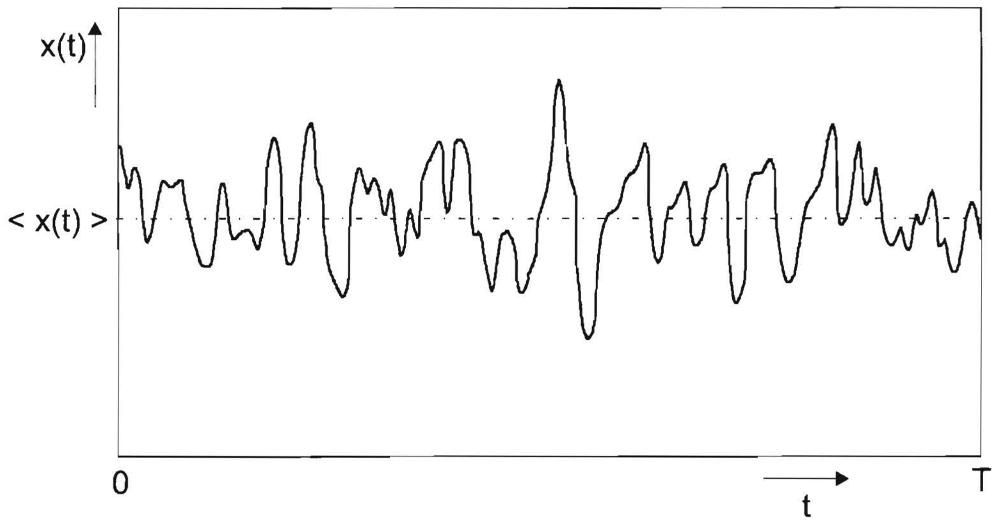


Figure 2.1. Example of a random signal $x(t)$ as measured over a time interval T.

conditions are denoted by $x_1(t), x_2(t), x_3(t)$, ..., etc. The ensemble of all these random *signals* is said to constitute a random *process* $\{x(t)\}$. If the conditions of all the trips are identical, and if the conditions remain steady during each trip, we can in general assume that the process is both stationary and ergodic.

By *stationary* we mean that the probability distribution of the quantities $x_1(t), x_2(t), x_3(t)$, ...,etc. at any instant of time t_1 is independent on the choice of t_1. In other words: all statistical averages are constant in time.

By *ergodic* we mean that the probability distribution of the process $\{x(t)\}$ is equal to the distributions of all the member functions that constitute the whole process.

So if we restrict our attention to stationary and ergodic random processes, by determining the statistical properties of a single member function $x(t)$, we can define the statistical properties of the whole random process $\{x(t)\}$ of which $x(t)$ is a member.

2.2 Ensemble and time averaging

The mean, or average, value of a random signal has been discussed before. A distinction can be made between the *ensemble average* $\langle x(t) \rangle$ and the *time average* value $\overline{x(t)}$ of a random signal. Imagine that we can perfectly isolate a random process in a lab experiment. The process is turned on at time $t = 0$. We then wait until a certain time t has passed and at this time we measure quantity x. This is repeated N times, see figure 2.2. The ensemble average or *statistical moment* is now calculated by taking the sum of all recorded values divided by N. We can thus write down the following definition for the ensemble average:

$$\langle x(t) \rangle = \lim_{N \to \infty} \frac{\sum_{i=1}^{N} x_i(t)}{N}. \tag{2.1}$$

The time average is calculated by averaging the value of the signal over a long period of time:

$$\overline{x(t)} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x_i(t) dt. \tag{2.2}$$

In the special case of a stationary, ergodic process, time averages and ensemble averages are equal $\langle x(t) \rangle = \overline{x(t)}$.
The above can be repeated for the k-th moment:

$$\langle x^k(t) \rangle = \lim_{N \to \infty} \sum_{i=1}^{N} \frac{x_i^k(t)}{N}, \tag{2.3}$$

$$\overline{x^k(t)} = \lim_{T \to \infty} \frac{1}{T} \int_0^T x_i^k(t) dt. \tag{2.4}$$

For stationary ergodic processes:

$$\langle x^k(t) \rangle = \overline{x^k(t)}. \tag{2.5}$$

The probability $P(X)$ is defined as the number of realizations for which at time t the value of $x(t)$ is smaller than a prescribed value X divided by the total number of realizations N and letting $N \to \infty$. So, at some time t, one has simply to count the number of realizations for which $x(t) < X$ and divide this number by N; see figure 2.2. This can be repeated for any other value of X and then results in a complete distribution of $P(X)$ at time t. If the process

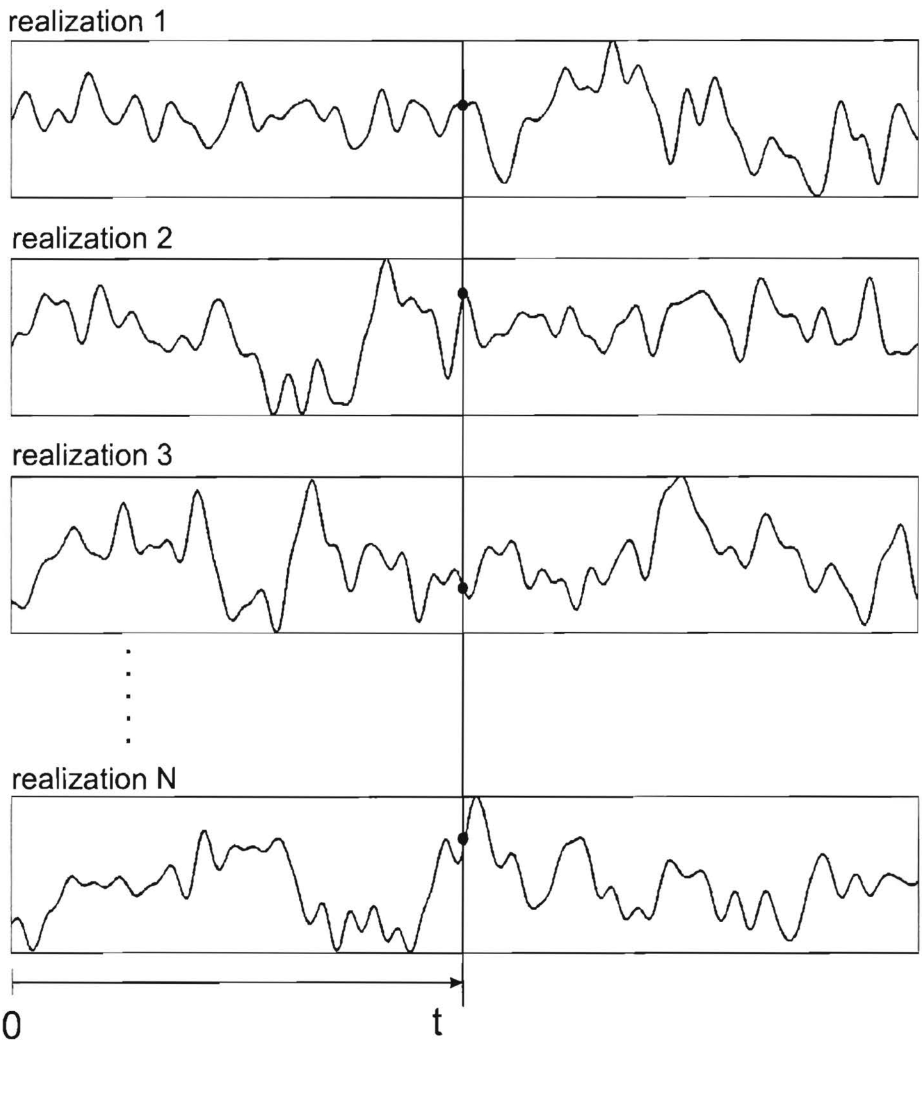


Figure 2.2. Illustration of the principle of ensemble averaging over N realizations.

is non-stationary, the distribution of $P(X)$ will vary with time t! However, if the process is stationary, $P(X)$ will be the same at any time. Furthermore, if the process is also ergodic, one can construct the probability distribution of a single time record. In this case, $P(X)$ is the amount of time that $X < x(t)$ divided by the total time T and letting $T \rightarrow \infty$: see figure 2.3. In general, for stationary ergodic process the statistics of $x(t)$ can be derived from a single time record of the process.

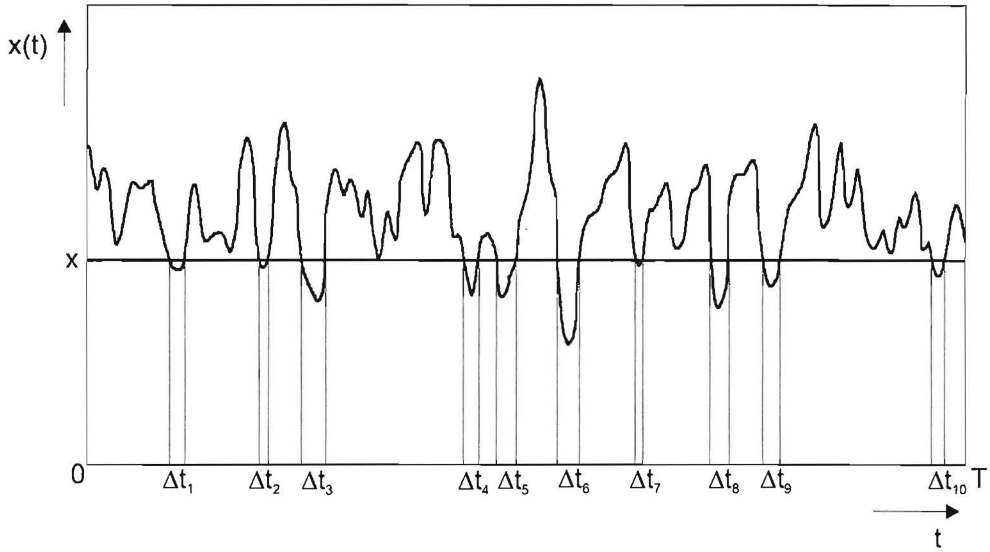


Figure 2.3. Determination of probability distribution from a stationary time record according to $P(x) = Pr(X < x) = \frac{1}{T}\sum_i \Delta t_i$.

2.3 Correlations in time

The extent to which two random variables $x(t)$ and $y(t)$ are correlated can be quantitatively expressed in the magnitude of the *covariance* $\sigma_{xy}^2 = \langle x(t)y(t)\rangle$, which we already encountered in section 1.4. In the case that $x(t)$ and $y(t)$ are completely independent of each other $\sigma_{xy}^2 = 0$. The same concept can also be used to assess the correlation of signals measured at different times with the definition of the *cross-correlation*

$$R_{xy}(\tau) = \langle x(t)y(t+\tau)\rangle. \tag{2.6}$$

τ is called the time separation *. For a stationary process, the cross-correlation only depends on τ because statistical properties of such a process do not change with time. In the same way, the correlation of a variable $x(t)$ at time t with itself at time $t+\tau$ is defined as

$$R_{xx}(\tau) = \langle x(t)x(t+\tau)\rangle \tag{2.7}$$

and is called the *auto-correlation function*. The calculation process is as follows: we multiply the value of a (zero-mean) quantity $x(t)$ at time t with the value of the same quantity when a τ amount of time has passed, we do this for all realizations and then take the ensemble average of all such products. If we assume to be dealing with a stationary ergodic process, $R_{xx}(\tau)$ will be the same for all member functions of the process. In that case, $\langle x(t)x(t+\tau)\rangle = \overline{x(t)x(t+\tau)}$; the latter can be calculated from a single realization by time averaging according to equation (2.2). That is, we multiply $x(t)$ with its value $x(t+\tau)$ when a time τ has passed and treat the product as x_i in equation (2.2). Definition (2.7) shows that the autocorrelation for zero time-separation is equal to the MSV of $x(t)$, $R_{xx}(0) = \langle x^2(t)\rangle$. The auto-correlation function, and correlation functions in general, is therefore often normalized as follows:

$$\overline{R}_{xx}(\tau) = \frac{\langle x(t)x(t+\tau)\rangle}{\sigma_{xx}^2} \tag{2.8}$$

*Definition (2.6) applies to zero-mean random variables. In case of non-zero-mean $x(t)$ and $y(t)$, we have to subtract their mean values when calculating their correlations. Correlations are then indicated by double brackets and are defined as: $\langle\langle x(t)y(t+\tau)\rangle\rangle = \langle [x(t) - \langle x(t)\rangle][y(t+\tau) - \langle y(t+\tau)\rangle]\rangle = \langle x(t)y(t+\tau)\rangle - \langle x(t)\rangle\langle y(t+\tau)\rangle$. The general definition of correlation is thus similar to that of cumulant: cf. eq. (1.13)

An example shape of the auto-correlation function is shown in figure 2.4. For very large time separation τ the value of $R_{xx}(\tau)$ must be zero as the following consideration will show. For very large time separation τ, the two values $x(t)$ and $x(t+\tau)$ will be quite uncorrelated to each other, so that some products $x(t)x(t+\tau)$ will be positive and some negative, the values being symmetrically scattered on each side of the value zero. The ensemble average of these values will therefore be zero, resulting in $R_{xx}(\infty)$=0.

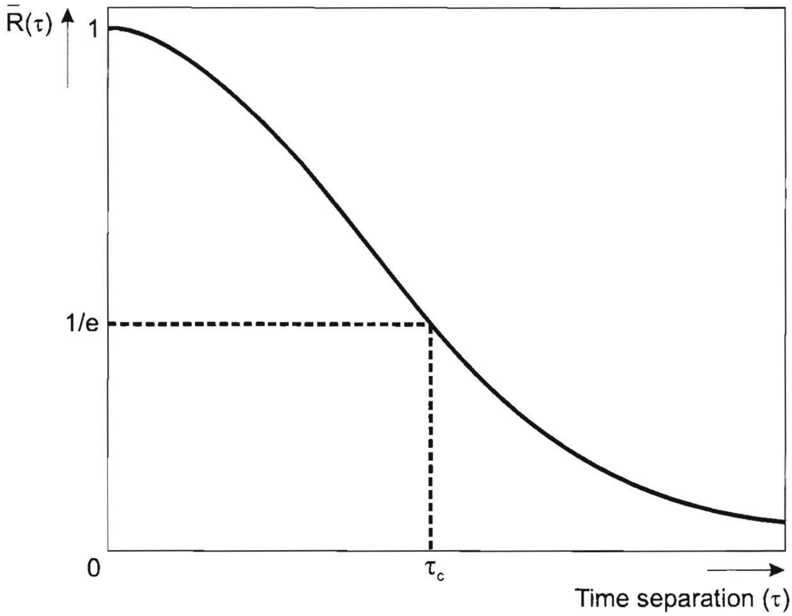


Figure 2.4. Example of a correlation function

The shape of the autocorrelation function supplies us with information about a random signal. The value of τ at which $R(\tau)$ effectively falls to zero gives an indication for the suddenness of the fluctuations in the signal. The lower this value of τ, the more sudden the fluctuations in the signal, in other words: the memory of the system gets shorter. This idea can be quantified with the use of the typical *correlation time*, τ_c, of the process. This can be defined as the time at which an autocorrelation function normalized with the MSV drops to a value of $1/e$ as illustrated in the figure.

Another property of the autocorrelation function, for a stationary random process, is symmetry around $\tau = 0$:

$$R_{xx}(\tau) = R_{xx}(-\tau), \tag{2.9}$$

which logically results in:

$$\frac{\partial R_{xx}(0)}{\partial \tau} = 0. \tag{2.10}$$

Property (2.9) can easily be demonstrated as follows:

$$R_{xx}(\tau) = \langle x(t)x(t+\tau)\rangle = \langle x(t-\tau)x(t)\rangle = \langle x(t)x(t-\tau)\rangle = R_{xx}(-\tau). \tag{2.11}$$

where, in step 2, we used the property that statistical averages of a stationary process do not change with a shift in time.

Chapter 3

Fourier transform in case of random vibration

A powerful method for analyzing the time-dependent behavior of linear systems is the *Fourier transform*. With some amendments, this method can also be applied to random vibrations.

3.1 Fourier integrals and transforms

Any periodic signal can be expressed as a series of harmonically varying quantities, called a *Fourier series*. Non-periodic functions, such as a transient loading, can only be expressed as a Fourier series if we consider it to be periodic, with infinite period. This gives rise to the concept of the *Fourier integral*. The following set of equations give the Fourier integral expression for the non-periodic function $x(t)$:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A(\omega)e^{i\omega t}d\omega \qquad (3.1)$$

$$A(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \qquad (3.2)$$

The quantity $A(\omega)$ is called the *Fourier transform* of $x(t)$. It is in general complex and shows how $x(t)$ can be considered to be distributed over the frequency range. On its turn, $x(t)$ is said to be the *inverse transform* of $A(\omega)$. The two quantities together form a *Fourier transform pair*.

3.2 Spectral density

The theories of Fourier series and integrals can not be applied directly to random signals. This is because the periodic requirement for the Fourier series is not met which rules out the Fourier series. Moreover, if we consider a random signal to continue over infinite time, neither the real or imaginary part of the Fourier transform converges to a steady value which is why it is not possible to use the concept of Fourier integrals: see also the text subsequent to eq. (5.28). Instead, we will introduce a new quantity, the *spectral density*, which has no convergence problems.

Consider a *stationary* ergodic random process $x(t)$ which is assumed to have started at $t = -\infty$ and continue until $t = \infty$. Such a signal is not periodic and it is thus impossible to define its Fourier transform. It is possible, however, to determine the Fourier transform $A_T(i\omega)$ of a signal $x_T(t)$ which is equal to $x(t)$ over the interval $-\frac{T}{2} < t < \frac{T}{2}$ and zero at all other times. In line with Robson [1]:

$$\begin{aligned}\frac{1}{T}\int_{-T/2}^{T/2} x_T^2(t)dt &= \frac{1}{T}\int_{-\infty}^{\infty} x_T(t)x_T(t)dt \\ &= \frac{1}{T}\int_{-\infty}^{\infty} x_T(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} A_T(\omega)e^{i\omega t}d\omega\right]dt \\ &= \frac{1}{2\pi T}\int_{-\infty}^{\infty} A_T(\omega)\left[\int_{-\infty}^{\infty} x_T(t)e^{i\omega t}dt\right]d\omega \qquad (3.3)\end{aligned}$$

If we define the complex conjugate of $A(\omega)$ as $A^*(\omega) = \int_{-\infty}^{\infty} x(t)e^{i\omega t}dt$, equation (3.3) can be written as:

$$\begin{aligned}\frac{1}{T}\int_{-T/2}^{T/2} x_T^2(t)dt &= \frac{1}{2\pi T}\int_{-\infty}^{\infty} A_T(\omega)A_T^*(\omega)d\omega \\ &= \frac{1}{2\pi T}\int_{-\infty}^{\infty} |A_T(\omega)|^2 d\omega \\ &= \frac{1}{\pi T}\int_{0}^{\infty} |A_T(\omega)|^2 d\omega \qquad (3.4)\end{aligned}$$

where use is made of the fact that $|A_T(\omega)|^2$ is an even function of ω. If we now let $T \to \infty$, we obtain an expression for the MSV of the original signal

$x(t)$:

$$\langle x^2(t) \rangle = \overline{x^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt = \int_0^\infty \lim_{T\to\infty} \left(\frac{1}{\pi T} |A_T(\omega)|^2 \right) d\omega \tag{3.5}$$

The *spectral density* or *power density* of $x(t)$ is now defined as:

$$S(\omega) = \lim_{T\to\infty} \left(\frac{1}{\pi T} |A_T(\omega)|^2 \right) \tag{3.6}$$

so that

$$\sigma^2 = \langle x^2(t) \rangle = \int_0^\infty S(\omega)d\omega. \tag{3.7}$$

The power density indicates how the harmonic content of $x(t)$ is spread over the frequency domain. The amount of $\langle x^2(t) \rangle$ associated with a narrow frequency band $\Delta\omega$ is equal to $S(\omega)\Delta\omega$. Different realizations of a stationary ergodic random process have a common $S(\omega)$. The power density $S(\omega)$ is thus a constant, non-random, statistical parameter of the random process $x(t)$. This contrasts with the Fourier transform $A(\omega)$ of $x(t)$ which varies randomly itself: see also section 5.3. If a signal has a spectrum that is uniform (constant) over the whole frequency domain, the spectrum is said to be 'white' and the signal is referred to as *white noise*. Clearly, this is a theoretical abstraction; real power densities are never constant. But the abstraction can be useful to handle certain problems such as lightly damped resonance: see, amongst others, chapters 13 and 14. A more realistic power density spectrum is that of the algebraic function:

$$S(\omega) = \frac{2\sigma^2\Omega^{-1}}{\pi(1 + (\omega/\Omega)^2)}. \tag{3.8}$$

Here, Ω is a parameter which scales the frequency of the spectrum, while σ is standard deviation:

$$\int_0^\infty S(\omega)d\omega = \frac{2}{\pi} \int_0^\infty \frac{\sigma^2 d\eta}{1+\eta^2} = \frac{2\sigma^2}{\pi} \arctan(\eta) \Big|_0^\infty = \sigma^2. \tag{3.9}$$

Another, more peaked, spectrum is:

$$S(\omega) = \frac{2\sigma^2\Omega}{\omega^2} e^{-2\Omega/\omega}. \tag{3.10}$$

In this expression, Ω is a parameter which determines at which frequency the power density has its largest value. By differentiation with respect to ω, one can show that $S(\omega)$ reaches its maximum at $\omega = \Omega$. The standard deviation is represented by σ. One can easily verify that the integral of $S(\omega)$ according to equation (3.10) equals σ^2 as should be (see equation (3.7)). The above mentioned spectra are shown in figure 3.1.

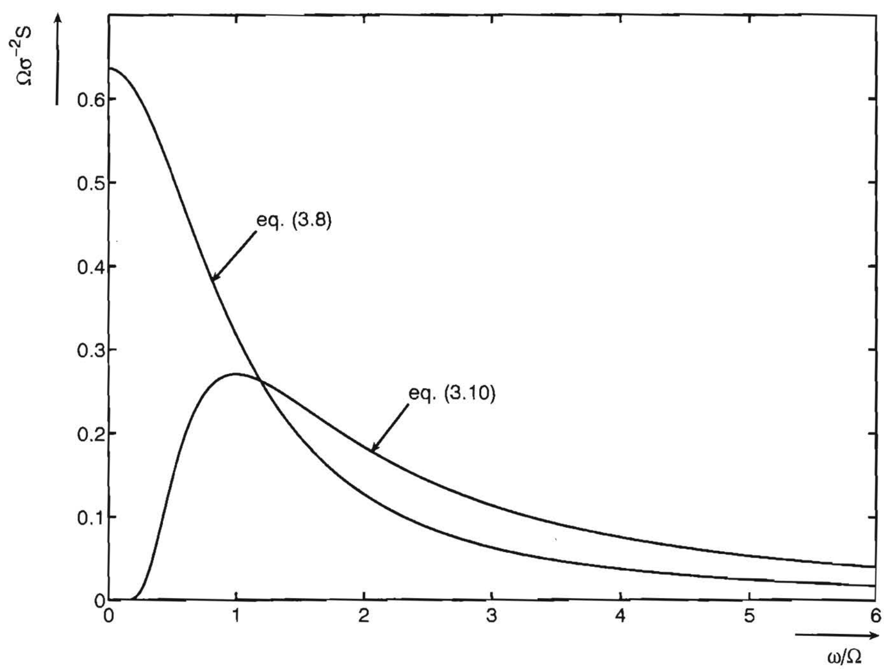


Figure 3.1. The power density spectra according to eqs. (3.8) and (3.10).

3.3 The Fourier transform of the autocorrelation function

There is a direct analytical relationship between the autocorrelation function and the spectral density of a stationary random signal. To show this, we express the autocorrelation as:

$$R(\tau) = \langle x(t)x(t+\tau)\rangle = \overline{x(t)x(t+\tau)} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt. \quad (3.11)$$

For its Fourier transform we have

$$\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau = \lim_{T\to\infty}\left[\frac{1}{T}\int_{-\infty}^{\infty} d\tau \int_{-T/2}^{T/2} x(t)x(t+\tau)e^{-i\omega\tau}dt\right]. \quad (3.12)$$

To evaluate the term between square brackets, we take the same approach as in the previous section: we consider a signal $x_T(t)$ which is equal to $x(t)$ over the interval $-T/2 \le t \le T/2$ and zero at all other times. We then have

$$\begin{aligned}
&\frac{1}{T}\int_{-\infty}^{\infty} d\tau \int_{-T/2}^{T/2} x(t)x(t+\tau)e^{-i\omega\tau}dt \\
&= \frac{1}{T}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_T(t)x_T(t+\tau)e^{-i\omega\tau}dt\right]d\tau \\
&= \frac{1}{T}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_T(t)x_T(t+\tau)e^{i\omega t}e^{-i\omega(t+\tau)}dt\right]d\tau \\
&= \frac{1}{T}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_T(t)x_T(s)e^{i\omega t}e^{-i\omega s}dt\right]ds \\
&= \frac{1}{T}\int_{-\infty}^{\infty} x_T(t)e^{i\omega t}dt \int_{-\infty}^{\infty} x_T(s)e^{-i\omega s}ds \\
&= \frac{1}{T}A_T^*(\omega)A_T(\omega) \\
&= \frac{1}{T}\left|A_T(\omega)\right|^2 \qquad (3.13)
\end{aligned}$$

Implementing (3.13) in the right-hand side of (3.12), dividing by π and using the definition of the power spectrum density (cf. eq. (3.6)) we find:

$$\begin{aligned}
S(\omega) &= \frac{1}{\pi}\int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau}d\tau \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} R(\tau)\cos(\omega\tau)d\tau \\
&= \frac{2}{\pi}\int_{0}^{\infty} R(\tau)\cos(\omega\tau)d\tau. \qquad (3.14)
\end{aligned}$$

There is thus a Fourier transform relationship between spectral density and the autocorrelation function, $S(\omega)$ being the Fourier transform of $\pi^{-1}R(\tau)$.

From the definition of the Fourier transform of section 3.1 it follows that:

$$R(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau}d\omega = \int_{0}^{\infty} S(\omega)e^{i\omega\tau}d\omega = \int_{0}^{\infty} S(\omega)\cos(\omega\tau)d\omega. \tag{3.15}$$

As for a stationary random signal $R(\tau)$ is a symmetrical function with respect to τ (see eq. (2.9)), $S(\omega)$ is symmetric as well: $S(\omega) = S(-\omega)$. These symmetries were used in eqs. (3.14) and (3.15). In case of white noise, i.e. $S(\omega)$ is constant with respect to ω, $R(\tau)$ will behave like Dirac's δ-function: $R(\tau) = \delta(\tau)$. White noise processes are therefore also called δ-correlated processes. The correlation functions associated with the spectra of eqs. (3.8) and (3.10) are shown in figure 3.2. The correlation function associated with spectrum (3.8) is a simple exponential function: $R(\tau) = \sigma^2 e^{-\tau/\Omega^{-1}}$: see Stratonovich [3], Vol.I, Table 1 of §2.1, where a number of analytical relationships for power density spectra and correlation functions are given. Note that Stratonovich applied a different normalization to $S(\omega)$; his $S(\omega)$ equals the present one multiplied with the factor 2π. The present normalization corresponds to that of Van Kampen [4], §III.3.

The correlation function associated with eq. (3.10) has been calculated numerically. It is seen that also this correlation function decays with time. While decaying, it becomes negative and then approaches zero. A behavior of decaying and oscillating around the neutral axis is not uncommon to correlation functions.

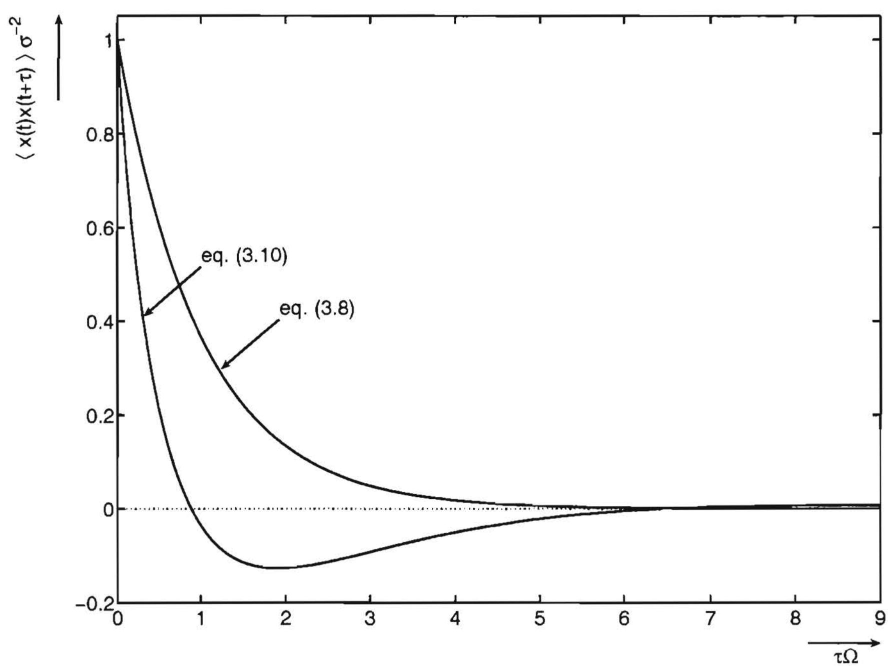


Figure 3.2. The auto-correlation functions associated with the spectra (3.8) and (3.10).

Chapter 4

Single degree of freedom system: analysis in the frequency domain

Many systems encountered in mechanical engineering practice can be modeled by a *single degree of freedom system*. This holds for simple spring-mass systems, but also for multi-degree of freedom systems, which are decomposed in their natural modes: see chapters 7-9. In the present chapter, we shall pay attention to an ordinary spring-mass system with linear damping and subject to random forcing: see figure 4.1. The equation of motion of such systems follows directly from Newton's second law (the sum of all forces equals mass times acceleration):

$$m\ddot{x} + d\dot{x} + cx = P(t) \tag{4.1}$$

where $x(t)$ is displacement response, m mass, c linear spring stiffness, d linear damping constant and $P(t)$ excitation force.

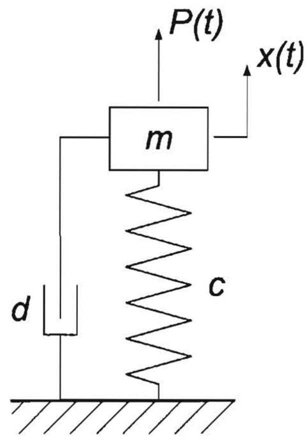


Figure 4.1. A spring-mass system.

4.1 Fourier transform

Using the concepts of Fourier transform introduced in section 3.1, we can write the excitation force as a Fourier integral:

$$P(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_P(\omega) e^{i\omega t} d\omega. \quad (4.2)$$

The response of the system, $x(t)$, can be written in a similar form with a different, frequency dependent, amplitude A_x:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_x(\omega) e^{i\omega t} d\omega. \quad (4.3)$$

If we now differentiate equation (4.3) with respect to time we get the following equations for the time derivatives of the response:

$$\dot{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A_x(\omega) i\omega e^{i\omega t} d\omega \quad (4.4)$$

$$\ddot{x}(t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} A_x(\omega) \omega^2 e^{i\omega t} d\omega. \quad (4.5)$$

Substituting equations (4.2)-(4.5) into the governing equation for this system (4.1):

$$-m\int_{-\infty}^{\infty} A_x(\omega)\omega^2 e^{i\omega t}d\omega + d\int_{-\infty}^{\infty} A_x(\omega) i\omega e^{i\omega t}d\omega + c\int_{-\infty}^{\infty} A_x(\omega)e^{i\omega t}d\omega = \int_{-\infty}^{\infty} A_P(\omega)e^{i\omega t}d\omega \quad (4.6)$$

This can of course be simplified to:

$$\int_{-\infty}^{\infty}\left[\left(-m\omega^2 + di\omega + c\right)A_x(\omega) - A_p(\omega)\right]e^{i\omega t}d\omega = 0 \quad (4.7)$$

$$\Rightarrow \left[\left(-m\omega^2 + di\omega + c\right)A_x(\omega) - A_p(\omega)\right]e^{i\omega t} = 0$$

$$\Rightarrow A_x(\omega) = \frac{A_P(\omega)}{-m\omega^2 + di\omega + c}. \quad (4.8)$$

which, in combination with equation (4.3), gives a complete description of the system response $x(t)$.

For further analysis of the system, it is convenient to introduce the dimensionless frequency $\omega^* = \frac{\omega}{\omega_0}$ and the *natural frequency* or *eigenfrequency* of the system $\omega_0 = \sqrt{\frac{c}{m}}$. With the introduction of these quantities, the expression for the Fourier amplitude of the system response can be rewritten as:

$$\begin{aligned} A_x(\omega) &= A_P(\omega)\left(-m\frac{\omega^2}{\omega_0^2}\omega_0^2 + id\frac{\omega}{\omega_0}\omega_0 + c\right)^{-1} \\ &= A_P(\omega)\left(-m\omega_0^2\omega^{*2} + id\omega_0\omega^* + m\omega_0^2\right)^{-1} \\ &= A_P(\omega)\left(\left(-\omega^{*2} + \frac{id\omega^*}{m\omega_0} + 1\right)m\omega_0^2\right)^{-1}. \end{aligned} \quad (4.9)$$

An even further simplification is possible after normalizing A_P with $m\omega_0^2$, $A_P^n(\omega^*) = \frac{A_P(\omega)}{m\omega_0^2}$, and introducing a dimensionless damping coefficient $\delta = \frac{d}{m\omega_0}$:

$$A_x(\omega^*) = \frac{A_P^n(\omega^*)}{-\omega^{*2} + i\delta\omega^* + 1}. \quad (4.10)$$

If we multiply this equation with the complex conjugate of $A_x(\omega)$, $A_x^*(\omega)$, we can derive an expression for the spectral density of the response of the system:

$$\begin{aligned} S_x(\omega^*) &= \lim_{T\to\infty} \frac{1}{\pi T} \left|A_x(\omega)\right|^2 \\ &= \lim_{T\to\infty} \frac{1}{\pi T} \left| \frac{A_p^n(\omega^*)}{-\omega^{*2} + i\delta\omega^* + 1} \right|^2 \\ &= \frac{S_p^n}{\left(1-\omega^{*2}\right)^2 + \delta^2\omega^{*2}}, \end{aligned} \tag{4.11}$$

because

$$S_p^n = \lim_{T\to\infty} \frac{1}{\pi T} \left|A_p^n(\omega^*)\right|^2. \tag{4.12}$$

4.2 Types of system response

Objective is to analyse the types of response we can expect from a spring-mass system under conditions of random excitation using solution (4.11). For this purpose, we shall use the spectrum of equation (3.10) to represent the power density of excitation. Our spring-mass system is linear: the magnitude of response will be linearly related to the magnitude of excitation. Therefore, the standard deviation of response is directly and linearly proportional to the standard deviation of excitation. It is of no use to study this relationship any further: hence, we put $\sigma = 1$ in eq. (3.10) (the same would be achieved by dividing the left- and right-hand side of solution (4.11) by σ^2 and re-normalizing S_x by σ^{-2}). The example excitation spectrum we henceforth use is:

$$S_p^n(\omega^*) = \frac{2\Omega^*}{\omega^{*2}} e^{-2\Omega^*/\omega^*} \quad , \quad \Omega^* = \Omega/\omega_0 \quad , \quad \int_0^\infty S_p^n(\omega^*) d\omega^* = 1. \tag{4.13}$$

The *power spectrum of response* is given by (solution (4.11)):

$$S_x(\omega^*) = \frac{S_p^n(\omega^*)}{(1-\omega^{*2})^2 + \delta^2\omega^{*2}}. \tag{4.14}$$

System response is now governed by two parameters: δ and Ω^*. The parameter δ determines the magnitude of damping; its effect will be analyzed below. The parameter Ω^* determines the frequency where most excitation energy occurs in comparison with the natural frequency ω_0 of the spring-mass system. The shape of $S_p^n(\omega^*)$ versus $\omega^*/\Omega^* = \omega/\Omega$ was shown in figure 3.1 (curve corresponding to eq. (3.10)). Most excitation energy is located around $\omega^*/\Omega^* = \mathcal{O}(1)$. So if $\Omega^* \ll 1$, i.e. $\Omega \ll \omega_0$, most energy is at low values of the dimensionless frequency ω^*; so at low frequencies compared to the natural frequency of the system. For $\Omega^* \gg 1$, i.e. $\Omega \gg \omega_0$, the opposite is true: excitation occurs mainly at frequencies that are large in comparison to the natural frequency. Depending on the value of Ω^*, three types of system response can now be distinguished. These are discussed in the subsequent sections.

4.3 Quasi-static response ($\Omega^* \ll 1$)

The first type of response is known as *quasi-static* or *sub-critical* and is illustrated in figure 4.2 for $\Omega^* = 0.25$ and $\delta = 1$. The spectrum S_p^n can only reach high values in case $\omega^*/\Omega^* \sim 1$, and thus $\omega^* \sim 0.25$. The major part of the system response is therefore found at small dimensionless frequencies where the denominator in solution (4.14) is practically equal to 1. When ω^* is very small, we can approximate S_x by

$$S_x \sim S_p^n = \frac{S_P}{c^2}. \tag{4.15}$$

This corresponds to a spring-mass system with negligible mass and damping. Energy of excitation is concentrated at low frequencies where the response of the system is governed by stiffness only! This is illustrated in figure 4.2: S_p^n and S_x according to (4.14) are almost equal.

4.4 Dynamical response ($\Omega^* \gg 1$)

The second type of response is called *dynamical* or *super-critical response* and is illustrated in figure 4.3 for $\Omega^* = 4$ and $\delta = 1$. For this situation, phenomena occur at dimensionless frequencies $\omega^* \gg 1$. Energy of excitation occurs predominantly at frequencies that are large in comparison with the natural

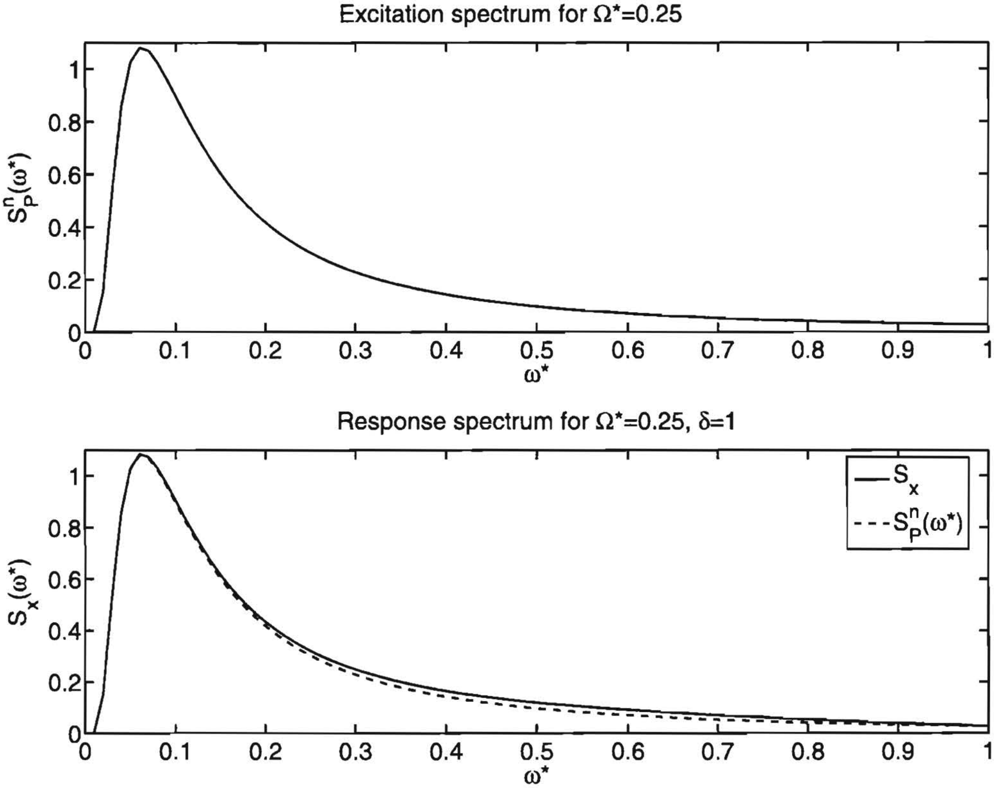


Figure 4.2. Quasi-static or sub-critical response.

(eigen)frequency of the spring-mass system. The term involving ω^{*4} in the denominator of solution (4.14) will dominate. Therefore, response is governed by inertia forces only: i.e., for large values of ω^* the system response can be approximated by

$$S_x \sim \frac{S_p^n}{\omega^{*4}}, \tag{4.16}$$

which is depicted in the figure with the dashed line. Note the scale on the horizontal axes in comparison with those of figures 4.2 and 4.4.

4.5 Resonant response ($\Omega^* = \mathcal{O}(1), \delta \ll 1$)

The third response type is known as *resonant response*. The spectra are plotted in figure 4.4 for $\Omega^* = 1$ and $\delta = 1$ and $\delta = 0.25$. Note the different scales on

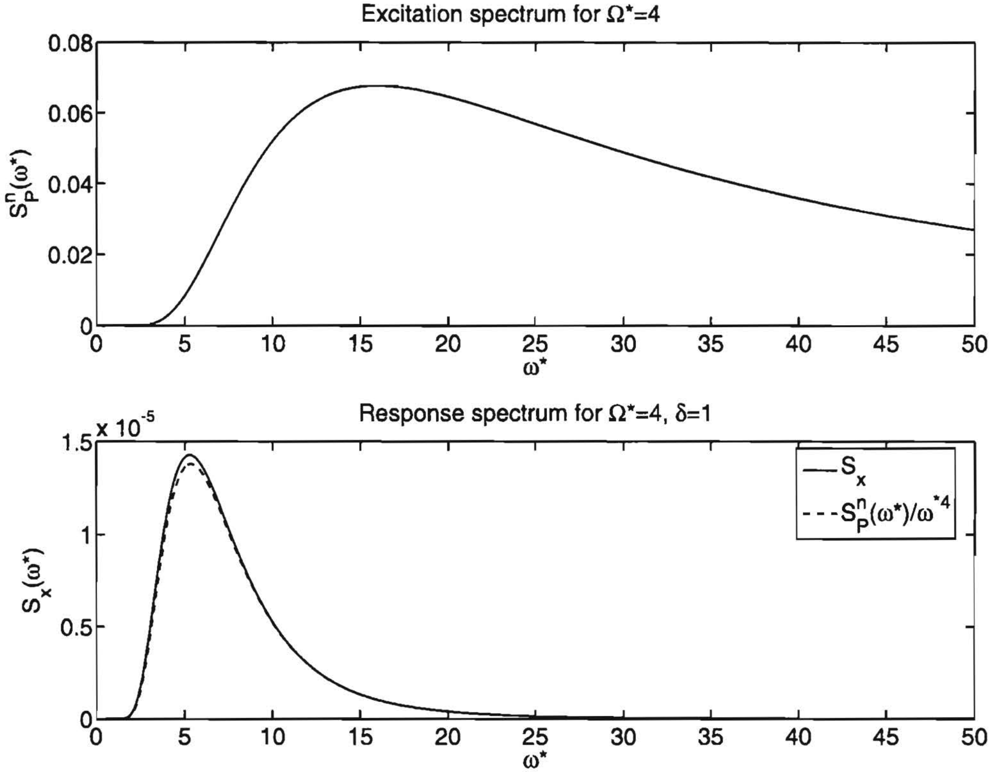


Figure 4.3. Dynamical or super-critical response.

the vertical axes.

To analyze the behavior of resonant response, consider the response spectrum $S_x(\omega^*)$ of equation (4.14) when $\delta \ll 1$. Disregarding terms containing δ one has: $S_x \rightarrow \frac{S_p^n}{(1-\omega^{*2})^2}$. Clearly, this expression explodes as $\omega^* \rightarrow 1$, that is: for frequencies close to the natural frequency of the system. The explosion is a consequence of dropping the damping term in the denominator of the response spectrum. For $\omega^* \rightarrow 1$, the damping term cannot be neglected anymore. To show this in mathematical terms, we introduce a 'new' frequency v^* such that: $\omega^* = 1 + \delta v^*$. We consider now the situation of ω^* close to the natural frequency, such that $v^* = \mathcal{O}(1)$. Then we have for the denominator of eq. (4.14):

$$(1 - \omega^{*2})^2 = 4\delta^2 v^{*2} + \mathcal{O}(\delta^3), \tag{4.17}$$

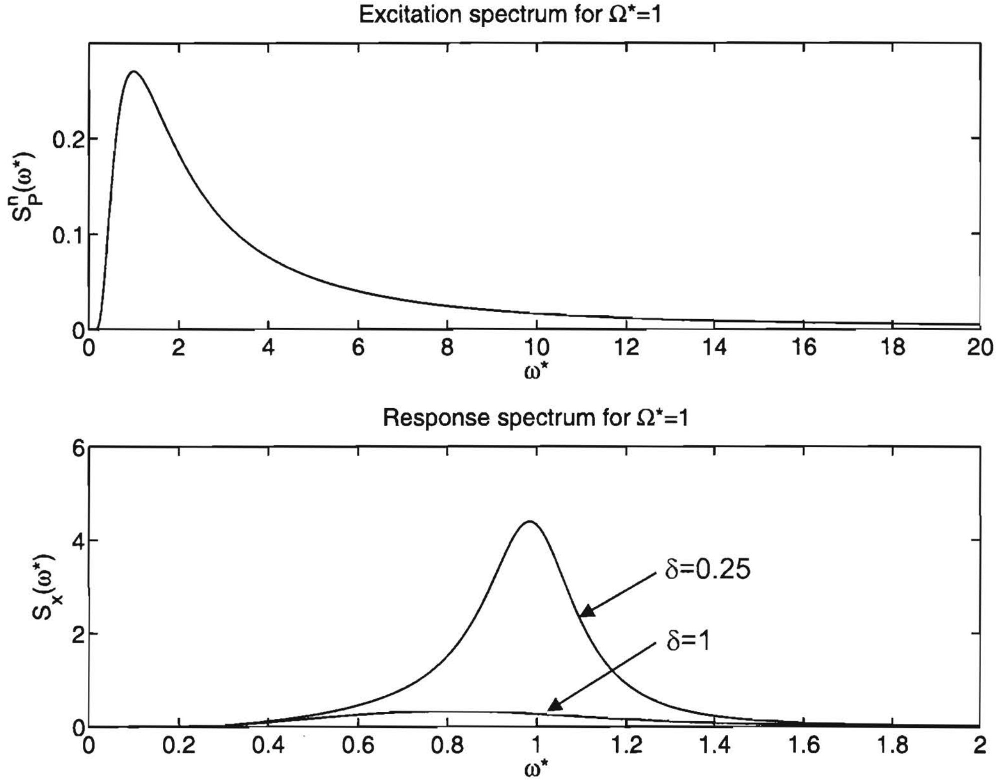


Figure 4.4. Resonant response.

$$\delta^2\omega^{*2} = \delta^2 + \mathcal{O}(\delta^3) \tag{4.18}$$

and

$$S_p^n(\omega^*) = S_p^n(\omega^* = 1) + \mathcal{O}(\delta) \tag{4.19}$$

The result is that, with a relative error of $\mathcal{O}(\delta)$:

$$S_x(\omega^*) = \delta^{-2}\frac{S_p^n(\omega^* = 1)}{4v^{*2} + 1}. \tag{4.20}$$

This result is shown in figure 4.5. The response peaks at $v^* = 0$, i.e. $\omega^* = 1$. The height of the peak is $\delta^{-2}S_p^n(\omega^* = 1)$; the width is δ, so the mean square response related to resonance scales as $\delta^{-1}S_p^n(\omega^* = 1)$. The exact value of the

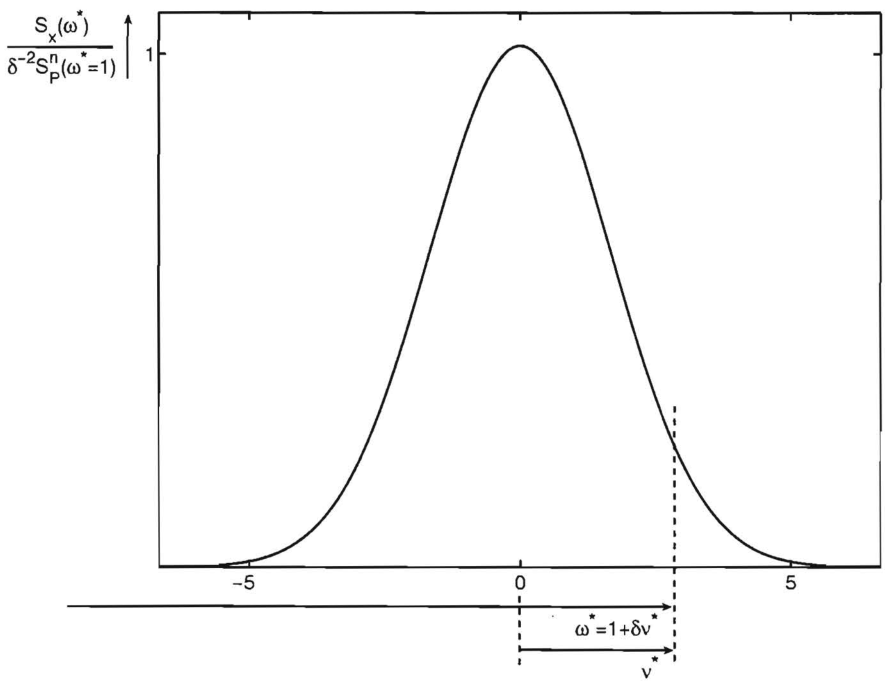


Figure 4.5. The response spectrum as a function of frequencies v^* and ω^*.

mean square response can be calculated using relation (3.7):

$$\sigma^2 = \int_0^\infty S_x(\omega^*)d\omega^* = \int_0^\infty \frac{S_p^n}{(1-\omega^{*2})^2 + \delta^2\omega^{*2}} d\omega^* \tag{4.21}$$

As we have seen, in case of resonant response, most of the response energy is concentrated around the natural frequency ($\omega^* = 1$). Therefore, we can approximate S_p^n by $S_p^n(\omega^* = 1)$, so that

$$\sigma_{res}^2 = \int_0^\infty \frac{S_p^n(\omega^* = 1)}{(1-\omega^{*2})^2 + \delta^2\omega^{*2}} d\omega^* = S_p^n(\omega^* = 1) \int_0^\infty \frac{d\omega^*}{(1-\omega^{*2})^2 + \delta^2\omega^{*2}} \tag{4.22}$$

An exact solution can be found for this integral when using the following transformation: $\omega^{*2} = x$ and thus $\omega^* = x^{1/2}$ and $d\omega^* = 1/2x^{-1/2}dx$. Using integral rule 3.252.12 in Gradshteyn & Ryzhik (which gives a thorough overview of exact integral solutions) [5], we can write:

$$\begin{aligned}\int_0^\infty \frac{d\omega^*}{(1-\omega^{*2})^2+\delta^2\omega^{*2}} &= \frac{1}{2}\int_0^\infty \frac{x^{-1/2}dx}{(1-x)^2+\delta^2 x} \\ &= \frac{1}{2}\int_0^\infty \frac{x^{-1/2}dx}{1+x^2-(2-\delta^2)x} \\ &= \frac{\pi}{2\delta}\end{aligned} \tag{4.23}$$

The expression for the resonant part of the variance of the system response therefore simplifies to:

$$\sigma_{res}^2 = \frac{\pi}{2\delta} S_P^n(\omega^* = 1) \tag{4.24}$$

To check whether the system response is dominated by resonance we have to compare the value of σ_{res} with the total value of the variance, σ_{tot}^2. Even for systems with an eigenfrequency that differs a lot from the typical excitation frequency Ω, the response can still be dominated by σ_{res}^2 if only δ is small enough. This is shown in figure 4.6 where the response of a system with high natural frequency ($\Omega^* \ll 1$) and light damping ($\delta \ll 1$) is shown (in this case $\Omega^* = 0.25$ and $\delta = 0.25$). The response spectrum reveals two peaks: one close to $\omega^* = 0.1$ where the response is quasi-static ($S_x = S_P^n$) and one around $\omega^* = 1$ which is due to resonance. The total variance of the system is:

$$\sigma_{tot}^2 = \sigma_{q-s}^2 + \sigma_{res}^2 \tag{4.25}$$

where the variance due to the quasi-static response is:

$$\sigma_{q-s}^2 = \int_0^\infty S_P^n(\omega^*)d\omega^* \tag{4.26}$$

Although $S_P^n(\omega^* = 1)$ is small as $\Omega^* \ll 1$ (see equation (4.13)), the contribution of the resonance peak can be large if $\delta \ll 1$! With the help of equation (4.24) one can calculate the minimum required damping necessary to suppress resonance under random excitation.

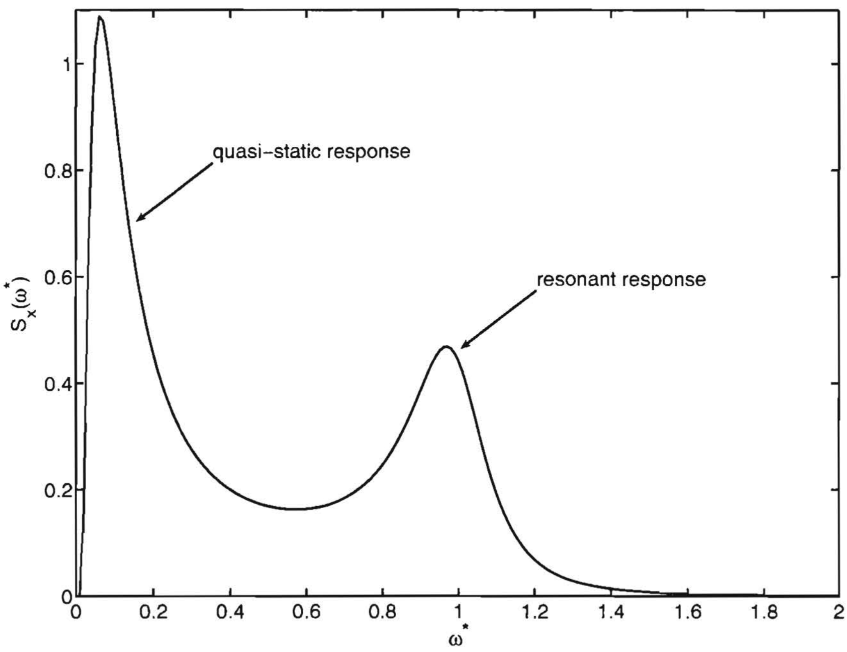


Figure 4.6. Quasi-static system with little damping applied.

Chapter 5

The description of a stationary Gaussian signal in the time-domain

An alternative to the frequency domain analysis of the previous chapter is analysis in the time-domain. To be able to perform such an approach, it is necessary to have explicit descriptions of the external noise in the time-domain. These descriptions are presented in this chapter.

5.1 Random signal described by sinusoidal waves with random phase angles

A conventional way to describe a stationary Gaussian random time signal is by addition of large numbers of sinusoidal waves with random phase angles:

$$\begin{aligned} x(t) &= \sum_{n=1}^{N} \left(2S(\omega_n)\Delta\omega\right)^{1/2} \cos(\omega_n t + \varphi_n), \\ \omega_n &= n\Delta\omega \quad , \quad 0 \leq t \leq T \quad , \quad T = 2\pi/\Delta\omega \end{aligned} \tag{5.1}$$

The amplitude of each wave is defined in accordance with $\left(2S(\omega_n)\Delta\omega\right)^{1/2}$ where $S(\omega_n)$ is the value of the power density taken at the frequency ω_n of the corresponding sinusoidal wave, and $\Delta\omega_n$ is the width of the parts in which the power spectrum is broken up: see figure 5.1. The phase angle φ_n

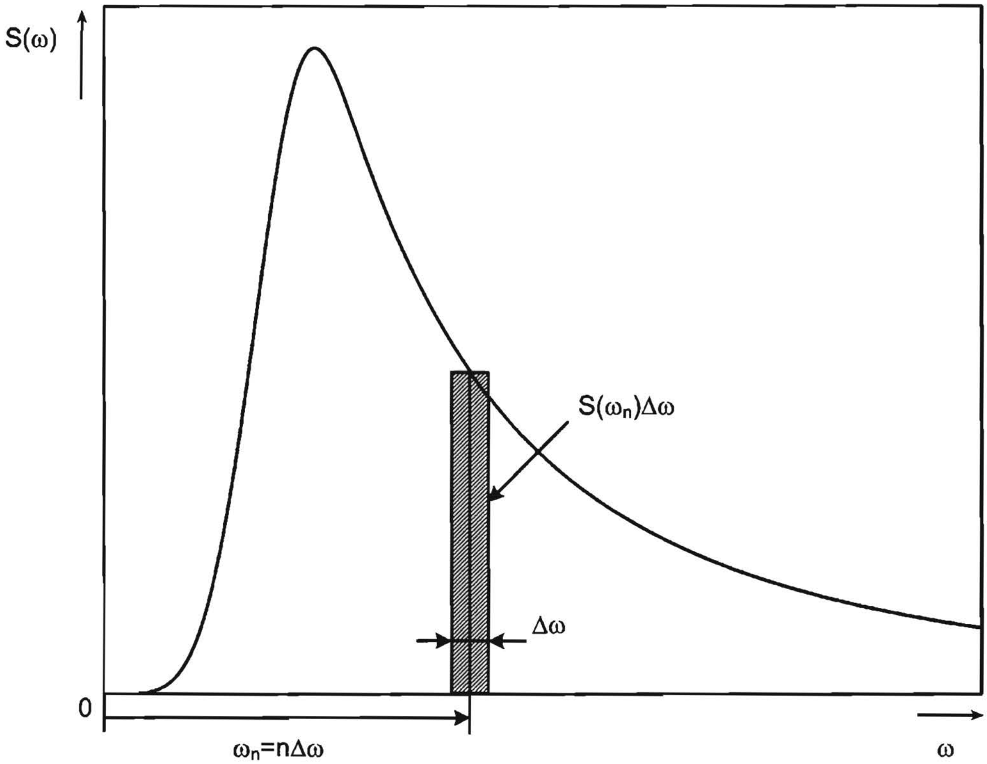


Figure 5.1. Break-up of the power density spectrum in discrete parts to generate a random time-series.

is random. For each n its value is chosen at-random according to a probability which is constant and equal to $1/(2\pi)$ in the range $0 \leq \varphi_n \leq 2\pi$. The randomly selected value of φ_n is constant in time. The duration of the time-series is limited to $2\pi/\Delta\omega$ in order to avoid repetition. This implies: to achieve long time duration, the parts in which the spectrum is divided, have to be sufficiently small. The argument of the sinusoidal wave can apart from φ_n be expressed as

$$\omega_n t = n\Delta\omega t = 2\pi nt/T = 2\pi nz \quad , \quad 0 \leq z \leq 1. \tag{5.2}$$

It shows that the series representation of eq. (5.1) corresponds to a Fourier series expansion of the signal over the time interval for which it is valid. Such a representation of a random signal can also be found in Van Kampen [4], §III.3. In the subsequent sections we shall show that the signal generated according to equation (5.1) approaches for $N \to \infty$ a stationary Gaussian process with power density equal to $S(\omega)$.

5.2 Gaussianity of the time-series

The series of sinusoidal waves can also be written as

$$x(t) = \sum_{n=1}^{N} a_n x_n \tag{5.3}$$

where

$$a_n = (2S(\omega_n)\Delta\omega)^{1/2} \tag{5.4}$$

and

$$x_n = \cos(\omega_n t + \varphi_n)\,. \tag{5.5}$$

Consider now a fixed moment in time. The variable x_n is a random variable of which the probability distribution can be derived from the probability distribution of φ_n using transformation of variables invoking relation (5.5). The variable φ_n is two-valued for $-1 \leq x_n \leq +1$: for every x_n there exist two values of φ_n in the interval $0 \leq \varphi_n \leq 2\pi$. Let us first restrict attention to the interval $0 \leq \varphi_n \leq \pi$ where the connection between x_n and φ_n is single-valued. In this case, the probability that $X_n \leq x_n$ is equal to the probability that $\Phi_n \leq \varphi_n$:

$$P(x_n) = P(\varphi_n). \tag{5.6}$$

Using the connection between cumulative probability and probability density (cf. equation (1.2)), we have

$$p(x_n)dx_n = p(\varphi_n)d\varphi_n, \tag{5.7}$$

or

$$p(x_n) = p(\varphi_n)\left|\frac{d\varphi_n}{dx_n}\right|. \tag{5.8}$$

Now $p(\varphi_n)$ is constant and equal to $1/(2\pi)$, while from equation (5.5) it follows that

$$\frac{dx_n}{d\varphi_n} = \sin(\omega_n t + \varphi_n) = \left[1 - \cos^2(\omega_n t + \varphi)\right]^{1/2} = (1 - x_n^2)^{1/2}. \qquad (5.9)$$

so that (5.8) becomes

$$p(x_n) = \frac{1}{2\pi(1 - x_n^2)^{1/2}} \quad , \quad -1 \le x_n \le +1. \qquad (5.10)$$

We only considered the interval $0 \le \varphi_n \le \pi$. Because the connection between x_n and φ_n in the interval $0 \le \varphi_n \le 2\pi$ is the same, we just have to multiply the right hand side of (5.10) by 2 to obtain the complete probability distribution of $p(x_n)$:

$$p(x_n) = \frac{1}{\pi(1 - x_n^2)^{1/2}} \quad , \quad -1 \le x_n \le +1. \qquad (5.11)$$

The surface underneath this probability density distribution is 1 as should be. Because $p(x_n)$ is a symmetrical function of x_n, uneven moments of x_n are zero. The second moment can be calculated as follows:

$$\begin{aligned} \int_{-1}^{+1} x_n^2 p(x_n) dx_n &= \frac{1}{\pi}\int_0^\pi \cos^2\phi d\phi \\ &= \frac{1}{\pi}\int_0^\pi \left(\frac{1}{2} + \frac{1}{2}\cos 2\phi\right) d\phi \\ &= \frac{1}{2}. \end{aligned} \qquad (5.12)$$

For the fourth moment, we can write

$$\begin{aligned} \int_{-1}^{+1} x_n^4 p(x_n) dx_n &= \frac{1}{\pi}\int_0^\pi \cos^4\phi d\phi \\ &= \frac{1}{\pi}\int_0^\pi \left(\frac{3}{8} + \frac{1}{2}\cos 2\phi + \frac{1}{8}\cos 4\phi\right) d\phi \\ &= \frac{3}{8}. \end{aligned} \qquad (5.13)$$

We can now conclude from representation (5.3) that at any moment in time x equals the sum of a large number of independent stochastic variables x_n, each having the same probability distribution. According to the *central* *limit theorem* (Van Kampen [4], §1.7), the probability distribution of x approaches the Gaussian distribution. This is even true if x_n, n fixed, is non-Gaussian, as is the case above. This feature is responsible for the dominant role of the Gaussian distribution in all fields of statistics.

To make the Gaussianity of x plausible, consider the so-called *characteristic function* of $p(x_n)$, defined as:

$$G_{x_n}(k) = \int_{-1}^{+1} e^{ikx_n} p_x(x_n) dx_n. \tag{5.14}$$

Expanding the exponential term, noting that uneven moments of x_n are zero and using eqs. (5.12) and (5.13), we can write

$$\begin{aligned} G_{x_n}(k) &= \int_{-1}^{+1} \left(1 + ikx_n - \frac{1}{2}k^2x_n^2 - \frac{i}{3!}k^3x_n^3 + \frac{1}{4!}k^4x_n^4 + ...\right) p_x(x_n)dx_n \\ &= 1 - \frac{1}{4}k^2 + \frac{1}{64}k^4 + \mathcal{O}(k^6). \end{aligned} \tag{5.15}$$

The characteristic function associated with $a_n x_n$ then reads

$$G_{a_n x_n}(k) = 1 - \frac{1}{4}k^2a_n^2 + \frac{1}{64}k^4a_n^4 + \mathcal{O}(k^6). \tag{5.16}$$

The characteristic function of the sum of random variables equals the product of the characteristic function of the individual terms. Therefore,

$$\begin{aligned} G_x(k) &= G_{a_1x_1}(k)G_{a_2x_2}(k)...G_{a_nx_n}(k) \\ &= \prod_{n=1}^{N} G_{a_nx_n}(k) \\ &= \prod_{n=1}^{N} \left(1 - \frac{1}{4}k^2a_n^2 + \frac{1}{64}k^4a_n^4 + \mathcal{O}(k^6)\right) \\ &= \prod_{n=1}^{N} \exp\left(-\frac{k^2}{4}a_n^2 - \frac{k^4}{64}a_n^4 + \mathcal{O}(k^6)\right) \\ &= \exp\left(-\frac{k^2}{4}\sum_{n=1}^{N} a_n^2 - \frac{k^4}{64}\sum_{n=1}^{N} a_n^4 + \mathcal{O}(k^6)\right). \end{aligned} \tag{5.17}$$

Here,

$$\sum_{n=1}^{N} a_n^2 = 2\sum_{n=1}^{N} S(\omega_n)\Delta\omega = 2\sigma^2, \tag{5.18}$$

if $N \rightarrow \infty$, while

$$\sum_{n=1}^{N} a_n^4 \sim \frac{\sigma^4}{N}, \tag{5.19}$$

where N is the number of parts of width $\Delta\omega$ into which the power density spectrum is subdivided. We now have,

$$G_x(k) = \exp\left(-\frac{k^2}{2}\sigma^2 + \mathcal{O}\left(\frac{k^4\sigma^4}{N}\right)\right), \tag{5.20}$$

so that

$$G_x(k) \sim \exp\left(-\frac{k^2}{2}\sigma^2\right) \qquad \text{as } N \rightarrow \infty. \tag{5.21}$$

This is the characteristic function of the Gaussian distribution with zero-mean and standard deviation σ. In conclusion, the distribution of x approaches that of a stationary Gaussian distribution with the desired properties (zero-mean, standard deviation equal to σ). The larger the number of terms in the series representation, i.e. the smaller $\Delta\omega$ in the break-down of the power spectrum, the smaller the higher order terms will be in comparison to the term retained: the series representation will represent more and more a Gaussian distribution with increasing N or decreasing $\Delta\omega$.

5.3 Power density spectrum of the time-series

The Fourier transform of $x(t)$ can be expressed as:

$$A(\omega_m) = \int_{-T/2}^{T/2} x(t)e^{-i\omega_m t}dt, \tag{5.22}$$

where, in line with (5.1), we have discretised the frequencies: i.e. $\omega_m = m\Delta\omega = m2\pi/T$. Substituting (5.1), we can write:

$$
\begin{aligned}
A(\omega_m) &= \sum_{n=1}^{N} (2S(\omega_n)\Delta\omega)^{1/2} \int_{-T/2}^{+T/2} e^{-i\omega_m t}\cos(\omega_n t + \varphi_n)dt \\
&= \sum_{n=1}^{N} (2S(\omega_n)\Delta\omega)^{1/2} \int_{-T/2}^{+T/2} (\cos\omega_m t - i\sin\omega_m t) \\
&\quad (\cos\varphi_n \cos\omega_n t - \sin\varphi_n \sin\omega_n t)dt \\
&= \sum_{n=1}^{N} (2S(\omega_n)\Delta\omega)^{1/2} \left\{ \int_{-T/2}^{+T/2} (\cos\varphi_n \cos\omega_m t \cos\omega_n t \right. \\
&\quad +i\sin\varphi_n \sin\omega_m t \sin\omega_n t)\, dt \\
&\quad - \int_{-T/2}^{+T/2} (\sin\varphi_n \cos\omega_m t \sin\omega_n t \\
&\quad \left. +i\cos\varphi_n \sin\omega_m t \cos\omega_n t)\, dt \right\}.
\end{aligned}
\tag{5.23}
$$

The term in the first integral is an even function with respect to t, the term in the second is uneven. Hence,

$$
\begin{aligned}
A(\omega_m) &= \sum_{n=1}^{N} (2S(\omega_n)\Delta\omega)^{1/2} \int_{-T/2}^{+T/2} (\cos\varphi_n \cos\omega_m t \cos\omega_n t \\
&\quad +i\sin\varphi_n \sin\omega_m t \sin\omega_n t)\, dt \\
&= \sum_{n=1}^{N} (2S(\omega_n)\Delta\omega)^{1/2} \left[\frac{1}{2}\cos\varphi_n \int_{-T/2}^{+T/2} \{\cos(\omega_m - \omega_n)t \right. \\
&\quad + \cos(\omega_m + \omega_n)t\}\, dt \\
&+ \left. \frac{1}{2} i \sin\varphi_n \int_{-T/2}^{+T/2} \{\cos(\omega_m - \omega_n)t - \cos(\omega_m + \omega_n)t\}\, dt \right] \\
&= \sum_{n=1}^{N} (2S(\omega_n)\Delta\omega)^{1/2} \left[\frac{1}{2}(cos\varphi_n + i\sin\varphi_n) \left. \frac{\sin(\omega_m - \omega_n)t}{(\omega_m - \omega_n)} \right|_{-T/2}^{+T/2} \right. \\
&+ \left. \frac{1}{2}(cos\varphi_n - i\sin\varphi_n) \left. \frac{\sin(\omega_m + \omega_n)t}{(\omega_m + \omega_n)} \right|_{-T/2}^{+T/2} \right] \\
&= \frac{1}{2}(2S(\omega_m)\Delta\omega)^{1/2}\, T(\cos\varphi_m + i\sin\varphi_m),
\end{aligned}
\tag{5.24}
$$

where we used the properties

$$(\omega_m - \omega_n)T/2 = (m-n)\pi, \tag{5.25}$$

$$(\omega_m + \omega_n)T/2 = (m+n)\pi, \tag{5.26}$$

$$\begin{aligned} \frac{\sin((\omega_m - \omega_n)T/2)}{\omega_m - \omega_n} &= T/2 \qquad \text{for } n = m \\ \frac{\sin((\omega_m - \omega_n)T/2)}{\omega_m - \omega_n} &= 0 \qquad \text{for } n \neq m \end{aligned} \tag{5.27}$$

$$\frac{\sin((\omega_m + \omega_n)T/2)}{\omega_m + \omega_n} = 0. \tag{5.28}$$

From (5.24) it can be concluded that the Fourier transform of a random function varies itself randomly through its dependency on φ_m. Furthermore, the Fourier transform grows unboundedly with time T. Quite different is the situation if we consider the power density of $x(t)$ which is related to $A(\omega)$ as:

$$S(\omega) = \lim_{T\to\infty} \frac{1}{\pi T} |A(\omega)|^2. \tag{5.29}$$

Substituting (5.24), we obtain

$$S(\omega) = \lim_{T\to\infty} \left[\frac{T}{2\pi} S(\omega_m)\Delta\omega(\cos^2\varphi_m + \sin^2\varphi_m) \right] = S(\omega_m). \tag{5.30}$$

because $T\Delta\omega/(2\pi) = 1$. So, we have shown that $S(\omega_n)$ in series expression (5.1) corresponds to the discretised value of the power density $S(\omega)$ of $x(t)$ at $\omega = \omega_n$.

Chapter 6

Single degree of freedom system: analysis in the time-domain

Methods are presented to analyze a single degree of freedom system in the time-domain. The approach is complementary to the frequency-domain analysis presented in chapter 4. Methods and results given can be used to back-up direct numerical simulations of systems in the time-domain.

6.1 Time-domain description of the excitation

In the previous chapter, a description in the time-domain was presented for a stationary Gaussian process. This representation is used to generate a time signal for the excitation of the spring-mass system we intend to analyze:

$$\begin{aligned} P^*(t^*) &= \sum_{n=1}^{N} \left(2S_P^n(\omega_n^*)\Delta\omega^*\right)^{1/2} \cos(\omega_n^* t^* + \varphi_n), \\ \omega_n^* &= n\Delta\omega^* \quad , \quad 0 \leq t^* \leq T \quad , \quad T = 2\pi/\Delta\omega^* \end{aligned} \tag{6.1}$$

The break-up of the power density spectrum in discrete parts to generate the above time-series was shown in figure 5.1. An illustration of a time-domain realization of the excitation according to equations (6.1)is shown in figure 6.1.

The power spectrum was described according to eq. (4.13). MATLAB was used. We took $\Omega^* = 1$, $\Delta\omega^* = 3 \cdot 10^{-2}$, $T = 200$, $N = 10^3$. Random phase angles were generated by a random generator, normalized to values between 0 and 2π. From the generated signal, various interesting statistical properties can be calculated. For this purpose, one can adopt the time averaging methods of chapter 2. In this way, results for standard deviation, auto-correlation and probability distribution can be calculated. Note that the standard deviation should be equal to 1, the auto-correlation should be equal to the correlation shown in figure 3.2 and the probability distribution should be close to the Gaussian distribution. The accuracy of the results can be assessed as a function of the value of $\Delta\omega^*$. Also a range of different realizations can be generated from which ensemble averages can be determined. Stationarity and ergodicity can thus be verified.

An illustration of the Gaussianity of series representation (6.1) with the power density spectrum according to eq. (4.13) has been given in figure 6.2. The probability distribution was calculated according to the procedure outlined in figure 2.3. While $\Omega^* = 1$, the spectrum was broken down taking $\Delta\omega = 3$ and $\Delta\omega = 3 \cdot 10^{-3}$ with $N = 10$ and $N = 10^3$, respectively. It is seen that Gaussianity improves with increasing N as forecasted by eqs. (5.17)-(5.21) in the previous chapter.

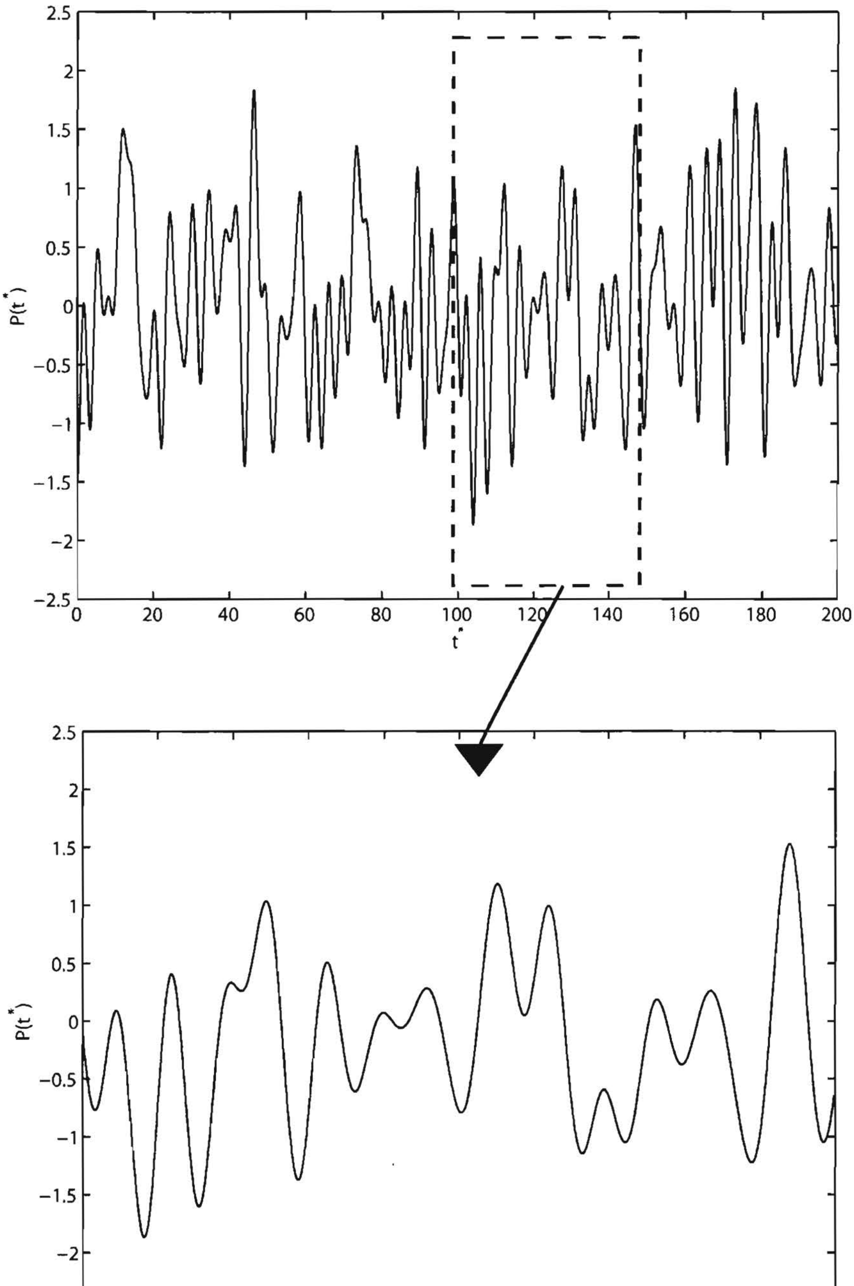


Figure 6.1. Illustration of a time-domain realization of the excitation for $\Omega^* = 1$.

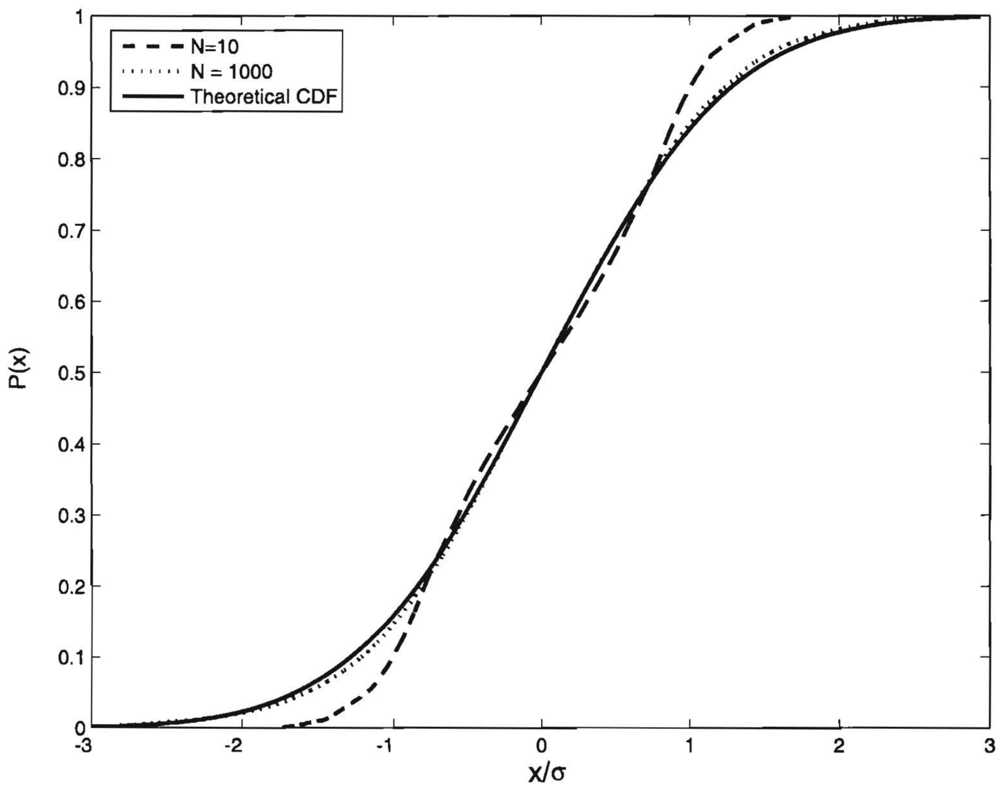


Figure 6.2. Probability distribution of excitation according to representation (6.1) and spectrum (4.13) with $\Omega^* = 1$.

6.2 Solution in the time-domain

We shall analyze the response behavior of the spring-mass system formulated in dimensionless form:

$$\ddot{x} + \delta\dot{x} + x = P^*(t^*), \tag{6.2}$$

where

$$P^*(t^*) = P(t)/c, \tag{6.3}$$

$$t^* = \omega_0 t. \tag{6.4}$$

The dots denote differentiation with respect to dimensionless time t^*, c is the stiffness or spring constant and ω_0 is the eigenfrequency of the system. Eq. (6.2) follows from eq. (4.1) using (6.3), (6.4) and

$$\omega_0 = \sqrt{c/m} \qquad \delta = d/(m\omega_0). \tag{6.5}$$

Given $P^*(t^*)$ as described by equation (6.1) and disregarding any transient response, the solution of equation (6.2) can be found by describing x as:

$$x = \sum_{i=n}^{N} \alpha_n \cos(\omega_n^* t^* + \varphi_n) + \sum_{n=1}^{N} \beta_n \sin(\omega_n^* t^* + \varphi_n). \tag{6.6}$$

Substitute (6.6) in (6.2) and equate coefficients of terms like $\cos(\omega_n^* t^* + \varphi_n)$ and $\sin(\omega_n^* t^* + \varphi_n)$:

$$-\alpha_n \omega_n^{*2} + \delta\beta_n\omega_n^* + \alpha_n = (2S_P^n(\omega_n^*)\Delta\omega^*)^{1/2}, \tag{6.7}$$

$$-\beta_n\omega_n^{*2} + \delta\alpha_n\omega_n^* + \beta_n = 0, \tag{6.8}$$

from which it follows that:

$$\alpha_n = \beta_n \frac{1 - \omega_n^{*2}}{\delta\omega_n^*}, \tag{6.9}$$

$$\beta_n = (2S_P^n(\omega_n^*)\Delta\omega^*)^{1/2} \frac{\delta\omega_n^*}{(1 - \omega_n^{*2})^2 + \delta^2\omega_n^{*2}}. \tag{6.10}$$

Using the property (we sometimes repeat basic algebra to brush-up the analytical mind; for an overview on algebraic rules and properties of known mathematical functions, see Abramowitz & Stegun [6] which is now also available in digital form; property (6.11) can be found under 4.3.17 of [6]):

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2. \tag{6.11}$$

equation (6.6) can be written as:

$$x = \sum_{n=1}^{N} \left(\alpha_n^2 + \beta_n^2\right)^{1/2} \cos\left(\omega_n^* t^* + \phi_n - \arctan(\beta_n/\alpha_n)\right), \tag{6.12}$$

where

$$\alpha_n^2 + \beta_n^2 = \frac{2S_p^n \Delta\omega^*}{(1-\omega_n^{*2})^2 + \delta^2\omega_n^{*2}}, \tag{6.13}$$

and

$$\beta_n/\alpha_n = \frac{\delta\omega_n^*}{1-\omega_n^{*2}}. \tag{6.14}$$

From (6.12) and (6.13) it can be seen that the spectral density of the response equals solution (4.11) (as should be!).

The above results can be used to back-up direct numerical simulation results of the spring-mass system. The equation, i.e. eq. (6.2), can be split-up into two first order equations:

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\delta y - x - P^*(t^*). \end{aligned} \tag{6.15}$$

Implementing the excitation according to eq. (6.1), the above two equations can be integrated numerically by standard routines. As initial conditions, one can take: $x = y = 0$ at $t = 0$. The numerically calculated result will exhibit a transient response which decays as $e^{-\delta t^*}$: for further details on transient response, see section 6.4. Once the transient response has vanished, the result should, in statistical sense, be the same as the analytical solution (6.12) which does not exhibit transient behavior.

6.3 Resonant response

In section 4.5 we have seen that resonant response ($\Omega^* = \mathcal{O}(1)$, $\delta \ll 1$) is governed by a small band of frequencies around $\omega^* = 1$. To describe this behavior, we write

$$\omega_n^* = 1 + \delta v_m^* \quad , \quad n = m + N_0 \quad , \quad N_0 = 1/\Delta\omega. \tag{6.16}$$

Oscillations at natural frequency $\omega^* = 1$ thus correspond to $n = N_0$ or $m = 0$ in the series representations of the solution. Substituting this in solution (6.12) and dropping terms of magnitude δ directly compared with order unity we have

$$\begin{aligned} x &= \left(\frac{\pi S_p^n(\omega^* = 1)}{2\delta}\right)^{1/2} \sum_{m=-N_0}^{N-N_0} \left(\frac{4\Delta v^*}{\pi(1+4v_m^{*2})}\right)^{1/2} \\ &\quad \cos\left\{t^* + v_m^* T^* + \varphi_m + \arctan\left(\frac{1}{2v_m^*}\right)\right\}, \end{aligned} \tag{6.17}$$

where

$$T^* = \delta t^*. \tag{6.18}$$

Note that $v_m^* = m\Delta v^*$ and that v_m^* extends from $-N_0\Delta v^*$ to $+(N - N_0)\Delta v^*$. Solution (6.17) can also be written as:

$$\begin{aligned} x &= \left(\frac{\pi S_p^n(\omega^* = 1)}{2\delta}\right)^{1/2} \{f_1(T^*)\cos t^* - f_2(T^*)\sin t^*\} \\ &= \left(\frac{\pi S_p^n(\omega^* = 1)}{2\delta}\right)^{1/2} a(T^*)\cos\{t^* + \psi(T^*)\}, \end{aligned} \tag{6.19}$$

where:

$$a(T^*) = \{f_1^2(T^*) + f_2^2(T^*)\}^{1/2}, \tag{6.20}$$

$$\psi(T^*) = \arctan\{f_2(T^*)/f_1(T^*)\}. \tag{6.21}$$

Here, $f_1(T^*)$ and $f_2(T^*)$ are Gaussian random signals:

$$f_1(T^*) = \sum_{m=-N_0}^{N-N_0} \left(\frac{4\Delta v^*}{\pi(1+4v_m^{*2})}\right)^{1/2} \cos\left(v_m^* T^* + \varphi_m + \arctan\left(\frac{1}{2v_m^*}\right)\right), \tag{6.22}$$

$$f_2(T^*) = \sum_{m=-N_0}^{N-N_0} \left(\frac{4\Delta v^*}{\pi(1+4v_m^{*2})}\right)^{1/2} \sin\left(v_m^* T^* + \varphi_m + \arctan\left(\frac{1}{2v_m^*}\right)\right), \tag{6.23}$$

which are stochastically equivalent to:

$$f_1(T^*) = \sum_{n=1}^{N} \left(\frac{8\Delta v^*}{\pi(1+4v_n^{*2})} \right)^{1/2} \cos\left(v_n^* T^* + \varphi_n\right), \tag{6.24}$$

$$f_2(T^*) = \sum_{n=1}^{N-N_0} \left(\frac{8\Delta v^*}{\pi(1+4v_n^{*2})} \right)^{1/2} \sin\left(v_n^* T^* + \varphi_n\right). \tag{6.25}$$

These signals hold over the time interval

$$0 \le T^* \le T_0^* \quad , \quad T_0^* = 2\pi/\Delta v^*. \tag{6.26}$$

The power densities of the signals equal $4/(\pi(1+4v^{*2}))$ implying that their standard deviations are equal to unity:

$$\langle f_1^2(T^*) \rangle = \langle f_2^2(T^*) \rangle = \int_0^\infty \frac{4dv^*}{\pi(1+4v^{*2})} = \frac{2}{\pi} \arctan(2v^*) \Big|_0^\infty = 1. \tag{6.27}$$

Furthermore, $f_1(T^*)$ and $f_2(T^*)$ are uncorrelated. This can be shown as follows:

$$\begin{aligned} \langle f_1(T^*) f_2(T^*) \rangle &= \lim_{T_0^* \to \infty} \frac{1}{T_0^*} \int_0^{T_0^*} f_1(T^*) f_2(T^*) dT^* \\ &= \lim_{T_0^* \to \infty} \sum_{n=0}^{N} \sum_{m=0}^{N} \frac{8\Delta v^*}{\pi(1+4v_n^{*2})(1+4v_m^{*2})T_0^*} \\ &\quad \times \int_0^{T_0^*} \cos\left(2\pi n \frac{T^*}{T_0^*} + \varphi_n\right) \sin\left(2\pi m \frac{T^*}{T_0^*} + \varphi_m\right) dT^* \end{aligned} \tag{6.28}$$

Here,

$$\begin{aligned} & \frac{1}{T_0^*} \int_0^{T_0^*} \cos\left(2\pi n \frac{T^*}{T_0} + \varphi_n\right) \sin\left(2\pi m \frac{T^*}{T_0} + \varphi_m\right) dT^* \\ = & \int_0^1 \cos(2\pi n z + \varphi_n) \sin(2\pi m z + \varphi_m) dz \\ = & \frac{1}{2} \int_0^1 \left[\sin(2\pi(m-n)z + \varphi_m - \varphi_n) + \sin(2\pi(m+n)z + \varphi_m + \varphi_n)\right] dz \\ = & \frac{1}{2} \int_0^1 [\sin(\varphi_m - \varphi_n)\cos(2\pi(m-n)z) + \cos(\varphi_m - \varphi_n)\sin(2\pi(m-n)z) \\ & + \sin(\varphi_m + \varphi_n)\cos(2\pi(m+n)z) + \cos(\varphi_m + \varphi_n)\sin(2\pi(m+n)z)]\, dz \\ = & \ 0. \end{aligned} \tag{6.29}$$

because the integrals of the oscillatory terms are zero over the interval $0 \leq z \leq 1$. Hence,

$$\langle f_1(T^*) f_2(T^*) \rangle = 0. \tag{6.30}$$

The mean-square value of response can now be calculated as follows (using eq. (6.19), (6.27) and (6.30)):

$$\begin{aligned}
\sigma_0^2 = \langle x^2(t) \rangle &= \frac{\pi S_p^n(\omega^* = 1)}{2\delta} \langle (f_1(T*) \cos t^* - f_2(T^*) \sin t^*)^2 \rangle \\
&= \frac{\pi S_p^n(\omega^* = 1)}{2\delta} [\langle f_1^2(T^*) \rangle^2 \cos^2 t^* + \langle f_2^2(T^*) \rangle \sin^2 t^* \\
&\quad -2\langle f_1(T^*) f_2(T^*) \rangle \cos t^* \sin t^*] \\
&= \frac{\pi S_p^n(\omega^* = 1)}{2\delta}
\end{aligned} \tag{6.31}$$

which is the same as eq. (4.24)!

From solution (6.19) it is seen that resonant response to random excitation consists of quasi-deterministic sinusoidal behavior around the natural frequency with an amplitude and phase which vary randomly and slowly with time ($T^* = \delta t^*$). An illustration of this behavior is given in figure 6.3, taking $\delta = 0.02$ and $T_0^* = 600$.

Calculating resonant response by direct numerical integration of the differential equation of the spring-mass system can be quite cumbersome. The damping term in the equation is very small and at the same time, the determining factor for the magnitude of the resonant response: a recipe for problems! While performing numerical integration, round-off errors are made. These can create an artificial form of damping, which is larger than the real damping. To avoid this, very small time steps are required which leads to long calculation times.

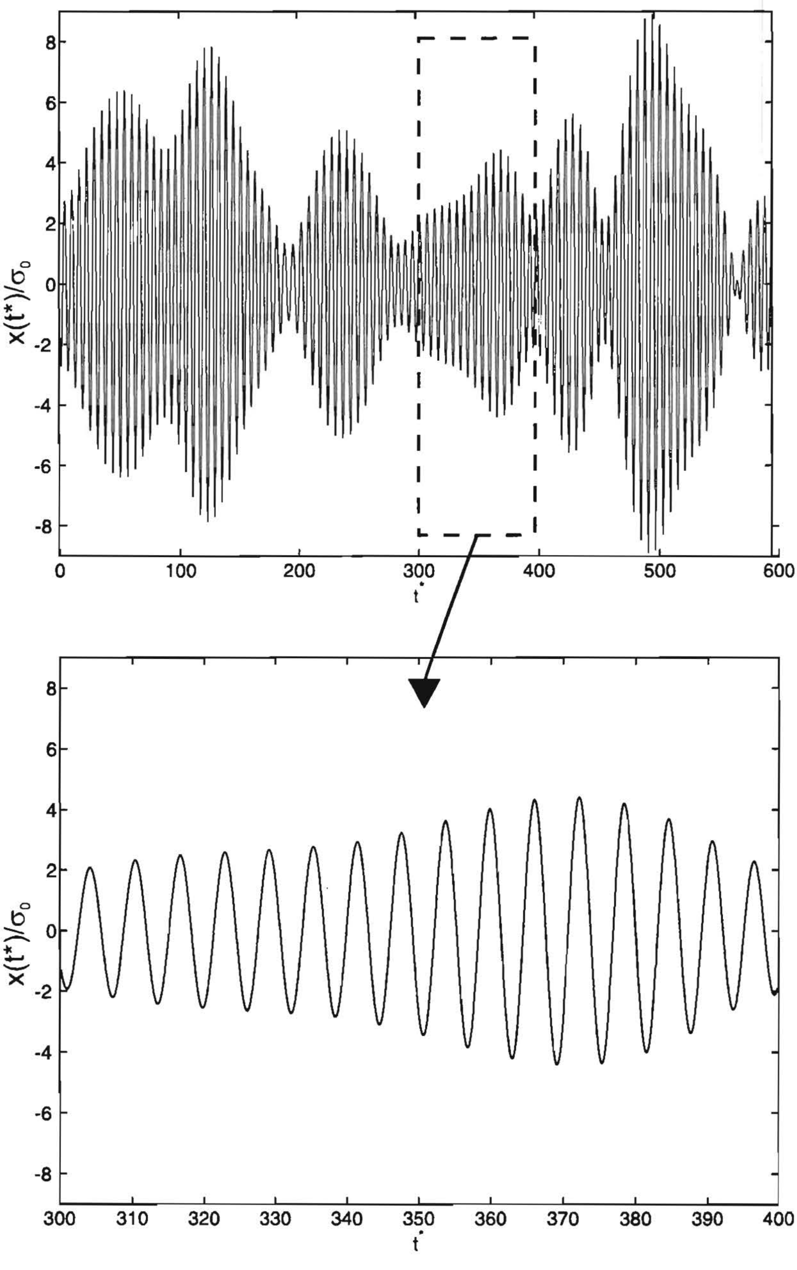


Figure 6.3. Resonant response ($\Omega^* = 1, \delta = 0.02$)

6.4 Transient response

The solutions presented so far did not take transient behavior into account. Both the solutions for response in the frequency-domain presented in chapter 4 and the solutions in the time-domain given in the previous sections describe response that is statistically stationary *. In the present section, we shall present descriptions for transient behavior due to some initial condition imposed at time zero. More specifically, we shall consider the case where displacement and velocity are zero at time zero.

A description in the time-domain for transient response is simply obtained by using the linearity of the governing differential equation: cf. eq. (6.2). A solution can be written as

$$x(t^*) = x_H(t^*) + x_S(t^*), \tag{6.32}$$

where $x_S(t^*)$ is the stationary solution given by eqs. (6.12)-(6.14). Because $x_S(t^*)$ satisfies eq. (6.2) with random right hand side $P^*(t^*)$, $x_H(t^*)$ will be the solution of the homogeneous problem

$$\ddot{x}_H + \delta \dot{x}_H + x_H = 0. \tag{6.33}$$

The two basic solutions of this problem are:

$$x_H = Ae^{-\delta t^*/2} \cos\left((1 - \delta^2/4)^{1/2} t^*\right) + Be^{-\delta t^*/2} \sin\left((1 - \delta^2/4)^{1/2} t^*\right). \tag{6.34}$$

At time zero, we require that displacement and velocity are zero:

$$x(t^*) = \dot{x}(t^*) = 0 \quad \text{at} \quad t^* = 0 \tag{6.35}$$

From eq. (6.32) it then follows that

$$x_H(t^*) = -x_S(0) \quad , \quad \dot{x}_H(t^*) = -\dot{x}_S(0) \quad \text{at} \quad t^* = 0, \tag{6.36}$$

where $x_S(0)$ and $\dot{x}_S(0)$ are specified by eqs. (6.12)-(6.14) substituting $t^* = 0$. Given conditions (6.36), the constants A and B of solution (6.33) can be evaluated to:

$$A = -x_S(0) \tag{6.37}$$

*With stationary response we mean response of which the statistics (mean value, standard deviation, etc.) do not vary with time. The response signal itself obviously varies with time (in a random way)

$$B = \left(1-\delta^2\right)^{-1/2}\left(-\dot{x}_S(0) - \frac{\delta}{2}x_S(0)\right) \tag{6.38}$$

The statistics of response can be determined as follows. The excitation $P^*(t^*)$ is a Gaussian process. In that case, any variable that is linearly related to $P^*(t^*)$ will be Gaussian as well. Also variables that suffer from a time-lag with respect to $P^*(t^*)$ because of inertia and/or damping. Therefore, the probability distribution of $x(t^*)$ will be Gaussian. All that remains to be specified are the parameters that determine this distribution, i.e. mean and standard deviation. The mean of $x(t^*)$ equals the mean of $x_H(t^*)$ plus the mean of $x_S(t^*)$, which are both equal to zero (because $\langle P^*(t^*)\rangle$ equals zero). Hence,

$$\langle x(t^*)\rangle = 0. \tag{6.39}$$

The standard deviation follows from the mean-square response as:

$$\begin{aligned}\sigma^2 &= \langle x^2(t^*)\rangle = \langle (x_H(t^*) + x_S(t^*))^2\rangle \\ &= \langle x_H^2(t^*) + x_S^2(t^*) + 2x_H(t^*)x_S(t^*)\rangle \\ &= \langle x_H^2(t^*)\rangle + \langle x_S^2(t^*)\rangle + 2\langle x_H(t^*)x_S(t^*)\rangle\end{aligned} \tag{6.40}$$

Substituting solutions (6.34) and (6.37)-(6.38) we end up with moments of products of $x_S(t)$ with $x_S(0)$ and $\dot{x}_S(0)$. The determination of these correlations, though not impossible, can be quite laborious, depending on the shape of the power density spectrum of $P^*(t^*)$. A more simple situation arises in case of resonant response. Such response occurs when damping is light, $\delta \ll 1$, and the total response is governed by the resonance peak in the power density spectrum. When $\delta \ll 1$, the time domain description for transient response given by eqs. (6.34) and (6.37)-(6.38) can be simplified to:

$$x_H(t^*) = -x_S(0)e^{-T^*/2}\cos t^* - \dot{x}_S(0)e^{-T^*/2}\sin t^*, \tag{6.41}$$

where, as before, $T^* = \delta t^*$. For resonant response, $x_S(t^*)$ can be described by the right hand side of eq. (6.19):

$$x_S(t^*) = \sigma_0\left\{f_1(T^*)\cos t^* - f_2(T^*)\sin t^*\right\}, \tag{6.42}$$

where $f_1(T^*)$ and $f_2(T^*)$ are uncorrelated Gaussian signals described by eqs. (6.24)-(6.25). The total response (transient and stationary) is then given by:

$$x(t^*) = \sigma_0\left(f_1(T^*) - f_1(0)e^{-T^*/2}\right)\cos t^* - \sigma_0\left(f_2(T^*) - f_2(0)e^{-T^*/2}\right)\sin t^*. \tag{6.43}$$

The standard deviation can be assessed as:

$$\begin{aligned}
\sigma^2 &= \langle x^2(t^*)\rangle \\
&= \sigma_0^2\left\langle\left(f_1(T^*) - f_1(0)e^{-T^*/2}\right)^2\right\rangle \cos^2 t^* \\
&\quad +\sigma_0^2\left\langle\left(f_2(T^*) - f_2(0)e^{-T^*/2}\right)^2\right\rangle \sin^2 t^* \\
&\quad -2\sigma_0^2\left\langle\left(f_2(T^*) - f_2(0)e^{-T^*/2}\right)\left(f_1(T^*) - f_1(0)e^{-T^*/2}\right)\right\rangle \\
&\quad \times \sin t^* \cos t^*.
\end{aligned} \tag{6.44}$$

Now

$$\begin{aligned}
\left\langle\left(f_1(T^*) - f_1(0)e^{-T^*/2}\right)^2\right\rangle &= \langle f_1^2(T^*)\rangle + \langle f_1^2(0)\rangle e^{-T^*} \\
&\quad -2\langle f_1(T^*)f_1(0)\rangle e^{-T^*/2}.
\end{aligned} \tag{6.45}$$

Here,

$$\langle f_1^2(T^*)\rangle = \langle f_1^2(0)\rangle = 1 : \tag{6.46}$$

see eq. (6.27). The auto-correlation of $f_1(T^*)$ follows from its power density spectrum which is $4/\pi(1+4v^{*2}$. The Fourier transform of this spectrum (cf. eq. (3.15)) yields the autocorrelation function being $e^{-T^*/2}$: this result is easily obtained from eq. (3.8) setting $\Omega = 1/2$, $\sigma = 1$ and noting that the auto-correlation associated with eq. (3.8) is $\sigma^2 e^{-\Omega T^*}$ (see text subsequent to eq. (3.15)). The net result for eq. (6.45) is now:

$$\left\langle\left(f_1(T^*) - f_1(0)e^{-T^*/2}\right)^2\right\rangle = 1 - e^{-T^*}. \tag{6.47}$$

Clearly, the same result is obtained for the second term on the right hand side of eq. (6.44):

$$\left\langle\left(f_2(T^*) - f_2(0)e^{-T^*/2}\right)^2\right\rangle = 1 - e^{-T^*}. \tag{6.48}$$

For the third term we can write:

$$\begin{aligned}
&\left\langle\left(f_1(T^*) - f_1(0)e^{-T^*/2}\right)\left(f_2(T^*) - f_2(0)e^{-T^*/2}\right)\right\rangle \\
&\quad = \langle f_1(T^*)f_2(T^*)\rangle + \langle f_1(0)f_2(0)\rangle e^{-T^*} \\
&\quad - \left(\langle f_1(0)f_2(T^*)\rangle + \langle f_2(0)f_1(T^*)\rangle\right) e^{-T^*/2}
\end{aligned} \tag{6.49}$$

In view of result (6.30),

$$\langle f_1(T^*)f_2(T^*)\rangle = \langle f_1(0)f_2(0)\rangle = 0. \tag{6.50}$$

Furthermore, using a procedure similar to that of eqs. (6.28)-(6.29) one can show that:

$$\langle f_1(0)f_2(T^*)\rangle + \langle f_2(0)f_1(T^*)\rangle = 0 \tag{6.51}$$

Here it is noted that the separate correlations are not equal to zero! These values can be calculated in the following way:

$$\langle f_1(0)f_2(T^*)\rangle = \langle f_1(T)f_2(T+T^*)\rangle = \lim_{T_0^*\to\infty}\int_0^{T_0^*} f_1(T)(f_2(T+T^*)dT. \tag{6.52}$$

The right hand side can be evaluated upon substituting series representations (6.24)-(6.25), using addition formulae for trigonometric functions to bring the terms back to products of terms like $\sin(2m\pi T/T_0^*)$ and $\cos(2m\pi T/T_0^*)$ and evaluating the integrals; all this similar to the procedure of eq. (6.29). The result is

$$\langle f_1(0)f_2(T^*)\rangle = \sum_{n=1}^{N}\frac{2\Delta v\sin(v_nT^*)}{\pi(1+4v_n^2)} = \int_0^\infty \frac{2\sin(vT^*)dv}{\pi(1+4v^2)} \tag{6.53}$$

Similarly it can be shown that

$$\langle f_2(0)f_1(T^*)\rangle = -\int_0^\infty \frac{2\sin(vT^*)dv}{\pi(1+4v^2)} \tag{6.54}$$

The above cross-correlation functions are zero if $T^* = 0$ but have nonzero but opposite values for $T^* > 0$.

The standard deviation now follows from eq. (6.44) as

$$\sigma^2(T^*) = \sigma_0^2(1-e^{-T^*}). \tag{6.55}$$

The Gaussian probability density distribution for transient response is then given by

$$p = \frac{1}{\sigma(T^*)\sqrt{2\pi}}e^{-x^2/(2\sigma^2(T^*))} \tag{6.56}$$

The distribution approaches a Dirac δ-function around $x = 0$ as $t \longrightarrow \infty$. For $T^* \longrightarrow \infty$ the distribution evolves to a stationary Gaussian distribution with

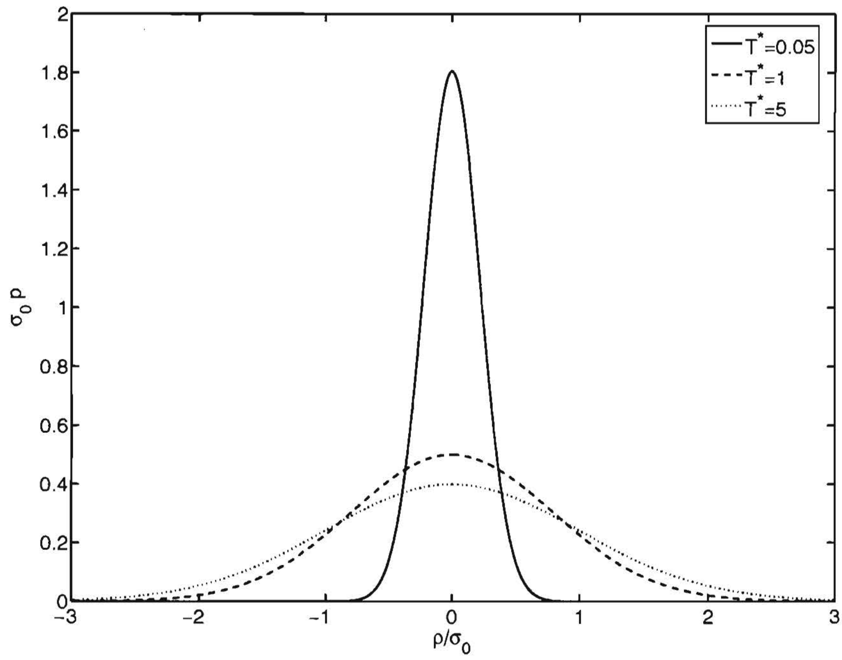


Figure 6.4. Transient cumulative Gaussian response behavior in case of zero initial response.

standard deviation σ_0. An illustration of the distribution is given is figure 6.4.

The cumulative Gaussian distribution for transient response reads as

$$P(x) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x}{\sigma(T^*)\sqrt{2}}\right)\right), \tag{6.57}$$

where erf is the error function: cf. eq. (1.9). An illustration of the cumulative distribution is given in figure 6.5. It is seen that the cumulative distribution evolves from a Heaviside function at $T^* = 0$ to the stationary Gaussian distribution as $T^* \longrightarrow \infty$.

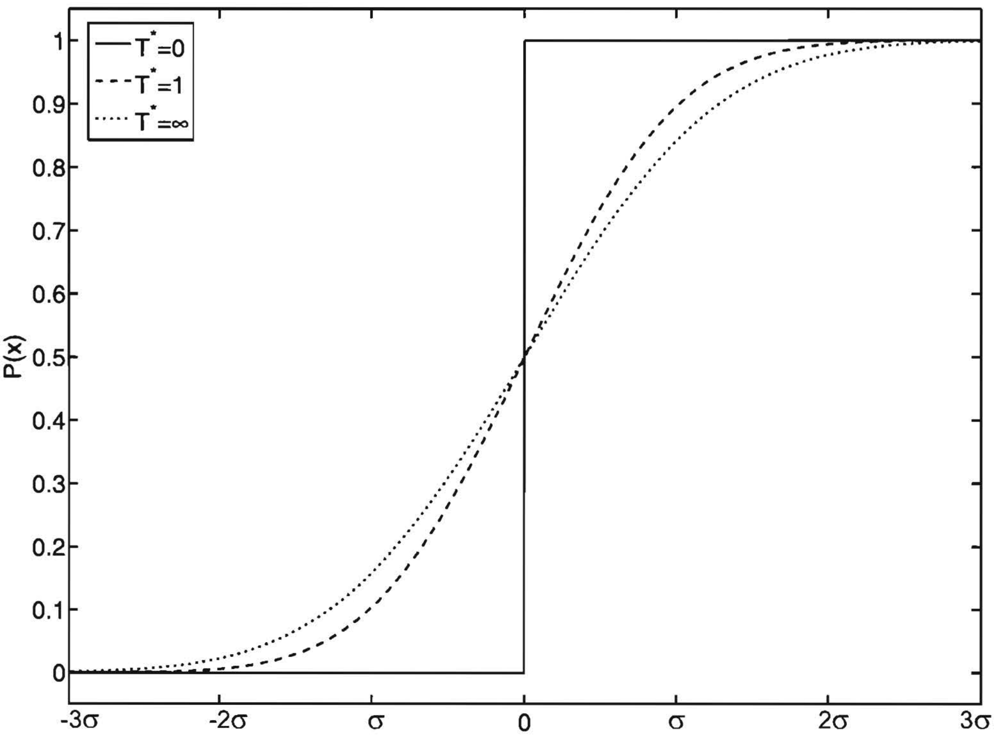


Figure 6.5. Transient cumulative Gaussian response behavior in case of zero initial response.

Chapter 7

Multi degree of freedom system: the tensioned beam

The beam is a basic element of many constructions of mechanical engineering. Its stiffness against lateral deflection can be increased by applying longitudinal tension at both ends. A guitar string is a nice example of such application. In off-shore structures we find beams and tensioned beams at various places. For example, as elements of steel jacket structures; as risers in which oil or gas flows from the well at the sea floor to the production platform (Brouwers [8]); or as anchoring device keeping a floating platform at its position, see figure 7.1 for examples of such structures. Direct wave loading and wave-induced lateral displacement via a floating structure are the cause of random response behavior in time. Analyzing the random behavior of the tensioned beam is the objective of subsequent chapters. In the present chapter, we shall focus on the formulation of the governing partial differential equations. Furthermore, we shall identify the conditions under which bending stiffness and stiffness by pre-tensioning prevail. In chapter 8, analysis is focussed to the case of a tensioned string, in chapter 9 to the beam.

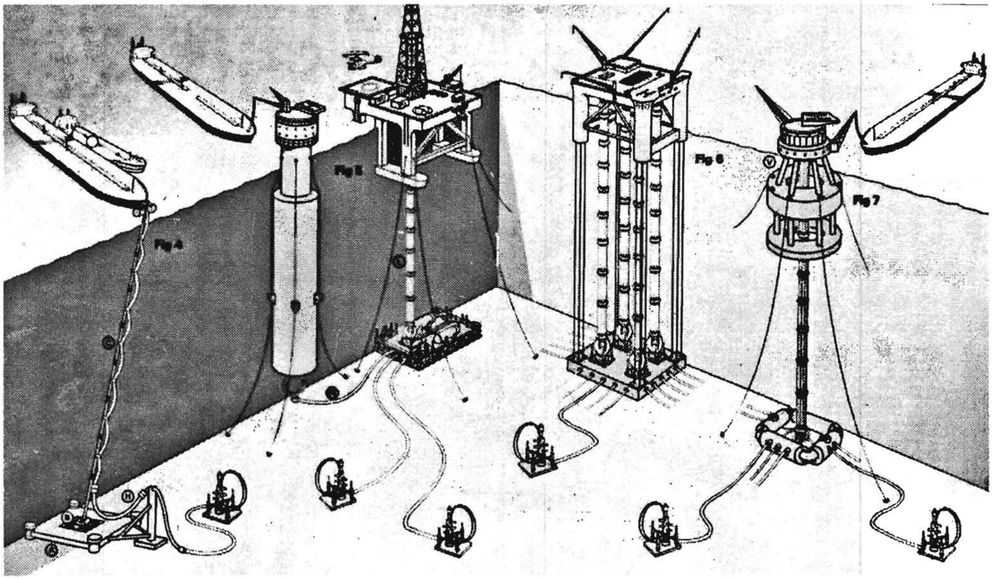


Figure 7.1. Illustration of various types of off-shore structures; by courtesy of Royal Dutch Shell Exploration & Production

7.1 Basis equations

Attention is focussed on the so-called marine riser which connects a floating off-shore structure to the oil or gas well at the sea bottom: Brouwers [8]. The riser can be modeled as a tensioned beam which undergoes lateral deflection as a result of direct wave forcing and lateral motion of the floating structure, see the sketch in figure 7.2.

Basic equations can be formulated when applying Newton's second law in horizontal direction to an infinitesimal element of the beam, see figure 7.3:

$$-dD + d(T_r\theta) + Fdz = \left(m\frac{\partial^2 x}{\partial t^2} + d\frac{\partial x}{\partial t}\right) dz. \tag{7.1}$$

Equilibrium of moments yields:

$$dM = Ddz. \tag{7.2}$$

The bending moment is related to deformation according to the *Bernoulli-*

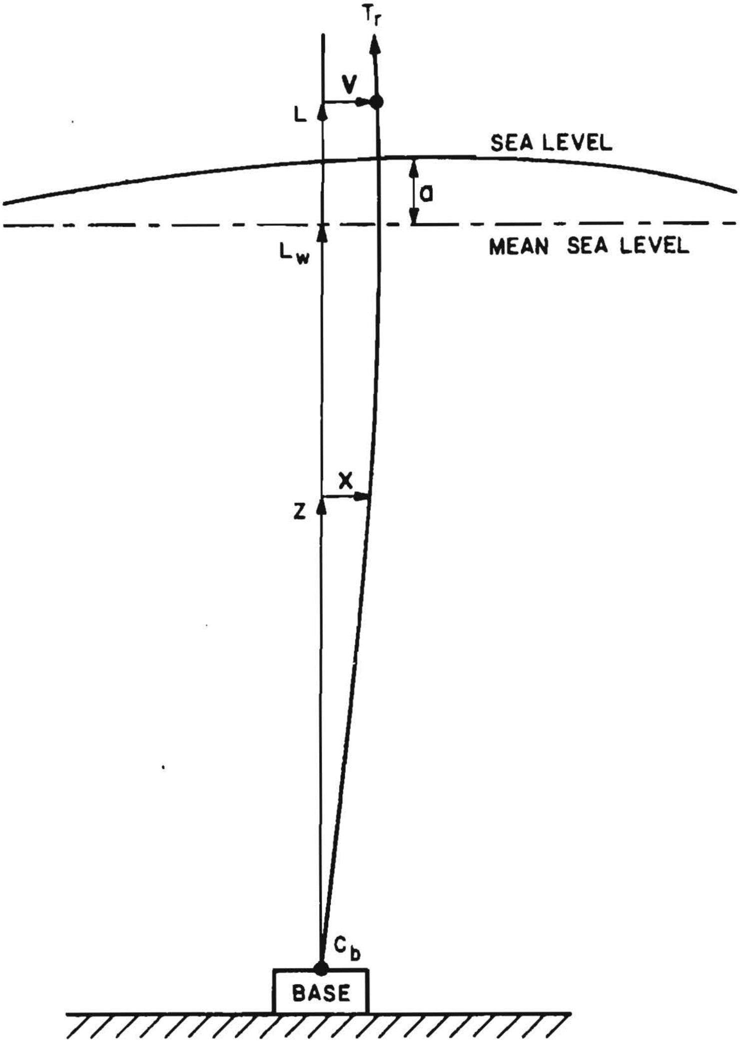


Figure 7.2. Riser configuration.

Euler relation

$$M = EI\frac{\partial^2 x}{\partial z^2}, \tag{7.3}$$

while for small angles, angle and displacement are related to each other by the geometrical relation

$$\theta = \frac{\partial x}{\partial z}. \tag{7.4}$$

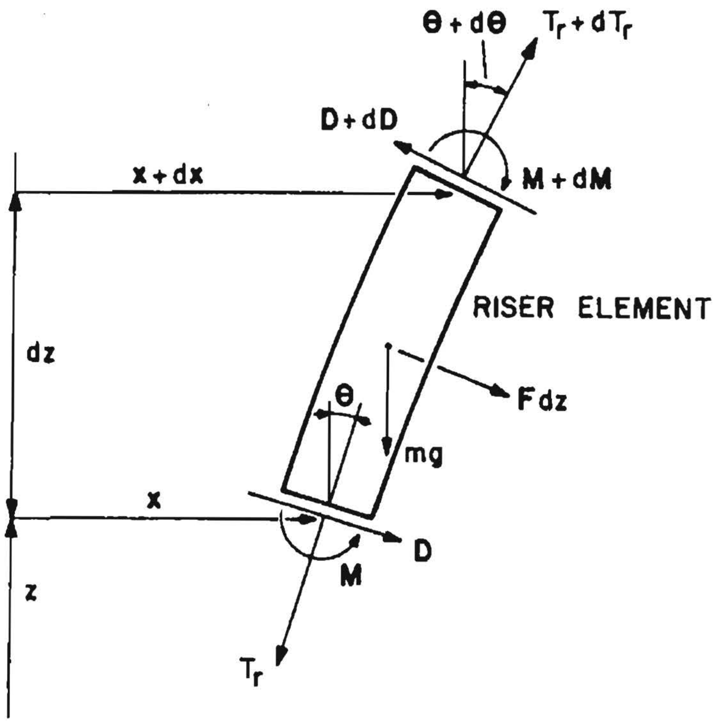


Figure 7.3. Infinitesimal element of a tensioned beam.

Substituting equations (7.2)-(7.4) into equation (7.1) yields the partial differential equation of the tensioned beam:

$$EI\frac{\partial^4 x}{\partial z^4} - \frac{\partial}{\partial z}\left(T_r(z)\frac{\partial x}{\partial z}\right) + m\frac{\partial^2 x}{\partial t^2} + d\frac{\partial x}{\partial t} = F(z,t), \quad (7.5)$$

where:

x = horizontal deflection
z = vertical distance from riser base
t = time
E = modulus of elasticity
I = moment of stiffness against bending
T_r = riser tension ($T_r > 0$)
m = mass per unit length
d = damping constant per unit length
F = external force per unit length

The bottom end of the riser is assumed to be fixed to the base:

$$x = 0 \quad \text{at} \quad z = 0 \tag{7.6}$$

and subject to a rotational constraint

$$EI\frac{\partial^2 x}{\partial z^2} = C_b \frac{\partial x}{\partial z} \quad \text{at} \quad z = 0, \tag{7.7}$$

where C_b is the rotational stiffness. At the top, the riser is connected to the floating structure by a hinge:

$$x = V(t) \quad \text{at} \quad z = L, \tag{7.8}$$

$$EI\frac{\partial^2 x}{\partial z^2} = 0 \quad \text{at} \quad z = L, \tag{7.9}$$

where $V(t)$ is the time-dependent horizontal displacement of the floating structure and L is the length of the riser.

7.2 Tensioned string versus beam

Two kinds of stiffness can be distinguished in the tensioned beam: bending stiffness and stiffness due to tensioning. The question now is: which is the important one? To answer this question, consider a beam of length L subject to a typical lateral deflection of magnitude a. The magnitudes of the bending and tension terms in equation (7.5) are then:

$$EI\frac{\partial^4 x}{\partial z^4} \approx \frac{EIa}{L^4}, \tag{7.10}$$

$$T\frac{\partial^2 x}{\partial z^2} \approx \frac{Ta}{L^2}. \tag{7.11}$$

Hence, bending and tension forces compare to each other as:

$$D_1 = \frac{EI}{TL^2}. \tag{7.12}$$

In case of low pre-tension, D_1 will be large and bending stiffness dominates. The tension forces can be disregarded and we end up with the problem of the beam: see chapter 9.

On the other hand, when $D_1 \ll 1$, the stiffness of the beam against lateral deflection is mainly due to pre-tension (the guitar string!) and bending forces can then be disregarded. However, when neglecting bending forces, the governing partial differential equation reduces from fourth- to second-order in z, so that 2 out of 4 boundary conditions have to be dropped. A singular perturbation problem results (just as the resonance problem in case of light damping in the spring-mass system; see section 4.5). To solve this problem, boundary layers have to be introduced at both ends of the tensioned string (so boundary layers not only occur in fluid mechanics!). At the bottom end, the boundary layer is described by a local coordinate:

$$\eta = \epsilon^{-1}\frac{z}{L} \quad ; \quad \eta = \mathcal{O}(1), \epsilon \ll 1. \tag{7.13}$$

In terms of this coordinate, we have:

$$EI\frac{\partial^4 x}{\partial z^4} \approx \frac{EI}{L^4\epsilon^4}\frac{\partial^4 x}{\partial \eta^4}, \tag{7.14}$$

$$T\frac{\partial^2 x}{\partial z^2} \approx \frac{T}{L^2\epsilon^2}\frac{\partial^2 x}{\partial \eta^2}. \tag{7.15}$$

Bending and tension forces become of the same magnitude as:

$$\frac{EI}{L^4\epsilon^4} = \frac{T}{L^2\epsilon^2} \quad \Rightarrow \quad \epsilon = D_1^{1/2}. \tag{7.16}$$

For large pre-tension, such that $\epsilon \ll 1$ or $D_1 \ll 1$, bending forces will only be important in small areas of length $L\epsilon = (EI/T)^{1/2}$ at both ends of the beam: $(EI/T)^{1/2} \ll L$ when $D_1 \ll 1$. Outside these areas, i.e. the main section of the beam, lateral deflection, angles and curvatures are governed by pre-tension forces rather than bending forces. The boundary layers at the top and bottom serve to adjust deflections, angles and curvatures to the values prescribed by the boundary conditions. Because bending and tension forces are 4th and 2nd order derivatives with respect to vertical distance, these terms can be assumed to dominate in comparison to all other terms in the boundary layers. To show this in explicit terms, introduce the boundary layer coordinate η defined by equation (7.13) into basic equation (7.5). This gives, after some re-ordering of terms:

$$\frac{\partial^4 x}{\partial \eta^4} - \frac{\partial^2 x}{\partial \eta^2} = \epsilon^4 \frac{L^4}{EI} \left(F - m\frac{\partial^2 x}{\partial t^2} - d\frac{\partial x}{\partial t} \right) . \tag{7.17}$$

For small ϵ, the right hand side becomes small and can be disregarded. Because the boundary layer is very short in length, inertia, damping and external forces are relatively small. The boundary layers are areas where response is quasi-static in nature. Solutions describe exponential decaying behavior over the distance $(EI/T)^{1/2}$. In case of clamped ends ($C_b \rightarrow \infty$ in equation (7.7)) they reveal local bending stress increase due to increasing curvature. To avoid any local stress increase, hinges or ball-joints should be incorporated at the ends of the tensioned string (like the guitar string). In the subsequent analysis, we shall pay no further attention to the boundary layers at the top and bottom: for further details, see Brouwers [8]. We shall consider the case of the beam without pre-tension (chapter 9) and the string without bending stiffness (chapter 8) only.

Chapter 8

Random Vibrations of a tensioned string

As we have shown in the previous chapter, for cases where the dimensionless ratio D_1 given by eq. (7.12) is small, $D_1 \ll 1$, forces due to bending stiffness can be disregarded. Furthermore, the pre-tension is assumed to be constant with respect to z: $T_r(z) = T_r$ (which is justified if the total weight of the riser is much less than the pre-tension applied at the top). The governing equation of the tensioned beam, i.e. eq. (7.5) can then be simplified to that of the tensioned string:

$$-T_r \frac{\partial^2 x}{\partial z^2} + m \frac{\partial^2 x}{\partial t^2} + d \frac{\partial x}{\partial t} = F(z,t) \tag{8.1}$$

We disregard lateral displacement imposed at the top. The boundary conditions are:

$$x(z,t) = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = L. \tag{8.2}$$

The effects of boundary conditions imposed on angle and curvature are accommodated by boundary layer type solutions at the top and bottom, see section 7.2 . These are not considered here.

8.1 Separation of variables

Drop the excitation and damping forces in equation (8.1):

$$T_r \frac{\partial^2 x(z,t)}{\partial z^2} = m \frac{\partial^2 x(z,t)}{\partial t^2} \tag{8.3}$$

To solve this equation we will apply *separation of variables* [7]. This involves expressing the solution $x(z,t)$ as a product of two functions which only depend on z and t, respectively:

$$x(z,t) = g(z)h(t) \tag{8.4}$$

Substitution in (8.3) and dividing through $g(z)h(t)$ gives:

$$T_r \frac{1}{g(z)} \frac{\partial^2 g(z)}{\partial z^2} = m \frac{1}{h(t)} \frac{\partial^2 h(t)}{\partial t^2} \tag{8.5}$$

The above equation can only hold if the terms on the left- and right-hand side are equal to a constant $-\lambda$ (the minus sign is chosen for convenience). This results in the following ordinary differential equations:

$$\frac{d^2 g(z)}{dz^2} = -\frac{\lambda}{T_r} g(z) \tag{8.6}$$

$$\frac{d^2 h(t)}{dt^2} = -\frac{\lambda}{m} h(t) \tag{8.7}$$

We will start with finding a solution for the first differential equation (8.6), using $x(z,t) = 0$ at $z = 0, L$ as boundary conditions. These boundary conditions can now be expressed as $g(z) = 0$ at $z = 0, L$ because $h(t) = 0$ leads to a trivial solution $x = 0$ for all times t. The general solution for such an equation is:

$$g(z) = A \sin\left(\sqrt{\frac{\lambda}{T_r}} z\right) + B \cos\left(\sqrt{\frac{\lambda}{T_r}} z\right) \tag{8.8}$$

The constant A and B can be determined using the boundary conditions. The first boundary condition ($g(0) = 0$ leads to $B = 0$). The second boundary condition only leads to non-trivial solutions in case $\sqrt{\frac{\lambda}{T_r}} L = n\pi$ with n an integer. This leads to the following eigenvalues:

$$\lambda_n = T_r \left(\frac{n\pi}{L}\right)^2 \quad , \qquad n = 1, 2, 3, ..., \infty \tag{8.9}$$

with the corresponding eigenfunctions or natural modes:

$$g_n(z) = \sin\left(\frac{n\pi z}{L}\right) \quad , \qquad n = 1, 2, 3, ..., \infty \tag{8.10}$$

Following the same procedure for the second differential equation (8.7) only leads to trivial solutions using the initial conditions $h(0) = 0$ and $dh(0)/dt = 0$: $h(t) = 0$. This means that the total solution $x(z,t) = g(z)h(t)$ equals zero in this case. A non-zero solution is only obtained by introducing a forcing. This will be considered in the next section.

8.2 Expansion in eigenfunctions

The solution of equation (8.1) is written as an infinite series of eigenfunctions with time-dependent amplitudes:

$$x(z,t) = \sum_{n=1}^{\infty} x_n(t) g_n(z) \tag{8.11}$$

with $g_n(z) = \sin(n\pi z/L)$. The equation for the amplitudes x_n can be found by substituting the above equation into the PDE:

$$\sum_{n=1}^{\infty} \left[T_r \frac{n^2\pi^2}{L^2} x_n(t) \sin\left(\frac{n\pi z}{L}\right) + \left(m\frac{d^2x_n(t)}{dt^2} + d\frac{dx_n}{dt} \right) \sin\left(\frac{n\pi z}{L}\right) \right] = F(z,t) \tag{8.12}$$

This equation can be solved for $x_n(t)$ using well-known properties of orthogonality of eigenfunctions g_k and g_n:

$$\int_0^L g_n(z) g_k(z) dz = 0 \quad \text{if} \quad k \neq n \tag{8.13}$$

and

$$\int_0^L g_n(z) g_n(z) dz = \frac{L}{2}. \tag{8.14}$$

In order to use this property we multiply (8.12) with g_k and integrate from $z = 0$ to $z = L$:

$$\int_0^L \sum_{n=1}^{\infty} \left[T_r \frac{n^2\pi^2}{L^2} x_n(t) \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{k\pi z}{L}\right) + \right.$$
$$\left. \left(m\frac{d^2x_n(t)}{dt^2} + d\frac{dx_n}{dt} \right) \sin\left(\frac{n\pi z}{L}\right) \sin\left(\frac{k\pi z}{L}\right) \right] dz$$
$$= \int_0^L F(z,t) \sin\left(\frac{k\pi z}{L}\right) dz. \tag{8.15}$$

Using the orthogonality properties this can be simplified to:

$$\left(T_r \frac{n^2\pi^2}{L^2} x_n(t) + m\frac{d^2 x_n(t)}{dt^2} + d\frac{dx_n}{dt}\right)\frac{L}{2} = \int_0^L F(z,t)\sin\left(\frac{n\pi z}{L}\right)dz. \tag{8.16}$$

Equation (8.16) can also be written in a form which is equal to that of a spring-mass system (equation (4.1)):

$$m\ddot{x}_n + d\dot{x}_n + c_n x_n = P_n(t), \tag{8.17}$$

where:

$$c_n = n^2\pi^2 \frac{T_r}{L^2}, \tag{8.18}$$

$$P_n(t) = \frac{2}{L}\int_0^L F(z,t)\sin\left(\frac{n\pi z}{L}\right)dz. \tag{8.19}$$

We see that by expanding the solution in terms of eigenfunctions of the tensioned string, the amplitude of each eigenfunction can be described by the equation of a single degree of freedom spring-mass system. In chapters 4 and 6, we have presented methods for analyzing the behavior of $x_n(t)$ in case of random excitation.

An interesting question is how the overall response is dominated by one of the eigenfunctions or eigenmodes. One important factor is the axial shape of the excitation. If the axial shape of $F(z,t)$ equals one of the eigenfunctions, say $F(z,t) = g_k(z)f(t)$, then $P_n(t)$ will only have a value different from zero if $n = k$. This follows directly from property (8.13)-(8.14). Furthermore, if $F(z,t)$ is constant with respect to z, $F(z,t) = F(t)$, then:

$$P_n(t) = \frac{4}{n\pi}F(t) \quad \text{for} \quad n \quad \text{odd}, \tag{8.20}$$

$$P_n(t) = 0 \quad \text{for} \quad n \quad \text{even}. \tag{8.21}$$

Another factor which can lead to domination of one eigenmode are the values of the natural frequency of the string:

$$\omega_{n0} = \left(\frac{c_n}{m}\right)^{1/2} = \frac{n\pi}{L}\left(\frac{T_r}{m}\right)^{1/2}, \qquad n = 1, 2, 3, ..., \infty. \tag{8.22}$$

If one particular natural frequency, say ω_k, has a value which is in the center of the power density spectrum of excitation and the others (ω_{k-1}, ω_{k+1}, etc.) have values where energies of excitation are much less, the standard deviation of $x_k(t)$ will dominate over the others. This follows directly from the results presented in chapters 4 and 6. In other words: as designers we can influence response with the value of the eigenfrequencies which depend on, amongst others, the amount of pre-tension which is applied.

Chapter 9

Random vibrations of a beam

As was shown in chapter 7, whenever $D_1 \gg 1$, bending forces dominate over tension forces in differential equation (7.5). Disregarding the tension and external forces, we have:

$$EI\frac{\partial^4 x}{\partial z^4} + m\frac{\partial^2 x}{\partial t^2} + d\frac{\partial x}{\partial t} = 0 \tag{9.1}$$

The boundary conditions at $z = 0$ are:

$$x(0,t) = 0 \qquad \text{and} \qquad EI\frac{\partial^2 x}{\partial z^2} = 0, \tag{9.2}$$

and at $z = L$:

$$x(L,t) = V(t) \qquad \text{and} \qquad EI\frac{\partial^2 x}{\partial z^2} = 0. \tag{9.3}$$

The second boundary conditions in eqs. (9.2) and (9.3) describe the situation of zero bending stress apparent in case of frictionless ball-joints or hinges. Condition (9.3a) describes prescribed random motion at the top (for example the motion of a floating structure to which the beam is connected via a hinge).

9.1 Quasi-static response

Assume for the time-being that response is quasi-static, denoted by x_p. In that case, we can disregard all dynamical terms in equation (9.1), so that:

$$EI\frac{\partial^4 x_p}{\partial z^4} = 0. \tag{9.4}$$

Integrating this expression twice gives:

$$\frac{\partial^2 x_p}{\partial z^2} = c_1 z + c_2. \tag{9.5}$$

The constants c_1 and c_2 are determined using the following boundary conditions:

$$EI\frac{\partial^2 x_p}{\partial z^2} = 0 \quad \text{at} \quad z = 0, \tag{9.6}$$

$$EI\frac{\partial^2 x_p}{\partial z^2} = 0 \quad \text{at} \quad z = L, \tag{9.7}$$

which yields $c_1 = c_2 = 0$. Hence:

$$\frac{\partial^2 x_p}{\partial z^2} = 0 \tag{9.8}$$

Again, we integrate this equation twice:

$$x_p(z) = c_3 z + c_4 \tag{9.9}$$

Apply the two remaining boundary conditions:

$$x_p = 0 \quad \text{at} \quad z = 0 \tag{9.10}$$

$$x_p = V \quad \text{at} \quad z = L \tag{9.11}$$

which yields $c_3 = V/L$ and $c_4 = 0$, so that:

$$x_p(z) = V \cdot \frac{z}{L}. \tag{9.12}$$

The total solution of equation (9.1) is written as:

$$x = x_p + x_h. \tag{9.13}$$

Substituting this in equation (9.1) and using equation (9.12) yields for x_h:

$$EI\frac{\partial^4 x_h}{\partial z^4} + m\frac{\partial^2 x_h}{\partial t^2} + d\frac{\partial x_h}{\partial t} = -\frac{z}{L}\left(m\ddot{V} + d\dot{V}\right). \tag{9.14}$$

The boundary conditions for this problem at $z = 0$ as well as $z = L$ are:

$$x_h = \frac{\partial^2 x_h}{\partial z^2} = 0. \tag{9.15}$$

In conclusion, by adding the solution for quasi-static response we have transformed our problem from a homogeneous PDE with inhomogeneous boundary conditions (equations (9.2)-(9.3)) to a PDE with prescribed excitation at the right-hand side and homogeneous boundary conditions: equations (9.14)-(9.15). This problem is similar to that treated in the previous chapter, except that we now have a fourth-order spatial derivative representing bending stiffness instead of a second order spatial derivative representing stiffness due to tensioning.

9.2 Separation of variables

To apply separation of variables, drop the damping and excitation terms:

$$EI\frac{\partial^4 x_h}{\partial z^4} + m\frac{\partial^2 x_h}{\partial t^2} = 0 \tag{9.16}$$

Now write $x(z,t)$ as a product of functions $g(z)$ and $h(t)$:

$$x(z,t) = g(z)h(t) \tag{9.17}$$

Differential equation (9.16) then becomes:

$$EI\frac{1}{g(z)}\frac{\partial^4 g(z)}{\partial z^4} = -m\frac{1}{h(t)}\frac{\partial^2 h(t)}{\partial t^2} \tag{9.18}$$

This equation can only be valid if both functions are equal to a constant $+\lambda$. This results in the following ordinary differential equations

$$\frac{d^4 g(z)}{dz^4} = \frac{\lambda}{EI}g(z) \tag{9.19}$$

$$\frac{d^2 h(t)}{dt^2} = \frac{-\lambda}{m}h(t) \tag{9.20}$$

In order to obtain the eigenfunctions we solve equation (9.19). Therefore, we choose $e^{\alpha z}$ as basic solution for the differential equation. It then follows that $\alpha^2 = \pm\sqrt{\lambda/EI}$ and we end up with the following four values for α:

$$\alpha_1 = \left(\frac{\lambda}{EI}\right)^{\frac{1}{4}} \quad \alpha_2 = -\left(\frac{\lambda}{EI}\right)^{\frac{1}{4}} \quad \alpha_3 = i\left(\frac{\lambda}{EI}\right)^{\frac{1}{4}} \quad \alpha_4 = -i\left(\frac{\lambda}{EI}\right)^{\frac{1}{4}} \tag{9.21}$$

The solution of $g(z)$ can then be written as:

$$g(z) = A_1 e^{\alpha_1 z} + A_2 e^{\alpha_2 z} + A_3 \cos\left[\left(\frac{\lambda}{EI}\right)^{\frac{1}{4}} z\right] + A_4 \sin\left[\left(\frac{\lambda}{EI}\right)^{\frac{1}{4}} z\right] \qquad (9.22)$$

Applying 3 out of the 4 boundary conditions, we find

$$g(z) = A_4 \sin(\left(\frac{\lambda}{EI}\right)^{\frac{1}{4}} z) \qquad (9.23)$$

Implementing the boundary condition $g = 0$ at $z = L$, one gets $\left(\frac{\lambda}{EI}\right)^{1/4} L = n\pi$ with n an integer. The eigenvalues thus are:

$$\lambda_n = EI\left(\frac{n^4\pi^4}{L^4}\right), \qquad n = 1, 2, 3, ..., \infty. \qquad (9.24)$$

The associated eigenfunctions are:

$$g_n(z) = \sin\left(\frac{n\pi z}{L}\right) \qquad (9.25)$$

9.3 Eigenfunction expansion

If we now add the excitation and damping force, we can solve eq. (9.14) by expansion in eigenfunctions:

$$EI\frac{\partial^4 x}{\partial z^4} + m\frac{\partial^2 x}{\partial t^2} + d\frac{\partial x}{\partial t} = -\frac{z}{L}\left(m\ddot{V} + d\dot{V}\right). \qquad (9.26)$$

Substitute $x(z,t) = \sum_{n=1}^{\infty} x_n(t) g_n(z)$ into equation (9.26) to obtain:

$$\sum_{n=1}^{\infty}\left[EI\frac{n^4\pi^4}{L^4}x_n \sin\left(\frac{n\pi z}{L}\right) + \left(m\frac{d^2x_n}{dt^2} + d\frac{dx_n}{dt}\right)\sin\left(\frac{n\pi z}{L}\right)\right] = -\frac{z}{L}\left(m\ddot{V} + d\dot{V}\right). \qquad (9.27)$$

As in the previous chapter, multiply the left- and right-hand side of this equation with $g_k(z)$ and integrate over z from $z = 0$ to $z = L$. Using the orthogonality property

$$\int_0^L g_n(z) g_k(z) dz = 0 \quad \text{for} \quad k \neq n \qquad (9.28)$$

$$\int_0^L g_n(z) g_k(z) dz = \frac{L}{2} \quad \text{for} \quad k = n \qquad (9.29)$$

we have:

$$EI\frac{n^4\pi^4}{L^4}x_n + m\frac{d^2x_n}{dt^2} + d\frac{dx_n}{dt} = -\frac{2}{L}\left(m\ddot{V} + d\dot{V}\right)\int_0^L \frac{z}{L}\sin\left(\frac{n\pi z}{L}\right)dz. \quad (9.30)$$

The integral can be evaluated as:

$$\begin{aligned}
\frac{1}{L}\int_0^L \frac{z}{L}\sin\left(\frac{n\pi z}{L}\right)dz &= \int_0^1 x\sin(n\pi x)dx \\
&= -\frac{1}{n\pi}\int_0^1 xd[\cos(n\pi x)] \\
&= -\frac{x\cos(n\pi x)}{n\pi}\bigg|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)dx \\
&= -\frac{\cos(n\pi)}{n\pi} + \frac{1}{(n\pi)^2}\sin(n\pi x)\bigg|_0^1 \\
&= -\frac{\cos(n\pi)}{n\pi} = \frac{(-1)^{n+1}}{n\pi}
\end{aligned} \quad (9.31)$$

so that equation (9.30) becomes

$$EI\frac{n^4\pi^4}{L^4}x_n + m\frac{d^2x_n}{dt^2} + d\frac{dx_n}{dt} = \frac{2(-1)^n}{n\pi}\left(m\ddot{V} + d\dot{V}\right). \quad (9.32)$$

Each amplitude of an eigenfunction is thus described according to an ordinary mass-spring system with random excitation. The eigenfrequencies are:

$$\omega_{n0} = \frac{n^2\pi^2}{L^2}\sqrt{\frac{EI}{m}}. \quad (9.33)$$

Note the difference with the eigenfrequencies of the tensioned string that are given in equation (8.22). Equation (9.32) can be solved further according to the methods presented in chapters 4 and 6.

9.4 The general problem of the tensioned beam

The general problem of the tensioned beam as described by equation (7.5) and boundary conditions (7.3)-(7.4) can be solved by combining the methods presented in chapter 6 and the previous sections of the present chapter:

1. Derive a quasi-static solution associated with the prescribed motion at the top:
$$EI\frac{\partial^4 x_p}{\partial z^4} - \frac{\partial}{\partial z}\left(T_r(z)\frac{\partial x_p}{\partial z}\right) = 0 \tag{9.34}$$
subject to the boundary conditions as given in equations (9.2)-(9.3). The solution will be of the form:
$$x_p = f(z)V(t). \tag{9.35}$$

2. The total solution is:
$$x = x_p + x_h \tag{9.36}$$
where x_h is a solution of the problem
$$EI\frac{\partial^4 x_h}{\partial z^4} - \frac{\partial}{\partial z}\left(T_r(z)\frac{\partial x_h}{\partial z}\right) + m\frac{\partial^2 x_h}{\partial t^2} + d\frac{\partial x_h}{\partial t} = F(z,t) - f(z)\left(m\ddot{V} + d\dot{V}\right) \tag{9.37}$$
subject to the homogeneous boundary conditions:
$$x_h = \frac{\partial^2 x_h}{\partial z^2} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = L. \tag{9.38}$$

3. Expand x_h in eigenfunctions $E_n(z)$ with eigenvalues λ_n which are the solutions of
$$EI\frac{d^4 E_n}{dz^4} - \frac{d}{dz}\left(T_r(z)\frac{dE_n}{dz}\right) - \lambda_n E_n = 0 \tag{9.39}$$
subject to the four boundary conditions:
$$E_n = \frac{d^2 E_n}{dz^2} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = L \tag{9.40}$$

4. The solutions of equation (9.37) are an infinite series of eigenfunctions, $n = 1, 2, 3, ..., \infty$ with prescribed values for λ_n, $n = 1, 2, 3, ..., \infty$; the eigenvalues. The solution for x_h can be written as:
$$x_h = \sum_{n=1}^{\infty} x_n(t)E_n(z). \tag{9.41}$$

The amplitude of each eigenfunction is described by an equation which is similar to that of the single degree spring-mass system. It is obtained by substituting equation (9.41) into (9.37), multiplying the left-and right-hand side by the adjoined eigenfunction $\overline{E_k}(z)$, integrating with respect to z from $z = 0$ to $z = L$ and using the orthogonality property:

$$\int_0^L E_n(z)\overline{E_k}(z)dz = 0 \quad \text{for} \quad k \neq n \tag{9.42}$$

$$\int_0^L E_n(z)\overline{E_k}(z)dz = \alpha_n \quad \text{for} \quad k = n \tag{9.43}$$

Note that $\overline{E_k}(z)$ is not necessarily the same as $E_n(z)$. Methods for deriving E_k or its governing adjoined differential equation can be found in Courant & Hilbert [7].

5. The equation for each amplitude of the eigenfunction is now obtained by substituting equation (9.41) in PDE (9.37), multiplying left and right hand with $\overline{E}_k(z)$ and integrating with respect to z from $z = 0$ to $z = L$:

$$\int_0^L \sum_{n=1}^{\infty} \left\{ x_n(t) \left(EI\frac{d^4E_n(z)}{dz^4} - \frac{d}{dz}T_r(z)\frac{dE_n(z)}{dz} \right) \overline{E}_k(z) \right.$$
$$\left. + \left(m\frac{d^2x_n}{dt^2} + d\frac{dx_n}{dt} \right) E_n(z)\overline{E}_k(z) \right\} dz$$
$$= \int_0^L F(z,t)\overline{E}_k(z)dz - (m\ddot{V} + d\dot{V}) \int_0^L f(z)\overline{E}_k(z)dz \tag{9.44}$$

Substituting equation (9.39) into the first term, using the orthogonality property (9.42)-(9.43) and dividing all terms by α_n, we have for each amplitude of the eigenfunctions the ordinary spring-mass type problem.

$$\lambda_n x_n + m\frac{d^2x_n}{dt^2} + d\frac{dx_n}{dt} = P_n(t), \tag{9.45}$$

where

$$P_n(t) = \alpha_n^{-1} \int_0^L F(z,t)\overline{E}_n(z)dz - \alpha_n^{-1} (m\ddot{V} + d\dot{V}) \int_0^L f(z)\overline{E}_n(z)dz. \tag{9.46}$$

The natural frequencies are $\omega_{n0} = (\lambda_n/m)^{1/2}$, $n = 1,2,3,...,\infty$. This problem can be further handled in the usual manner.

Chapter 10

Peak and extreme statistics

Given a Gaussian random signal of the variable $x(t)$, the probability density of all possible values of x is given by a Gaussian distribution. The Gaussian distribution is determined by the mean value and standard deviations, see equation (1.7). The previously presented methods enable specification of the standard deviation of $x(t)$; its mean value (if relevant) can be calculated by standard methods of *static structural analysis*. So the probability distribution of instantaneous values of x, that is the percentage of time that $x(t)$ lies between certain values, is fully specified. Given a record of random fluctuations $x(t)$ in time (see figure 10.1), however, the *statistics of the peak values* of $x(t)$ are different from those of the instantaneous values. Furthermore, the *extreme value* of $x(t)$, i.e. the value of the largest peak in a given record of certain duration is subject to uncertainty; it will be different for different realizations of the process. Also, the statistics of extreme values will be different from those of Gaussianly distributed instantaneous values. Peak values are important for assessing *fatigue damage* in mechanical structures; extreme values are important for analyzing the possibility of yield. This means that there is a great need in deriving expressions for these statistics in mechanical engineering practice. Deriving these expressions is the objective of this chapter.

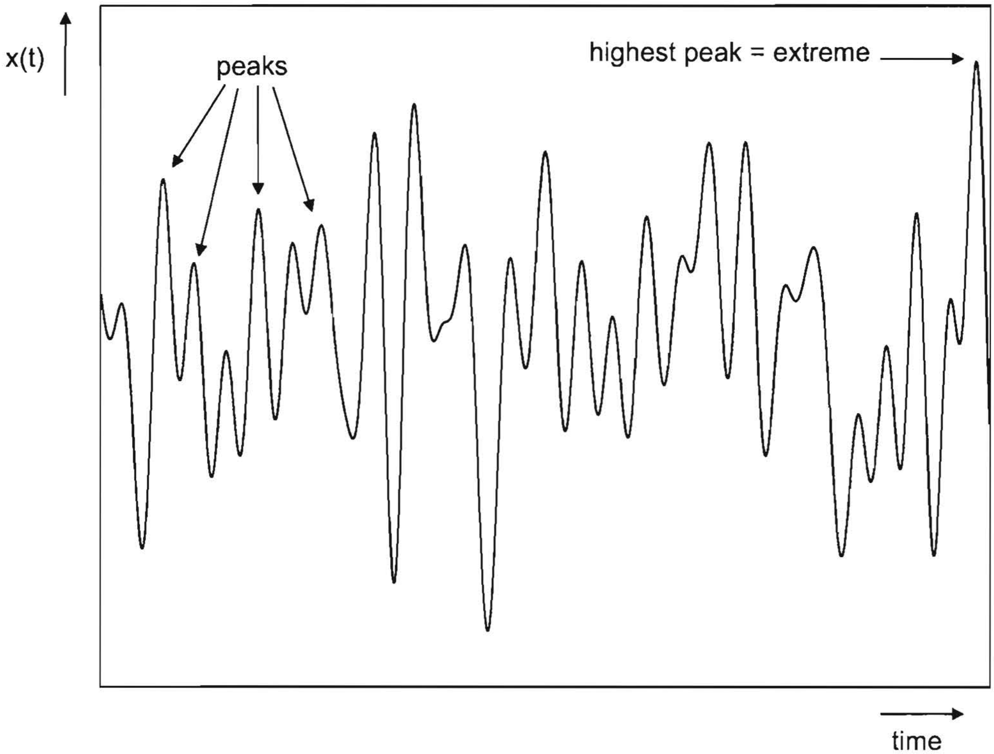


Figure 10.1. Peaks and extreme of a random record.

10.1 Peak statistics

The simplest case for determining the statistics of peaks is that of *narrow-band response*. That is, response which is governed by a narrow-band of frequencies as, for example, is the case for resonant response: sections 4.5 and 6.3. Response can then be described by a sinusoidal wave of which the amplitude varies slowly and randomly with time. Clearly, the statistics of peaks in such a signal are those of the amplitude: see figure 6.3. To derive its probability distribution, we write (see also equation (6.19)):

$$x = \rho(T^*) \sin \{t^* + \psi(T^*)\}, \tag{10.1}$$

where T^* is 'slow' time defined as:

$$T^* = \delta t^*, \tag{10.2}$$

with $\delta \ll 1$. For the time derivative, we have:

$$\begin{aligned} \dot{x} &= \rho(T^*) \cos\{t^* + \psi(T^*)\} + \delta \frac{d\rho(T^*)}{dT^*} \sin\{t^* + \psi(T^*)\} \\ &+ \delta \frac{d\psi(T^*)}{dT^*} \rho(T^*) \cos\{t^* + \psi(T^*)\}. \end{aligned} \quad (10.3)$$

Because $\delta \ll 1$ we can approximate $\dot{x}$ with a relative error of $\mathcal{O}(\delta)$ by:

$$\dot{x} = \rho(T^*) \cos\{t^* + \psi(T^*)\}. \quad (10.4)$$

Response x and its time derivative are uncorrelated, i.e. their covariance is zero, because for stationary processes we can write:

$$\begin{aligned} \langle x\dot{x} \rangle = \overline{x\dot{x}} &= \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t^*)\dot{x}(t^*)dt^* \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T x dx \\ &= \lim_{T\to\infty} \frac{1}{T} \left(\frac{1}{2} x^2(t^*) \right)_0^T = 0. \end{aligned} \quad (10.5)$$

The standard deviation of $\dot{x}$ is equal to that of x. This can be seen as follows. The standard deviation of x is related to the power density of x by (see equation (3.7)):

$$\sigma_x^2 = \int_0^\infty S_x d\omega. \quad (10.6)$$

According to the theory of chapters 3 and 4, we have for the power density of the time derivative of x:

$$S_{\dot{x}} = \omega^2 S_x, \quad (10.7)$$

so that the standard deviation of $\dot{x}$ is given by

$$\sigma_{\dot{x}}^2 = \int_0^\infty \omega^2 S_x d\omega. \quad (10.8)$$

In case of lightly damped ($\delta \ll 1$) resonant response, the power density of x peaks sharply in a narrow band of frequencies near the natural frequency. As ω is made dimensionless with the natural frequency, S_x will peak at $\omega = 1$. So the value of the integral in equation (10.8) is mainly determined by values of S_x at and very close to $\omega = 1$. Therefore, for $\delta \ll 1$,

$$\sigma_{\dot{x}}^2 = \int_0^\infty S_x d\omega = \sigma_x^2. \quad (10.9)$$

The joint Gaussian distribution of x and $\dot{x}$ can now be expressed as (cf. equation (1.11)):

$$p(x,\dot{x}) = \frac{1}{2\pi\sigma_x^2} e^{-\frac{x^2+\dot{x}^2}{2\sigma_x^2}}. \tag{10.10}$$

The value of ρ and ψ, $0 \leq \rho \leq \infty$, $0 \leq \psi \leq 2\pi$ is uniquely related to that of x and $\dot{x}$, $-\infty \leq x \leq \infty$, $-\infty \leq \dot{x} \leq \infty$. That is, for any particular value of x and $\dot{x}$ there is only one corresponding value for ρ and ψ, and vice versa. In this case, the joint probability that $X \leq x$ and $\dot{X} \leq \dot{x}$ is equal to that of $\rho \leq \rho$ and $\Psi \leq \psi$:

$$P(\rho,\psi) = P(x,\dot{x})\Big|_{x=x(\rho,\psi),\dot{x}=\dot{x}(\rho,\psi)} \tag{10.11}$$

Cumulative probability is related to probability density as:

$$p(\rho,\psi) = \frac{\partial^2 P(\rho,\psi)}{\partial\rho\partial\psi} \quad , \quad p(x,\dot{x}) = \frac{\partial^2 P(x,\dot{x})}{\partial x\partial\dot{x}}. \tag{10.12}$$

so that we can derive from (10.11) the following relation for probability densities:

$$p(\rho,\psi) = \left\{ p(x,\dot{x}) \left| \frac{dx d\dot{x}}{d\rho d\psi} \right| \right\}_{x=x(\rho,\psi),\dot{x}=\dot{x}(\rho,\psi)}. \tag{10.13}$$

The Jacobian can be evaluated as:

$$\begin{aligned} \left| \frac{dx d\dot{x}}{d\rho d\psi} \right| = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \psi} \\ \frac{\partial \dot{x}}{\partial \rho} & \frac{\partial \dot{x}}{\partial \psi} \end{vmatrix} &= \left| \frac{\partial x}{\partial \rho}\frac{\partial \dot{x}}{\partial \psi} - \frac{\partial x}{\partial \psi}\frac{\partial \dot{x}}{\partial \rho} \right| \\ &= \rho\left[\cos^2(t^*+\psi) + \sin^2(t^*+\psi)\right] = \rho. \end{aligned} \tag{10.14}$$

Noting that $x^2 + \dot{x}^2 = \rho^2$, the joint distribution of ρ and ψ now becomes

$$p(\rho,\psi) = \frac{\rho}{2\pi\sigma_x^2} e^{-\rho^2/(2\sigma_x^2)}. \tag{10.15}$$

This can also be written as the product of two independent probability densities of amplitude and phase:

$$p(\rho,\psi) = p(\rho)p(\psi). \tag{10.16}$$

where

$$p(\rho) = \frac{\rho}{\sigma_x^2} e^{-\rho^2/(2\sigma_x^2)} \qquad 0 \leq \rho \leq \infty. \tag{10.17}$$

and

$$p(\psi) = \frac{1}{2\pi} \qquad 0 \leq \psi \leq 2\pi. \tag{10.18}$$

Note that

$$\int_0^\infty p(\rho)d\rho = 1 \qquad \text{and} \qquad \int_0^{2\pi} p(\psi)d\psi = 1 : \tag{10.19}$$

the total surface underneath the probability density must be equal to one as should be (because the cumulative probability must become equal to one at the upper boundary of the domain, see equation (1.1)).

The probability density according to equation (10.17) is known as the *Rayleigh distribution*. It is shown in figure 10.2.

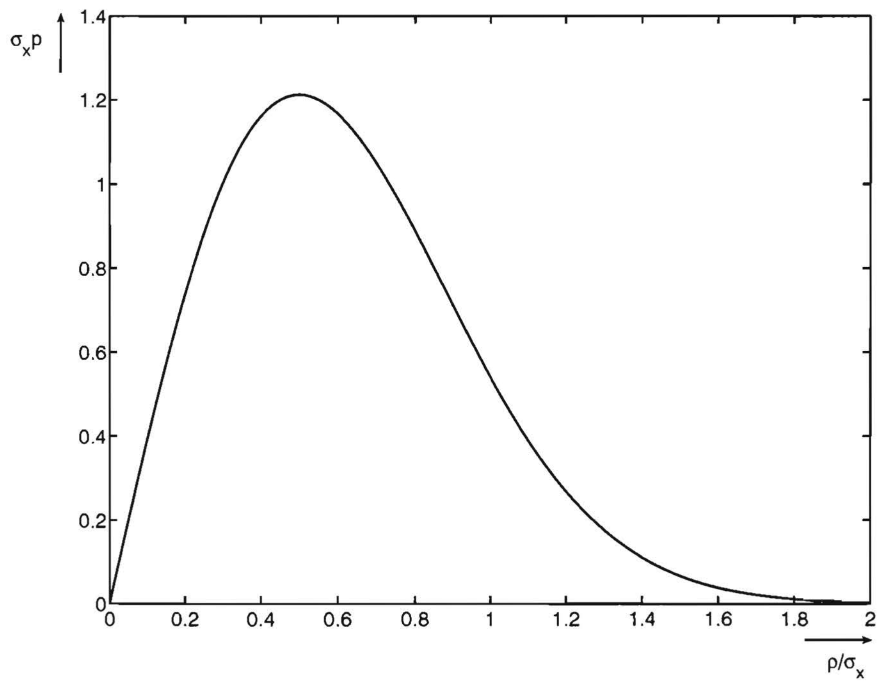


Figure 10.2. The Rayleigh distribution: equation (10.17).

The above shown distribution for peaks applies to a narrow-band process as is the case in, for example, resonant response. For more general non-narrow-band processes, the peak distribution is more complicated. In fact, it can then be described by *Rice's distribution*: see Robson [1]. The high-tails end of this distribution, that is the probability density of the high values of ρ in Rice's distribution, can be conservatively approximated by the Rayleigh distribution [1]. Therefore, the Rayleigh distribution represents a conservative estimate to the real peak distribution. In this sense it can be used to estimate fatigue damage due to a large number of stress peaks apparent in a mechanical structure.

10.2 Fatigue damage

The cumulative damage which occurs at some place in a mechanical engineering structure due to cyclic stresses can be assessed using *Miner's hypothesis*. It implies that the total damage equals the sum of the damage due to each individual stress cycle. The damage d_i due to a single cycle with stress amplitude ρ_i is given by

$$d_i = \beta \rho_i^\lambda, \tag{10.20}$$

where β and λ are material constants. Their values are assessed from cyclic fatigue tests. The total damage is:

$$d = \sum_{i=1}^{n} \beta \rho_i^\lambda. \tag{10.21}$$

To avoid failure during lifetime,

$$d < 1. \tag{10.22}$$

The coefficient λ, $\lambda > 1$, in equation (10.21) reflects the fact that larger stress peaks contribute more than smaller ones. For welds, λ can have values of about 4: American Welding Society. In that case, a stress amplitude which is two times larger leads to an accumulated damage which is 16 times larger, implying that the total number of cycles before failure will be 16 times smaller.

In our problem, the stress cycle is random in nature. We can write for large numbers of cycles n:

$$\begin{aligned} d &= \beta \sum_{i=1}^{n} \rho_i^\lambda = \beta n \left(\frac{1}{n} \sum_{i=1}^{n} \rho_i^\lambda \right) \\ &= \beta n \langle \rho^\lambda \rangle \\ &= \beta n \int_0^\infty \rho^\lambda p(\rho) d\rho, \end{aligned} \tag{10.23}$$

where $p(\rho)$ is probability density of peaks. The number of cycles can be calculated from the average frequency of the random stress record:

$$\overline{\omega}^2 = \frac{\int_0^\infty \omega^2 S d\omega}{\int_0^\infty S d\omega}, \tag{10.24}$$

where S is the power density of the stress record. The number of cycles now is:

$$n = \frac{T}{2\pi / \overline{\omega}}, \tag{10.25}$$

where $2\pi/\overline{\omega}$ is average time between peaks and T is total time.

The damage can now be calculated once we know the probability density of peaks. Because λ is relatively large, the contribution to the integral in equation (10.23) will mainly come from large values of ρ, i.e. the large tails end of $p(\rho)$. For a Gaussian random process, this part of the peak distribution can be conservatively described by Rayleigh's distribution: cf. equation (10.17). In this case,

$$\int_0^\infty \rho^\lambda p(\rho) d\rho = \int_0^\infty \frac{\rho^{\lambda+1}}{\sigma^2} e^{-\rho^2/(2\sigma^2)} d\rho. \tag{10.26}$$

Taking $\eta = \rho/\sigma$

$$\int_0^\infty \frac{\rho^{\lambda+1}}{\sigma^2} e^{-\rho^2/(2\sigma^2)} d\rho = \sigma^\lambda \int_0^\infty \eta^{\lambda+1} e^{\eta^2/2} d\eta. \tag{10.27}$$

Taking $t = (1/2)\eta^2$:

$$\begin{aligned} \sigma^\lambda \int_0^\infty \eta^{\lambda+1} e^{1/2\eta^2} d\eta &= \sigma^\lambda 2^{\lambda/2} \int_0^\infty t^{\lambda/2} e^{-t} dt \\ &= \sigma^\lambda 2^{\lambda/2} \Gamma(1 + \lambda/2) \end{aligned} \tag{10.28}$$

where Γ is the *Gamma function* which can be found in Abramowitz & Stegun [6], chapter 6. The fatigue damage then becomes:

$$d = \beta n \sigma^\lambda 2^{\lambda/2} \Gamma(1 + \lambda/2). \tag{10.29}$$

For $\lambda = 4$, $\Gamma(1+\lambda/2) = 2$ in which case $d = 8\beta n\sigma^4$.

The above relations can also be used to calculate the amplitude of an equivalent deterministic sinusoidal stress cycle: that is, a sinusoidal stress cycle which leads to the same fatigue damage as that caused by the random cycle. The equivalent amplitude ρ_{eq} follows from the equality

$$\beta n \rho_{eq}^{\lambda} = \beta n \int_0^{\infty} \rho^{\lambda} p(\rho) d\rho, \tag{10.30}$$

which can be evaluated to

$$\rho_{eq} = \left(\int_0^{\infty} \rho^{\lambda} p(\rho) d\rho \right)^{1/\lambda}. \tag{10.31}$$

Substituting the expression for the integral given by eqs. (10.26)-(10.28), one has

$$\rho_{eq} = \sigma\sqrt{2}\,(\Gamma(1+\lambda/2))^{1/\lambda}. \tag{10.32}$$

For $\lambda = 4$, this becomes $\rho_{eq} = 1.68\sigma$.

10.3 Extreme statistics

Consider the probability $P(\rho)$ that any of the peaks out of a time record having n peaks will be less than a particular value $\hat{\rho}$. The probability that all of the n peaks are less than $\hat{\rho}$ is then (see also Longuet-Higgins [9]):

$$P_{\hat{\rho}}(\hat{\rho}) = P^n(\rho)\Big|_{\rho=\hat{\rho}} \tag{10.33}$$

where $P(\rho)$ is the CDF of peaks:

$$P(\rho) = \int_0^{\rho} p(\rho) d\rho. \tag{10.34}$$

As shown in section 10.1, for narrow-band processes peaks are described according to the Rayleigh distribution (which represents, as mentioned before, also a conservative estimate to the distribution of the larger peaks in case of more general non-narrow-band Gaussian processes):

$$P(\rho) = 1 - e^{-\rho^2/(2\sigma^2)}, \tag{10.35}$$

so that

$$P_{\hat{\rho}}(\hat{\rho}) = \left(1 - e^{-\hat{\rho}^2/(2\sigma^2)}\right)^n \tag{10.36}$$

The PDF of extremes is then

$$p_{\hat{\rho}}(\hat{\rho}) = \frac{dP_{\hat{\rho}}}{d\hat{\rho}} = n\left(1 - e^{-\frac{\hat{\rho}^2}{2\sigma^2}}\right)^{n-1} \frac{\hat{\rho}}{\sigma^2} e^{-\hat{\rho}^2/(2\sigma^2)} \tag{10.37}$$

In figure 10.3 we have shown the density of extremes for number of peaks $n = 10^3$ and $n = 10^4$. The density shifts to higher values when n increases, as to be expected! Knowing the duration of a stationary state of random excitation, i.e. the number of peaks n, we can quantify the possible values of the largest peak which are some multiple of the standard deviation. This enables us to estimate the likelihood of overstressing leading to yield at sensitive places in a mechanical engineering structure. In off-shore structures, one considers in general the so-called 100-years storm: a stationary sea-state lasting about 3 hours and having a probability of occurring once in 100 years. The number of peaks in such a period is typically 10^3. Methods of random analysis as presented in previous sections are used to calculate the standard deviation of stress during such a sea-state. The statistics of extremes are subsequently used to analyze the possibilities of overstressing.

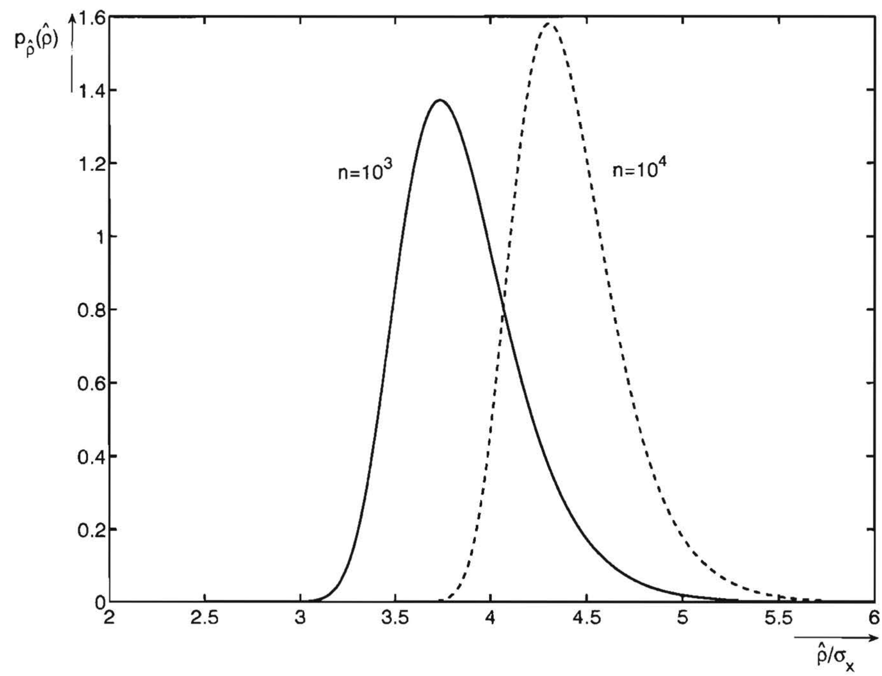


Figure 10.3. Probability density of the extreme peak normalized with standard deviation for different values of n (number of peaks).

Chapter 11

Non-linear analysis: Some general observations and numerical time-domain simulation

The methods of analysis presented in the previous chapters were concerned with linear systems subject to Gaussian random excitation. In case of linear systems subject to Gaussian excitation, response is also Gaussian. Governing probability distributions of response variables are thus known; all that is necessary is specification of the parameters of these distributions, e.g. standard deviations, using the methods presented in the previous chapters. If the system is non-linear, however, the situation is rather different. The probability distributions of instantaneous response become non-Gaussian, peak statistics become non-Rayleigh or non-Rice, etc. For each particular non-linear system, the probability distributions as well as their parameters have to be determined.

A crude method to handle non-linear systems is by numerical time-domain simulation. It is in fact straightforward application of the results presented in chapter 5. Gaussian excitation is generated by a series of sinusoidal waves with randomly chosen phase angle: equation (5.1). The differential equation which describes the system to which the excitation is applied is solved numerically in the time-domain (and space-domain if necessary). Statistical parameters of response can be derived from the generated random record using previously presented methods. Ensuring that instationary transients are removed from the time-record, the probability distribution of instantaneous response can be assessed according to the procedure presented in figure 2.3. Moments of response can be calculated according to equation (2.4). By generating a large series of different realizations, statistical parameters can also be gathered from ensemble averaging: see figure 2.2. But in all these cases, the numerical approach is lengthy and time-consuming; in particular, if one wants to achieve accuracy in the assessment of the tails of the distributions, such tails being of prime importance for the evaluation of fatigue damage and extreme response. Moreover, numerical results yield only data; obtaining insight into the effect of design variables on design criteria from these data is rather cumbersome. Analytical methods for handling random processes are therefore the preferred way for the designer. But such methods only exist for specific non-linear problems. Some of these that are particularly important for mechanical engineering practice, will be considered in the next chapters.

Chapter 12

Non-linear quasi-static response

Whenever there is a direct non-linear relation between two stochastic variables, the statistical relationships between these two variables can be derived using well-known methods of probability theory. With direct relationship we mean a relation without time-delay. In dynamical systems such as the spring-mass system of chapters 4 and 6 and the tensioned beam of chapters 7, 8 and 9, a direct relationship will occur when the inertia and damping forces play a minor role. This will happen if the center of gravity of the power density spectrum of excitation is at frequencies which are much lower than the natural frequency of the system. Response is then primarily governed by excitation forces and static restoring forces. The restoring forces can be non-linear as a result of non-linear, non-elastic behavior of the structure. But non-linearity can also occur through the excitation force. An example of the latter is the non-linear drag force acting on members of off-shore structures due to water velocities associated with random waves on the sea surface. These drag forces are particularly important in case of large waves acting on relatively small members of off-shore structures. We shall treat this problem in some detail as an example of how to proceed in case of quasi-static non-linear behavior.

The drag force acting on a bluff body in fluid is proportional to the square of the velocity v of the fluid perpendicular to the body. In case of alternating fluid velocity, the force will be proportional to $v|v|$. For a linearly quasi-statically responding structure, the response variable will also be proportional to $v|v|$. Therefore,

$$x = \alpha v|v|, \tag{12.1}$$

where x is response variance and α constant of proportionality; the value of α can be calculated by applying static analysis under unit force. The velocity of the fluid due to wind-generated waves can, in general, be well-presented by a Gaussian distribution. To derive the distribution of x given relation (12.1) we note that there exists a one-to-one relationship between x and v: for each value of v there is only one value of x and vise versa. In this case, the probability that $X < x$ is equal to the probability that $V < v(x)$:

$$P(x) = P_v(v)\Big|_{v=v(x)}, \tag{12.2}$$

where $v = v(x)$ follows from the inverse of relationship (12.1).

$$v = (\alpha|x|)^{-1/2}x, \tag{12.3}$$

(to derive this relation consider first $v > 0$, $x > 0$ and subsequently $v < 0$, $x < 0$ and combine the results). Cumulative probabilities are related to probability densities as

$$p_v(v) = \frac{dP_v(v)}{dv}, \quad p(x) = \frac{dP(x)}{dx}, \tag{12.4}$$

so that we can derive from equation (12.2):

$$p(x) = \left\{ p_v(v)\left|\frac{dv}{dx}\right| \right\}_{v=v(x)}. \tag{12.5}$$

For $p_v(v)$ we take the Gaussian distribution

$$p_v(v) = \frac{1}{\sigma_v\sqrt{2\pi}}e^{-v^2/(2\sigma_v^2)}. \tag{12.6}$$

Furthermore,

$$\left|\frac{dv}{dx}\right| = \frac{1}{2}(\alpha|x|)^{-1/2}. \tag{12.7}$$

Relation (12.4) can now be evaluated to the following probability density of x:

$$p(x) = \frac{1}{2\sigma_v(2\pi\alpha|x|)^{1/2}}e^{-|x|/(2\alpha\sigma_v^2)}. \tag{12.8}$$

The standard deviation σ_x of x follows from:

$$\sigma_x^2 = \int_{-\infty}^{+\infty} x^2 p(x) dx = \frac{(2\alpha)^2 \sigma_v^4}{\pi^{1/2}} \int_0^{\infty} t^{3/2} e^{-t} dt = \frac{(2\alpha)^2 \sigma_v^4}{\pi^{1/2}} \Gamma\left(\frac{5}{2}\right) = 3\alpha^2 \sigma_v^4. \tag{12.9}$$

where we took $x = 2\alpha\sigma_v^2 t$. In (12.9) Γ is the Gamma function: Abramowitz & Stegun [6], chapter 6. In terms of σ_x, equation (12.8) becomes:

$$p(x) = \left(\frac{8\pi\sigma_x|x|}{\sqrt{3}}\right)^{-1/2} e^{-|x|\sqrt{3}/(2\sigma_x)}. \tag{12.10}$$

For the cumulative probability density we can write:

$$P(x) = \int_{-\infty}^{x} p(x) dx = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{t=\sqrt{3}x/(2\sigma_x)} |t|^{-1/2} e^{-|t|} dt, \tag{12.11}$$

with $x = 2\sigma_x t/\sqrt{3}$. The integral on the right-hand side can also be expressed in terms of the *incomplete Gamma function* $\gamma(x_1, x_2)$: Abramowitz & Stegun [6], chapter 6:

$$P(x) = \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \frac{|x|}{x} \gamma\left(\frac{1}{2}, \frac{1}{2}\sqrt{3}\frac{|x|}{\sigma_x}\right). \tag{12.12}$$

In fact, this result could have been directly obtained by substituting eq. (12.3) in eq. (12.2) using eq. (1.10) and noting that the incomplete Gamma function can be expressed in the error function: Abramowitz & Stegun [6], property 6.5.16. In figure 12.1, the above distribution has been shown and compared with the Gaussian distribution. It is seen that for large values of x/σ_x, the probability of not being exceeded is less in case of the above distribution. It reflects a larger probability of obtaining large values at equal standard deviation due to quadratic loading.

Given a non-linear direct relationship between wave loading and response, it is also possible to derive descriptions for the probability distributions of peaks invoking the narrow-band model. In this way it is possible to derive analytical expressions for fatigue damage and extreme statistics in case of non-linear quasi-static behavior: Brouwers and Verbeek [11]. This approach can be applied to many other non-linear problems as long as the non-linear behavior is quasi-static.

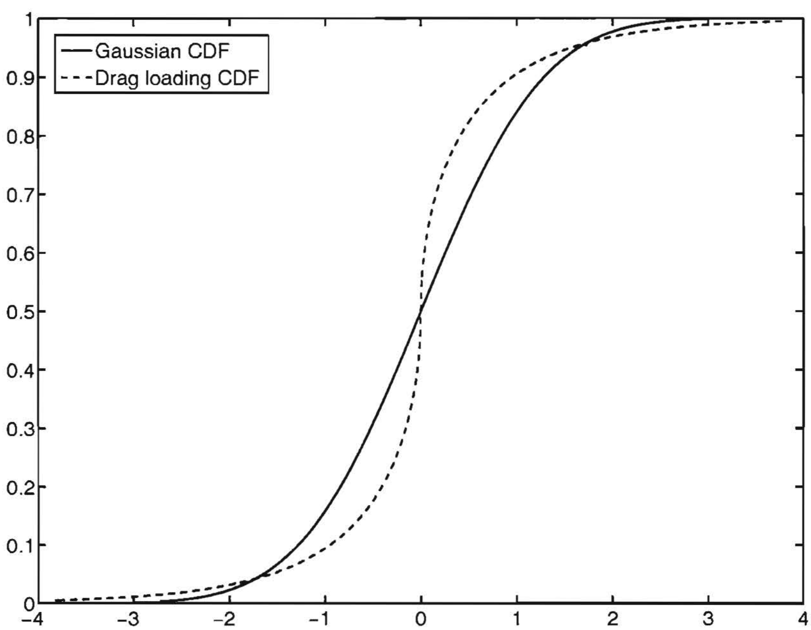


Figure 12.1. CDF for Gaussian process and for drag loading.

Chapter 13

Non-linearly damped resonance

As mentioned before, analytical methods for handling non-linear problems of random vibration are only available for specific cases. One of these cases is non-linearly damped resonance of a mass-spring system. In case of light damping, the governing equation can be reduced using two-scale expansions. The resulting stochastic problem can be treated by a Fokker-Planck equation.

13.1 Two-scale expansions

Attention is focussed to a mass-spring system with non-linear power-law damping: i.e. the damping force equals $d\dot{x}|\dot{x}|^{\alpha}$. In dimensionless form applying the non-dimensioning of eqs. (6.2)-(6.5), the equation of motion is:

$$\ddot{x} + \delta\dot{x}|\dot{x}|^{\alpha} + x = P^*(t^*), \tag{13.1}$$

which becomes equal to eq. (6.2) for $\alpha = 0$ (linear damping). The excitation is a stationary Gaussian process described according to equation (6.1). Whenever $\delta \ll 1$, damping will be unimportant in solutions of eq. (13.1) except for frequencies close to the natural frequency $\omega^* = 1$. This becomes apparent when disregarding damping and analyzing the behavior of the solution of the resulting equation in the frequency domain: see section 4.5. Therefore, when focussing on the solution appropriate for resonance, we can limit attention to the behavior of the excitation in a narrow band of frequencies around

$\omega^* = 1$. This can be modeled as:

$$\omega^* = 1 + \delta^\beta \nu^* \quad , \quad \nu^* = \mathcal{O}(1). \tag{13.2}$$

The exponent β, $\beta > 0$, is 1 in case of linear damping; its value appropriate for non-linear damping will be determined below. The excitation described by eq. (6.1) can now be approximated by (analogous to the procedure of eqs. (6.16)-(6.25)):

$$P(t^*) = 2\delta^{\beta/2} S_1^{1/2} \sum_{m=-N_0}^{+N-N_0} \left(\frac{\Delta\nu^*}{2}\right)^{1/2} \cos(t^* + \nu_m^* T^* + \varphi_m), \tag{13.3}$$

where

$$T^* = \delta^\beta t^* \quad , \quad \nu_m^* = m\Delta\nu^* \quad , \quad S_1 = S_p^n(\omega^* = 1). \tag{13.4}$$

Eq. (13.3) can also be written as (see again eqs. (6.16)-(6.25))

$$P(t^*) = 2\delta^{\beta/2} S_1^{1/2} \left(f_1(T^*) \cos t^* + f_2(T^*) \sin t^*\right), \tag{13.5}$$

where

$$f_1(T^*) = \sum_{n=1}^{N} (\Delta\nu^*)^{1/2} \cos(\nu_n^* T^* + \varphi_n), \tag{13.6}$$

$$f_2(T^*) = -\sum_{n=1}^{N} (\Delta\nu^*)^{1/2} \sin(\nu_n^* T^* + \varphi_n). \tag{13.7}$$

Here, $f_1(T^*)$ and $f_2(T^*)$ are Gaussian white-noise processes of power density 1/2 (compare the above equations with eq. (5.1)). Furthermore, $f_1(T^*)$ and $f_2(T^*)$ are uncorrelated:

$$\langle f_1 f_2 \rangle = \overline{f_1 f_2} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{+T/2} f_1 f_2 dT^* = 0 : \tag{13.8}$$

the prove of this is similar to the procedure of eqs. (6.28)-(6.30).

The excitation consists of sinusoidal waves with randomly and slowly varying amplitudes: cf. eqs. (13.5)-(13.7). In view of the two times t^* and T^* apparent in the excitation, we also assume that the solution of eq. (13.1) exhibits the two times t^* and T^*:

$$x = x(t^*, T^*), \tag{13.9}$$

so that

$$\dot{x} = \frac{\partial x}{\partial t^*} + \delta^\beta \frac{\partial x}{\partial T^*}, \tag{13.10}$$

$$\ddot{x} = \frac{\partial^2 x}{\partial t^{*2}} + 2\delta^\beta \frac{\partial^2 x}{\partial t^* \partial T^*} + \mathcal{O}(\delta^{2\beta}). \tag{13.11}$$

Substituting eqs. (13.9)-(13.11) and (13.5) into (13.1) yields:

$$\begin{aligned} \frac{\partial^2 x}{\partial t^{*2}} &+ x + 2\delta^\beta \frac{\partial^2 x}{\partial t^* \partial T^*} + \delta \frac{\partial x}{\partial t^*} \left| \frac{\partial x}{\partial t^*} \right|^\alpha \\ &= 2\delta^{\beta/2} S_1^{1/2} \left(f_1(T^*) \cos t^* + f_2(T^*) \sin t^* \right) \\ &+ \mathcal{O}(\delta^{2\beta}) + \mathcal{O}(\delta^{1+\beta}) + \mathcal{O}(\delta^{1+\alpha\beta}). \end{aligned} \tag{13.12}$$

For small δ we can disregard all terms except the first two:

$$\frac{\partial^2 x}{\partial t^{*2}} + x = 0 \qquad \text{if} \qquad \delta \to 0, \tag{13.13}$$

the solution of which is:

$$x = \delta^\gamma a(T^*) \cos(t^* + \phi(T^*)), \tag{13.14}$$

where amplitude and phase can vary with the second and slow time T^*. The value of the exponent γ has yet to be determined.

A problem like eq. (13.12) gives rise to supposing a solution for x in terms of a perturbation expansion in powers of δ as:

$$x = \delta^\gamma (x_1 + \delta^\mu x_2 + ...) \quad , \quad \mu > 0 \tag{13.15}$$

Equations for x_1, x_2, etc. are obtained by substituting expansion (13.15) in eq. (13.12) and equating terms of equal order in δ. The largest or leading order are the terms formed by the first two terms in eq. (13.12) expressed in x_1:

$$\frac{\partial^2 x_1}{\partial t^{*2}} + x_1 = 0, \tag{13.16}$$

which is of course the problem formed by eq. (13.13) with solution: $x_1 = a(T^*) \cos(t^* + \phi(T^*))$. The next order of terms lead to the problem:

$$\begin{aligned} \frac{\partial^2 x_2}{\partial t^{*2}} + x_2 &= -2\delta^{\beta-\mu} \frac{\partial^2 x_1}{\partial t^* \partial T^*} - \delta^{1+\alpha\gamma-\mu} \frac{\partial x_1}{\partial t^*} \left| \frac{\partial x_1}{\partial t^*} \right|^\alpha \\ &+ 2\delta^{\beta/2-\gamma-\mu} S_1^{1/2} \left(f_1(T^*) \cos t^* + f_2(T^*) \sin t^* \right) \end{aligned} \tag{13.17}$$

Now, meaningful solutions are only obtained if the terms on the right hand side are all of order δ^0. Hence: $\beta - \mu = 1 + \alpha\gamma - \mu = \beta/2 - \gamma - \mu = 0$, so that

$$\mu = \frac{2}{2+\alpha} \quad , \quad \beta = \frac{2}{2+\alpha} \quad , \quad \gamma = -\frac{1}{2+\alpha}. \tag{13.18}$$

In terms of these values and substituting the solution for x_1, eq. (13.17) becomes:

$$\begin{aligned}\frac{\partial^2 x_2}{\partial t^{*2}} + x_2 &= 2\frac{\partial}{\partial T^*}\left(a\sin(t^*+\phi)\right) + a^{1+\alpha}\sin(t^*+\phi)|\sin(t^*+\phi)|^{\alpha} \\ &\quad +2S_1^{1/2}(f_1\cos t^* + f_2 \sin t^*). \end{aligned} \tag{13.19}$$

Here, the term that is due to damping can be expanded in a Fourier series:

$$\sin(t^*+\phi)|\sin(t^*+\phi)|^{\alpha} = \alpha_1\sin(t^*+\phi) + \alpha_2\sin 2(t^*+\phi) + \ldots \tag{13.20}$$

where

$$\begin{aligned}\alpha_1 &= \frac{2}{\pi}\int_0^{\pi}\sin^2\eta|\sin\eta|^{\alpha}d\eta \\ &= \frac{4}{\pi}\int_0^{\pi/2}\sin^{\alpha+2}\eta d\eta \\ &= \frac{2}{\pi}B\left(\frac{\alpha+3}{2},\frac{1}{2}\right) \\ &= \frac{2\Gamma((\alpha+3)/2)}{\pi^{1/2}\Gamma((\alpha+4)/2)}\end{aligned} \tag{13.21}$$

where B is *Beta function* and Γ is Gamma function: Abramowitz & Stegun [6], chapter 6.

Implementing eq. (13.20) and using addition formulae for trigonometric functions, eq. (13.19) becomes

$$\begin{aligned}\frac{\partial^2 x_2}{\partial t^{*2}} + x_2 &= \left\{2\frac{d}{dT^*}(a\cos\phi) + \alpha_1 a^{\alpha+1}\cos\phi + 2S_1^{1/2}f_2(T^*)\right\}\sin t^* \\ &\quad + \left\{2\frac{d}{dT^*}(a\sin\phi) + \alpha_1 a^{\alpha+1}\sin\phi + 2S_1^{1/2}f_1(T^*)\right\}\cos t^* \\ &\quad +\alpha_2\cos 2\phi\sin 2t^* + \alpha_2\sin 2\phi\cos 2t^* + \ldots\end{aligned} \tag{13.22}$$

The basic solutions of the homogeneous part of the above differential equation are $\sin t^*$ and $\cos t^*$. Terms like $\sin t^*$ and $\cos t^*$ on the right hand side of eq. (13.22) then give rise to particular solutions of the type $t^* \sin t^*$ and $t^* \cos t^*$. These grow unboundedly with increasing t^*. To arrive at meaningful solutions, we have to avoid this behavior; that is, we have to set the coefficients of the terms like $\sin t^*$ and $\cos t^*$ on the right-hand side of eq. (13.22) equal to zero. This criterion for avoiding *secular behavior* is at the heart of the *two-scale method*. It leads to specification of the equations governing amplitude and phase:

$$2\frac{d}{dT^*}(a\cos\phi) + \alpha_1 a^{1+\alpha}\cos\phi = -2S_1^{1/2} f_2 \tag{13.23}$$

$$2\frac{d}{dT^*}(a\sin\phi) + \alpha_1 a^{1+\alpha}\sin\phi = -2S_1^{1/2} f_1 \tag{13.24}$$

The above equations determine the amplitude and phase of solution (13.14).

13.2 Derivation of the Fokker-Planck equation

Eqs. (13.23) and (13.24) are two-coupled first order equations for amplitude and phase with white-noise excitation. These equations can be transformed into a so-called Fokker-Planck equation for the transient joint probability density distribution of amplitude and phase. Advantage of this transformation is that the resulting Fokker-Planck equation is a linear partial differential equation, in contrast to eqs. (13.23) and (13.24), which are non-linear for general values of α. As we shall see later, the Fokker-Planck equation associated with eqs. (13.23)-(13.24) can be solved yielding explicit descriptions of the stationary probability distribution of amplitude and phase. The Fokker-Planck equation was initially derived in connection with models of random motion of molecules. In later years, the approach has been extended to non-linear problems: e.g. Stratonovich [3] and Van Kampen [4]. Below, we shall present a heuristic derivation of the Fokker-Planck equation associated with eqs. (13.23)-(13.24). These two equations can be formulated as:

$$\frac{d\mathbf{u}}{dT^*} = -\mathbf{g}(\mathbf{u}) + S_1^{1/2}\mathbf{f}(T^*), \tag{13.25}$$

where

$$\mathbf{u} = (u_1, u_2) = (a\cos\phi, a\sin\phi), \tag{13.26}$$

$$\mathbf{g}(\mathbf{u}) = \frac{1}{2}\alpha_1 \mathbf{u}(u_1^2 + u_2^2)^{\alpha/2}, \tag{13.27}$$

$$\mathbf{f}(T^*) = -(f_2(T^*), f_1(T^*)), \tag{13.28}$$

where f_1 and f_2 are white noise or δ-correlated processes.

Objective is to derive an equation that describes the probability density $p_u(\mathbf{u}, T^*)$ given equation (13.25). From (13.25) it follows that

$$\mathbf{u}(T^* + \Delta T^*) = \mathbf{u}(T^*) - \mathbf{g}(\mathbf{u}(T^* + \Delta T^*))\Delta t + S_1^{1/2}\mathbf{F}(\Delta T^*) + o(\Delta T^*), \tag{13.29}$$

where $o(\Delta T^*)$ is used to indicate a rest term that goes to zero more rapidly than ΔT^* if $\Delta T^* \to 0$. The function $\mathbf{F}(\Delta T^*)$ is defined as

$$\mathbf{F}(\Delta T^*) = \int_{T^*}^{T^*+\Delta T^*} \mathbf{f}(\tau) d\tau. \tag{13.30}$$

The white-noise is zero-mean. Hence,

$$\langle \mathbf{F}(\Delta T^*) \rangle = 0. \tag{13.31}$$

Furthermore,

$$\begin{aligned}
\langle F_m F_n \rangle &= \int_{T^*}^{T^*+\Delta T^*} d\tau_1 \int_{T^*}^{T^*+\Delta T^*} d\tau_2 \langle f_m(\tau_1) f_n(\tau_2) \rangle \\
&= \frac{\pi}{2}\delta_{mn} \int_{T^*}^{T^*+\Delta T^*} d\tau_1 \int_{T^*}^{T^*+\Delta T^*} d\tau_2 \delta(\tau_2 - \tau_1) \\
&= \frac{\pi}{2}\delta_{mn} \int_{0}^{\Delta T^*} d\tau_1 \int_{0}^{\Delta T^*} d\tau_2 \delta(\tau_2 - \tau_1) \\
&= \frac{\pi}{2}\delta_{mn} \int_{0}^{\Delta T^*} d\tau_1 \int_{-\tau_1}^{-\tau_1+\Delta T^*} \delta(\tau) d\tau \\
&= \frac{\pi}{2}\delta_{mn} \Delta T^*,
\end{aligned} \tag{13.32}$$

where we used the property that f_1 and f_2 are mutually uncorrelated white noise processes of power density $1/2$, in which case the autocorrelation functions of f_1 and f_2 are Dirac δ-functions with intensity $\frac{\pi}{2}$: see eq. (3.14). Furthermore, we note that higher order moments of F_m are $o(\Delta T^*)$.

Equation (13.29) provides a relation between the statistical values of $\mathbf{u}(T^* + \Delta T^*)$ denoted by $\mathbf{u}'$ and of $\mathbf{u}(T^*)$ and $\mathbf{F}(\Delta T^*)$ denoted by $\mathbf{u}''$ and F:

$$\mathbf{u}'' = \mathbf{u}' + \mathbf{g}(\mathbf{u}')\Delta T^* - S_1^{1/2}\mathbf{F} \tag{13.33}$$

Using methods of transformations of random variables (Van Kampen, §I5 [4]), the following relation between the probability densities of $\mathbf{u}'$, $\mathbf{u}''$ and $\mathbf{F}$ denoted by $p_u(\mathbf{u}', T^* + \Delta T^*)$, $p_u(\mathbf{u}'', T^*)$ and $p_F(\mathbf{F}, \Delta T^*)$ can be given:

$$p_u(\mathbf{u}', T^* + \Delta T^*) dV_{u'} = \int p_F(\mathbf{F}, \Delta T^*) p_u(\mathbf{u}'', T^*) dV_F dV_{u''} \tag{13.34}$$

where use was made of the property that $\mathbf{F}$ is statistically independent of $\mathbf{u}''$. Invoking (13.33) we can expand as

$$\begin{aligned} p_u(\mathbf{u}'', T^*) &= p_{\mathbf{u}}(\mathbf{u}' + \mathbf{g}(\mathbf{u}')\Delta T^* - S_1^{1/2}\mathbf{F}, T^*) \\ &= p_u(\mathbf{u}', T^*) + \sum \frac{\partial p_u(\mathbf{u}', T^*)}{\partial u'_k} (g_k(\mathbf{u}')\Delta T^* - S_1^{1/2} F_k) \\ &\quad + \frac{1}{2} S_1 \sum \frac{\partial^2 p_u(\mathbf{u}')}{\partial u'_j \partial u'_k} F_j F_k + o(\Delta T^*). \end{aligned} \tag{13.35}$$

Expanding the left-hand side of equation (13.34) for small ΔT^* and implementing (13.35) in the right-hand side, we have

$$\begin{aligned} p_u(\mathbf{u}', T^*) &+ \frac{\partial p_u(\mathbf{u}', T^*)}{\partial T^*} \Delta T^* = \frac{dV_{u''}}{dV_{u'}} \int p_F(\mathbf{F}, \Delta T^*) \left\{ p_u(\mathbf{u}', T^*) \right. \\ &+ \sum \frac{\partial p_u(\mathbf{u}', T^*)}{\partial u'_k} (g_k(\mathbf{u})\Delta T^* - S_1^{1/2} F_k) \\ &\left. + \frac{1}{2} S_1 \sum \frac{\partial^2 p_u(\mathbf{u}', T^*)}{\partial u'_j \partial u'_k} F_j F_k \right\} dV_F + o(\Delta T^*) \\ &= \left| \frac{d\mathbf{u}''}{d\mathbf{u}'} \right| \left\{ p_u(\mathbf{u}', T^*) + \sum \frac{\partial p_u(\mathbf{u}', T^*)}{\partial u'_k} g_k(\mathbf{u}) \Delta T^* \right. \\ &\left. + \frac{1}{4} \pi S_1 \sum \frac{\partial^2 p_u(\mathbf{u}', T^*)}{\partial u'^2_k} \Delta T^* \right\} + o(\Delta T^*). \end{aligned} \tag{13.36}$$

where we used equations (13.31)-(13.32) to evaluate moments of $\mathbf{F}$. The Jacobian can be assessed as follows:

$$\left| \frac{d\mathbf{u}''}{d\mathbf{u}'} \right| = \begin{vmatrix} 1 + \frac{\partial g_1}{\partial u'_1} \Delta T^* & \frac{\partial g_1}{\partial u'_2} \Delta T^* \\ \frac{\partial g_2}{\partial u'_1} \Delta T^* & 1 + \frac{\partial g_2}{\partial u'_2} \Delta T^* \end{vmatrix} = 1 + \sum \frac{\partial g_k}{\partial u'_k} \Delta T^* + o(\Delta T^*). \tag{13.37}$$

Leaving out the primes and collecting terms like ΔT^*, one finds from equations (13.36)and (13.37):

$$\frac{\partial p_u(\mathbf{u}, T^*)}{\partial T^*} = \sum \frac{\partial}{\partial u_k} (g_k(\mathbf{u}) p_u(\mathbf{u}, T^*)) + \frac{\pi}{4} S_1 \sum \frac{\partial^2 p_u(\mathbf{u}, T^*)}{\partial u_k^2}, \tag{13.38}$$

which, in vector notation, reads as

$$\frac{\partial p_u}{\partial T^*} = \nabla \cdot (\mathbf{g} p_u) + \frac{\pi}{4} S_1 \Delta p_u. \tag{13.39}$$

The above equation is the Fokker-Planck equation associated with the fluctuation equation (13.25). It describes the evolution of the joint probability density of u_1 and u_2 as a function of time given some initial distribution at $T^* = 0$. For example, if u_1 and u_2 have fixed deterministic values u_{10} and u_{20} at time zero, the initial conditions for equation (13.39) are:

$$p_u(\mathbf{u}, T^*) = \delta(u_1 - u_{10})\delta(u_2 - u_{20}) \qquad \text{at} \qquad T^* = 0, \tag{13.40}$$

where δ is Dirac delta function. The solution of Fokker-Planck equation (13.39) then describes the evolution of a fixed initial value to an uncertain future value described by a probability distribution.

13.3 Solutions for stationary response

If the solution of eq. (13.39) evolves to a stationary probability distribution, this probability distribution is described by:

$$\nabla \cdot (\mathbf{g} p_u) + \frac{\pi}{4} S_1 \Delta p_u = 0. \tag{13.41}$$

In terms of the cylindrical variables a and ϕ related to u_1 and u_2 by eq. (13.27), this becomes

$$\frac{\partial}{\partial a}(g_a p) + \frac{\pi}{4} S_1 \left(\frac{\partial}{\partial a} a \frac{\partial}{\partial a}\left(\frac{p}{a}\right) + \frac{1}{a^2}\frac{\partial^2 p}{\partial \phi^2} \right) = 0 \tag{13.42}$$

where

$$p = p(a, \phi) = p_u(u_1, u_2) \left| \frac{du_1 du_2}{da d\phi} \right| = a p_u(u_1, u_2) \tag{13.43}$$

and

$$g_a = \frac{1}{2}\alpha_1 a^{1+\alpha} \tag{13.44}$$

The solution of (13.42) can be written as the product of two independent probability distributions of a and ϕ:

$$p = p(a)p(\phi), \tag{13.45}$$

where

$$p(\phi) = \frac{1}{2\pi} \quad , \quad 0 \leq \phi \leq 2\pi \tag{13.46}$$

while $p(a)$ satisfies

$$\frac{\partial}{\partial a}\left(g_a p + \frac{\pi}{4} S_1 a \frac{\partial}{\partial a}\left(\frac{p}{a}\right)\right) = 0. \tag{13.47}$$

Integrating once and requiring that p tends to zero exponentially fast as $a \to \infty$, we have

$$g_a p + \frac{\pi}{4} S_1 a \frac{\partial}{\partial a}\left(\frac{p}{a}\right) = 0. \tag{13.48}$$

The solution of this equation is:

$$p = C_0 a \exp\left(-\frac{4}{\pi S_1}\int^a g_a da\right) = C_0 a \exp\left(-\frac{2\alpha_1 a^{\alpha+2}}{\pi(\alpha+2)S_1}\right) \tag{13.49}$$

where the integration constant follows from the requirement that $\int_0^\infty p da = 1$:

$$C_0^{-1} = \int_0^\infty a \exp\left(-\frac{2\alpha_1 a^{\alpha+2}}{\pi(\alpha+2)S_1}\right) da. \tag{13.50}$$

Substituting

$$\frac{2\alpha_1 a^{\alpha+2}}{\pi(\alpha+2)S_1} = t \tag{13.51}$$

eq. (13.50) can be evaluated to:

$$\begin{aligned} C_0^{-1} &= \left(\frac{\pi(\alpha+2)S_1}{2\alpha_1}\right)^{\frac{2}{\alpha+2}} \frac{1}{\alpha+2}\int_0^\infty t^{\frac{2}{\alpha+2}-1} e^{-t} dt \\ &= \left(\frac{\pi(\alpha+2)S_1}{2\alpha_1}\right)^{\frac{2}{\alpha+2}} \frac{\Gamma(2/(\alpha+2))}{\alpha+2} \end{aligned} \tag{13.52}$$

With this result, we have obtained an expression in closed-form for the probability distribution of amplitudes for non-linearly damped resonant response.

The mean-square response can be assessed as follows:

$$\begin{aligned} \sigma_x^2 &= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{+T/2} x^2(t) dt \\ &= \lim_{T\to\infty} \frac{\delta^{-\frac{2}{2+\alpha}}}{T}\int_{-T/2}^{+T/2} a^2(T^*)\sin^2(t^* + \phi(T^*)) dt^* \\ &= \lim_{T\to\infty} \delta^{-\frac{2}{2+\alpha}}\left(\frac{1}{\pi}\int_{-\pi/2}^{+\pi/2} \sin^2\eta d\eta\right)\frac{1}{T}\int_{-T/2}^{+T/2} a^2(T^*) dT^* \\ &= \frac{1}{2}\delta^{-\frac{2}{2+\alpha}}\int_0^\infty a^2 p(a) da, \end{aligned} \tag{13.53}$$

where we substituted eqs. (13.14) and (13.18) and made use of the property that the sinusoidal wave $\sin(t+\phi(T^*))$ varies much more rapidly (if $\delta \ll 1$) than the random variables a and ϕ. Upon substituting the distribution found for $p(a)$ (cf. eqs. (13.49) and (13.52)), we have

$$\begin{aligned}\sigma_x^2 &= \frac{1}{2}\delta^{-\frac{2}{2+\alpha}} C_0 \int_0^\infty a^3 \exp\left(-\frac{2\alpha_1 a^{\alpha+2}}{\pi(\alpha+2)S_1}\right) da \\ &= \frac{\delta^{-\frac{2}{2+\alpha}}}{2(\alpha+2)} C_0 \left(\frac{\pi(\alpha+2)S_1}{2\alpha_1}\right)^{\frac{4}{\alpha+2}} \int_0^\infty t^{\frac{4}{\alpha+2}-1} e^{-t} dt \\ &= \frac{\Gamma(4/(\alpha+2))}{2\Gamma(2/(\alpha+2))} \left(\frac{\pi^{3/2}(\alpha+2)\Gamma((\alpha+4)/2) S_p^n(\omega^*=1)}{4\delta\Gamma((\alpha+3)/2)}\right)^{\frac{2}{\alpha+2}} \qquad (13.54)\end{aligned}$$

where we also used the latter of eqs. (13.4). Eq. (13.54) describes the standard deviation of response in case of non-linearly damped resonance. For the special case of linear damping, $\alpha = 0$, the right-hand side of eq (13.54) becomes equal to the previous result obtained by Fourier transform: cf. eq. (4.24) (as should be).

The probability distribution of peak response, $\rho = a\delta^{-1/(2+\alpha)}$ (cf. eqs. (13.15) and (13.18)), normalized with standard deviation of instantaneous response, σ_x, now becomes

$$p(\rho) = \frac{(\alpha+2)}{2} \frac{\Gamma[4/(\alpha+2)]}{\Gamma^2[2/(\alpha+2)]} \frac{\rho}{\sigma_x^2} \exp - \left[\left\{\frac{\Gamma[4/(\alpha+2)]}{2\Gamma[2/(\alpha+2)]}\right\}^{1/2} \frac{\rho}{\sigma_x}\right]^{\alpha+2}, \qquad (13.55)$$

In figure 13.1 we have shown plots of the probability density of amplitude or peak value considering the cases $\alpha = -1$ (Coulomb or dry friction), $\alpha = 0$ (linear damping) and $\alpha = 1$ (quadratic damping). For $\alpha = 0$, the solution becomes equal to the Rayleigh distribution (as should be). With increasing value of α, the probability density of larger values of ρ/σ_x becomes less in magnitude: higher peak values are better damped with higher values of α!

With the distribution of amplitudes, it becomes possible to derive analytical expressions for statistical parameters that govern non-linearly damped resonant response of mechanical structures subject to random excitation. An example is lightly damped response of a mooring riser of a floating oil production system: Brouwers [12]. Here, response is governed by resonance in

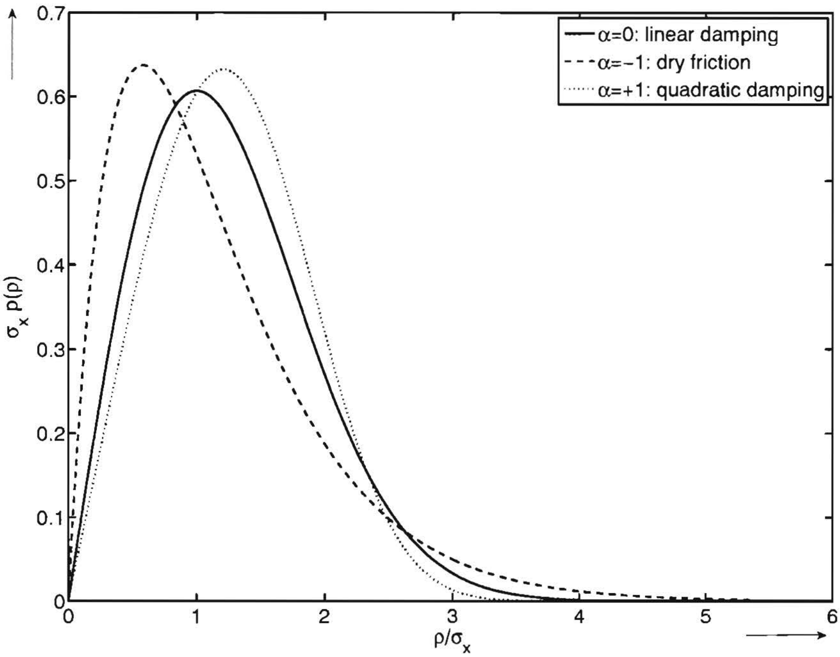


Figure 13.1. Probability density of peaks or amplitudes for a lightly damped resonant system with power law damping.

the first natural mode. The amplitude of this mode can be described by an equation equivalent to a non-linearly damped mass-spring system. Using the methods of the previous sections, it is possible to assess parameters as expected fatigue damage and expected extreme response under conditions of non-linear damping.

To illustrate the possibilities to assess fatigue damage consider the amplitude of the equivalent deterministic stress cycle discussed in section 10.2: i.e. the sinusoidal cycle with deterministic amplitude ρ_{eq} described by eq. (10.31) which leads to the same fatigue damage as that caused by the random cycle with probability according to eq. (13.55) above. Substituting eq. (13.55) into

Table 13.1. Amplitude of deterministic stress cycle leading to equivalent fatigue damage for the case that $\lambda = 4$ in Miner's law. Considered are Coulomb friction $\alpha = -1$, linear damping $\alpha = 0$ and quadratic damping $\alpha = 1$.

α	ρ_{eq}/σ_x
-1	1.91
0	1.68
+1	1.57

eq. (10.31) one has

$$\rho_{eq} = \sigma_x \left\{ \frac{(\alpha+2)}{2} \frac{\Gamma(4/(\alpha+2))}{[\Gamma(2/(\alpha+2))]^2} \int_0^\infty x^{1+\lambda} \exp\left(- \left[\frac{\Gamma(4/(\alpha+2))x^2}{2\Gamma(2/(\alpha+2))} \right]^{\frac{\alpha+2}{2}} \right) dx \right\}^{1/\lambda} . \qquad (13.56)$$

Implementing

$$\left[\frac{\Gamma(4/(\alpha+2))x^2}{2\Gamma(2/(\alpha+2))} \right]^{\frac{\alpha+2}{2}} = t \qquad (13.57)$$

we obtain

$$\rho_{eq} = \sigma_x \frac{\sqrt{2}[\Gamma(2/(\alpha+2))]^{1/2-1/\lambda}}{[\Gamma(4/(\alpha+2))]^{1/2}} \left\{ \int_0^\infty t^{\frac{\lambda+2}{\alpha+2}-1} e^{-t} dt \right\}^{1/\lambda}$$

$$= \sigma_x \left[\frac{2\Gamma(2/(\alpha+2))}{\Gamma(4/(\alpha+2))} \right]^{1/2} \left[\frac{\Gamma[(\lambda+2)/(\alpha+2)]}{\Gamma(2/(\alpha+2))} \right]^{1/\lambda} , \qquad (13.58)$$

The right hand side of this equation has been calculated taking a power $\lambda = 4$ in Miner's law for fatigue damage: cf. eq. (10.20). Furthermore, damping powers were taken as $\alpha = -1$, $\alpha = 0$ and $\alpha = 1$. Results have been plotted in table 13.1. It is seen that with increasing α, ρ_{eq} decreases. Because with increasing power of damping, the probability density of higher peaks becomes less (figure 13.1), fatigue damage normalized with standard deviation decreases as well.

The probability density distribution of the extreme peak in a record of n peaks can be calculated according to eqs. (10.33)-(10.34) substituting for $p(\rho)$ the distribution according to eq. (13.55). The result has been shown in figure 13.2 taking a record of 1000 peaks and considering $\alpha = -1$, $\alpha = 0$ and $\alpha = 1$. The extreme value normalized with standard deviation shifts to lower values with increasing α.

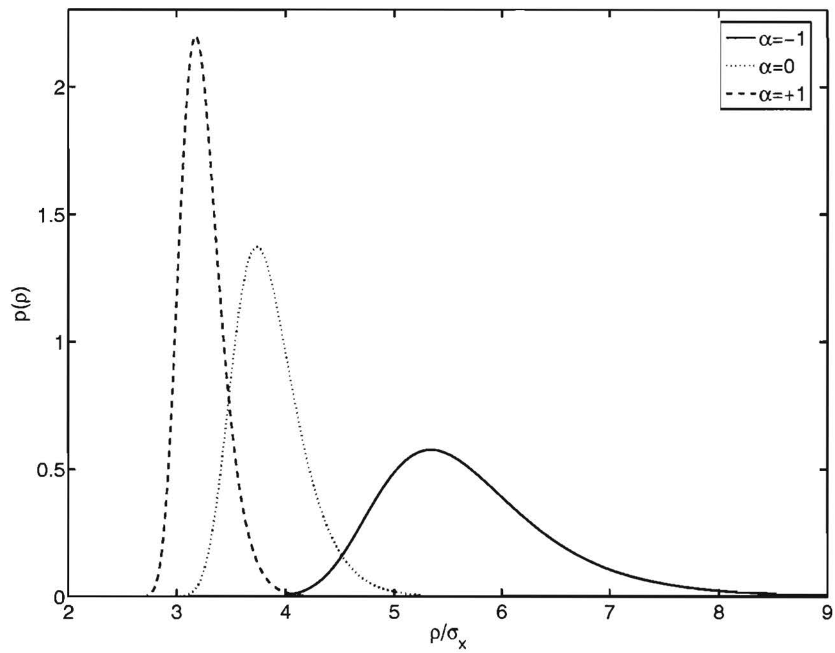


Figure 13.2. Probability densities of extremes for Coulomb friction $\alpha = -1$, linear damping $\alpha = 0$ and quadratic damping $\alpha = 1$.

13.4 Transient response behavior

So far, emphasis was laid on solutions of the Fokker-Planck equation for stationary response. In this section, attention is focussed on non-stationary solutions. We shall restrict ourselves to linear damping. It will be shown that the time-dependent solutions of the Fokker-Planck equation for linear damping are consistent with the previous results for non-stationary resonant response of section 6.4. Time-dependent solutions of more complex versions of the Fokker-Planck equation will be dealt with in the next chapter.

We shall return to the formulation of the Fokker-Planck equation in terms of the variables u_1 and u_2: cf. eq. (13.39). In case of linear damping, $\alpha = 0$ and $\alpha_1 = 1$ so that $\mathbf{g} = 1/2\mathbf{u}$: see eqs (13.1), (13.21) and (13.27). Eq. (13.39) now becomes:

$$\frac{\partial p_u}{\partial T^*} = \frac{1}{2}\frac{\partial}{\partial u_1}(u_1 p_u) + \frac{1}{2}\frac{\partial}{\partial u_2}(u_2 p_u) + \frac{\pi S_1}{4}\left(\frac{\partial^2 p_u}{\partial u_1^2} + \frac{\partial^2 p_u}{\partial u_2^2}\right) \tag{13.59}$$

In line with the situation considered in section 6.4, we assume zero displacement and velocity at time zero. Noting that (cf. eqs. (13.14), (13.18) and (13.26))

$$x = \delta^{1/2}(u_1 \sin t + u_2 \cos t), \tag{13.60}$$

$$\dot{x} = \delta^{1/2}(u_1 \cos t - u_2 \sin t), \tag{13.61}$$

the initial conditions taken for eq. (13.59) are:

$$u_1 = u_2 = 0 \quad \text{at} \quad T^* = 0. \tag{13.62}$$

Eq. (13.59) is the Fokker-Planck equation appropriate for a Gaussian process: Van Kampen [4], §VIII6. The solution of eq. (13.59) is thus given by the two-dimensional Gaussian distribution (eq. (1.11)):

$$p_{\mathbf{u}} = \frac{1}{2\pi\sqrt{\sigma_{u_1}^2\sigma_{u_2}^2 - \sigma_{u_1u_2}^4}} \exp\left(-\frac{\sigma_{u_2}^2 {u_1}^2 - 2\sigma_{u_1u_2}^2 u_1 u_2 + \sigma_{u_1}^2 {u_2}^2}{2(\sigma_{u_1}^2\sigma_{u_2}^2 - \sigma_{u_1u_2}^4)}\right), \tag{13.63}$$

Here, σ_{u_1}, σ_{u_2} and $\sigma_{u_1u_2}$ are time-dependent standard deviations and covariance. Descriptions for σ_{u_1}, σ_{u_2} and $\sigma_{u_1u_2}$ can be obtained by substituting solution (13.63) into differential eq. (13.59) and equating coefficients of like terms. An alternative way is presented here. Therefore, multiply each term of eq. (13.59) with u_1^2 and integrate with respect to u_1 and u_2 from $-\infty$ to $+\infty$. Because

$$\sigma_{u_1}^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_1^2 p_u du_1 du_2, \tag{13.64}$$

the first term yields,

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_1^2 \frac{\partial p_u}{\partial T^*} du_1 du_2 = \frac{d\sigma_{u_1}^2}{dT^*} \tag{13.65}$$

Adopting partial integration, the integral of the first term on the right hand side yields:

$$\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} u_1^2 \frac{\partial}{\partial u_1}(u_1 p_u) du_1 \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} u_1^2 d(u_1 p_u) \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} du_2 (u_1^3 p_u)\Bigg|_{u_1=-\infty}^{u_1=+\infty} - \frac{1}{2}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} u_1 p_u du_1^2 \\
&= -\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} u_1^2 p_u du_1 \\
&= -\sigma_{u_1}^2
\end{aligned} \tag{13.66}$$

where we assumed that $u_1^3 p_u$ tends to zero as $u_1 \rightarrow \pm\infty$, which is likely, because probability densities generally tend exponentially fast to zero at the extreme ends of the distribution.

For the integral of the second term on the right hand side of eq. (13.59) one can write:

$$\frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_1^2 \frac{\partial}{\partial u_2}(u_2 p_u) du_1 du_2 = \frac{1}{2}\int_{-\infty}^{+\infty} u_1^2 du_1 (u_2 p_u)\Bigg|_{-\infty}^{+\infty} = 0 \tag{13.67}$$

Similarly, the fourth term is also found to be equal to zero. For the third term, we need to apply partial integration twice:

$$
\begin{aligned}
\frac{\pi S_1}{4}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} u_1^2 \frac{\partial^2 p_u}{\partial u_1^2} du_1 &= \frac{\pi S_1}{4}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} u_1^2 d\left(\frac{\partial p_u}{\partial u_1}\right) \\
&= \frac{\pi S_1}{4}\int_{-\infty}^{+\infty} du_2 \left(u_1^2 \frac{\partial p_u}{\partial u_1}\right)\Bigg|_{-\infty}^{+\infty} - \frac{\pi S_1}{4}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} \frac{\partial p_u}{\partial u_1} du_1^2 \\
&= -\frac{\pi S_1}{2}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} u_1 \frac{\partial p_u}{\partial u_1} du_1 \\
&= -\frac{\pi S_1}{2}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} u_1 dp_u \\
&= -\frac{\pi S_1}{2}\int_{-\infty}^{+\infty} du_2 (u_1 p_u)\Big|_{-\infty}^{+\infty} + \frac{\pi S_1}{2}\int_{-\infty}^{+\infty} du_2 \int_{-\infty}^{+\infty} p_u du_1 \\
&= \frac{\pi S_1}{2},
\end{aligned} \tag{13.68}
$$

because

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p_u du_1 du_2 = 1. \tag{13.69}$$

Collecting the above results, we obtain from eq. (13.59) an equation for σ_{u_1} which reads as

$$\frac{d\sigma_{u_1}^2}{dT^*} = -\sigma_{u_1}^2 + \frac{\pi S_1}{2}. \tag{13.70}$$

In view of condition (13.62), $\sigma_{u_1} = 0$ at $T^* = 0$. The solution to eq. (13.70) then is:

$$\sigma_{u_1}^2 = \frac{\pi S_1}{2}\left(1 - e^{-T^*}\right) \tag{13.71}$$

In a similar way, one can show that

$$\sigma_{u_2}^2 = \frac{\pi S_1}{2}\left(1 - e^{-T^*}\right). \tag{13.72}$$

Multiplying eq. (13.59) with $u_1 u_2$, integrating with respect to u_1 and u_2 over the range $-\infty \leq u_1 \leq +\infty$ and $-\infty \leq u_2 \leq +\infty$, and evaluating terms a.o. by partial integration as above, one arrives at:

$$\frac{d\sigma_{u_1 u_2}}{dT^*} = -\sigma_{u_1 u_2}. \tag{13.73}$$

Because of condition (13.62), $\sigma_{u_1 u_2} = 0$ at $T^* = 0$, so that eq. (13.73) yields

$$\sigma_{u_1 u_2} = 0. \tag{13.74}$$

Similarly, it can be shown that the mean values of u_1 and u_2 are zero, a feature already adopted in solution (13.63). The transient joint probability density of u_1 and u_2 now becomes:

$$p_u = \frac{1}{2\pi\sigma_u^2(T^*)} \exp\left(-\frac{u_1^2 + u_2^2}{2\sigma_u^2(T^*)}\right) \tag{13.75}$$

where

$$\sigma_u^2(T^*) = \sigma_0^{*2}(1 - e^{-T^*}) \quad , \quad \sigma_0^{*2} = \frac{\pi S_1}{2}, \tag{13.76}$$

where σ_0^* is standard deviation of stationary $\mathbf{u}$ valid as $T^* \rightarrow \infty$. From eqs. (13.60)-(13.61) one can verify that

$$u_1^2 + u_2^2 = \delta(x_1^2 + \dot{x}_1^2). \tag{13.77}$$

Moreover,

$$p_x = p_u \left|\frac{d\mathbf{u}}{d\mathbf{x}}\right| = \delta p_u, \tag{13.78}$$

so that

$$p_x = \frac{1}{2\pi\sigma^2(T^*)} \exp\left(-\frac{x^2 + \dot{x}^2}{2\sigma^2(T^*)}\right) \tag{13.79}$$

where $\sigma(T^*)$ is the standard deviation of transient resonant response given before (cf. eqs. (6.31) and (6.55)):

$$\sigma^2(T^*) = \frac{\sigma_u(T^*)}{\delta} = \sigma_0^2(1 - e^{-T^*}) \quad , \quad \sigma_0^2 = \frac{\sigma_0^{*2}}{\delta}, \tag{13.80}$$

σ_0 being standard deviation of stationary resonant response. Because x and $\dot{x}$ are uncorrelated, we can express the right hand side of eq. (13.79) as the product of two one-dimensional Gaussian distributions for x and $\dot{x}$ with standard deviations $\sigma(T^*)$. The result for x thus obtained is entirely consistent with the results presented in section 6.4: see eqs. (6.55)-(6.56).

Applying eqs. (13.26) and (13.43), one can derive an expression for the joint probability density of amplitude and phase from eqs. (13.75) and (13.76) as:

$$p(a, \phi) = \frac{a}{2\pi\sigma_u^2(T^*)} \exp\left(-\frac{a^2}{2\sigma_u^2(T^*)}\right). \tag{13.81}$$

Integrating over all possible phase angles in the range $0 \leq \phi \leq 2\pi$, we have for the one-dimensional probability density of amplitude the description:

$$p(a) = \frac{a}{\sigma_u^2(T^*)} \exp\left(-\frac{a^2}{2\sigma_u^2(T^*)}\right). \tag{13.82}$$

The evolution of the distribution with time has been shown in figure 13.3. It illustrates the development of a Dirac type distribution near $a = 0$ at $T^* = 0$ towards the Rayleigh distribution as $T^* \to \infty$.

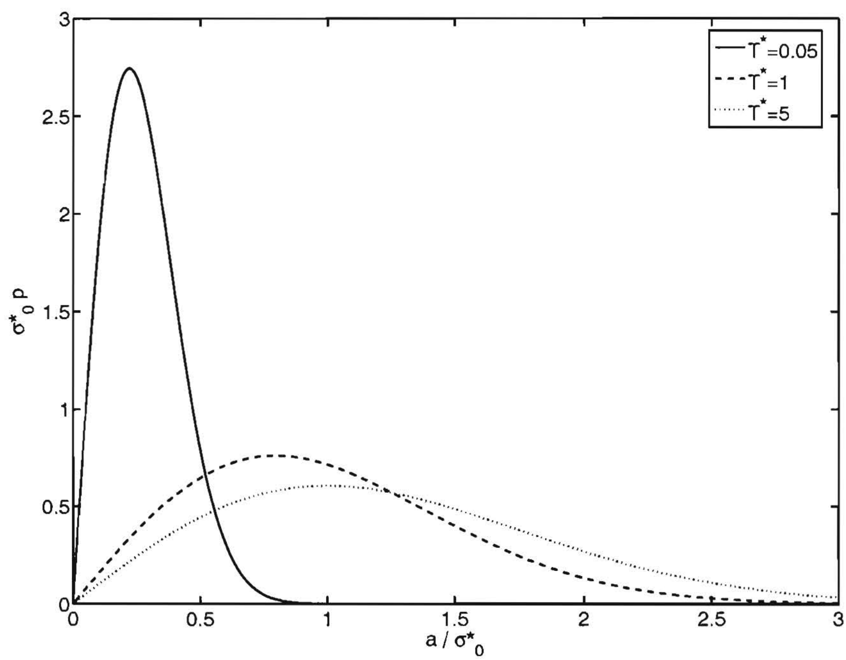


Figure 13.3. Transient probability density distributions of amplitude in case of linear damping.

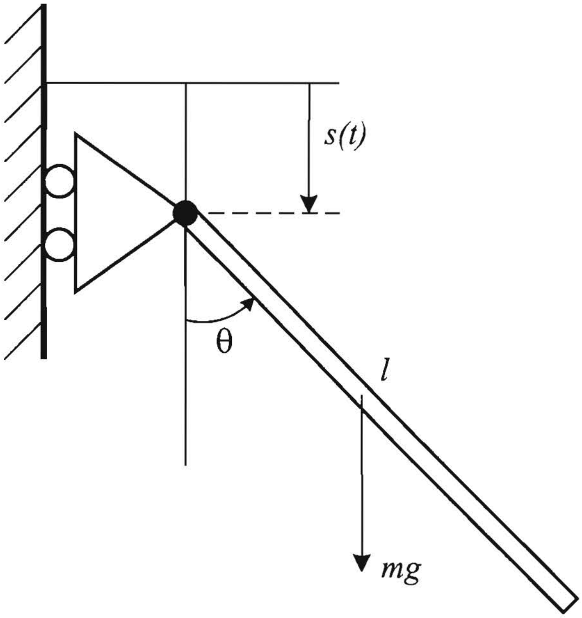
Chapter 14

Stability in case of randomly fluctuating restoring

It is known that a sinusoidally varying component in the stiffness term of a mass spring system can give rise to instability, even if the magnitude of the oscillation is small in comparison to the total stiffness: e.g. Verhulst [13]. The phenomenon is also known as *Mathieu instability*. It occurs for oscillations with a frequency which is twice the natural frequency of the system. Depending on the damping, the system can exhibit an oscillation solely due to the small oscillating component in the stiffness terms and without any external excitation applied.

A nice example is the pendulum mounted in a frame that oscillates vertically, see figure 14.1. For small swing angles of the pendulum, the equation of motion becomes that of a linear spring-mass system with a fluctuating component in the restoring force. For frequencies of oscillation of the frame that are twice the natural frequency, $\Omega = 2\omega_0$, a solution exists other than that of the pendulum at rest; i.e., a solution for which the pendulum exhibits a swinging motion, even if the amplitude of vibration of the frame is very small ($\epsilon \ll 1$; see figure 14.1). The amplitude of swing-oscillations grows with time unless sufficient damping is applied: Verhulst [13].

There are many examples of mechanical systems exhibiting an oscillating parametric component: Notably floating structures in water where restoring occurs according to Archimedes' law. When the water surface oscillates in height and as a result, the degree of submergence of the structure as well, restoring exhibits some degree of fluctuation. The system is prone to Mathieu instability. Examples are among others roll motion of ships and horizontal displacement of tensioned leg platforms for oil production: see figure 14.2. In these applications, the fluctuating component in the restoring forces is caused by waves that vary randomly in time. Question now is: does the phenomenon of instability also occur in case of random fluctuation of the restoring force? The answer will be given below. It is an illustration of how to apply stochastic theory in case of questions on stability.



Kinetic Energy: $T = \frac{1}{6} m l^2 \dot{\theta}^2$
Potential energy: $U = ml(g + \ddot{s}) \cos\theta$
Lagrangian: $L = T - U$
Lagrange's equation: $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\Rightarrow \ddot{\theta} + \omega_0^2 \left(1 + \frac{\ddot{s}}{g}\right) \sin\theta = 0,$$

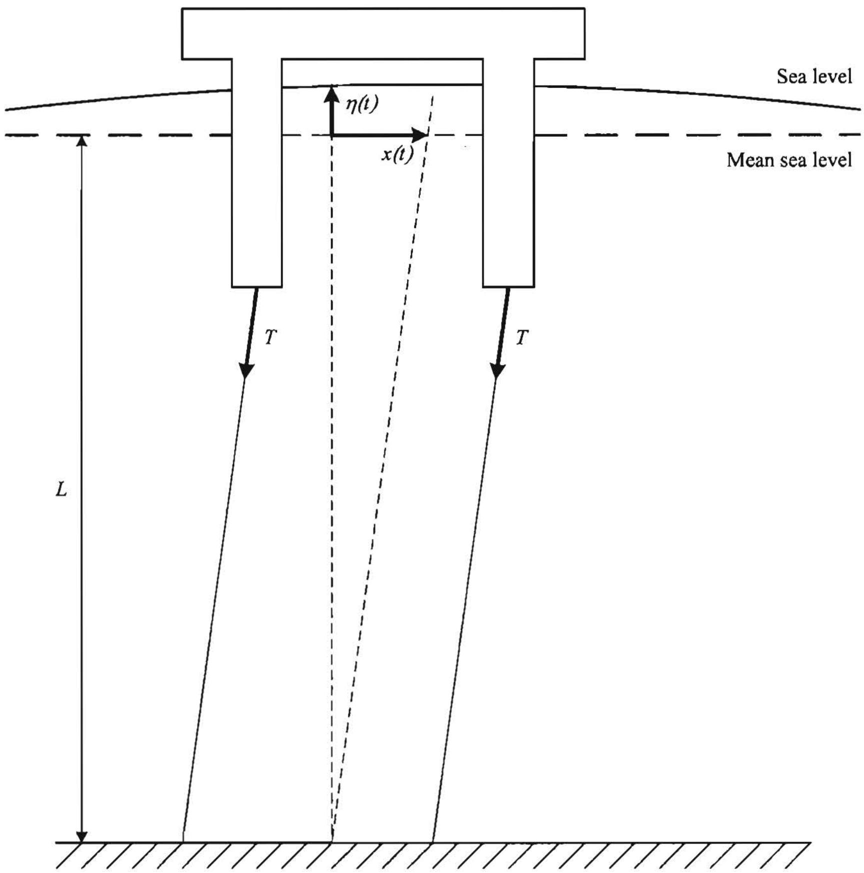
where $\omega_0^2 = \frac{3g}{l}$

If $\theta \ll 1$ and $s(t) = s_0 \sin \Omega t$:

$$\ddot{\theta} + \omega_0^2 (1 + \epsilon_1 \sin \Omega t)\theta = 0,$$

where $\epsilon_1 = -3\left(\frac{\Omega^2}{\omega_0^2}\right)\left(\frac{s_0}{l}\right)$

Figure 14.1. Pendulum with oscillating support.



Newton's second law in horizontal direction:

$$m\ddot{x} + 2T \sin \frac{x}{L} = 0$$

For $T = T_0(1 + \epsilon_2 \eta(t))$ and $x/L \ll 1$:

$$m\ddot{x} + \frac{2T_0}{L}(1 + \epsilon_2 \eta(t))x = 0$$

Figure 14.2. Horizontal motion of a tensioned leg platform.

14.1 Basic equations

The governing equation of motion can be written as:

$$\ddot{x} + \delta \dot{x}|\dot{x}|^{\alpha} + (1 + \delta^{1/2} f(t^*))x = 0. \tag{14.1}$$

The equation is in dimensionless form as before: cf. eqs. (6.3) and (13.1). Damping is light, $\delta \ll 1$, as well as parametric excitation. Now rewrite eq. (14.1) as:

$$\ddot{x} + x = -\delta \dot{x}|\dot{x}|^{\alpha} - \delta^{1/2} x f(t^*). \tag{14.2}$$

Setting $\delta = 0$ gives rise to the solution

$$x = a\cos(t^* + \phi) \tag{14.3}$$

Amplitude and phase are chosen such that

$$\dot{x} = -a\sin(t^* + \phi), \tag{14.4}$$

so that

$$a = (x^2 + \dot{x}^2)^{1/2} \tag{14.5}$$

$$\phi = -t^* - \arctan(\dot{x}/x) \tag{14.6}$$

Differentiating (14.5) and (14.6) with respect to t^* and implementing (14.2), one obtains:

$$a\frac{da}{dt^*} = \dot{x}(x + \ddot{x}) = -\delta|\dot{x}|^{\alpha+2} - \delta^{1/2} x\dot{x} f(t^*) \tag{14.7}$$

$$a^2\frac{d\phi}{dt^*} = -x(x + \ddot{x}) = \delta x\dot{x}|\dot{x}|^{\alpha} + \delta^{1/2} x^2 f(t^*) \tag{14.8}$$

Upon substituting (14.3)-(14.4) and dividing by a and a^2, respectively, we have the following equations for amplitude and phase

$$\frac{da}{dt^*} = -\delta a^{\alpha+1}|\sin(t^* + \phi)|^{\alpha+2} + \delta^{1/2} a\sin(t^* + \phi)\cos(t^* + \phi) f(t^*) \tag{14.9}$$

$$\frac{d\phi}{dt^*} = -\delta a^{\alpha}\cos(t^* + \phi)\sin(t^* + \phi)|\sin(t^* + \phi)|^{\alpha} + \delta^{1/2}\cos^2(t^* + \phi) f(t^*) \tag{14.10}$$

Question now is: how do amplitude and phase behave with increasing time t^* given some initial value at $t^* = 0$:

$$a = a_0 \quad , \quad \phi = \phi_0 \quad \text{at} \quad t^* = 0. \tag{14.11}$$

The answer to this question will give us a clue to whether the system is stable or not.

14.2 Fokker-Planck equation

Because $f(t^*)$ is random, a and ϕ will be random as well. Their behavior can be studied from the transient probability density of a and ϕ which is governed by a Fokker-Planck equation associated with eqs. (14.9)-(14.10). In section 13.2, we derived a Fokker-Planck equation for a fluctuating equation of the type of eq. (13.25). The present fluctuation equation is rather different: the derivation has to be revisited. For this purpose, rewrite eqs. (14.9)-(14.10) in the form:

$$\frac{dq_n}{dt^*} = -\delta g_n(\mathbf{q}, t^*) + \delta^{1/2} h_n(\mathbf{q}, t^*) f(t^*) \tag{14.12}$$

where $\mathbf{q} = (q_1, q_2) = (a, \phi)$, while

$$g_1 = q_1^{\alpha+1} |\sin(q_2 + t^*)|^{\alpha+2} \tag{14.13}$$

$$g_2 = q_1^{\alpha} \cos(q_2 + t^*) \sin(q_2 + t^*) |\sin(q_2 + t^*)|^{\alpha} \tag{14.14}$$

$$h_1 = q_1 \sin(q_2 + t^*) \cos(q_2 + t^*) \tag{14.15}$$

$$h_2 = \cos^2(q_2 + t^*) \tag{14.16}$$

Comparing eq. (14.12) with eq. (13.25) the following differences are observed. First of all, the functions g_n and h_n on the right hand side of eq. (14.12) vary with time. In fact, they oscillate with t^* which is fast compared to the change of $\mathbf{q}$ which is slow because the time derivative of $\mathbf{q}$ is small when $\delta \ll 1$. Secondly, the second term on the right hand side of eq. (14.12) contains products of functions of $\mathbf{q}$ and the random excitation $f(t^*)$. It reflects parametric excitation in contrast to direct excitation in eq. (13.25). Thirdly, the random excitation is a regular finite band-width Gaussian process. In eq. (13.25), the excitation is white noise being the outcome of applying a limit process involving $\delta \to 0$. Such a limit process still needs to be applied to eq. (14.12).

The derivation of the Fokker-Planck equation associated with a fluctuating equation like eq. (14.12) applying the limit process $\delta \to 0$ has been given by Stratonovich: Vol. I, §4.8-4.9 [3]. The derivation is not repeated here. In terms of the present variables, the result of the Stratonovich derivation, i.e. eq. (4.194) of Stratonovich; Vol. I, §4.9 [3], is:

$$\begin{aligned} \frac{\partial p}{\partial T^*} &= \sum \frac{\partial(\langle g_l \rangle p)}{\partial q_l} - \sum \frac{\partial}{\partial q_l} \sum \left(\int_0^\infty \left\langle \frac{\partial h_l}{\partial q_m} h_{m\tau} f f_\tau \right\rangle d\tau p \right) \\ &+ \sum \frac{\partial^2}{\partial q_l \partial q_m} \left(\int_0^\infty \langle h_l h_{m\tau} f f_\tau \rangle \, d\tau p \right), \end{aligned} \tag{14.17}$$

where $p = p(\mathbf{q}, T^*)$ is transient probability density of $\mathbf{q}$, $T^* = \delta t^*$ and $g_l = g_l(\mathbf{q}, t^*)$, $h_l = h_l(\mathbf{q}, t^*)$, $h_{m\tau} = h_m(\mathbf{q}, t^* + \tau)$, $f = f(t^*)$, $f_\tau = f(t^* + \tau)$. Angular brackets denote statistical averaging over fluctuations of f taking $\mathbf{q}$ fixed.

Compared to Fokker-Planck equation (13.38) derived in section 13.2 it is noted that coefficients in eq. (14.17) have yet to be averaged. This will be done in the paragraph below. Furthermore, it is seen that eq. (14.17) contains an extra term, viz. the second term on the right hand side, in comparison with eq. (13.38). This term is due to correlation between $\mathbf{q}$ and f when the term $h_n(\mathbf{q}, t)$ preceding the random excitation in eq. (14.12) varies with $\mathbf{q}$. In fact, the presence of this extra term has provoked debate amongst mathematicians. It is known as the *Itô -Stratonovich dilemma*: Van Kampen [4], IX5. It is now agreed that the derivation of the Fokker-Planck equation by Stratonovich is correct. It leads to consistent formulations which are invariant to the choice of the Markov variables $\mathbf{q}$.

To evaluate the statistical averages in the coefficients of the Fokker-Planck equation, we note that the net contribution of rapidly oscillatory terms can be calculated by averaging over one period of oscillation. Thereby, one can assume that $\mathbf{q}$ is constant because $\mathbf{q}$ varies over the much larger time scale

T^*. Accordingly,

$$\begin{aligned}\langle g_1\rangle &= \left\langle q_1^{\alpha+1}|\sin(q_2+t^*)|^{\alpha+2}\right\rangle \\ &= q_1^{\alpha+1}\frac{1}{2\pi}\int_0^{2\pi}|\sin\eta|^{\alpha+2}d\eta \\ &= \frac{1}{2}\alpha_1 a^{\alpha+1} \qquad (14.18)\end{aligned}$$

$$\begin{aligned}\langle g_2\rangle &= \langle q_1^{\alpha}\cos(q_2+t^*)\sin(q_2+t^*)|\sin(q_2+t^*)|^{\alpha}\rangle \\ &= q_1^{\alpha}\frac{1}{2\pi}\int_0^{2\pi}\cos\eta\sin\eta|\sin\eta|^{\alpha}d\eta = 0, \qquad (14.19)\end{aligned}$$

where α_1 is given by eq. (13.21).

The diffusion terms in Fokker-Planck equation (14.17) contain integrals of correlations of the second term on the right hand side of the fluctuation equation. To illustrate how to proceed in this case, consider:

$$\begin{aligned}&\int_0^{\infty}\langle h_1 h_{1\tau} f f_{\tau}\rangle d\tau = \int_0^{\infty} d\tau q_1^2\langle\sin(q_2+t^*)\cos(q_2+t^*)\sin(q_2+t^*+\tau) \\ &\qquad \cos(q_2+t^*+\tau)f(t^*)f(t^*+\tau)\rangle \\ &= \frac{1}{4}q_1^2\int_0^{\infty} d\tau\langle\sin(2t^*+2q_2)\sin(2t^*+2q_2+2\tau)f(t^*)f(t^*+\tau)\rangle \\ &= \frac{1}{8}q_1^2\int_0^{\infty} d\tau\langle[\cos(2\tau)-\cos(4t^*+4q_2+2\tau)]\, f(t^*)f(t^*+\tau)\rangle \\ &= \frac{1}{8}q_1^2\int_0^{\infty} d\tau\cos(2\tau)\langle f(t^*)f(t^*+\tau)\rangle \\ &= \frac{a^2}{16}\int_{-\infty}^{\infty}\langle f(t^*)f(t^*+\tau)\rangle\cos(2\tau)d\tau \\ &= \frac{\pi}{16}a^2 S_2 \qquad (14.20)\end{aligned}$$

where we used multiplication and multiple-angle formulae of trigonometric functions. Furthermore, we implemented eq. (3.14) relating auto-correlation to power density: S_2 is the value of the power density of $f(t^*)$ at $\omega^* = 2$, the frequency that corresponds to twice the undamped natural frequency of the system.

In a similar manner we can evaluate the other coefficients in the diffusion terms of eq. (14.17). We shall illustrate this for two terms:

$$\int_0^\infty \langle h_2 h_{2\tau} f f_\tau \rangle d\tau = \int_0^\infty d\tau \left\langle \cos^2(q_2 + t^*) \cos^2(q_2 + t^* + \tau) f(t^*) f(t^* + \tau) \right\rangle$$

$$= \frac{1}{4} \int_0^\infty d\tau \langle [1 + \cos(2q_2 + 2t^*)][1 + \cos(2q_2 + 2t^* + 2\tau)] f(t^*) f(t^* + \tau) \rangle$$

$$= \frac{1}{4} \int_0^\infty d\tau \langle [1 + \cos(2q_2 + 2t^*) + \cos(2q_2 + 2t^* + 2\tau) + \frac{1}{2}\cos(2\tau)$$

$$+ \frac{1}{2}\cos(4q_2 + 4t^* + 2\tau)] f(t^*) f(t^* + \tau) \rangle$$

$$= \frac{1}{4} \int_0^\infty d\tau [1 + \frac{1}{2}\cos(2\tau)] \langle f(t^*) f(t^* + \tau) \rangle$$

$$= \frac{\pi}{8} S_0 + \frac{\pi}{16} S_2, \tag{14.21}$$

where S_0 is the value of the power density of $f(t^*)$ at $\omega^* = 0$.

$$\int_0^\infty \left\langle \frac{\partial h_2}{\partial q_2} h_{2\tau} f f_\tau \right\rangle d\tau = \int_0^\infty d\tau \langle 2\cos(q_2 + t^*) \sin(q_2 + t^*)$$

$$\cos^2(q_2 + t^* + \tau) f(t^*) f(t^* + \tau) \rangle$$

$$= \frac{1}{2} \int_0^\infty d\tau \langle \sin(2q_2 + 2t^*) [1 + \cos(2q_2 + 2t^* + 2\tau)] f(t^*) f(t^* + \tau) \rangle$$

$$= \frac{1}{2} \int_0^\infty d\tau \left\langle \left[\sin(2q_2 + 2t^*) - \frac{1}{2}\sin(2\tau) + \frac{1}{2}\sin(4q_2 + 4t^* + 2\tau) \right] \right.$$

$$\left. f(t^*) f(t^* + \tau) \right\rangle$$

$$= -\frac{1}{4} \int_0^\infty \sin(2\tau) \langle f(t^*) f(t^* + \tau) \rangle d\tau$$

$$= -\frac{\pi}{8} \bar{S}_2 \tag{14.22}$$

where

$$\bar{S}(\omega) = \frac{2}{\pi} \int_0^{+\infty} R(\tau) \sin(\omega\tau) d\tau, \tag{14.23}$$

and

$$\bar{S}_2 = \bar{S}(\omega = 2). \tag{14.24}$$

The procedures for eliminating oscillatory terms can also be found in Vol. II of Stratonovich [3], chapter 10.

Eliminating all oscillatory terms in Fokker-Planck equation (14.17) according to the previously presented procedures and returning to the original variables a and ϕ, we obtain:

$$\frac{\partial p}{\partial T^*} = \frac{1}{2}\alpha_1 \frac{\partial}{\partial a}(a^{\alpha+1} p) + \frac{\pi S_2}{16}\frac{\partial}{\partial a}\left(a^3 \frac{\partial}{\partial a}\left(\frac{p}{a}\right)\right) + \frac{\pi}{16}(S_2 + 2S_0)\frac{\partial^2 p}{\partial \phi^2} + \frac{\pi}{8}\bar{S}_2 \frac{\partial p}{\partial \phi}, \tag{14.25}$$

where $p = p(a, \phi, T^*)$ is the joint probability of amplitude and phase as a function of time and $T^* = \delta t^*$. One-dimensional probability densities for amplitude and phase follow from p as:

$$p_a = \int_0^{2\pi} p d\phi \quad , \quad p_\phi = \int_0^{\infty} p da. \tag{14.26}$$

Integrating (14.25) with respect to ϕ and a respectively, and applying appropriate boundary conditions, i.e.: $p \to 0$ and $(\partial/\partial a)p \to 0$ as $a \to \infty$; $p(\phi = 0) = p(\phi = 2\pi)$, $(\partial/\partial\phi)p(\phi = 0) = (\partial/\partial\phi)p(\phi = 2\pi)$,

$$\frac{\partial p_a}{\partial T^*} = \frac{1}{2}\alpha_1 \frac{\partial}{\partial a}\left(a^{\alpha+1} p_a\right) + \frac{\pi S_2}{16}\frac{\partial}{\partial a}\left(a^3 \frac{\partial}{\partial a}\left(\frac{p_a}{a}\right)\right), \tag{14.27}$$

$$\frac{\partial p_\phi}{\partial T^*} = \frac{\pi}{8}\bar{S}_2 \frac{\partial p_\phi}{\partial \phi} + \frac{\pi}{16}(S_2 + 2S_0)\frac{\partial^2 p}{\partial \phi^2}, \tag{14.28}$$

subject to initial conditions

$$p_a = \delta(a - a_0) \quad , \quad p_\phi = \delta(\phi - \phi_0) \quad \text{at} \quad T^* = 0. \tag{14.29}$$

It is also possible to formulate fluctuation equations for amplitude and phase that correspond to eqs. (14.27)-(14.28): i.e.,

$$\frac{da}{dT^*} = -\frac{1}{2}\alpha_1 a^{1+\alpha} + \frac{\pi S_2}{8} a + \frac{\sqrt{2S_2}}{4} a f_1(T^*), \tag{14.30}$$

$$\frac{d\phi}{dT^*} = -\frac{\pi \bar{S}_2}{8} + \frac{1}{4}(2S_2 + 4S_0)^{1/2} f_2(T^*). \tag{14.31}$$

where $f_1(T^*)$ and $f_2(T^*)$ are uncorrelated Gaussian white noise processes of power density unity. By applying transformation rules (14.17), one can easily verify that the Fokker-Planck equations associated with eqs. (14.30) and (14.31) correspond to eqs. (14.27)-(14.28). In this way, it is possible to reconstruct fluctuation equations from a Fokker-Planck equation. The thus obtained equations, cf. eqs. (14.30) and (14.31) are stochastically equivalent to the original equations, viz. eqs. (14.9)-(14.10) taking the limit $\delta \to 0$. That is, they lead to the same statistics for transient behavior when $\delta \to 0$.

Unbounded growth of response will depend on the behavior of $a(T^*)$ as T^* increases. To study this behavior, we can focus attention to the description of $a(T^*)$ in the time domain by eq. (14.30) or on the probability distribution given by eq. (14.27). This will be done in the subsequent sections. Thereby, attention will be focussed on a combination of linear and non-linear damping: i.e. in eq. (14.1) we replace the damping force by:

$$\dot{x}|\dot{x}|^{\alpha} \to \beta_0 \dot{x} + \beta_\alpha \dot{x}|\dot{x}|^{\alpha}, \tag{14.32}$$

in which case,

$$\alpha_1 a^{1+\alpha} \to \beta_0 a + \bar{\beta}_\alpha a^{1+\alpha} \tag{14.33}$$

with

$$\bar{\beta}_\alpha = \beta_\alpha \alpha_1, \tag{14.34}$$

in eqs. (14.27) and (14.30). So we shall consider the problems

$$\frac{da}{dT} = -\frac{1}{2}\beta_0 a - \frac{1}{2}\bar{\beta}_\alpha a^{1+\alpha} + \frac{\pi S_2}{8} a + \frac{\sqrt{2S_2}}{4} a f_1(T)$$
$$a = a_0 \quad \text{at} \quad T = 0, \tag{14.35}$$

and

$$\frac{\partial p_a}{\partial T} = \frac{1}{2}\beta_0 \frac{\partial a p_a}{\partial a} + \frac{1}{2}\bar{\beta}_\alpha \frac{\partial a^{1+\alpha} p_a}{\partial a} + \frac{\pi S_2}{16}\frac{\partial}{\partial a}\left(a^3 \frac{\partial}{\partial a}\left(\frac{p_a}{a}\right)\right)$$
$$p_a = \delta(a - a_0) \quad \text{at} \quad T = 0; \tag{14.36}$$

for convenience, we left out the asterisk of T^* (hence $T = \delta t^*$). As we shall see in the next sections, the combination of linear and non-linear damping leads to a range of interesting phenomenae as $T \to \infty$.

14.3 Time-domain behavior

It appears possible to solve eq. (14.35) in closed form. The solution is given by (Brouwers [14]):

$$a = y \exp\left(\gamma \frac{\pi S_2}{8} T + \frac{\sqrt{2S_2}}{4} w(T)\right), \tag{14.37}$$

where

$$y = a_0 \quad \text{if} \quad \bar{\beta}_\alpha = 0, \tag{14.38}$$

i.e. for linear damping, and more general

$$y = \left[a_0^{-\alpha} + \frac{1}{2}\alpha\bar{\beta}_\alpha \int_0^T \exp\left(\alpha\gamma\frac{\pi S_2}{8}\tau + \alpha\frac{\sqrt{2S_2}}{4}w(\tau) \right) d\tau \right]^{-1/\alpha}, \tag{14.39}$$

i.e. for non-linear damping. In the above solutions,

$$\gamma = 1 - \frac{4\beta_0}{\pi S_2}, \tag{14.40}$$

while

$$w(T) = \int_0^T f_1(\tau)d\tau \tag{14.41}$$

is a *Wiener* or *Brownian motion process* of unit intensity. The correctness of the above solution can easily be checked by substitution in the original differential equation and boundary conditions.

Statements about the behavior of $a(T)$ as $T \to \infty$ can be made because it is known that for Brownian motion with probability 1, $|w(T)| \leq (2T \ln\ln T)^{1/2}$: Ariaratnam and Tam [15]. We shall first illustrate this for linear damping ($\bar{\beta}_\alpha = 0$). The first term in the exponent of solution (14.37), if $\gamma \neq 0$, grows linearly with T which is faster than $w(T)$. Therefore, the value of γ determines the behavior as $T \to \infty$. If $\gamma < 0$, a decays exponentially, if $\gamma > 0$, a grows exponentially. Hence, stability is ensured if $\gamma < 0$, i.e. if $\beta_0 > (\pi S_2)/4$. If the magnitude of linear damping is sufficiently large in comparison with the value of the power density at twice the natural frequency, the system is stable: any initial disturbance decays with time. Interestingly, also for deterministic sinusoidal parametric excitation, a threshold for the amount of linear damping necessary to ensure stability has been established: Verhulst [13].

We shall now investigate the case of non-linear damping. Distinction will be made between (A) $\alpha > 0$ and (B) $\alpha < 0$, as well as between (I) $\gamma > 0$ and (II) $\gamma < 0$.

(AI) $\alpha > 0$ ***and*** $\gamma > 0$

Inspection of the right hand side of eq. (14.39) shows that for $T \to \infty$, the second term in square brackets grows exponentially in magnitude in comparison with the first. The first term, which represents the effect of the initial condition, can thus be disregarded when $T \to \infty$. From eqs. (14.37) and (14.39) it then follows that

$$a \sim \left[\frac{1}{2}\alpha\bar{\beta}_\alpha \int_0^T \exp\left(\alpha\gamma\frac{\pi S_2}{8}(\tau - T) + \alpha\frac{\sqrt{2S_2}}{4}(w(\tau) - w(T))\right) d\tau\right]^{-1/\alpha}$$
$$\text{as} \quad T \to \infty. \qquad (14.42)$$

From this expression, it can be concluded that the amplitude of response remains finite with probability 1 as $T \to \infty$. Furthermore, the amplitude does not tend to zero with probability 1 as $T \to \infty$.

(AII) $\alpha > 0$ ***and*** $\gamma < 0$

In view of the negative value of γ, rather than growing exponentially, the integral on the right hand side of eq. (14.39) now approaches a constant as $T \to \infty$, with the result that

$$a \sim \left(a_0^{-\alpha} + \frac{1}{2}\alpha\bar{\beta}_\alpha C_0\right)^{-1/\alpha} \exp\left(\gamma\frac{\pi S_2}{8}T + \frac{\sqrt{2S_2}}{4}w(T)\right)$$
$$\text{as} \quad T \to \infty \qquad (14.43)$$

where

$$C_0 = \int_0^\infty \exp\left(\alpha\gamma\frac{\pi S_2}{8} + \alpha\frac{\sqrt{2S_2}}{4}w(\tau)\right) d\tau \qquad (14.44)$$

Bearing in mind that $|w(T)| \leq (2T \ln\ln T)^{1/2}$ as $T \to \infty$ with probability 1, it follows from eq. (14.43) that the amplitude of response decays exponentially to zero with probability 1 as $T \to \infty$. Furthermore, it is noted that, apart from the constant C_0, the asymptotic result of eq. (14.43) corresponds to the solution for linear damping given by eqs. (14.37) and (14.38). In fact, the right hand side of eq. (14.43) can be made a solution of the linear problem by adapting the initial value for amplitude imposed at time zero. As the value of C_0 varies with realizations of $w(T)$, the adapted initial value will also vary with realizations of $w(T)$. The asymptotic result of eq. (14.43) thus corresponds to the solution of the linear problem subject to variable initial condition at time zero.

(BI) $\alpha < 0$ ***and*** $\gamma < 0$
The integral on the right hand side of eq. (14.39) increases exponentially with time. Because of the negative value of α, for each realization there is a moment in time, T_0, defined by

$$a_0^{-\alpha} + \frac{1}{2}\alpha\bar{\beta}_\alpha \int_0^{T_0} \exp\left(\alpha\gamma\frac{\pi S_2}{8} + \alpha\frac{\sqrt{2S_2}}{4}w(\tau)\right) d\tau = 0, \tag{14.45}$$

for which y and hence the amplitude of response becomes zero. The solution given by eqs. (14.37) and (14.39) then only holds for $0 \leq T \leq T_0$.

The contribution of $a(T)$ as $T > T_0$ is as follows. For $-1 < \alpha < 0$ it can be shown that $a(T) \equiv 0$ as $T \geq 0$. This continuation satisfies eq. (14.35) and joins smoothly the solution appropriate for $T \leq T_0$. For $\alpha \leq -1$, however, we arrive at an irregularity. As can be verified from eq. (14.35) for $\alpha < 0$,

$$\frac{da}{dT} \sim -\frac{1}{2}\bar{\beta}_\alpha a^{\alpha+1} \text{ as } a \to 0 \tag{14.46}$$

from which it follows that $da/dT \neq 0$ as $a \to 0$ when $\alpha = -1$ and that $da/dT \to \infty$ as $a \to 0$ when $\alpha < -1$. Furthermore, for any positive value of a, the slope da/dT is negative. Hence, when $\alpha < 0$ and $\gamma < 0$, differential equation (14.35) tends to reduce the amplitude to zero but the value zero itself is not allowed when $\alpha \leq -1$.

The above mentioned irregularity suggests improper formulation. A description of the damping force with $\alpha < -1$ is not conceived as being realistic and will be excluded from the present analysis. For $\alpha = -1$, which represents Coulomb friction, the irregularity may be ascribed to reducing the original equation of motion (14.1) to approximate eq. (14.35). Such a reduction is justified as long as the damping force in equation (14.1) is small in comparison to the acceleration force; that is if:

$$\delta a^\alpha \ll 1. \tag{14.47}$$

For $\alpha < 0$, this condition is no longer satisfied for small values of a such that $a \sim \delta^{-1/\alpha}$. Solutions derived from eq. (14.35) can thus be expected to become inaccurate or even invalid if $a \to 0$ and $\alpha < 0$.

It can be concluded that for $-1 \leq \alpha < 0$ and $\gamma < 0$, the amplitude of response approaches in a finite time (T_0) some finite value close to zero ($\sim \delta^{1/\alpha}$).

***(BII)* $\alpha < 0$ and $\gamma > 0$**

Just as for case (AII), $\alpha > 0$ and $\gamma < 0$, the integral on the right hand side of eq. (14.39) approaches a constant as $T \to \infty$. The solution then reduces to that given by eq. (14.43), according to which response grows exponentially as $T \to \infty$. The asymptotic result corresponds to the solution for linear damping subject to variable initial condition at time zero.

It is noted, however, that the above result is only valid as long as the coefficient preceding the exponential term in solution (14.43) is positive of magnitude, i.e. if

$$-\frac{1}{2}\alpha\bar{\beta}_\alpha C_0 < a_0^{-\alpha}, \tag{14.48}$$

a condition which, in view of the negative value of α, is not satisfied when considering all possible realizations of the random process $w(T)$. For those realizations for which the above-mentioned condition is not satisfied, there is a moment in time, T_0, defined by eq. (14.45), for which the amplitude becomes zero. Solution (14.43) is then only valid if $0 \leq T \leq T_0$. The continuation of $a(T)$ as $T > T_0$ is equal to that discussed under the previous case (BI).

It can be concluded that for realizations of the excitation to the extent that condition (14.48) is satisfied, the amplitude of response increases exponentially as $T \to \infty$. On the other hand, for those realizations for which condition (14.48) is not satisfied the amplitude of response approaches in a finite time, T_0, some finite value close to zero ($\sim \delta^{1/\alpha}$).

14.4 Solutions for transient probability density and statistical moments

A general solution can be derived from partial differential equation (14.36): Brouwers [14]. To derive this solution, use is made of the *Laplace transformation* (Abramowitz & Stegun [6], chapter 29):

$$p_a(a,T_1) = \frac{1}{2\pi i}\int_{e-i\infty}^{e+i\infty} \tilde{p}_a(a,s)e^{sT_1}ds, \qquad T_1 = \frac{\pi S_2}{16}T, \tag{14.49}$$

For $\tilde{p}_a(a,s)$ we then have the ordinary differential equation:

$$s\tilde{p}_a - \delta(a-a_0) = \frac{8\bar{\beta}_\alpha}{\pi S_2}\frac{d}{da}(a^{1+\alpha}\tilde{p}_a) + \frac{d}{da}\left(-2\gamma a\tilde{p}_a + a\frac{d(a\tilde{p}_a)}{da}\right) \tag{14.50}$$

where γ is given by eq. (14.40) and δ is Dirac's delta function.

Solutions for $\tilde{p}_a(a,s)$ can be constructed analogous to the derivation of *Green's functions* (Morse and Feshbach [16]):

$$\tilde{p}_a(a,s) = \begin{cases} \tilde{p}_a^1(a,s) & \text{for } 0 \le a \le a_0 \\ \tilde{p}_a^2(a,s) & \text{for } a_0 \le a \le \infty \end{cases} \tag{14.51}$$

where $\tilde{p}_a^{1,2}(a,s)$ are solutions of the homogeneous equation

$$s\tilde{p}_a^{1,2} = \frac{8\bar{\beta}_\alpha}{\pi S_2}\frac{d}{da}(a^{1+\alpha}\tilde{p}_a^{1,2}) + \frac{d}{da}\left(-2\gamma a\tilde{p}_a^{1,2} + a\frac{d(a\tilde{p}_a^{1,2})}{da}\right) \tag{14.52}$$

As boundary conditions we require that $a\tilde{p}_a^1$ and $a\tilde{p}_a^2$ tend to zero as $a \to 0$ and $a \to \infty$, respectively. Furthermore, integrating eq. (14.50) over the interval $a_0 - \epsilon \le a \le a_0 + \epsilon$ and letting $\epsilon \to 0$, we have the additional conditions

$$\tilde{p}_a^2 = \tilde{p}_a^1 \qquad \text{at} \qquad a = a_0$$

$$\frac{d\tilde{p}_a^2}{da} = \frac{d\tilde{p}_a^1}{da} - a_0^{-2} \qquad \text{at} \qquad a = a_0 \tag{14.53}$$

For further analysis, it is useful to make distinction between the cases (A) $\alpha > 0$ and (B) $\alpha < 0$.

(A) $\alpha > 0$

Upon substitution of the variable η, defined by

$$\eta = \frac{8\bar{\beta}_\alpha}{\pi S_2 \alpha}a^\alpha \tag{14.54}$$

and

$$\tilde{p}_a^{1,2} = a^{\gamma-1}\eta^{-1/2}e^{-\eta/2}f(\eta), \tag{14.55}$$

eq. (14.52) reduces to

$$\frac{d^2f}{d\eta^2} + \left(-\frac{1}{4} + \frac{\kappa}{\eta} + \frac{1/4 - \mu^2}{\eta^2}\right)f = 0, \tag{14.56}$$

where

$$\mu = \frac{(s+\gamma^2)^{1/2}}{\alpha} \qquad ; \qquad \kappa = \frac{1}{2} + \frac{\gamma}{\alpha} \tag{14.57}$$

Eq. (14.56) is known as *confluent hypergeometric equation*, the two basic solutions of which are given by *Whittaker's functions* $W_{\kappa,\mu}(\eta)$ and $M_{\kappa,\mu}(\eta)$: Abramowitz & Stegun [6], chapter 13; see also [17]. Application of the boundary conditions then yields the solution,

$$\tilde{p}_a^1(a,s) = \frac{\Gamma(\mu - \gamma/\alpha)}{\alpha a \eta_0 \Gamma(1+2\mu)} \left(\frac{\eta}{\eta_0}\right)^{\gamma/\alpha - 1/2} e^{-\eta/2+\eta_0/2} W_{\kappa,\mu}(\eta_0) M_{\kappa,\mu}(\eta) \quad (14.58)$$

$$\tilde{p}_a^2(a,s) = \frac{\Gamma(\mu - \gamma/\alpha)}{\alpha a \eta_0 \Gamma(1+2\mu)} \left(\frac{\eta}{\eta_0}\right)^{\gamma/\alpha - 1/2} e^{-\eta/2+\eta_0/2} W_{\kappa,\mu}(\eta) M_{\kappa,\mu}(\eta_0) \quad (14.59)$$

where Γ is the gamma function [6] and η_0 follows from eq. (14.54) taking $a = a_0$.

The problem that now remains is to re-transform the description given above from the s-domain to the T-domain using eq. (14.49). In this connection it should be noted that the function $\tilde{p}_a(a,s)$ as defined by eqs. (14.51), (14.58) and (14.59) is a many-valued function of s with its branch line along $-\infty \leq s \leq -\gamma^2$: see figure 14.3. Furthermore, when $\gamma > 0$, $\tilde{p}_a(a,s)$ has simple poles at $s = n\alpha(n\alpha - 2\gamma)$, where $n = 0, 1, 2, ..., j$ with $j < \gamma/\alpha$.

Laplace integral (14.49) is now evaluated using Cauchy's theorem

$$\frac{1}{2\pi i}\int \tilde{p}_a(a,s)e^{sT_1}ds$$
$$= \sum \text{residues of } \tilde{p}_a(a,s)e^{sT_1} \text{ at the poles within the contour of integration.}$$

As contour of integration, we have chosen the path $ABCDEFA$ shown in figure 14.3 when $\gamma > 0$, and the path $AGOHFA$ when $\gamma < 0$. Here, the paths $BCDE$ and GOH are defined by the equations $(s+\gamma^2)^{1/2} = ir$ and $(s+\gamma^2)^{1/2} = ir + |\gamma|$, respectively, where r is real and $-\infty \leq r \leq \infty$. The resulting expressions for the probability density of amplitude are given below.

(AI) $\alpha > 0$ ***and*** $\gamma > 0$

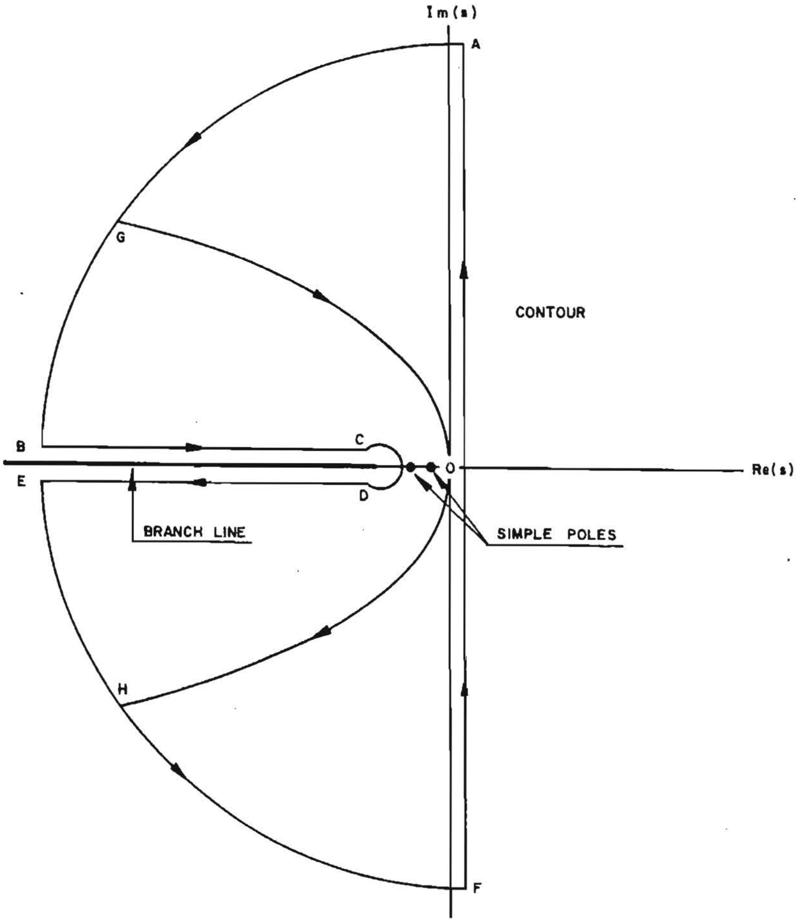


Figure 14.3. Branch line, poles and contours of integration in the complex plane.

In this case

$$\begin{aligned} p_a(a(\eta), T_1) &= 2a^{-1}\eta^{2\gamma/\alpha}e^{-\eta}\sum_{n=0}^{<\gamma/\alpha}\frac{(\gamma-n\alpha)n!(\eta\eta_0)^{-n}}{\Gamma(2\gamma/\alpha+1-n)} \\ &\qquad L_n^{(2\gamma/\alpha-2n)}(\eta)L_n^{(2\gamma/\alpha-2n)}(\eta_0)e^{-n\alpha(2\gamma-n\alpha)T_1} \\ &\quad+\frac{\alpha\eta}{2\pi a\eta^{\kappa}}e^{\eta_0/2-\eta/2}\int_0^{\infty}\frac{|\Gamma(ir-\gamma/\alpha)|^2}{|\Gamma(2ir)|^2} \\ &\qquad W_{\kappa,ir}(\eta)W_{\kappa,ir}(\eta_0)e^{-\gamma^2T_1-\alpha^2r^2T_1}dr, \end{aligned} \tag{14.60}$$

where $i = \sqrt{-1}$. The summation in the series term is up to γ/α (but excluding $n = \gamma/\alpha$). Furthermore, $L_n^{(2\gamma/\alpha-2)}(\eta)$ is Laguerre's polynomial and $W_{\kappa,ir}(\eta)$ is Whittaker's function [6].

According to eq. (14.60), the solution for the probability density consists of a finite series and an integral. The finite series corresponds to a discrete and finite number of eigenvalues associated with Fokker-Planck equation (14.36), whereas the integral corresponds to a continuous spectrum of eigenvalues. Such a hybrid situation is not special; it has also been encountered in a number of solutions of the *Schrödinger equation* in quantum mechanics [16].

From the series term in solution (14.60), it is noted that the first term ($n = 0$) describes a stationary probability density distribution. The other terms of the series, if present, decay exponentially with T_1 as $T_1 \to \infty$. Furthermore, using expansions of Whittaker's functions for $r \to 0$ [6, 17], the integral in solution (14.60) can be shown to decay as $T_1^{1/2}e^{-\gamma^2 T_1}$ when $T_1 \to \infty$. Solution (14.60) thus describes a transient probability density which decays exponentially with time to a stationary form given by

$$p_a(a, T_1) \sim \frac{\alpha}{\Gamma(2\gamma/\alpha)} \left(\frac{8\bar{\beta}_\alpha}{\pi S_2 \alpha}\right)^{2\gamma/\alpha} a^{2\gamma-1} \exp\left(-\frac{8\bar{\beta}_\alpha a^\alpha}{\pi S_2 \alpha}\right) \quad \text{as } T_1 \to \infty \qquad (14.61)$$

Statistical moments of amplitude, defined by

$$E[a^k] = \int_0^\infty a^k p_a(a, T_1)da, \qquad k \geq 0, \qquad (14.62)$$

can be shown to approach a constant as $T_1 \to \infty$. The steady-state value is equal to the value obtained when substituting the result of eq. (14.61) into the right hand side of eq. (14.62).

(AII) $\alpha > 0$ ***and*** $\gamma < 0$

Using the variables and parameters η, T_1 and κ, the probability density of amplitude can be described as

$$p_a(a(\eta), T_1) = \frac{(\eta/\eta_0)^\kappa}{2\pi a \eta} e^{\eta_0/2-\eta/2} \int_{-\infty}^{+\infty} \frac{\Gamma(-2\gamma/\alpha + ir/\alpha)}{\Gamma(-2\gamma/\alpha + 2ir/\alpha)} H(\eta, r) e^{-r^2 T_1 - 2\gamma i r T_1} dr, \qquad (14.63)$$

where

$$H(\eta, r) = \begin{cases} M_{\kappa,\mu}(\eta) W_{\kappa,\mu}(\eta_0) & \text{for } \eta \leq \eta_0 \\ M_{\kappa,\mu}(\eta_0) W_{\kappa,\mu}(\eta) & \text{for } \eta \geq \eta_0 \end{cases} \qquad (14.64)$$

with

$$\mu = ir/\alpha - \gamma/\alpha. \tag{14.65}$$

Applying *method of steepest descent* to the right hand side of eq. (14.63), it is possible to derive an asymptotic expression for the probability density which is valid for $T_1 \longrightarrow \infty$: see 14.6 Appendix. The asymptotic expression can be written as

$$p_a(a, T_1) \sim \frac{1}{2a(\pi T_1)^{1/2}} \int_0^\infty G(r, a_0) e^{-(2\gamma T_1 - \ln(a/r))^2/(4T_1)} dr \text{ as } T_1 \longrightarrow \infty, \tag{14.66}$$

where

$$G(r, a_0) = \frac{i\alpha\eta_0}{4\pi r} \left(\frac{a_0}{r}\right)^\alpha e^{(1/2)\eta_0} \int_C \frac{(2t_1)^{2+2\gamma/\alpha}(1+4t_1^2(a_0/r)^\alpha)^{-1/2}}{[-1+(1+4t_1^2(a_0/r)^\alpha)^{1/2}]^{1+2\gamma/\alpha}} e^{-\eta_0 t_1 - (1/2)\eta_0(1+4t_1^2(a_0/r)^\alpha)^{1/2}} dt_1. \tag{14.67}$$

The path of integration C in eq. (14.67) starts at $+\infty$ on the real axis, circles the origin in the counterclockwise direction and returns to the starting point.

The asymptotic result given by eq. (14.66) is equal to the solution for linear damping, i.e. the solution of eq. (14.36) with $\bar{\beta}_\alpha = 0$ subject to the initial condition that $p_a(a, T_1) = G(a_0, a)$ at time zero. It represents the probabilistic description of an exponentially decaying amplitude. The result is in line with the asymptotic expression for time-domain solutions given by eqs. (14.43) and (14.44).

For the kth moment of the amplitude, defined by eq. (14.62), one can derive the asymptotic expressions

$$E[a^k] \sim \frac{a_0^k \Gamma^2(-\gamma/\alpha) e^{\eta_0/2}}{2\pi^{1/2}\alpha^3 \eta_0^{\kappa+k/\alpha}} W_{\kappa,0}(\eta_0) \frac{e^{-\gamma^2 T_1}}{T_1^{3/2}} \int_0^\infty \eta^{-3/2+\kappa/\alpha+\gamma/\alpha} e^{-\eta/2} W_{\kappa,0}(\eta) d\eta$$
$$\text{as } T_1 \to \infty \text{ if } k > -\gamma \tag{14.68}$$

and

$$E[a^k] \sim \frac{a_0^k \Gamma(-2\gamma/\alpha - k/\alpha) e^{\eta_0/2}}{\Gamma(-2k/\alpha - 2\gamma/\alpha)\eta_0^{\kappa+k/\alpha}} W_{\kappa,-\gamma/\alpha-k/\alpha}(\eta_0) e^{k(k+2\gamma)T_1}$$
$$\text{as } T_1 \to \infty \text{ if } k < -\gamma. \tag{14.69}$$

This shows that any statistical moment, except for the zero's, decays exponentially to zero as time increases. The result presented in eq. (14.69) is in line with that for pure linear damping where the kth moment is found to be proportional to $\exp(k(k+2\gamma)T_1)$ as $T_1 \to \infty$ when $k > 0$ [15]. The result given in eq. (14.68) indicates that the presence of a non-linear term in the damping force which increases more than linearly with velocity causes statistical moments with $k > -2\gamma$ also to decay to zero (as opposed to the result for pure linear damping).

(B) $\alpha < 0$

A convenient transformation of variables is one involving q, defined by

$$q = -\frac{8\bar{\beta}_\alpha}{\pi S_2 \alpha} a^2 \tag{14.70}$$

and

$$\bar{p}_a^{1,2} = a^{\gamma-1} q^{-1/2} e^{q/2} f(q). \tag{14.71}$$

Eq. (14.52) then reduces to the confluent hypergeometric equation

$$\frac{d^2 f}{dq^2} + \left(-\frac{1}{4} + \frac{\kappa_1}{q} + \frac{1/4 - \mu_1^2}{q^2} \right) f = 0, \tag{14.72}$$

where

$$\kappa_1 = \gamma/\alpha_1 - 1/2 \quad , \quad \mu_1 = (s+\gamma^2)^{1/2}/\alpha_1 \quad , \quad \alpha_1 = -\alpha \tag{14.73}$$

Applying conditions (14.53) and the boundary conditions imposed at $a = 0$ and $a = \infty$, we obtain the solutions

$$\bar{p}_a^1(a,s) = \frac{\Gamma(1-\gamma/\alpha_1+\mu_1)}{\alpha_1 a q_0 \Gamma(1+2\mu_1)} (q/q_0)^{-\gamma/\alpha_1 - 1/2} e^{q/2 - q_0/2} W_{\kappa_1,\mu_1}(q_0) M_{\kappa_1,\mu_1}(q), \tag{14.74}$$

$$\bar{p}_a^2(a,s) = \frac{\Gamma(1-\gamma/\alpha_1+\mu_1)}{\alpha_1 a q_0 \Gamma(1+2\mu_1)} (q/q_0)^{-\gamma/\alpha_1 - 1/2} e^{q/2 - q_0/2} W_{\kappa_1,\mu_1}(q) M_{\kappa_1,\mu_1}(q_0). \tag{14.75}$$

It is noted that $\bar{p}_a^{1,2}(a,s)$ is a many-valued function of s with its branch line along $-\infty \le s \le -\gamma^2$: see figure 14.3. Furthermore, when $\gamma > \alpha_1$, $\bar{p}_a^{1,2}(a,s)$ has simple poles at $s = n\alpha_1(n\alpha_1 - 2\gamma)$, where $n = 1, 2, 3, ..., j$ with $j < \gamma/\alpha_1$. Note that there is no pole at $s = 0$!

To re-transform the descriptions given above into the time-domain using Cauchy's theorem, the contours discussed under the case (A) above and shown in figure 14.3 have been used again: contour $ABCDEFA$ for $\gamma < 0$ and contour $AGOHFA$ for $\gamma > 0$. The resulting expressions for the probability density of amplitude are given below.

(BI) $\alpha > 0$ ***and*** $\gamma > 0$

In terms of the variables q and T_1, the probability density of amplitude can be expressed as

$$\begin{aligned} p_a(a(q), T_1) &= \frac{\alpha_1 (q_0/q)^{\kappa_1}}{2\pi a q} e^{q/2 - q_0/2} \int_0^\infty \frac{|\Gamma(1-\gamma/\alpha_1 + ir)|^2}{|\Gamma(2ir)|^2} \\ &\qquad W_{\kappa_1, ir}(q) W_{\kappa_1, ir}(q_0) e^{-(r^2\alpha^2+\gamma^2)T_1} dr. \end{aligned} \tag{14.76}$$

From solution (14.76) it can be verified that

$$\begin{aligned} p_a(a(q), T_1) &\sim \frac{(q_0/q)^{\kappa}}{2\pi^{1/2}\alpha_1^2 a q} e^{q/2 - q_0/2} \Gamma^2(1-\gamma/\alpha_1) W_{\kappa_1,0}(\eta) W_{\kappa_1,0}(\eta_0) T_1^{-3/2} e^{-\gamma^2 T_1} \\ &\quad \text{as} \quad T_1 \to \infty. \end{aligned} \tag{14.77}$$

According to this result, the probability density decreases to zero as $T_1 \to \infty$ so that $\int_0^\infty p_1(a, T_1) da \to 0$. Apparently, probability 'escapes' as time increases. This may be explained by the time-domain solutions given in the previous section. According to these solutions, for $\alpha < 0$ and $\gamma > 0$, any realization of $a(T)$ approaches zero (or almost zero) in a finite time. The above-mentioned probability distribution now describes this process except for the behavior at $a = 0$ itself. The phenomenon of 'escape of probability' is not unfamiliar for stochastic processes where variables reach, in a finite time, some deterministic value [18].

The kth order moment of response can be shown to be

$$\begin{aligned} E[a^k] &\sim \frac{a_0^k \Gamma^2(1-\gamma/\alpha_1)}{2\pi^{1/2} q_0^2 \alpha_1^3} W_{\kappa_1,0}(q_0) e^{-q_0/2} \frac{e^{-\gamma^2 T_1}}{T_1^{3/2}} \\ &\quad \int_0^\infty (q/q_0)^{-\gamma/\alpha_1 - k/\alpha_1 - 3/2} W_{\kappa_1,0}(q) e^{q/2} dq \\ &\quad \text{as} \quad T_1 \to \infty \qquad \text{if} \quad k < -\gamma \end{aligned} \tag{14.78}$$

and

$$E[a^k] \sim \frac{a_0^k \Gamma(1+k/\alpha_1) e^{-q_0/2}}{\Gamma(2k/\alpha_1 + 2\gamma/\alpha_1)} W_{\kappa_1, k/\alpha_1+\gamma/\alpha_1}(q_0) e^{k(k+2\gamma)T_1}$$
$$\text{as} \quad T_1 \to \infty \quad \text{if} \quad k > -\gamma \qquad (14.79)$$

From these expressions, it is seen that the kth moment decreases exponentially to zero whenever $k < -2\gamma$. The kth moment, however, grows exponentially if $k > -2\gamma$. A similar conclusion can be drawn for pure linear damping where the kth moment is found to be proportional to $\exp(k(k+2\gamma)T_1)$ as $T_1 \to \infty$ [15]. In other words, provided that $\gamma < 0$, the presence of a non-linear term in the damping force which increases less than linearly with velocity ($\alpha < 0$) does not alter, in a qualitative sense, the conclusions with regard to the behavior of $E[a^k]$ as $T_1 \to \infty$ obtained for pure linear damping.

(BII) $\alpha < 0$ ***and*** $\gamma > 0$

Using the variables and parameters q, T_1, κ_1 and α_1, the probability density of amplitude can be described as

$$p_a(a(q), T_1) = \frac{(q_0/q)^{\kappa_1}}{2\pi a q} e^{q/2 - q_0/2} \int_{-\infty}^{+\infty} \frac{\Gamma(1+ir/\alpha_1)}{\Gamma(2\gamma/\alpha_1 + 2ir/\alpha_1)} H(q,r) e^{-r^2 T_1 - 2\gamma i r T_1} dr, \qquad (14.80)$$

where

$$H(q,r) = \begin{cases} M_{\kappa_1,\mu_1}(q) W_{\kappa_1,\mu_1}(q_0) & \text{for } q \le q_0 \\ M_{\kappa_1,\mu_1}(q_0) W_{\kappa_1,\mu_1}(q) & \text{for } q \ge q_0 \end{cases} \qquad (14.81)$$

with

$$\mu_1 = ir/\alpha_1 + \gamma/\alpha_1. \qquad (14.82)$$

Applying methods of steepest descent, it can be verified from solution (14.80) that the asymptotic expression for $p_a(a, T_1)$ valid as $T_1 \to \infty$ is equal to that given in eq. (14.66) but with $G(r, a_0)$ defined as

$$G(a_0, r) = \frac{i\alpha q_0}{4\pi r} \left(\frac{a_0}{r}\right)^{\alpha} e^{(1/2)q_0} \int_C \frac{(2t_1)^{2+2\gamma/\alpha} (1 + 4t_1^2 (a_0/r)^\alpha)^{-1/2}}{[-1 + (1 + 4t_1^2(a_0/r)^\alpha)^{1/2}]^{1+2\gamma/\alpha}}$$
$$e^{-q_0 t_1 - (1/2) q_0 (1+4t_1^2(a_0/r)^\alpha)^{1/2}} dt_1. \qquad (14.83)$$

The path of integration C is equal to that described for the integral in eq. (14.67).

As discussed before, the asymptotic result given by (14.66) corresponds to the solution for linear damping, subject to the initial condition that $p_a(a, T_1) = G(a_0, a)$ at time zero, but with $G(a_0, a)$ defined by eq. (14.83) above. In view of the positive value of γ in the exponent of eq. (14.66), the result represents the probabilistic description of an exponentially increasing amplitude. The result is in line with the asymptotic expressions for time-domain behavior of amplitude as given by eqs. (14.43) and (14.44) in the previous section.

As we have seen in the previous section, however, apart from growing exponentially with time when $\alpha < 0$ and $\gamma > 0$, there is also a chance of the amplitude approaching zero or some value close to zero as $T_1 \to \infty$. Indeed, if we integrate the right hand side of eq. (14.66) over the entire a range and take account of the result given in eq. (14.83), we find

$$\int_0^\infty p_a(a, T_1)da \sim 1 - P(2\gamma/\alpha_1, q_0) \text{ as } T_1 \to \infty \tag{14.84}$$

where P is an *incomplete Gamma function* [5]. Noting that $0 < P < 1$, it follows from eq. (14.84) that the integral is less than unity, which implies that the solution given by eqs. (14.66) and (14.83) does not describe the entire probability density of amplitude. As found under case (BI) above, the given probability density distributions do not describe the behavior of amplitude at $a = 0$.

The kth moment, $k \geq 0$, associated with the above-mentioned probability distribution can be shown to be

$$E[a^k] \sim \frac{a_0^k \Gamma(1 + k/\alpha_1)}{\Gamma(2k/\alpha_1 + 2\gamma/\alpha_1)} e^{-q_0/2} q_0^{\kappa_1 + k/\alpha_1} W_{\kappa_1, k/\alpha_1 + \gamma/\alpha_1}(q_0) e^{k(k+2\gamma)T_1} \tag{14.85}$$

$$\text{as } T_1 \to \infty$$

According to this description, any statistical moment, except from the zero's, grows exponentially as $T_1 \to \infty$.

14.5 Conclusions

Analytical descriptions have been derived for the non-stationary response of a randomly parametrically excited second-order system. Damping consisted of a combination of linear and non-linear power-law damping. Response was initiated by some disturbance at time zero.

The description of the response consists of a sinusoidal wave with frequency equal to the undamped natural frequency of the system. Amplitude and phase of the sinusoidal wave vary slowly, instationary and randomly with time. Amplitude and phase are described by fluctuation equations with the time as independent variable and by Fokker-Planck equations for the probability density of amplitude and phase. These descriptions hold asymptotically in the limit of small damping and small parametric excitation. Fluctuation equation and Fokker-Planck equation for amplitude have subsequently been solved in closed-form using methods of mathematical physics involving higher transcendental functions. Although lengthy and complicated, the procedure was worthwhile performing: it led to concrete results enabling clear and unequivocal conclusions to be drawn on stability of the system. The designer can use this information to design safe structures. Results derived for the behavior of time-domain realizations of amplitude, of probability distributions of amplitude and of statistical moments of amplitude, as time increases, have been summarized in table 14.1.

The asymptotic results for time domain realizations and for probability distributions as time tends to infinity are complementary to each other. From these results, three types of response can be identified.

(a) When the linear part of the total damping force is sufficiently large in comparison to the parametric excitation force, i.e. when $\gamma < 0$ or $\beta_0 > \pi S_2/4$, response decays to zero as time increases. In other words, there is no apparent effect due to parametric excitation when $\gamma < 0$ or $\beta_0 > \pi S_2/4$.

(b) For small linear part of damping force or large parametric excitation to the extent that $\gamma > 0$ or $\beta_0 < \pi S_2/4$, on the other hand, every time-domain realization of response remains finite only if the non-linear part of the damping force increases more than linearly with velocity ($\alpha > 0$). Response then tends with time to a stationary state of random finite-amplitude response. The magnitude of this 'limit-cycle' response type is independent of the initial disturbances imposed at time zero but is strongly determined by the magnitude of the parametric excitation force and the non-linear part of the damping force.

(c) For small linear part of damping force in combination with a non-linear part of the damping force which increases less than linearly with velocity, i.e. for $\gamma > 0$ and $\alpha < 0$, however, time-domain realizations of amplitudes may either tend in a finite time to zero (or almost zero) or grow exponentially. Either possibility has a finite chance of occurrence.

Parametric excitation can be minimized by appropriately designing the natural frequency such that S_2 is small and $\gamma < 0$, i.e. $S_2 < 4\beta_0/\pi$, where β_0 is the coefficient of linear damping. In many applications of mechanical engineering, linear damping is provided by a fluid surrounding the vibrating structure. But the Reynolds numbers of the fluid flow are generally large, so that linear damping must come from boundary layer behavior. The magnitude of this damping is generally small [19]. So it may well be that $S_2 > 4\beta_0/\pi$. In that case, a type of damping that is stronger than linear damping is necessary to ensure that response does not grow unboundedly. Such damping is provided by fluid drag occurring on bluff bodies moving over distances comparable to their own cross-sectional dimension. It leads to quadratic damping and the resulting response is of a stationary type as described under case (b) above. The magnitude of the quadratic damping can be enhanced by mounting plates to the vibrating structure in a direction perpendicular to its motion. Examples are bilge keels on ships and small diameter braces on floating marine structures. They can generate the necessary quadratic damping even at relatively small excursions of the structure. The solutions given under case $\gamma > 0$ and $\alpha > 0$ can be used to quantify this response. Parameters as fatigue damage and extreme response can be calculated as a function of design variables.

Table 14.1. Behavior of time-domain realizations of amplitude $a(T)$, of probability distributions of amplitude $p_a(a,T)$ and of statistical moments of amplitude $E[a^k]$ as time increases.

		$a(T)$ as $T \to \infty$	$p_a(a,T)$ as $T \to \infty$	$E[a^k]$ as $T \to \infty$ with $k > 0$
	$\alpha > 0$	Decays exponentially to zero	Describes an amplitude which tends to zero	Decays exponentially to zero
$\gamma < 0$	$\bar{\beta}_\alpha = 0$	Decays exponentially to zero	Describes an amplitude which tends to zero	Decays to zero if $k > -2\gamma$; grows exponentially if $k < -2\gamma$
	$\alpha < 0$	Decays in a finite time to zero (or almost zero)	Vanishes and does not describe the behavior at $a = 0$	Decays to zero if $k < -2\gamma$; grows exponentially if $k > -2\gamma$
	$\alpha > 0$	Neither grows unboundedly nor decays to zero	Becomes a stationary distribution	Becomes a constant
$\gamma > 0$	$\bar{\beta}_\alpha = 0$	Grows exponentially	Describes an amplitude which tends to ∞	Grows exponentially
	$\alpha < 0$	For certain realizations exponential increase; for others decay in a finite time to zero or almost zero	Describes an amplitude which tends to ∞; does not describe the behavior at $a = 0$	Grows exponentially

14.6 Appendix: Derivation of asymptotic expressions for probability density and statistical moments

Procedures are described for deriving expressions for probability density and statistical moments of amplitude which are asymptotically valid as $T_1 \to \infty$. Use will be made of techniques to compute the asymptotic behavior of integrals such as the method of steepest descent or *saddle-point method*: e.g. see Morse and Feshbach [16], §4.6 and De Bruin [20]. Results presented apply to the case where $\alpha > 0$ and $\gamma < 0$, as presented in section 14.4. Similar procedures can be adopted to derive asymptotic results valid for the other parameter ranges of α and γ.

Probability density as $T_1 \to \infty$

From the time-domain solutions, it is known that for $\alpha > 0$ and $\gamma < 0$ the amplitude of response decays exponentially to zero as $T_1 \to \infty$. Accordingly, when studying the behavior as $T_1 \to \infty$, our main interest concerns the behavior of $p_a(a(\eta), T_1)$ given by eqs. (14.63) and (14.64) at $\eta \ll 1$. Using the asymptotic expression [5]:

$$M_{\kappa,\mu} \sim \eta^{1/2+ir/\alpha-\gamma/\alpha}\left(1+\frac{ir-2\gamma}{\alpha-2\gamma+2ir}\eta+\ldots\right) \qquad \text{as } \eta \to 0, \tag{14.86}$$

solution (14.63) reduces to

$$p_a(a(\eta), T_1) \sim \frac{e^{\eta_0/2}}{2\pi a \eta_0^{1/2+\gamma/\alpha}} \int_{-\infty}^{+\infty} \frac{\Gamma(-2\gamma/\alpha+ir/\alpha)}{\Gamma(-2\gamma/\alpha+2ir/\alpha)} \eta^{ir/\alpha} W_{\kappa,\mu}(\eta_0) \, e^{-r^2T_1-2\gamma irT_1} dr \qquad \text{as } \eta \to 0 \tag{14.87}$$

Furthermore, use will be made of *Hankel's Contour Integral* [5]

$$\Gamma^{-1}(-2\gamma/\alpha+2ir/\alpha) = \frac{i}{2\pi}\int_C (-t_1)^{2\gamma/\alpha-2ir/\alpha} e^{-t_1} dt_1, \tag{14.88}$$

where the contour C is described in the discussion of eq. (14.67), and the integral representation for Whittaker's function [5]

$$W_{\kappa,\mu}(\eta_0) = \frac{e^{-\eta_0/2}\eta_0^{1/2+\gamma/\alpha}}{\Gamma(ir/\alpha-2\gamma/\alpha)} \int_0^{\infty} e^{-t_2}(1+t_2/\eta_0)^{ir/\alpha} t_2^{ir/\alpha-2\gamma/\alpha-1} dt_2. \tag{14.89}$$

Upon substituting the above relations into the right hand side of eq. (14.87) and introducing the coordinate transformation

$$r = -i\left[\gamma - \frac{1}{2\alpha T_1}\ln\left(\eta\left(1 + t_2/\eta_0\right)t_2/t_1^2\right)\right] + T_1^{-1/2}\nu, \tag{14.90}$$

we obtain, after some algebraic manipulation, the expression

$$p_a(a(\eta), T_1) \sim \frac{i}{(2\pi)^2 a T_1^{1/2}}\int_C\int_{t_2=0}^{\infty}(-t_1)^{2\gamma/\alpha}t_2^{-1-2\gamma/\alpha}e^{-t_1-t_2-F^2(t_1,t_2)}dt_2 t_1$$
$$\int_{\nu=-\infty}^{+\infty} e^{-v^2}dv \qquad \text{as} \quad \eta \to 0, \tag{14.91}$$

where

$$F(t_1, t_2) = \gamma T_1^{1/2} - \frac{1}{2\alpha T_1^{1/2}}\ln\left(\eta\left(1 + t_2/\eta_0\right)t_2/t_1^2\right). \tag{14.92}$$

It is noted that the right hand side of eq. (14.91) will reach a maximum value for values of η where $|F(t_1, t_2)|$ is of unit order of magnitude. These values of η define the location and extent of the main part of the probability density function. From eq. (14.92), it can then be verified that the main part of the probability density function shifts exponentially to zero as $T_1 \to \infty$. The same result would have been obtained if we had considered the complete description of the probability density, as done in the next section, rather than the simplified expression valid for small values of η only, as given by eq. (14.87).

As the main part of the probability density shifts exponentially to zero as $T_1 \to \infty$, we can replace the limit $\eta \to 0$ on the right hand side of eq. (14.91) by $T_1 \to \infty$. Furthermore,

$$\int_{\nu=-\infty}^{+\infty} e^{-\nu^2}d\nu = \pi^{1/2}. \tag{14.93}$$

Applying the coordinate transformations

$$t_1 = \eta_0 t_1^* \quad , \quad t_2 = -\frac{1}{2}\eta_0 + \frac{1}{2}\eta_0(1 + 4t_1^{*2}(a_0/r)^{\alpha})^{1/2}, \tag{14.94}$$

and dropping the asterisk of t_1^* then enables the asymptotic result presented by eqs. (14.66) and (14.67) to be derived from eqs. (14.91) and (14.92).

Statistical moments as $T_1 \to \infty$

Making use of the integral representations for the Whittaker functions given by eq. (14.89) and [5]

$$M_\kappa(\eta) = \frac{e^{\eta/2}\eta^{1/2+ir/\alpha-\gamma/\alpha}\Gamma(1+2ir/\alpha-2\gamma/\alpha)}{\Gamma(ir/\alpha-2\gamma/\alpha)\Gamma(1+ir/\alpha)} \int_{t_1=0}^{1} e^{-\eta t_1} t_1^{ir/\alpha}(1-t_1)^{ir/\alpha-2\gamma/\alpha-1}dt_1 \quad (14.95)$$

substitution of the expression for the probability density of amplitude given by eqs. (14.63) and (14.64) into eq. (14.62) yields

$$E[a^k] = \frac{a_0^k e^{\eta_0/2}}{2\pi\alpha\eta_0^{k/\alpha+\kappa}}(I_1+I_2), \quad (14.96)$$

where

$$I_1 = \int_{r=-\infty}^{+\infty}\int_{\eta=0}^{\eta_0}\int_{t_1=0}^{1} \frac{2(ir-\gamma)W_{\kappa,\mu}}{\alpha\Gamma(1+ir/\alpha)}\eta^{k/\alpha-1+ir/\alpha}t_1^{ir/\alpha} (1-t_1)^{-1-2\gamma/\alpha+ir/\alpha}e^{-\eta t_1-r^2T_1-2\gamma irT_1}dt_1 d\eta dt, \quad (14.97)$$

and

$$I_2 = \int_{r=-\infty}^{+\infty}\int_{\eta=\eta_0}^{\infty}\int_{t_2=0}^{\infty} \frac{M_{\kappa,\mu}(\eta_0)\eta^{\gamma/\alpha+k/\alpha-1}}{T(-2\gamma/\alpha+2ir/\alpha)}t_2^{-2\gamma/\alpha-1+ir/\alpha} (1+t_2/\eta)^{ir/\alpha}e^{-t_2-\eta-r^2T_1-2\gamma irT_1}dt_2 d\eta dr. \quad (14.98)$$

Upon applying the coordinate transformation

$$r = i\left(-\gamma+\frac{1}{2\alpha T_1}\ln(\eta t_1(1-t_1))\right)+\frac{\nu}{T_1^{1/2}} \quad (14.99)$$

eq. (14.97) becomes

$$I_1 = \frac{2e^{k(k+2\gamma)T_1}}{\alpha T_1^{1/2}}\int_{\eta=0}^{\eta_0}\int_{\nu=-\infty}^{+\infty}\int_{t_1=0}^{1}\frac{(ir-\gamma)W_{\kappa,\mu}(\eta_0)}{\Gamma(1+ir/\alpha)} (1-t_1)^{-1-2\gamma/\alpha-k/\alpha}t_1^{-k/\alpha}e^{-\eta t_1-F_1^2-\nu^2}dt_1 d\nu\frac{d\eta}{\eta} \quad (14.100)$$

where

$$F_1 = (\gamma+k)T_1^{1/2}-\frac{1}{2\alpha T_1^{1/2}}\ln(\eta t_1(1-t_1)). \quad (14.101)$$

Furthermore, introducing the transformations

$$r = i\left(-\gamma + \frac{1}{2\alpha T_1}\ln\left(t_2(1+t_2/\eta)\right)\right) + \frac{\nu}{T_1^{1/2}} \tag{14.102}$$

and

$$t_2 = -\eta + \eta t_1 \tag{14.103}$$

the integral I_2 becomes

$$\begin{aligned} I_2 \quad = \quad & \frac{e^{-\gamma^2 T_1}}{T_1^{1/2}} \int_{\eta=\eta_0}^{\infty} \int_{\nu=-\infty}^{+\infty} \int_{t_1=1}^{\infty} \frac{M_{\kappa,\mu}(\eta_0) t_1^{\gamma/\alpha}}{\Gamma(-2\gamma/\alpha + 2ir/\alpha)} (t_1-1)^{-\gamma/\alpha-1} \eta^{k/\alpha-1} \\ & \exp\left(-\nu^2 - \eta t_1 - \frac{1}{4\alpha^2 T_1}\ln^2(t_1(t_1-1)\eta)\right) dt_1 d\nu d\eta. \end{aligned} \tag{14.104}$$

To derive asymptotic expressions valid as $T_1 \to \infty$ from the descriptions of I_1 and I_2 given by eqs. (14.100) and (14.104), distinction is made between the cases $k < -\gamma$ and $k > -\gamma$.

$k < -\gamma$

The contribution to integration with respect to r on the right hand side of eq. (14.100) will be largest for values of η where $|F_1|$ is of unit order of magnitude. From eq. (14.101) it then follows that, provided that $k < -\gamma$, as $T_1 \to \infty$, the contribution to the integral stems predominantly from values of η close to zero. Furthermore, taking $|F_1| = \mathcal{O}(1)$, r as defined by eq. (14.99), reduces to ik as $T_1 \to \infty$. Using these properties and eq. (14.93) we can write

$$\begin{aligned} I_1 \quad \sim \quad & -\frac{2(k+\gamma)\pi^{1/2} e^{k(k+2\gamma)T_1}}{\alpha T_1^{1/2}} W_{\kappa,-k/\alpha-\gamma/\alpha}(\eta_0) \\ & \int_{\eta=0}^{\infty} \int_{t_1=0}^{1} (1-t_1)^{-1-2\gamma/\alpha-k/\alpha} t_1^{-k/\alpha} e^{-F_1^2} dt_1 \frac{d\eta}{\eta} \\ & \text{as} \quad T_1 \to \infty. \end{aligned} \tag{14.105}$$

The integrals on the right hand side of this equation can be evaluated by changing from the coordinate η to the coordinate F_1 defined by eq. (14.101). Integration with respect to t_1 subsequently reduces to a Beta Function [5]. Noting that I_2 decays to zero as $T_1^{-1/2} e^{-\gamma^2 T_1}$, it then follows from eqs. (14.96) and (14.105) that $E[a^k]$ approaches zero as $T_1 \to \infty$ in the manner described by eq. (14.69). This result would also have been obtained when the asymptotic result for the probability density given by eqs. (14.66) and (14.67) would have been substituted into the right hand side of eq. (14.62).

$k > -\gamma$

Inspection of the integrals on the right hand sides of eqs. (14.100) and (14.104) shows that for $k > -\gamma$, there is no local area of η-values where the contribution to integration with respect to r is dominant. Accordingly, all values of η between 0 and ∞ have to be considered. An asymptotic approximation can then be derived by applying the transformation

$$r = -i\gamma + \frac{v}{T_1^{1/2}} \tag{14.106}$$

to the integral on the right hand side of eq. (14.63). Making use of the relations

$$\mu = \frac{iv}{\alpha T_1^{1/2}} \tag{14.107}$$

and

$$M_{\kappa,\mu}(\eta) \sim -\mu W_{\kappa,0}(\eta)\Gamma(-\gamma/\alpha) \qquad \text{as } T_1 \to \infty \tag{14.108}$$

$$\Gamma(2\mu) \sim (2\mu)^{-1} \qquad \text{as } T_1 \to \infty \tag{14.109}$$

one finds from eqs. (14.63) and (14.64) that

$$\begin{aligned} p(a(\eta), T_1) \quad &\sim \quad \frac{e^{-\gamma^2 T_1}(\eta/\eta_0)^{\kappa} e^{\eta_0/2-\eta/2}}{\pi\alpha^2 T_1^{3/2} a\eta} \Gamma(-\gamma/\alpha) W_{\kappa,0}(\eta) W_{\kappa,0}(\eta_0) \\ &\times \int_{-\infty}^{+\infty} v^2 e^{-v^2} dv \qquad \text{as} \quad T_1 \to \infty. \end{aligned} \tag{14.110}$$

Noting that

$$\int_{-\infty}^{+\infty} v^2 e^{-v^2} dv = \frac{\pi^{1/2}}{2}, \tag{14.111}$$

substitution of eq. (14.110) into eq. (14.62) yields the asymptotic result given by eq. (14.68). Note that this result would not have been obtained when the asymptotic expression for $p_a(a(\eta), T_1)$ given by eqs. (14.66) and (14.67) would have directly been substituted into the right hand side of eq. (14.62). Hence, while evaluating the asymptotic value of the right hand side of eq. (14.62), it is not allowed to interchange integration with respect to a and the limit process involving $T_1 \to \infty$!

Abstract

Stochastic or random vibrations occur in a variety of applications of mechanical engineering. Examples are: the dynamics of a vehicle on an irregular road surface; the variation in time of thermodynamic variables in municipal waste incinerators due to fluctuations in heating value of the waste; the vibrations of an airplane flying through turbulence; the fluctuating wind loads acting on civil structures; the response of off-shore structures to random wave loading.

Attention is focussed on problems of external noise. That is, models of mechanical engineering structures are considered where the source of random behavior comes from outside: e.g. a prescribed random force or a prescribed random displacement. The structures inhibit inertia, damping and restoring, linear and non-linear. Questions addressed are: how do these structures respond to random excitation, and how to quantify the random behavior of response variables in a manner that an engineer is able to make rational design decisions.

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