On the degree of approximation of functions in $C^1 [0,1]$ by Bernstein polynomials

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by

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Abstract

Let f be a real function defined on the interval [0,1] and let $B_n(f;x)$ denote its n-th order Bernstein polynomial. The object of this paper is to study the exact degree of approximation with Bernstein polynomials for functions in $C^1[0,1]$. We estimate the difference $|B_n(f;x) - f(x)|$ in terms of $\omega(f';\delta)$, the modulus of continuity of f', with $\delta = \frac{1}{\sqrt{n}}$. Starting-point of our considerations is a theorem of Lorentz ([5], p. 21). Similar work on the degree of approximation with Bernstein polynomials for functions in C[0,1] has been done by Sikkema ([10],[11]) and Esseen [1]. Results for functions in $C^1[0,1]$ and $\delta = \frac{1}{n}$ may be found in [8].

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1. Introduction and summary

Let $C[0,1]$ be the set of real continuous functions defined on $[0,1]$. The expression

$$B_n(f;x) := \sum_{k=0}^{n} f(\tfrac{k}{n})p_{n,k}(x) ,$$

where $f \in C[0,1]$ and

$$p_{n,k}(x) = \binom{n}{k}x^k(1-x)^{n-k} \quad (x \in [0,1];\ n = 1,2,\ldots;\ k=0,1,\ldots,n) ,$$

is called the Bernstein polynomial of order n of the function f. Bernstein proved as early as 1912 that

$$(1.1) \qquad B_n(f;x) \to f(x) \quad (n \to \infty) ,$$

uniformly on $[0,1]$. For a proof of this result the reader is referred to [5], pp. 5-6. We note that B_n is a positive linear operator, i.e. $f \geq 0$ on $[0,1]$ implies $B_n f \geq 0$ on $[0,1]$. This property can be used to give an elegant proof of (1.1) (cf. [3], pp. 28-30). There is an extensive literature on the rapidity with which $B_n(f;x)$ tends to $f(x)$ as $n \to \infty$. As an illustration we cite here a result of Popoviciu [6], who proved that

$$(1.2) \qquad \max_{0\leq x\leq 1} |B_n(f;x) - f(x)| \leq \frac{3}{2}\,\omega(f;\frac{1}{\sqrt{n}})$$

for all $f \in C[0,1]$ and all $n \in \mathbb{N}$. Here $\omega(f;\delta)$ denotes the modulus of continuity of f, i.e.

$$\omega(f;\delta) = \max_{|x-y|\leq\delta} |f(x) - f(y)| \quad (x,y \in [0,1];\ \delta > 0) .$$

A refinement of (1.2) can be found in [5], p. 20. There also the problem was raised of determining the *best* constant in the right-hand side of (1.2). This problem was solved by Sikkema in a couple of papers ([10], [11]). He proved that for all $f \in C[0,1]$ and all $n \in \mathbb{N}$

$$(1.3) \qquad \max_{0\leq x\leq 1} |B_n(f;x) - f(x)| \leq \kappa\omega(f;\frac{1}{\sqrt{n}}) ,$$

where

$$(1.4) \qquad \kappa = \frac{4306 + 837\sqrt{6}}{5832} = 1.089887^{*)} ,$$

*) Here and elsewhere the numbers are rounded to the last digit shown.

and that κ in (1.3) cannot be replaced by any number smaller than the one given in (1.4) without invalidating the inequality.
Esseen [1] proved that for all $f \in C[0,1]$

$$(1.5) \qquad \overline{\lim_{n\to\infty}} \frac{\max_{0\le x\le 1} |B_n(f;x) - f(x)|}{\omega(f;\frac{1}{\sqrt{n}})} \le A ,$$

with

$$(1.6) \qquad A = 2 \sum_{j=0}^{\infty} (j+1)\{\Phi(2j+2) - \Phi(2j)\} = 1.045564 ,$$

where

$$(1.7) \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt ,$$

and he showed that the number A in (1.5) cannot be replaced by any number smaller than the one given in (1.6).
This paper deals with similar problems. Here the setting is the space $C^1[0,1)$ of real functions that have a continuous derivative on $[0,1]$. Starting-point of our considerations is a result of Lorentz ([5], p. 21) concerning the degree of approximation with Bernstein polynomials for this class of functions. His theorem reads as follows.

<u>Theorem 1.1</u> (Lorentz). Let $f \in C^1[0,1]$ and let $\omega_1(f;\delta) := \omega(f';\delta)$ be the modulus of continuity of f', then for $n \in \mathbb{N}$ one has

$$(1.8) \qquad \max_{0\le x\le 1} |B_n(f;x) - f(x)| \le C \frac{1}{\sqrt{n}} \omega_1(f;\frac{1}{\sqrt{n}}) ,$$

with $C = 3/4$.

As in the case of $f \in C[0,1]$, one may ask for the best constant in (1.8). To be more precise, for each fixed $n \in \mathbb{N}$ let (cf. remark 1.2 on p. 4)

$$(1.9) \qquad c_n := \sup_{f\in C^1[0,1]} \frac{\sqrt{n} \max_{0\le x\le 1} |B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})} .$$

The problem then is to determine

$$(1.10)\qquad c^{(1)} := \sup_{n\geq 1} c_n .$$

To this end we introduce the functions $c_n(x)$ defined by

$$(1.11)\qquad c_n(x) = \sup_{f\in C^1[0,1]} \frac{\sqrt{n}\,|B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})} .$$

We shall derive explicit expressions for the $c_n(x)$, and obtain $c^{(1)}$, making use of the obvious equality

$$(1.12)\qquad \sup_{f\in C^1[0,1]} \sup_{0\leq x\leq 1} \frac{|B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})} = \sup_{0\leq x\leq 1} \sup_{f\in C^1[0,1]} \frac{|B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})}$$

where, in fact, on both sides of (1.12) $\sup\limits_{0\leq x\leq 1}$ may be replaced by $\max\limits_{0\leq x\leq 1}$.

We now give a sketch of the contents of the various sections of this report. Section 2 contains some preliminary results that will be needed later. In order to make the paper reasonably self-contained, we start section 3 with Lorentz's proof of theorem 1.1. By a slight modification of this proof we obtain a small improvement of the estimate (1.8). Then it is shown by elementary means that $c^{(1)} < \frac{1}{2}$. In section 4 the so-called extremal functions are introduced; these play a fundamental role in determining $c_n(x)$ as defined in (1.11), and hence in determining c_n and $c^{(1)}$. In section 5 we calculate c_n for $n = 1,2,\ldots,5$. A simple proof of the fact that $c^{(1)} = c_1 = \frac{1}{4}$ is given in section 6, using the positivity of the operators B_n. In section 7 we obtain

$$(1.13)\qquad c^{(2)} := \sup_{n\geq 2} c_n ,$$

and, finally, in section 8 we derive $\lim\limits_{n\to\infty} c_n(x)$ and $\lim\limits_{n\to\infty} c_n$, and we give some numerical information concerning the numbers c_n.

<u>Remark 1.1</u>. In [8] similar problems are treated for functions $f \in C^1[0,1]$ normed by $\omega_1(f;\frac{1}{n})$ instead of $\omega_1(f;\frac{1}{\sqrt{n}})$. There it is proved that for $n \in \mathbb{N}$ the smallest constant d_n satisfying the inequality

$$(1.14)\qquad \max_{0\leq x\leq 1} |B_n(f;x) - f(x)| \leq d_n\omega_1(f;\tfrac{1}{n})$$

for all $f \in C^1[0,1]$, is given by

$$d_n = \begin{cases} \frac{1}{8} + \frac{1}{8(n+1)} & \text{if } n \text{ is even}, \\ \frac{1}{8} + \frac{1}{8n} & \text{if } n \text{ is odd}. \end{cases}$$

It is, of course, possible to consider norming by $\omega_1(f;n^{-\alpha})$ for, say, $0 < \alpha \leq 1$. It seems, however, that the case $\alpha = \frac{1}{2}$ is the most interesting, and the most natural from an asymptotic point of view. The case $\alpha = 1$ is by far the most tractable.

Remark 1.2. Inequalities of the type (1.8) and (1.14) are satisfied by *linear functions* (which are left unchanged by B_n) for all positive values of the constant C or d_n. It follows that the linear functions are of no interest for the problem we are concerned with. As the right-hand side of (1.9), and similar expressions elsewhere, are undefined for linear functions, in the remaining part of this report we shall often disregard these functions, without explicitly indicating this in our notation.

2. Preliminary results

This section contains three lemmas, the contents of which will be needed later. We start out with a well-known result that may be found in [4], p. 122 or [5], p. 14.

Lemma 2.1. Let

$$T_{n,s}(x) := \sum_{k=0}^{n} (k - nx)^s p_{n,k}(x) \qquad (n = 1,2,\ldots;\ s = 0,1,2,\ldots) . \tag{2.1}$$

Then one has the following recursion formula

$$T_{n,s+1}(x) = x(1-x)\{T'_{n,s}(x) + nsT_{n,s-1}(x)\} , \tag{2.2}$$

where

$$T_{n,0}(x) = 1,\ T_{n,1}(x) = 0 . \tag{2.3}$$

Corollary 2.1. If $x(1-x)$ is denoted by X, then in particular

(2.4) $$T_{n,2}(x) = nX,\quad T_{n,4}(x) = 3n^2X^2 + nX(1-6X)\ ,$$

(2.5) $$T_{n,6}(x) = 15n^3X^3 + 5n^2X^2(5-26X) + nX(1-30X+120X^2)\ .$$

The proof of lemma 2.1 is omitted. Corollary 2.1 is a straightforward consequence of (2.2), using (2.3).
The next lemma deals with a particular sum that plays a prominent role in the calculation of the functions $c_n(x)$ as defined in (1.11); we list some of its properties.

Lemma 2.2. Defining

(2.6) $$S_n(x) = \tfrac{1}{2}\sqrt{n}\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|p_{n,k}(x) \qquad (x\in[0,1];\ n=1,2,\ldots)\ ,$$

one has

$$S_n(x) = S_n(1-x)\ .$$

If $[a]$ denotes the largest integer not exceeding a and if $\|S_n\| := \max_{x\in[0,1]}|S_n(x)|$, then

(2.7) $$S_n(x) = \frac{1}{\sqrt{n}}(n-r)\binom{n}{r}x^{r+1}(1-x)^{n-r} \qquad (r=[nx])\ ,$$

(2.8) $$\max_{x\in[0,\frac{1}{n}]} S_n(x) < \max_{x\in[\frac{1}{n},\frac{2}{n}]} S_n(x) < \ldots < \max_{x\in[\frac{r^*}{n},\frac{r^*+1}{n}]} S_n(x) =$$

$$= \max_{x\in[\frac{r^*}{n},\frac{1}{2}]} S_n(x) = \|S_n\| \qquad (r^* = [\tfrac{n-1}{2}])\ ,$$

(2.9) $$\begin{cases}\frac{1}{4} = \|S_1\| > \|S_3\| > \|S_5\| > \ldots\ , \\ \frac{4}{27}\sqrt{2} = \|S_2\| > \|S_4\| > \|S_6\| > \ldots\ ,\end{cases}$$

(2.10) $$\lim_{n\to\infty} S_n(x) = \sqrt{\frac{x(1-x)}{2\pi}} =: S(x)\ ,$$

$$(2.11)\qquad \|S_n\| \to S(\tfrac{1}{2}) = \frac{1}{2\sqrt{2\pi}} = 0.19947114 \qquad (n \to \infty)\ .$$

Proof. We shall first establish formula (2.7). Let $x \in [0,1]$ and let $r = [nx]$. Taking into account the second part of (2.3) we have

$$\tfrac{1}{2}\sum_{k=0}^{n} \left|\frac{k}{n} - x\right| p_{n,k}(x) = \tfrac{1}{2}\sum_{k=0}^{r} \left(x - \frac{k}{n}\right) p_{n,k}(x) + \tfrac{1}{2}\sum_{k=r+1}^{n} \left(\frac{k}{n} - x\right) p_{n,k}(x) =$$

$$= \sum_{k=0}^{r} \left(x - \frac{k}{n}\right) p_{n,k}(x) =: f_r(n,x)\ .$$

In order to evaluate the $f_r(n,x)$ we consider r to be independent of n and x for the moment, and we take generating functions. Changing the order of summation and using (2.3) again it is easily verified that one has

$$\sum_{r=0}^{n} f_r(n,x) z^r = \sum_{k=0}^{n} \left(x - \frac{k}{n}\right) p_{n,k}(x) \sum_{r=k}^{n} z^r = \sum_{k=0}^{n} \left(x - \frac{k}{n}\right) p_{n,k}(x) \frac{z^k - z^{n+1}}{1 - z} =$$

$$= \frac{1}{1-z}\{x(xz + 1 - x)^n - \frac{1}{n}\sum_{k=0}^{n} k\binom{n}{k}(xz)^k (1-x)^{n-k}\} =$$

$$= \frac{1}{1-z}\{x(xz + 1 - x)^n - xz\sum_{k=1}^{n} \binom{n-1}{k-1}(xz)^{k-1}(1-x)^{n-k}\} =$$

$$= \frac{1}{1-z}\{x(xz + 1 - x)^n - xz(xz + 1 - x)^{n-1}\} = x(1-x)(xz+1-x)^{n-1}\ .$$

Expanding the last expression in powers of z we obtain

$$x(1-x)(xz + 1 - x)^{n-1} = x(1-x)\sum_{r=0}^{n-1} \binom{n-1}{r}(xz)^r (1-x)^{n-1-r} =$$

$$= \sum_{r=0}^{r} \binom{n-1}{r} x^{r+1}(1-x)^{n-r} z^r\ .$$

Equating the coefficients of z^r, and taking into account the definitions of $f_r(n,x)$ and $S_n(x)$, it follows that (with $r = [nx]$ again)

$$S_n(x) = \sqrt{n}\binom{n-1}{r} x^{r+1}(1-x)^{n-r} = \frac{1}{\sqrt{n}}(n-r)\binom{n}{r} x^{r+1}(1-x)^{n-r}\ .$$

This proves (2.7). We omit verification of $S_n(x) = S_n(1-x)$; it is an easy consequence of (2.7). We note that a different proof of (2.7) can be given by making use of Hilfssatz 1 in [10].

The monotonicity of the various maxima of $S_n(x)$ on the interval $[0,\frac{1}{2}]$ can be shown as follows. Obviously, for fixed r the maximum of $S_n(x)$ on the interval $[\frac{r}{n},\frac{r+1}{n}]$ is attained at $x = \frac{r+1}{n+1} = \frac{[nx]+1}{n+1}$. In order to prove (2.8) it is therefore sufficient to show that this maximum, i.e.

$$(2.12) \qquad \frac{(n+1)}{\sqrt{n}} \binom{n}{r} \left(\frac{r+1}{n+1}\right)^{r+1} \left(\frac{n-r}{n+1}\right)^{n-r+1}$$

is an increasing function of r on $\{0,1,2,\dots,[\frac{n-1}{2}]\}$.

The quotient of two successive maxima is equal to

$$\frac{\binom{n}{r}\left(\frac{r+1}{n+1}\right)^{r+1}\left(\frac{n-r}{n+1}\right)^{n-r+1}}{\binom{n}{r-1}\left(\frac{r}{n+1}\right)^{r}\left(\frac{n-r+1}{n+1}\right)^{n-r+2}} = \left(\frac{r+1}{r}\right)^{r+1}\left(\frac{n-r}{n-r+1}\right)^{n-r+1} =$$

$$= \frac{\left(1-\frac{1}{n-r+1}\right)^{n-r+1}}{\left(1-\frac{1}{r+1}\right)^{r+1}} .$$

As $(1-\frac{1}{x})^x$ is an increasing function for $x > 1$, this ratio is at least one as long as $n-r+1 \ge r+1$, i.e. $r \le \frac{n}{2}$. Taking into account the range of r, it thus follows that for n even the largest maximum of $S_n(x)$ is attained when $r = \frac{n}{2} - 1$. In case n is odd $S_n(x)$ attains its largest maximum when $r = \frac{n-1}{2}$. This proves (2.8). As a consequence we have

$$(2.13) \qquad \|S_n\| = \begin{cases} S_n\left(\frac{n}{2n+2}\right) & \text{if } n \text{ is even ,} \\ S_n\left(\frac{1}{2}\right) & \text{if } n \text{ is odd .} \end{cases}$$

We proceed with the proof of (2.9). As for the first part of it, this amounts to showing (cf. (2.12) and (2.13)) that for $n = 2m+1$ we have

$$\frac{2m+2}{\sqrt{2m+1}} \binom{2m+1}{m} \left(\frac{1}{2}\right)^{2m+3} > \frac{2m+4}{\sqrt{2m+3}} \binom{2m+3}{m+1} \left(\frac{1}{2}\right)^{2m+5} \qquad (m = 0,1,2,\dots) .$$

This inequality is equivalent to

$$2(m+1) > \sqrt{(2m+1)(2m+3)} \qquad (m = 0,1,2,\dots) ,$$

which is apparently true.

The verification of the second part of (2.9) is more tedious. Assuming $n = 2m$ and taking into account formulae (2.12) and (2.13), it is easily verified that we have to show that

$$\sqrt{m+1}\, m^{m+\frac{1}{2}}(2m+3)^{2m+3} > 2(2m+1)^{2m+2}(m+2)^{m+2} \qquad (m = 1,2,\dots) .$$

Taking logarithms of both sides we have to establish that for $m = 1,2,\dots$

$$(2.14) \qquad \tfrac{1}{2}\log m(m+1) + m\log m + (2m+3)\log(2m+3) - \log 2 - (2m+2)\log(2m+1) + - (m+2)\log(m+2) > 0 ,$$

which is easily seen to be true for $m = 1$ and $m = 2$. It can be shown that the derivative of the left-hand side of (2.14) is negative for $m \geq 2$. This observation, together with the fact that (2.14) holds for large m (as can be seen from its expansion in powers of $\frac{1}{m}$), assures that the second assertion of (2.9) holds. We omit all computational details. Finally, an application of the central limit theorem easily yields (2.10). Assertion (2.11) then is an immediate consequence. For details we refer to section 8, where similar limits are computed. This completely proves lemma 2.2. ∎

In table 2.1 we show the numerical values of $\|S_n\|$, $n = 1,2,\dots,30$, together with the corresponding values of x, where the maxima are attained.

We proceed with a simple lemma that will be used in sections 3 and 6.

Lemma 2.3. If c_n is defined as in (1.9), then

$$c_1 = 1/4 .$$

Proof. Using the mean value theorem and the definition of the modulus of continuity we have

$$|B_1(f;x) - f(x)| = |x(1-x)f'(\xi_0) - x(1-x)f'(\xi_1)| =$$

$$= x(1-x)|f'(\xi_0) - f'(\xi_1)| \leq \tfrac{1}{4}\omega_1(f;1) .$$

Taking $f(x) = \frac{1}{2}|x - \frac{1}{2}|$, $0 \leq x \leq 1$, it follows that $c_1 = \frac{1}{4}$. The fact that f is not differentiable at $x = \frac{1}{2}$ does not, of course, affect the argument. ∎

n	xmax	$\|S_n\|$	n	xmax	$\|S_n\|$
1	0.500000	0.250000	16	0.470588	0.202246
2	0.333333	0.209513	17	0.500000	0.202425
3	0.500000	0.216506	18	0.473684	0.201969
4	0.400000	0.207360	19	0.500000	0.202112
5	0.500000	0.209631	20	0.476190	0.201743
6	0.428571	0.205586	21	0.500000	0.201859
7	0.500000	0.206699	22	0.478261	0.201554
8	0.444444	0.204419	23	0.500000	0.201650
9	0.500000	0.205078	24	0.480000	0.201394
10	0.454545	0.203614	25	0.500000	0.201475
11	0.500000	0.204050	26	0.481481	0.201256
12	0.461538	0.203031	27	0.500000	0.201326
13	0.500000	0.203340	28	0.482759	0.201137
14	0.466667	0.202590	29	0.500000	0.201198
15	0.500000	0.202821	30	0.483871	0.201033

Table 2.1

3. An upper bound for $c^{(1)}$

In the introductory section we have formulated theorem 1.1 of Lorentz. As theorems of this type are the central theme of this report, for the sake of completeness, we here reproduce the proof of Lorentz' theorem as given by him in [5], p. 21.

Proof of theorem 1.1. We have

$$(3.1) \qquad f(x_1) - f(x_2) = (x_1 - x_2)f'(\xi) =$$

$$= (x_1 - x_2)f'(x_1) + (x_1 - x_2)\{f'(\xi) - f'(x_1)\} \quad (x_1 < \xi < x_2) \ .$$

Let $x \in [0,1]$ be arbitrary and fixed, and let δ be an arbitrary positive number. In view of (3.1) and the second part of (2.3), we deduce, using a well-known property of the modulus of continuity, that we have

$$|B_n(f;x) - f(x)| = \Big|\sum_{k=0}^{n} \{f(x) - f(\tfrac{k}{n})\}p_{n,k}(x)\Big| \le$$

$$\le \Big|\sum_{k=0}^{n}(x-\tfrac{k}{n})f'(x)p_{n,k}(x)\Big| + \omega_1(f;\delta)\{\sum_{k=0}^{n}|\tfrac{k}{n}-x|p_{n,k}(x) + \frac{1}{\delta}\sum_{|\frac{k}{n}-x|>\delta}(\tfrac{k}{n}-x)^2 p_{n,k}(x)\} \le$$

$$\le \omega_1(f;\delta)\{\sum_{k=0}^{n}|\tfrac{k}{n}-x|p_{n,k}(x) + \frac{1}{\delta}\sum_{k=0}^{n}(\tfrac{k}{n}-x)^2 p_{n,k}(x)\} \le$$

$$\le \omega_1(f;\delta)\{\sqrt{\sum_{k=0}^{n}(\tfrac{k}{n}-x)^2 p_{n,k}(x)} + \frac{1}{\delta}\sum_{k=0}^{n}(\tfrac{k}{n}-x)^2 p_{n,k}(x)\} ,$$

by Schwarz' inequality. By the first part of (2.4) we have

$$(3.2) \qquad \sum_{k=0}^{n}(\tfrac{k}{n}-x)^2 p_{n,k}(x) = \frac{x(1-x)}{n} \le \frac{1}{4n} \quad (x \in [0,1]) ,$$

and hence

$$|B_n(f;x) - f(x)| \le \omega_1(f;\delta)\{\frac{1}{2\sqrt{n}} + \frac{1}{4n\delta}\} .$$

Putting $\delta = \frac{1}{\sqrt{n}}$ here, we obtain theorem 1.1 ∎

We next show that by a slight modification of the above proof it is possible to improve on the constant $\frac{3}{4}$.

Theorem 3.1.

$$c^{(1)} := \sup_{n\ge 1}\ \sup_{f\in C^1[0,1]} \frac{\sqrt{n}\max\limits_{0\le x\le 1}|B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})} < \frac{11}{16} .$$

Proof. Proceeding as in the proof of theorem 1.1 one has

$$|B_n(f;x) - f(x)| \le \omega_1(f;\delta)\{\sum_{k=0}^{n}|\tfrac{k}{n}-x|p_{n,k}(x) + \frac{1}{\delta}\sum_{|\frac{k}{n}-x|>\delta}(\tfrac{k}{n}-x)^2 p_{n,k}(x)\} \le$$

$$(3.3) \qquad \le \omega_1(f;\delta)\{\sum_{k=0}^{n}|\tfrac{k}{n}-x|p_{n,k}(x) + \frac{1}{\delta^3}\sum_{|\frac{k}{n}-x|>\delta}(\tfrac{k}{n}-x)^4 p_{n,k}(x)\} \le$$

$$\le \omega_1(f;\delta)\{\sqrt{\sum_{k=0}^{n} (\frac{k}{n}-x)^2 p_{n,k}(x)} + \frac{1}{\delta^3}\sum_{k=0}^{n} (\frac{k}{n}-x)^4 p_{n,k}(x)\} .$$

Using (2.4) and taking into account that $T_{n,4}(x)$ is maximal at $x = \frac{1}{2}$ for all $n \ge 2$, it follows that

$$(3.4) \qquad \sum_{k=0}^{n} (\frac{k}{n}-x)^4 p_{n,k}(x) = \frac{3x^2(1-x)^2}{n^2} + \frac{x(1-x)}{n^3}\{1 - 6x(1-x)\} \le$$

$$\le \frac{3}{16n^2} - \frac{1}{8n^3} < \frac{3}{16n^2} \quad (n = 1,2,\ldots) ,$$

where the case $n = 1$ has to be verified separately. Using this and (3.2) we obtain

$$|B_n(f;x) - f(x)| \le \omega_1(f;\delta)\{\frac{1}{2\sqrt{n}} + \frac{3}{16n^2\delta^3}\} .$$

Taking $\delta = \frac{1}{\sqrt{n}}$ it follows that for all $x \in [0,1]$

$$|B_n(f;x) - f(x)| \le \frac{11}{16}\,\frac{1}{\sqrt{n}}\,\omega_1(f;\frac{1}{\sqrt{n}}) \quad (n = 1,2,\ldots) .$$

This proves theorem 3.1. ∎

Remark 3.1. As is obvious from the considerations above, we also have

$$|B_n(f;x) - f(x)| \le \omega_1(f;\delta)\{\sum_{k=0}^{n} |\frac{k}{n}-x| p_{n,k}(x) + \frac{1}{\delta^5}\sum_{k=0}^{n} (\frac{k}{n}-x)^6 p_{n,k}(x)\} .$$

The second sum in the right-hand side can be evaluated by using (2.5). However, it turns out that this yields a constant that is worse than the constant of theorem 3.1.

Remark 3.2. Instead of applying Schwarz' inequality to the first sum in the right-hand side of (3.3) one can use the estimates (2.9) of lemma 2.2. In this way, treating the case $n = 1$ separately (cf. lemma 2.3), one can improve slightly further on the upper bound for $c^{(1)}$. We shall not pursue this, but instead improve on this upper bound by a more effective method.

We have

Theorem 3.2.

$$c^{(1)} := \sup_{n\geq 1} \sup_{f\in C^1[0,1]} \frac{\sqrt{n}\max_{0\leq x\leq 1}|B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})} < \frac{1}{2}.$$

Proof. Let $n \geq 2$, let $x \in [0,1]$ and let δ be positive. In view of (2.3) and using a well-known property of the modulus of continuity it is easily verified that one has

$$|B_n(f;x) - f(x)| = |\sum_{k=0}^{n} \{f(\tfrac{k}{n}) - f(x)\}p_{n,k}(x)| = |\sum_{k=0}^{n} p_{n,k}(x)\int_x^{k/n}\{f'(t) - f'(x)\}dt| \leq$$

$$\leq \sum_{|\frac{k}{n}-x|\leq\delta} |\int_x^{k/n}(f'(t) - f'(x))dt|p_{n,k}(x) + \sum_{|\frac{k}{n}-x|>\delta} |\int_x^{k/n}(f'(t) - f'(x))dt|p_{n,k}(x) \leq$$

$$\leq \omega_1(f;\delta)\sum_{|\frac{k}{n}-x|\leq\delta}|\tfrac{k}{n}-x|p_{n,k}(x) + \omega_1(f;\delta)\sum_{|\frac{k}{n}-x|>\delta}|\int_x^{k/n}(\frac{|t-x|^3}{\delta^3}+1)dt|p_{n,k}(x) =$$

$$= \omega_1(f;\delta)\sum_{k=0}^{n}|\tfrac{k}{n}-x|p_{n,k}(x) + \frac{\omega_1(f;\delta)}{\delta^3}\sum_{|\frac{k}{n}-x|>\delta}|\int_x^{k/n}|t-x|^3dt|p_{n,k}(x) =$$

$$= \omega_1(f;\delta)\{\sum_{k=0}^{n}|\tfrac{k}{n}-x|p_{n,k}(x) + \frac{1}{4\delta^3}\sum_{|\frac{k}{n}-x|>\delta}(\tfrac{k}{n}-x)^4p_{n,k}(x)\} \leq$$

$$\leq \omega_1(f;\delta)\{\sum_{k=0}^{n}|\tfrac{k}{n}-x|p_{n,k}(x) + \frac{1}{4\delta^3}\sum_{k=0}^{n}(\tfrac{k}{n}-x)^4p_{n,k}(x)\}. \tag{3.5}$$

Putting $\delta = \frac{1}{\sqrt{n}}$ and taking into account definitions (2.1) and (2.6), we obtain

$$|B_n(f;x) - f(x)| \leq \frac{1}{\sqrt{n}}\omega_1(f;\frac{1}{\sqrt{n}})\{2S_n(x) + \frac{1}{4n^2}T_{n,4}(x)\}. \tag{3.6}$$

The expression between brackets in (3.6) can be evaluated by means of the second part of (2.4) and formulae (2.7), (2.9) of lemma 2.2. Using these results and observing (3.4), by straightforward calculation one has

$$\|S_n\| \le \|S_3\| = \frac{1}{8}\sqrt{3} \quad (n = 2,3,\dots) ,$$

$$\frac{1}{4n^2}\|T_{n,4}\| = \frac{1}{4n^2} \max_{0\le x\le 1} \sum_{k=0}^{n} (k - nx)^4 p_{n,k}(x) < \frac{3}{64} \quad (n = 2,3,\dots) .$$

Consequently, in view of (3.6) and lemma 2.3 for the case $n = 1$, it follows that $c^{(1)} < \frac{1}{4}\sqrt{3} + \frac{3}{64} < \frac{1}{2}$. ∎

<u>Remark 3.3</u>. Considering the proof of theorem 3.2, the following inequality apparently also holds:

$$|B_n(f;x) - f(x)| \le$$

$$\le \omega_1(f;\delta)\{\sum_{k=0}^{n} |\tfrac{k}{n}-x|p_{n,k}(x) + \frac{1}{\delta^{2s-1}} \sum_{|\frac{k}{n}-x|>\delta} |\int_x^{k/n} |t-x|^{2s-1}dt|p_{n,k}(x)\},$$

where s is an arbitrary positive number. It turns out that $s = 2$ is a suitable choice when one sets out to prove that $c^{(1)} < \frac{1}{2}$. Taking $s = 1$ gives rise to simpler calculations, but then a few cases corresponding to small values of n have to be treated separately. Choosing $s = 3$, one can use (2.5), but the calculations become somewhat more intricate.

4. <u>The extremal functions</u>

Up to now we have not made use of the functions $c_n(x)$ defined in section 1, formula (1.11), but instead we have obtained a (rather crude) upper bound for $c^{(1)}$. In this section we derive an explicit expression for $c_n(x)$, which will be used in the following sections to determine the quantities c_n ($n = 1,2,\dots,5$), $c^{(1)}$ and $c^{(2)}$ as defined in (1.9), (1.10) and (1.13). We first slightly simplify the notation and define

$$(4.1) \qquad \Delta_n(f;x) = B_n(f;x) - f(x) .$$

We shall make use of the representation (cf. (3.5))

$$(4.2) \qquad \Delta_n(f;x) = \sum_{k=0}^{n} p_{n,k}(x) \int_x^{k/n} f'(t)dt ,$$

and of the fact that for every linear function ℓ we have

(4.3) $$\Delta_n(f+\ell,x) = \Delta_n(f;x) .$$

The main object of this section is to prove the following theorem.

Theorem 4.1. For each $n \in \mathbb{N}$, for each $x_0 \in [0,1]$ and each $\delta > 0$

(4.4) $$\sup_{f\in C^1[0,1]} \frac{|\Delta_n(f;x_0)|}{\omega_1(f;\delta)} = \Delta_n(\tilde{f};x_0) ,$$

where $\tilde{f}$, which depends on x_0 and δ, is defined for all real x by the conditions

(4.5) $$\begin{cases} \tilde{f}(x_0) = 0 , \\ \tilde{f}'(x) = j+\tfrac{1}{2} \quad (j\delta < x - x_0 \le (j+1)\delta;\ j = 0,\pm1,\pm2,\ldots) . \end{cases}$$

The functions $\tilde{f}$ will be called *extremal functions*. We shall prove theorem 4.1 in a number of small steps, stated as lemmas, which gradually narrow down the class of functions to be considered. We first slightly widen the class $C^1[0,1]$ to the class K_δ of functions on $(-\infty,\infty)$ defined as follows:

(4.6) $$K_\delta = \left\{ f \;;\; \begin{array}{l} f \text{ is continuous, } f' \text{ is bounded, } f' \text{ has finitely many} \\ \text{jump discontinuities on finite intervals and no other} \\ \text{discontinuities, } 0 < \omega_1(f;\delta) \le 1 . \end{array} \right\} .$$

The restriction $\omega_1(f;\delta) > 0$ excludes the linear functions (cf. remark 1.2 on p. 4), and the restriction $\omega_1(f;\delta) \le 1$ is simply a matter of scaling. We might, in fact, restrict ourselves to functions with $\omega_1(f;\delta) = 1$, but this is not practical for our purposes.

In order to avoid trivial, but troublesome, difficulties at the boundary points 0 and 1, we continue all functions to the interval $(-\infty,\infty)$, in such a way that their essential properties, e.g. convexity, extend to this interval.

We now state and prove our lemmas.

Lemma 4.1.

(4.7) $$\sup_{f\in C^1[0,1]} \frac{|\Delta_n(f;x_0)|}{\omega_1(f;\delta)} = \sup_{f\in K_\delta} \frac{|\Delta_n(f;x_0)|}{\omega_1(f;\delta)} .$$

Proof. On $[0,1]$ every $f \in K_\delta$ is the pointwise limit of functions in $C^1[0,1]$ with the same value of $\omega_1(\cdot;\delta)$, as is easily seen by approximating f' by continuous functions. The result then follows from the continuity of B_n with respect to pointwise convergence. ∎

Lemma 4.2.

$$\sup_{f\in K_\delta} \frac{|\Delta_n(f;x_0)|}{\omega_1(f;\delta)} = \sup_{\substack{f\in K_\delta \\ f \text{ convex}}} \frac{\Delta_n(f;x_0)}{\omega_1(f;\delta)} .$$

Proof. Without loss of generality we take, here and in the sequel, $f \in K_\delta$ such that $\Delta_n(f;x_0) \geq 0$. We define a function $\check{f}$ by (see figure 4.1)

$$\check{f}(x_0) = f(x_0) ,$$

$$\check{f}'(x) = \begin{cases} \inf\limits_{x\leq u\leq x_0} f'(u) & \text{if } x \leq x_0 , \\ \sup\limits_{x_0\leq u\leq x} f'(u) & \text{if } x \geq x_0 . \end{cases}$$

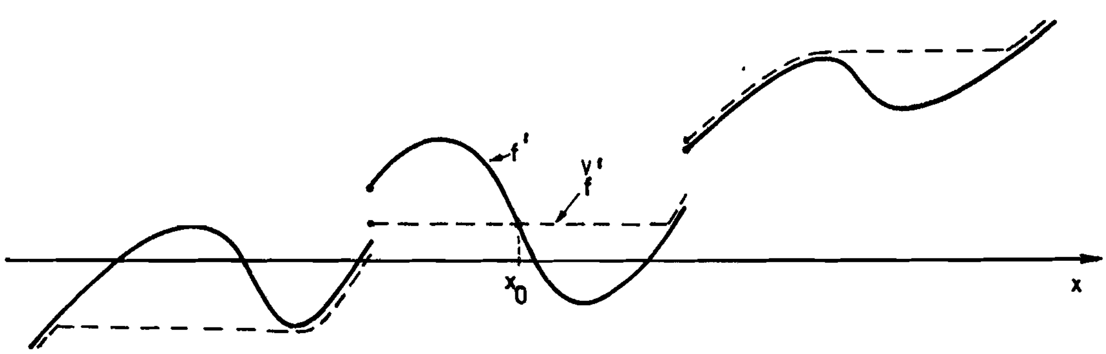

Figure 4.1.

Clearly $\check{f}'$ is nondecreasing, i.e. $\check{f}$ is convex. As $\check{f}' \leq f'$ on $(-\infty,x_0]$ and $\check{f}' \geq f'$ on $[x_0,\infty)$, it follows from (4.2) that $\Delta_n(\check{f};x_0) \geq \Delta_n(f;x_0)$. Moreover, $\omega_1(\check{f},\delta) \leq \omega_1(f;\delta)$. This can be seen as follows: if on $[x,x+\delta]$ the derivative $\check{f}'$ varies by ε, i.e. if $\check{f}'(x+\delta) - \check{f}'(x) = \varepsilon$, then by the definition of $\check{f}'$, for each $\eta > 0$, there exist y_1 and y_2 with $x \leq y_1 < y_2 \leq x + \delta$ such that $f'(y_2) - f'(y_1) > \varepsilon - \eta$. This proves that $\omega_1(\check{f};\delta) \leq \omega_1(f;\delta)$. It is easily verified that $\check{f}$ satisfies the remaining conditions for K_δ, and the lemma is proved. ∎

For arbitrary f on $(-\infty,\infty)$ we define f^* by

$$(4.8)\qquad \begin{cases} f^* \text{ is continuous} \\ f^*(x_0 + j\delta) = f(x_0 + j\delta) & (j = 0,\pm 1,\pm 2,\ldots) \\ f^* \text{ is linear on each interval } (x_0 + j\delta,\ x_0 + j\delta + \delta) . \end{cases}$$

Lemma 4.3. If f is convex and $f \in K_\delta$, then f^* is convex and $f^* \in K_\delta$.

Proof. The function f^* is trivially convex: its graph is a polygon inscribed in the graph of f. In order to prove that $f^* \in K_\delta$, we show that $\omega_1(f^*;\delta) \leq \omega_1(f;\delta)$ and hence $\omega_1(f^*;\delta) \leq 1$; the other conditions are easily checked. We proceed as follows. If t is not of the form $x_0 + j\delta$, then $f^{*'}(t)$ is well defined. For $t = x_0 + j\delta$ we define $f^{*'}(t)$ by continuity from the left. Now, for any two points t_1 and t_2 with $t_1 < t_2 \leq t_1 + \delta$ we have for some integer j

$$0 \leq f^{*'}(t_2) - f^{*'}(t_1) \leq f^{*'}(t_1 + \delta) - f^{*'}(t_1) =$$

$$= \frac{f(x_0 + j\delta + \delta) - f(x_0 + j\delta)}{\delta} - \frac{f(x_0 + j\delta) - f(x_0 + j\delta - \delta)}{\delta} =$$

$$= \frac{1}{\delta}\int_{j\delta}^{(j+1)\delta} f'(x_0 + t)dt - \frac{1}{\delta}\int_{(j-1)\delta}^{j\delta} f'(x_0 + t)dt =$$

$$= \frac{1}{\delta}\int_{j\delta}^{(j+1)\delta} \{f'(x_0 + t) - f'(x_0 + t - \delta)\}dt .$$

From this inequality it follows that $\omega_1(f^*;\delta) \leq \omega_1(f;\delta) \leq 1$, and the lemma is proved. ∎

Lemma 4.4. If $f \in K_\delta$ is convex, then

$$\frac{\Delta_n(f^*;x_0)}{\omega_1(f^*;\delta)} \geq \frac{\Delta_n(f;x_0)}{\omega_1(f;\delta)} .$$

Proof. As $f^*(x) \geq f(x)$ for all x, by the positivity of the operator B_n and the fact that $f^*(x_0) = f(x_0)$, we have $\Delta_n(f^*;x_0) \geq \Delta_n(f;x_0)$. From the proof of the preceding lemma we conclude that $\omega_1(f^*;\delta) \leq \omega_1(f;\delta)$, and the lemma follows. ∎

We now define a class K_δ^* of piecewise linear functions by

$$K_\delta^* = \{f; f \in K_\delta,\ f \text{ convex},\ f \equiv f^*,\ f(x_0) = 0,\ f'(x) = \tfrac{1}{2} \text{ for } x_0 < x \le x_0 + \delta\}\ ,$$

where the restrictions on $f(x_0)$ and $f'(x)$ are inessential because of (4.3). From the preceding four lemmas we now obtain

Lemma 4.5.

$$\sup_{f \in C^1[0,1]} \frac{|\Delta_n(f;x_0)|}{\omega_1(f;\delta)} = \sup_{f \in K_\delta^*} \frac{\Delta_n(f;x_0)}{\omega_1(f;\delta)}\ .$$

We are now ready for the proof of the main result of this section.

Proof of theorem 4.1. For $f \in K_\delta^*$ we have in view of (4.2)

$$(4.9)\qquad \frac{\Delta_n(f;x_0)}{\omega_1(f;\delta)} = \sum_{k=0}^{n} p_{n,k}(x_0) \int_{x_0}^{k/n} \frac{f'(t)}{\omega_1(f;\delta)}\, dt\ ,$$

where f' is a nondecreasing stepfunction with largest step equal to $\omega_1(f;\delta)$. It follows that $f'/\omega_1(f;\delta)$ is a nondecreasing stepfunction with largest step equal to 1, i.e. with modulus of continuity equal to 1. As is obvious from (4.9), $\Delta_n(f;x_0)/\omega_1(f;\delta)$ is maximal if all jumps of $f'/\omega_1(f;\delta)$ are equal to 1, i.e. if $f/\omega_1(f;\delta) = \tilde{f}$ as defined in (4.5). This proves the theorem. ∎

We conclude this section by giving explicit expressions for $\tilde{f}$ and $\Delta_n(\tilde{f};x_0)$. From (4.5) we have for $x > x_0$

$$\tilde{f}'(x) = \tfrac{1}{2} + \sum_{j=1}^{\infty} H(x - x_0 - j\delta)\ ,$$

where H denotes the unit stepfunction, taken to be continuous from the left. Hence, because $\tilde{f}$ is symmetric with respect to x_0,

$$(4.10)\qquad \tilde{f}(x) = \tfrac{1}{2}|x - x_0| + \sum_{j=1}^{\infty} (|x - x_0| - j\delta)_+\ ,$$

where $a_+ := \max(a,0)$. As $\tilde{f}(x_0) = 0$ we have

$$\Delta_n(\tilde{f};x_0) = B_n(\tilde{f};x_0)\ ,$$

and therefore

$$\Delta_n(f;x_0) = \tfrac{1}{2}\sum_{k=0}^{n} \left|\frac{k}{n}-x_0\right| p_{n,k}(x_0) + \sum_{k=0}^{n}\sum_{j=1}^{\infty} \left(\left|\frac{k}{n}-x_0\right| - j\delta\right)_+ p_{n,k}(x_0) \;,$$

or

$$\Delta_n(f;x_0) = \tfrac{1}{2}\sum_{k=0}^{n} \left|\frac{k}{n}-x_0\right| p_{n,k}(x_0) + \sum_{j=1}^{\infty} \sum_{\left|\frac{k}{n}-x_0\right| \geq j\delta} \left(\left|\frac{k}{n}-x_0\right| - j\delta\right) p_{n,k}(x_0) \;. \tag{4.11}$$

From a graph of $\tilde{f}$ (see figure 4.2) one easily obtains

$$\tilde{f}(x) = (\tilde{\ell} + \tfrac{1}{2})|x - x_0| - \tfrac{1}{2}\tilde{\ell}(\tilde{\ell} + 1)\delta \;,$$

where $\tilde{\ell} = [|x - x_0|/\delta]$. Hence we have

$$\Delta_n(\tilde{f};x_0) = B_n(\tilde{f};x_0) = \sum_{k=0}^{n} p_{n,k}(x_0)\{(\check{\ell} + \tfrac{1}{2})\left|\frac{k}{n}-x_0\right| - \tfrac{1}{2}\check{\ell}(\check{\ell} + 1)\delta\} \;,$$

with $\check{\ell} = [|\frac{k}{n}-x_0|/\delta]$. This can be rewritten as

$$\Delta_n(\tilde{f};x_0) = \sum_{k=0}^{n} p_{n,k}(x_0)\{\ell(\frac{k}{n} - x_0) - \tfrac{1}{2}\ell(\ell - 1)\delta\} \;, \tag{4.12}$$

with $\ell = [(\frac{k}{n} - x_0)/\delta] + 1$. Formula (4.12), with $\delta = \frac{1}{\sqrt{n}}$, has been used for the computer calculations (cf. table 8.1, p. 37).

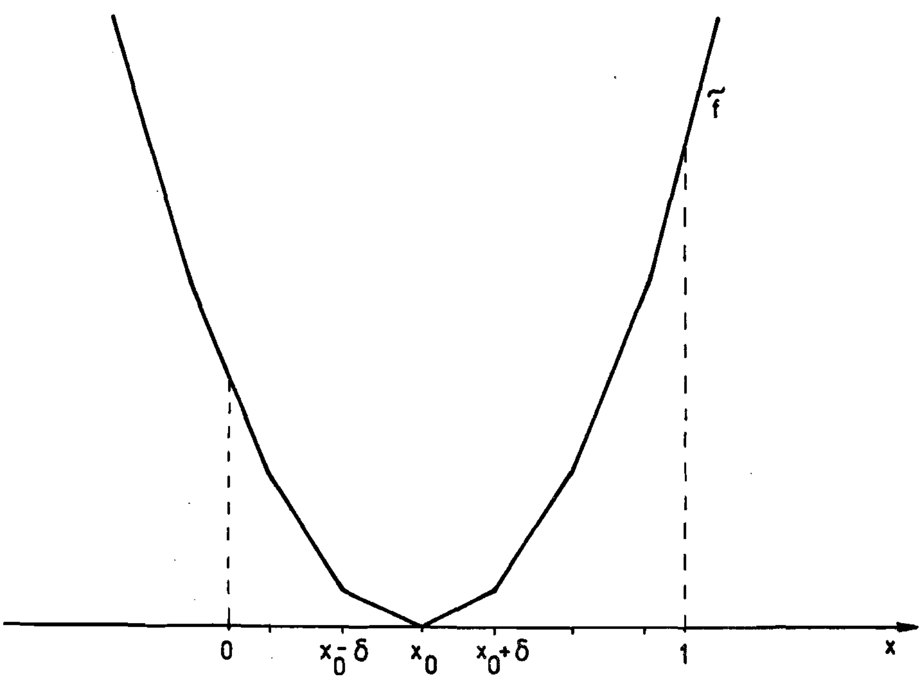

Figure 4.2.

5. Calculation of c_n for some small values of n

The object of this section is to determine the first few constants c_n by using the results of the preceding section. For that purpose we take $\delta = \frac{1}{\sqrt{n}}$ and we write $\tilde{f}_n$ instead of $\tilde{f}$. Furthermore, we shall restrict ourselves here to the cases n = 1,2,3,4,5. It turns out that for these small values of n the calculations involved to determine c_n are still manageable; for n = 5, however, the computational effort is already considerable. As will be clear from theorem 7.1 of section 7, the constant c_5 is the one we are particularly interested in. The exact determination of the constants c_n for $n \geq 6$ does not seem to be easy, in particular when n is even. In principle, it can be done in the same way as we are proceeding in this section. Ultimately, it amounts to determining the absolute maximum of a piecewise polynomial function on $[0,\frac{1}{2}]$, but for $n \geq 6$ the calculations involved become rather intricate. Therefore, in section 7 we use a method that yields estimates for the constants c_n $(n \geq 6)$, that are sufficiently sharp for our purposes. The values of c_n can also be obtained numerically; for these results we refer to table 8.1.

In order to determine $c_1,\ldots,c_5$ we recall that in section 4 we proved that

$$(5.1) \qquad \Delta_n(\tilde{f}_n;x_0) = B_n(\tilde{f}_n;x_0)$$

where, according to formula (4.10),

$$\tilde{f}_n(x) = \tfrac{1}{2}|x - x_0| + Q_n(x) ,$$

with

$$Q_n(x) = \sum_{j=1}^{\infty} \left(|x - x_0| - \frac{j}{\sqrt{n}}\right)_+ .$$

As $\omega_1(\tilde{f}_n;\frac{1}{\sqrt{n}}) = 1$, we have (cf. (1.11), (4.1) and (4.5))

$$(5.2) \qquad c_n(x_0) = \sqrt{n}\,\Delta_n(\tilde{f}_n;x_0) = \sqrt{n}\,B_n(\tilde{f}_n;x_0) ,$$

and hence by (4.11), writing x instead of x_0

$$(5.3) \qquad c_n(x) = \tfrac{1}{2}\sqrt{n}\sum_{k=0}^{n} \left|\frac{k}{n} - x\right| p_{n,k}(x) + \sqrt{n}\sum_{j=1}^{\infty}\ \sum_{\left|\frac{k}{n}-x\right| \geq \frac{j}{\sqrt{n}}} \left(\left|\frac{k}{n} - x\right| - \frac{j}{\sqrt{n}}\right)p_{n,k}(x) ,$$

or

$$(5.4) \qquad c_n(x) = S_n(x) + R_n(x) ,$$

with

$$R_n(x) = \sqrt{n}\, B_n(Q_n;x)$$

and $S_n(x)$ as defined in (2.6).

A precise evaluation of $R_n(x)$ is only feasible for small values of n. Together with lemma 2.2, formula (5.4) then allows one to determine the maximum c_n of $c_n(x)$ without lengthy calculations, say for $n \le 5$. In section 7 we shall obtain upper bounds for $R_n(x)$.

The calculation of the constants c_n rests completely upon the representation for $c_n(x)$ as given in (5.3). (We recall that c_1 was already determined in lemma 2.3.) In what follows, we shall consider the cases $n = 1,\dots,5$. Because of symmetry we restrict ourselves to $0 \le x \le \frac{1}{2}$.

<u>n = 1</u>. In this case the second contribution of (5.3) to $c_1(x)$ is zero, and in view of formula (2.7) we have

$$c_1(x) = x(1 - x) \qquad (0 \le x \le \tfrac{1}{2}) .$$

Hence

$$c_1 = \max_{0\le x\le \frac{1}{2}} c_1(x) = c_1(\tfrac{1}{2}) = \frac{1}{4} .$$

<u>n = 2</u>. There are two cases to be considered, viz. $0 \le x \le 1 - \frac{1}{2}\sqrt{2}$ and $1 - \frac{1}{2}\sqrt{2} \le x \le \frac{1}{2}$. According to (5.3) and using (2.7) we have

$$c_2(x) = \sqrt{2}\, x(1 - x)^2 + \sqrt{2}(1 - x - \tfrac{1}{2}\sqrt{2})x^2 \qquad (0 \le x \le 1 - \tfrac{1}{2}\sqrt{2}) ,$$

$$c_2(x) = \sqrt{2}\, x(1 - x)^2 \qquad (1 - \tfrac{1}{2}\sqrt{2} \le x \le \tfrac{1}{2}) .$$

One easily verifies that

$$\max_{0\le x\le 1-\frac{1}{2}\sqrt{2}} c_2(x) = \frac{\sqrt{2} - 1}{2} = 0.207107 ,$$

$$\max_{1-\frac{1}{2}\sqrt{2}\le x\le \frac{1}{2}} c_2(x) = c_2(\tfrac{1}{3}) = \frac{4}{27}\sqrt{2} = 0.209513 .$$

Hence

$$c_2 = \frac{4}{27}\sqrt{2} .$$

<u>n = 3</u>. In view of (5.3) and (2.7) one has

$$c_3(x) = \sqrt{3}\,x(1-x)^3 + \sqrt{3}\{(1-x-\tfrac{1}{3}\sqrt{3})x^3 + (2-3x-\sqrt{3})x^2(1-x)\} \quad (0 \le x \le \tfrac{2}{3}-\tfrac{1}{3}\sqrt{3}) ,$$

$$c_3(x) = \sqrt{3}\,x(1-x^3) + \sqrt{3}(1-x-\tfrac{1}{3}\sqrt{3})x^3 \quad (\tfrac{2}{3}-\tfrac{1}{3}\sqrt{3} \le x \le \tfrac{1}{3}) ,$$

$$c_3(x) = 2\sqrt{3}\,x^2(1-x)^2 + \sqrt{3}(1-x-\tfrac{1}{3}\sqrt{3})x^3 \quad (\tfrac{1}{3} \le x \le 1-\tfrac{1}{3}\sqrt{3}) ,$$

$$c_3(x) = 2\sqrt{3}\,x^2(1-x)^2 \quad (1-\tfrac{1}{3}\sqrt{3} \le x \le \tfrac{1}{2}) .$$

Again, one easily verifies that

$$\max_{0\le x\le \frac{2}{3}-\frac{1}{3}\sqrt{3}} c_3(x) < 0.120955, \quad \max_{\frac{2}{3}-\frac{1}{3}\sqrt{3}\le x\le\frac{1}{3}} c_3(x) < 0.213834 ,$$

$$\max_{\frac{1}{3}\le x\le 1-\frac{1}{3}\sqrt{3}} c_3(x) = 0.206267, \quad \max_{1-\frac{1}{3}\sqrt{3}\le x\le\frac{1}{2}} c_3(x) = c_3(\tfrac{1}{2}) = \tfrac{1}{8}\sqrt{3} = 0.216506,$$

and thus

$$c_3 = \tfrac{1}{8}\sqrt{3} .$$

<u>n = 4</u>. Obviously, there are two cases to be considered, viz. $0 \le x \le \frac{1}{4}$ and $\frac{1}{4} \le x \le \frac{1}{2}$. Taking into account (5.3) and (2.7) we find

$$c_4(x) = 2x(1-x)^4 + (1-2x)x^4 + 2(1-4x)x^3(1-x) \quad (0 \le x \le \tfrac{1}{4}) ,$$

$$c_4(x) = 6x^2(1-x)^3 + (1-2x)x^4 \quad (\tfrac{1}{4} \le x \le \tfrac{1}{2}) .$$

Elementary calculations show that

$$\max_{0\le x\le\frac{1}{4}} c_4(x) = c_4(\tfrac{1}{5}) = \frac{523}{3125} = 0.16736 ,$$

$$\max_{\frac{1}{4}\le x\le\frac{1}{2}} c_4(x) = c_4(\tfrac{2}{5}) = \frac{664}{3125} = 0.21248 .$$

Consequently,

$$c_4 = \frac{664}{3125} .$$

n = 5. A close examination of (5.3) (cf. figure 5.1) shows that one has to deal with the following expressions for $c_5(x)$.

$$c_5(x) = \sqrt{5}\,x(1-x)^5 + \sqrt{5}\{(1-x-\tfrac{1}{5}\sqrt{5})x^5 + (4-5x-\sqrt{5})x^4(1-x) + $$
$$+ (6-10x-2\sqrt{5})x^3(1-x)^2 + (1-x-\tfrac{2}{5}\sqrt{5})x^5\} \quad (0 \le x \le 1-\tfrac{2}{5}\sqrt{5}) ,$$

$$c_5(x) = \sqrt{5}\,x(1-x)^5 + \sqrt{5}\{(1-x-\tfrac{1}{5}\sqrt{5})x^5 + (4-5x-\sqrt{5})x^4(1-x) + $$
$$+ (6-10x-2\sqrt{5})x^3(1-x)^2\} \quad (1-\tfrac{2}{5}\sqrt{5} \le x \le \tfrac{3}{5}-\tfrac{1}{5}\sqrt{5}) ,$$

$$c_5(x) = \sqrt{5}\,x(1-x)^5 + \sqrt{5}\{(1-x-\tfrac{1}{5}\sqrt{5})x^5 + (4-5x-\sqrt{5})x^4(1-x)\} \quad (\tfrac{3}{5}-\tfrac{1}{5}\sqrt{5} \le x \le \tfrac{1}{5}) ,$$

$$c_5(x) = 4\sqrt{5}\,x^2(1-x)^4 + \sqrt{5}\{(1-x-\tfrac{1}{5}\sqrt{5})x^5 + (4-5x-\sqrt{5})x^4(1-x)\} \quad (\tfrac{1}{5} \le x \le \tfrac{4}{5}-\tfrac{1}{5}\sqrt{5}) ,$$

$$c_5(x) = 4\sqrt{5}\,x^2(1-x)^4 + \sqrt{5}\{(1-x-\tfrac{1}{5}\sqrt{5})x^5\} \quad (\tfrac{4}{5}-\tfrac{1}{5}\sqrt{5} \le x \le \tfrac{2}{5}) ,$$

$$c_5(x) = 6\sqrt{5}\,x^3(1-x)^3 + \sqrt{5}\{(1-x-\tfrac{1}{5}\sqrt{5})x^5\} \quad (\tfrac{2}{5} \le x \le \tfrac{1}{5}\sqrt{5}) ,$$

$$c_5(x) = 6\sqrt{5}\,x^3(1-x)^3 + \sqrt{5}\{(1-x-\tfrac{1}{5}\sqrt{5})x^5 + (x-\tfrac{1}{5}\sqrt{5})(1-x)^5\} \quad (\tfrac{1}{5}\sqrt{5} \le x \le \tfrac{1}{2}) .$$

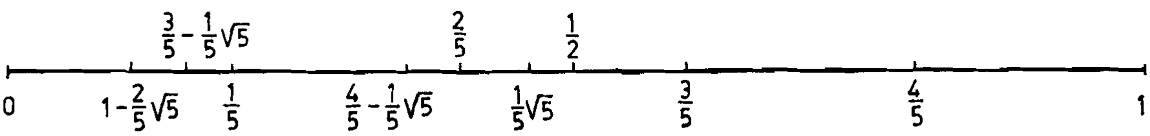

Figure 5.1.

For our purposes it is not necessary to determine the maxima of $c_5(x)$ on all the respective intervals; estimates will be sufficient. Elementary calculations show that the maximum of $c_5(x)$ on the interval $[\frac{1}{5}\sqrt{5}, \frac{1}{2}]$ is attained at $x = \frac{1}{2}$ and $c_5(\frac{1}{2}) = \frac{2\sqrt{5}-1}{16} = .217008$. Once this number is available we can compare it with (upper bounds for) the maxima of $c_5(x)$ on the remaining intervals. Proceeding in this way we arrive at the following results.

$$\max_{0\le x\le 1-\frac{2}{5}\sqrt{5}} c_5(x) < 0.1368, \qquad \max_{1-\frac{2}{5}\sqrt{5}\le x\le\frac{3-\sqrt{5}}{5}} c_5(x) < 0.1542 ,$$

$$\max_{\frac{3-\sqrt{5}}{5}\le x\le\frac{1}{5}} c_5(x) < 0.1558, \qquad \max_{\frac{1}{5}\le x\le\frac{4-\sqrt{5}}{5}} c_5(x) < 0.2011 ,$$

$$\max_{\frac{4-\sqrt{5}}{5}\le x\le\frac{2}{5}} c_5(x) < 0.1989, \qquad \max_{\frac{2}{5}\le x\le\frac{1}{5}\sqrt{5}} c_5(x) = 0.2069 .$$

In view of these results we conclude that

$$(5.5) \qquad c_5 = \frac{2\sqrt{5}-1}{16} .$$

The graph of the extremal function corresponding to the constant c_5 is shown in figure 5.2.

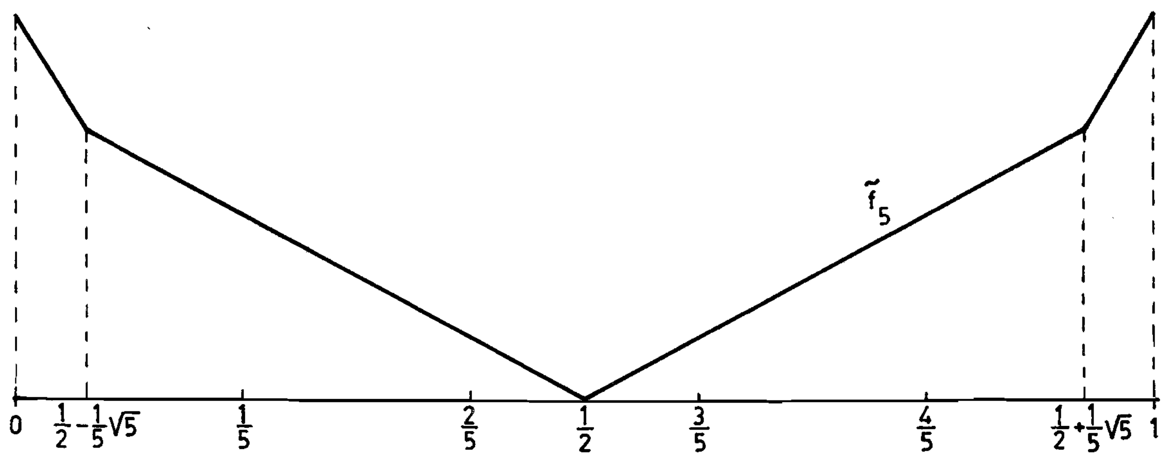

Figure 5.2.

Remark 5.1. As will be clear from the example treated above, the method with which the constants c_n can be determined, is straightforward and simple in principle. However, it is also obvious that the amount of computational work involved grows quite rapidly. Furthermore, certain numerical complications arise when determining the absolute maximum of the piecewise polynomial function $c_n(x)$ for large values of n. Most of these complications can be avoided, however, by using suitable estimates for $c_n(x)$ and $R_n(x)$ (cf. remark 6.3 and the contents of section 7).

The results of this section are collected in the following theorem.

Theorem 5.1. For c_n defined as in (1.9), we have

$$c_1 = c_1(\tfrac{1}{2}) = \tfrac{1}{4} = 0.250000 ,$$

$$c_2 = c_2(\tfrac{1}{3}) = \tfrac{4}{27}\sqrt{2} = 0.209513 ,$$

$$c_3 = c_3(\tfrac{1}{2}) = \tfrac{1}{8}\sqrt{3} = 0.216506 ,$$

$$c_4 = c_4(\tfrac{2}{5}) = \frac{664}{3125} = 0.212480 ,$$

$$c_5 = c_5(\tfrac{1}{2}) = \frac{2\sqrt{5} - 1}{16} = 0.217008 .$$

6. A simple proof of $c^{(1)} = 1/4$

In theorem 4.1 we obtained the extremal function $\tilde{f}$, depending on an arbitrary positive number δ. Since we wish to sharpen Lorentz' theorem 1.1, we take $\delta = \frac{1}{\sqrt{n}}$ and again write $\tilde{f}_n$ instead of $\tilde{f}$. In view of (4.1) one has

$$(6.1) \qquad \tilde{f}_n(x) = \tfrac{1}{2}|x - x_0| + \sum_{j=1}^{\infty} (|x - x_0| - \frac{j}{\sqrt{n}})_+ ,$$

where $a_+ = \max(a,0)$.

Using the functions $\tilde{f}_n$ we shall prove in an elementary way (cf. [9]) that $c_n \le \frac{1}{4}$ for all $n \in \mathbb{N}$. To this end we introduce a quadratic function q_n defined by

$$(6.2) \qquad q_n(x) = \frac{1}{8\sqrt{n}} + \tfrac{1}{2}\sqrt{n}(x - x_0)^2 .$$

The graph of q_n is a parabola that is tangent to the graph of $\tilde{f}_n$ in the midpoints of each of the linear pieces of that graph (cf. figure 6.1).

The properties of the function q_n are formally stated in the following lemma.

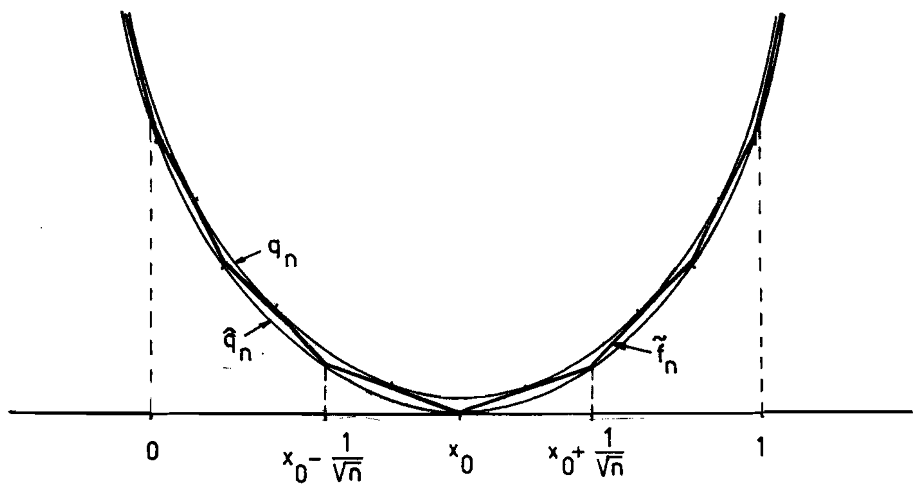

Figure 6.1.

Lemma 6.1. Let q_n be defined by (6.2) and let $\tilde{f}_n$ be the extremal function defined by (6.1), then we have

i) $$q_n(x_0 + \frac{2k+1}{2\sqrt{n}}) = \tilde{f}_n(x_0 + \frac{2k+1}{2\sqrt{n}}) = \frac{2k^2 + 2k + 1}{4\sqrt{n}} \quad (k = 0, \pm 1, \pm 2, \ldots) ,$$

ii) $$q_n'(x_0 + \frac{2k+1}{2\sqrt{n}}) = \tilde{f}_n'(x_0 + \frac{2k+1}{2\sqrt{n}}) = k + \tfrac{1}{2} \quad (k = 0, \pm 1, \pm 2, \ldots) ,$$

iii) $$q_n(x) \geq \tilde{f}_n(x) \quad (x \in [0,1]) ,$$

iv) $$\sqrt{n}\, B_n(q_n; x_0) = \frac{1}{8} + \frac{1}{2} x_0(1 - x_0) .$$

Proof. In view of the second part of (4.5) it follows by integration from x_0 to $x_0 + \frac{2k+1}{2\sqrt{n}}$ that for $k \geq 0$ we have

$$\tilde{f}_n(x_0 + \frac{2k+1}{2\sqrt{n}}) = \frac{1}{\sqrt{n}}(\frac{1}{2} + \frac{3}{2} + \ldots + \frac{2k-1}{2} + \frac{2k+1}{4}) = \frac{2k^2 + 2k + 1}{4\sqrt{n}} = q_n(x_0 + \frac{2k+1}{2\sqrt{n}}) .$$

By symmetry we obtain i) also for $k < 0$. From (6.1) and (6.2) we immediately have ii). Taking into account that $q_n(x_0) > \tilde{f}_n(x_0)$ and the fact that q_n is a convex function, property iii) now follows from i) and ii). Finally, iv) is an easy consequence of the first part of (2.4). This proves the lemma. ∎

We are now in a position to prove one of the main results of this report (cf. remark 1.2).

Theorem 6.1.

$$c^{(1)} := \sup_{n\geq 1} \sup_{f\in C^1[0,1]} \frac{\sqrt{n}\max\limits_{0\leq x\leq 1} |B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})} = \frac{1}{4}.$$

Proof. Noting that B_n is a positive operator, it follows from properties iii) and iv) of lemma 6.1 that for all $x_0 \in [0,1]$ one has in view of (5.2)

$$(6.3) \qquad c_n(x_0) = \sqrt{n}\, B_n(\tilde{f}_n;x_0) \leq \sqrt{n}\, B_n(q_n;x_0) = \frac{1}{8} + \frac{1}{2}x_0(1-x_0) .$$

Hence, $c_n \leq \frac{1}{4}$ for $n = 1,2,3,\ldots$. Taking into account lemma 2.3 and observing definition (1.10) of $c^{(1)}$ we obtain $c^{(1)} = c_1 = \frac{1}{4}$. ∎

Remark 6.1. In order to get a lower bound for $c_n(x_0)$ we consider the function $\hat{q}_n$ defined by

$$\hat{q}_n(x) = \frac{\sqrt{n}}{2}(x-x_0)^2 .$$

It is easily verified that one has (cf. figure 6.1)

$$\hat{q}_n(x_0 + \frac{k}{\sqrt{n}}) = \tilde{f}_n(x_0 + \frac{k}{\sqrt{n}}) = \frac{k^2}{2\sqrt{n}} \qquad (k = 0,\pm 1,\pm 2,\ldots) ,$$

$$\hat{q}_n(x) \leq \tilde{f}_n(x) \qquad (x \in [0,1]) .$$

Proceeding in the same way as in the proof of theorem 6.1 we deduce

$$(6.4) \qquad c_n(x_0) = \sqrt{n}\, B_n(\tilde{f}_n;x_0) \geq \sqrt{n}\, B_n(\hat{q}_n;x_0) = \tfrac{1}{2}x_0(1-x_0) .$$

Remark 6.2. The estimate (6.3) can be improved by using a function $\tilde{q}_n$, that differs slightly from the function q_n appearing in lemma 6.1. This function is chosen to be of the form

$$\tilde{q}_n(x) = \frac{a}{\sqrt{n}} + \frac{b\sqrt{n}}{2}(x-x_0)^2 ,$$

where the parameters a and b are chosen such that the graph of $\tilde{q}_n$ is tangent to the two linear pieces of $\tilde{f}_n$ adjacent to $x = x_0$, and such that $\sqrt{n}\, B_n(\tilde{q}_n;x_0)$ is minimal.

One finds

$$\tilde{q}_n(x) = \frac{1}{4}\left\{\sqrt{\frac{x_0(1-x_0)}{n}} + \sqrt{\frac{n}{x_0(1-x_0)}}\,(x-x_0)^2\right\} ;$$

the graph of this function is slightly steeper than that of q_n. As $\tilde{q}_n(x) \geq \tilde{f}_n(x)$ for all $x \in [0,1]$, one derives in a similar way as in the proof of theorem 6.1

$$c_n(x_0) = \sqrt{n}\, B_n(\tilde{f}_n;x_0) \leq \sqrt{n}\, B_n(\tilde{q}_n;x_0) = \frac{\sqrt{x_0(1-x_0)}}{2} < \frac{1}{8} + \frac{1}{2}\, x_0(1-x_0) \quad (x_0 \neq \tfrac{1}{2}). \tag{6.5}$$

We note that the functions q_n and $\tilde{q}_n$ are identical if $x_0 = \frac{1}{2}$.

Corollary 6.1. For all $x \in [0,1]$ and all $n \in \mathbb{N}$

$$\tfrac{1}{2}x(1-x) \leq c_n(x) \leq \tfrac{1}{2}\sqrt{x(1-x)} .$$

Proof. This is an immediate consequence of (6.4) and (6.5). ∎

Corollary 6.2. If $0 \leq x \leq 0.2517$ or $0.7483 \leq x \leq 1$, then

$$c_n(x) < c_5 = \frac{2\sqrt{5}-1}{16} = 0.217008 . \tag{6.6}$$

Proof. Using (5.5) the inequality in (6.6) easily follows from (6.5). ∎

Remark 6.3. We note that corollaries 6.1 and 6.2 are of some relevance for the numerical investigation of $\max_{x\in[0,\frac{1}{2}]} c_n(x)$: small values of x need not be taken into consideration. For instance, when n = 5 the first three cases of p. 22 can be disposed of immediately.

7. Determination of $c^{(2)}$

Having the extremal functions available, it is a comparatively simple matter to obtain the best constant in Lorentz' theorem 1.1, *when* n *runs through the set of all positive integers*. This was done in the preceding section. The simplicity of this problem is mainly due to the fact that $\sup_{n\geq 1} c_n = c_1$, and also to the fact that estimate (6.3) becomes an equality if $n = 1$. Thus, case $n = 1$ can be regarded as rather special, and it seems natural to ask for $c^{(2)} = \sup_{n\geq 2} c_n$, c_n being defined as in (1.9). This question will be answered in the present section. We recall that in section 5, formula (5.2), we established that

$$(7.1) \qquad c_n(x_0) = \sqrt{n}\, B_n(\tilde{f}_n;x_0) ,$$

where, according to theorem 4.1, for all x

$$\tilde{f}_n(x) = \tfrac{1}{2}|x - x_0| + Q_n(x) ,$$

with

$$Q_n(x) = \sum_{j=1}^{\infty} (|x - x_0| - \frac{j}{\sqrt{n}})_+ .$$

In what follows we shall obtain upper bounds for $R_n(x_0) = \sqrt{n}\, B_n(Q_n;x_0)$, which, together with lemma 2.2 and some numerical results, yield one of the main theorems of this report. We have

Theorem 7.1.

$$c^{(2)} := \sup_{n\geq 2} \sup_{f\in C^1[0,1]} \frac{\sqrt{n} \max_{0\leq x\leq 1} |B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})} = c_5 = \frac{2\sqrt{5} - 1}{16} = 0.217008497 .$$

Proof. In order to prove this theorem we use (7.1) and we write, replacing x by x_0 in (5.4),

$$(7.2) \qquad c_n(x_0) = S_n(x_0) + R_n(x_0) .$$

In lemma 2.2 it was proved that $S_n(x_0)$ has on $[0,\frac{1}{2}]$ a unique absolute maximum, denoted by $\|S_n\|$. For $1 \leq n \leq 30$, the values of $\|S_n\|$ are given in table 2.1, p. 9. We now proceed to obtain upper bounds R_n^* for $R_n(x_0)$. To this end we approximate Q_n by polynomials, $P_{n,s}$, of the form

$$P_{n,s}(x) = a_{n,s}(x - x_0)^{2s} \qquad (s = 1,2,3,\ldots) .$$

These polynomials are chosen in such a way that the graph of $P_{n,s}$ touches the (non-horizontal) linear pieces of the graph of Q_n, nearest to x_0 (see figure 7.1).

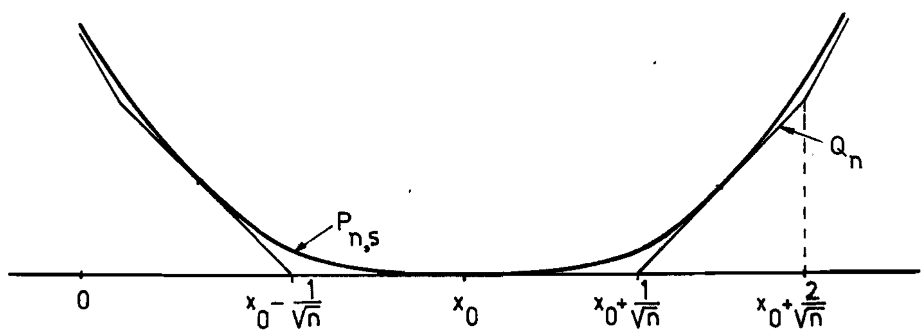

Figure 7.1.

One finds

$$(7.3) \qquad P_{n,s}(x) = \frac{(2s-1)^{2s-1}}{(2s)^{2s}} n^{s-\frac{1}{2}}(x-x_0)^{2s} .$$

It is easily verified that $P_{n,s}(x) \geq Q_n(x)$ for all x. Taking into account the positivity of the Bernstein operator B_n, we have the following upper bounds for $R_n(x_0)$.

$$(7.4) \qquad R_n(x_0) \leq \sqrt{n}\, B_n(P_{n,s};x_0) = \frac{(2s-1)^{2s-1}}{(2s)^{2s}} n^s B_n((x-x_0)^{2s};x_0) \quad (s=1,2,\ldots).$$

The best bound is obtained for s = 3, and using formulae (2.1) and (2.5) we get

$$(7.5) \qquad R_n(x_0) \leq \frac{5^5}{6^6 n^3} T_{n,6}(x_0) =$$

$$= \frac{5^5}{6^6}\{15X_0^3 + \frac{5}{n} X_0^2(5-26X_0) + \frac{1}{n^2} X_0(1-30X_0+120X_0^2)\} ,$$

where $X_0 = x_0(1-x_0)$.
As the last expression in (7.5) is increasing in X_0 for all $n \geq 4$, its maximum is attained at $X_0 = \frac{1}{4}$, i.e. at $x_0 = \frac{1}{2}$. It follows that

$$(7.6) \qquad R_n(x_0) \leq R_n^* := \frac{5^6}{2^{12}\, 3^5}\{1 - \frac{2}{n} + \frac{16}{15n^2}\} < 0.015699 \quad (n \geq 4) .$$

Taking into account formulae (7.1), (7.2), (7.4), to prove theorem 7.1 it is sufficient to show that for all $n \neq 5$ we have $\|S_n\| + R_n^* \leq 0.217008$, or, equivalently, that

(7.7) $$\| S_n \| \le 0.217008 - \frac{5^6}{2^{12}\,3^5}\{1 - \frac{2}{n} + \frac{16}{15n^2}\} =: \alpha_n \ .$$

Theorem 5.1 takes care of the cases n = 2,3,4. In table 7.1 the values of $\| S_n \|$ and α_n are given for $6 \le n \le 29$, and it turns out that inequality (7.7) does indeed hold for all these values, with the exception of n = 7,9,11.

n	$\| S_n \|$	α_n	n	$\| S_n \|$	α_n
6	0.205586	0.206077	18	0.201969	0.203002
7	0.206699	0.205453	19	0.202112	0.202916
8	0.204419	0.204973	20	0.201743	0.202838
9	0.205078	0.204591	21	0.201859	0.202767
10	0.203614	0.204282	22	0.201554	0.202702
11	0.204050	0.204026	23	0.201650	0.202643
12	0.203031	0.203810	24	0.201394	0.202589
13	0.203340	0.203626	25	0.201475	0.202539
14	0.202590	0.203467	26	0.201256	0.202492
15	0.202821	0.203328	27	0.201326	0.202450
16	0.202246	0.203207	28	0.201137	0.202410
17	0.202425	0.203099	29	9.201198	0.202372

Table 7.1.

As

$$\| S_{28} \| < 0.217008 - 0.015699$$

and

$$\| S_{29} \| < 0.217008 - 0.015699 \ ,$$

the values of $n \ge 30$ are taken care of by the monotonicity of $\| S_{2m} \|$ and $\| S_{2m+1} \|$, cf. (2.9). So, what remains to be done is a separate treatment of the cases n = 7,9,11.

<u>n = 7</u>. In order to show that $c_7(x) < c_5$ for all $x \in [0,1]$, it is sufficient to restrict x to the interval [0.48, 0.50]. This can be seen as follows. From (7.6) and table 7.1 it follows that

(7.8) $$R_7^* = 0.011555 \ .$$

The behaviour of the sum $S_7(x)$ can be dealt with by noting that it has a maximum 0.199588 at $x = \frac{3}{8}$ and, moreover, that $S_7(x)$ is decreasing on $[\frac{3}{8}, \frac{3}{7}]$ and increasing on $[\frac{3}{7}, \frac{1}{2}]$ (cf. figure 7.2 and also (2.8)).

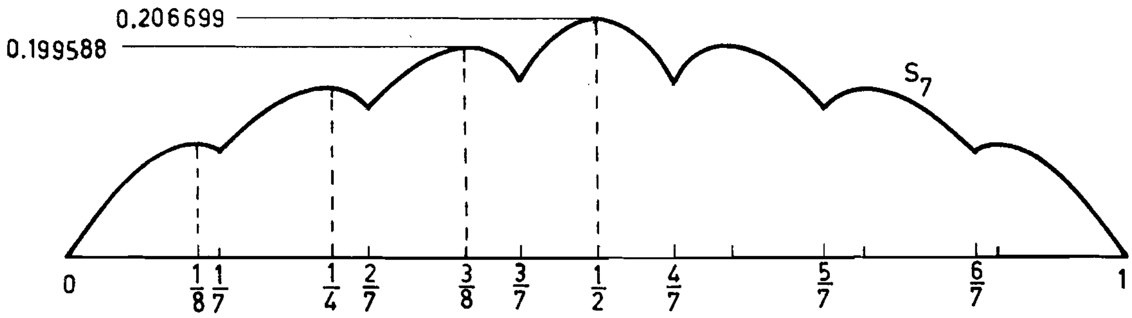

Figure 7.2.

As $S_7(0.48) = 0.205380$, one thus has in view of lemma 2.2 and (7.8) that for all $x \in [0,0.48]$

(7.9) $$S_7(x) + R_7^* \le 0.216935 < c_5 = 0.217008 .$$

To evaluate $c_7(x)$, $0.48 \le x \le 0.50$, we use formula (5.3). It is easily verified that for this range of x one has

$$c_7(x) = 20\sqrt{7}\, x^4(1-x)^4 + \sqrt{7}\{(1-x)^7(x-\tfrac{1}{7}\sqrt{7}) + x^7(1-x-\tfrac{1}{7}\sqrt{7})\} ,$$

which is maximal for $x = \frac{1}{2}$, with $c_7(\frac{1}{2}) = \frac{11\sqrt{7}-2}{128} = 0.211744 < c_5$. This, together with (7.9), proves that $c_7 < c_5$.

<u>n = 9</u>. Similarly, restricting x to $[\frac{4}{9}, \frac{1}{2}]$, we have in view of (5.3)

$$c_9(x) = 210x^5(1-x)^5 +$$
$$+ 3\{(1-x)^9(x-\tfrac{1}{3}) + (1-x)^8 x(9x-4) + x^8(1-x)(5-9x) + x^9(\tfrac{2}{3}-x)\},$$

which is again maximal for $x = \frac{1}{2}$, with $c_9(\frac{1}{2}) = \frac{109}{512} = 0.212891 < c_5$. This establishes that $c_9 < c_5$.

n = 11. This case can be covered in an analogous way as n = 7, n = 9. However, the expression for $c_{11}(x)$ as given by formula (5.3) becomes somewhat awkward to deal with, as there are contributions for k = 0,1,2,9,10,11. Case n = 11 can also be handled by improving slightly on the estimate (7.4). Considering the difference $P_{11,3}(x) - Q_{11}(x)$, we find, now restricting x to the interval [0.49,0.50] that $P_{11,3}(0) - Q_{11}(0) > 0.17$, and $P_{11,3}(1) - Q_{11}(1) > 0.20$. It follows that the estimates (7.5) and (7.6) can be improved by

$$\sqrt{11}\{0.17(1-x)^{11} + 0.20x^{11}\} > 0.000550 \quad (0.49 \le x \le 0.50) .$$

As $S_{11}(\frac{1}{2}) = 0.204050$ and $R^*_{11} = 0.012982$ (cf. table 7.1), this suffices to prove that $c_{11} < c_5$. This concludes the proof of theorem 7.1. ∎

Remark 7.1. We recall that by a considerable amount of computation we proved in section 5 that $c_5 = c_5(\frac{1}{2}) = \frac{2\sqrt{5}-1}{16}$. Using the methods of this section this result can be deduced in a much easier way. In fact, in examining $c_5(x)$ it is sufficient to restrict x to the interval [0.46,0.50], as it is easily verified that one has

$$S_5(x) \le 0.205632 \qquad (0 \le x \le 0.46) ,$$

$$R^*_5 = 0.010089 .$$

Consequently,

$$c_5(x) < 0.215721 \qquad (0 \le x \le 0.46) . \tag{7.10}$$

Using (5.3) and (2.7) we have on [0.46,0.50]

$$c_5(x) = 6\sqrt{5}\,x^3(1-x)^3 + \sqrt{5}\{(1-x-\tfrac{1}{5}\sqrt{5})x^5 + (x-\tfrac{1}{5}\sqrt{5})(1-x)^5\} ,$$

which attains its maximum at $x = \frac{1}{2}$, with $c_5(\frac{1}{2}) = \frac{2\sqrt{5}-1}{16} = 0.217008$. Because of (7.10) it then follows that $c_5 = c_5(\frac{1}{2})$.

Remark 7.2. It is perhaps appropriate to note that in dealing with the cases n = 7,9,11 as above, we have *not* shown that $c_7 = c_7(\frac{1}{2})$, $c_9 = c_9(\frac{1}{2})$, $c_{11} = c_{11}(\frac{1}{2})$, though this can be proved by carefully applying the method of section 5.

8. The limiting behaviour of $c_n(x)$

As we remarked in the introductory section 1, Esseen [1], complementing part of the work of Sikkema [10], determined the constant

$$\overline{\lim_{n\to\infty}} \sup_{f\in C[0,1]} \frac{\max_{0\le x\le 1} |B_n(f;x) - f(x)|}{\omega(f;\frac{1}{\sqrt{n}})},$$

whereas earlier Popoviciu [6] and then Sikkema [10] had given estimates for this quantity. In view of this it seems natural to put the analogous problem here, i.e. to ask for

$$\overline{\lim_{n\to\infty}} \sup_{f\in C^1[0,1]} \frac{\sqrt{n}\max_{0\le x\le 1} |B_n(f;x) - f(x)|}{\omega_1(f;\frac{1}{\sqrt{n}})}.$$

This section will be concerned with this kind of problem. In fact, using the central limit theorem we shall prove the following result, which is of a more detailed character.

Theorem 8.1. For $c_n(x)$ defined as in (1.11), we have

(8.1) $$c(x) := \lim_{n\to\infty} c_n(x) = \sqrt{\frac{X}{2\pi}} + 2\sqrt{X}\sum_{j=1}^{\infty}\int_{j/\sqrt{X}}^{\infty}\left(u-\frac{j}{\sqrt{X}}\right)\varphi(u)\,du \qquad (0 < x < 1),$$

(8.2) $$\lim_{n\to\infty}\max_{x\in[0,1]} c_n(x) = c(\tfrac{1}{2}) = \frac{1}{2\sqrt{2\pi}} + \sum_{j=1}^{\infty}\int_{2j}^{\infty}(u-2j)\varphi(u)\,du = 0.20796899,$$

where

(8.3) $$\varphi(u) = \frac{1}{\sqrt{2\pi}}\, e^{-u^2/2}, \quad X = x(1-x).$$

In order to prove this theorem we need two lemmas.

Lemma 8.1. If U is a nonnegative random variable with distribution function F, then for a ≥ 0

$$E(U - a)_+ = \int_a^\infty (1 - F(u))du ,$$

where E denotes expectation.

Proof.

$$E(U - a)_+ = \int_a^\infty (u - a)dF(u) ,$$

and the assertion of the lemma follows on integration by parts. ∎

Lemma 8.2. If V_n is a binomial random variable with expectation nx and variance nX, and if we put $U_n = \frac{V_n - nx}{\sqrt{nX}}$, then for the distribution function F_n of $|U_n|$ one has for all u ≥ 0 and all x ∈ (0,1)

$$(8.4) \qquad 1 - F_n(u) \le 2e^{-u^2x(1-x)} .$$

Proof. Following Lorentz ([5], pp. 18 - 19) and Rathore ([7], p. 123) one has

$$\Phi_n(v,x) := \sum_{k=0}^{n} p_{n,k}(x)e^{v(k-nx)} = \{xe^{v(1-x)} + (1-x)e^{-vx}\}^n .$$

Expanding $xe^{v(1-x)} + (1-x)e^{-vx}$ in powers of v, one obtains for $|v| \le \frac{3}{2}$ the inequality

$$xe^{v(1-x)} + (1-x)e^{-vx} \le 1 + \frac{v^2}{8}\{1 + \frac{|v|}{3} + \frac{v^2}{12(1-\frac{|v|}{5})}\} \le 1 + 0.221v^2 \le e^{0.221v^2} .$$

Defining

$$\psi_n(v,x) = \sum_{k=0}^{n} p_{n,k}(x)e^{v|k-nx|} ,$$

we therefore have

$$(8.5) \qquad \psi_n(v,x) \le \Phi_n(v,x) + \Phi_n(-v,x) \le 2e^{0.221nv^2} \quad (|v| \le \tfrac{3}{2}) .$$

Now, by a Chebyshev-type argument, one has

$$\sum_{\exp(v|k-nx|)\geq c\psi_n(v,x)} p_{n,k}(x) \leq \frac{1}{c} \quad (c > 0) ,$$

and therefore, by inequality (8.5)

$$\sum_{\exp(v|k-nx|)\geq 2c\,\exp(0.221nv^2)} p_{n,k}(x) \leq \frac{1}{c} .$$

Putting $c = \frac{1}{2}e^{n\delta^2}$ and $v = \frac{3}{2}\delta$ we have, as $\frac{9}{4}\,0.221 = 0.49725 < \frac{1}{2}$,

$$\sum_{|k-nx|\geq\delta n} p_{n,k}(x) \leq \sum_{\frac{3}{2}|k-nx|\delta\geq n\delta^2(1\,+\,0.49725)} p_{n,k}(x) \leq 2e^{-\delta^2 n} , \tag{8.6}$$

which yields (8.4) if we take $\delta = u\sqrt{\frac{x(1-x)}{n}}$. This proves the lemma. ∎

<u>Remark 8.1</u>. Inequality (8.6) is contained in the "Stellingen" section of Van de Ven's dissertation ([12], stelling X).

<u>Proof of theorem 8.1</u>. Using the notation of lemma 8.1, we have in view of (5.3)

$$c_n(x) = \tfrac{1}{2}\sqrt{n}\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|p_{n,k}(x) + \sqrt{n}\sum_{j=1}^{\infty}\sum_{k=0}^{n}\left(\left|\frac{k}{n}-x\right| - \frac{j}{\sqrt{n}}\right)_+ p_{n,k}(x) =$$

$$= \sqrt{X}\{\tfrac{1}{2}E(|U_n|) + \sum_{j=1}^{\infty} E(U_n - \frac{j}{\sqrt{X}})_+\} . \tag{8.7}$$

An application of lemma 8.1 yields

$$\begin{cases} E(|U_n|) = \displaystyle\int_0^{\infty} (1 - F_n(u))du , \\ E(|U_n| - \frac{j}{\sqrt{X}})_+ = \displaystyle\int_{j/\sqrt{X}}^{\infty} (1 - F_n(u))du . \end{cases} \tag{8.8}$$

Introduce

$$(8.9) \qquad \Phi(u) = \int_{-\infty}^{u} \varphi(v)dv ,$$

where $\varphi(v)$ is given by (8.3). Then by the Berry-Esseen theorem (cf. [2], p. 542) $1 - F_n(u)$ tends to $2\{1 - \Phi(u)\}$ as $n \to \infty$, uniformly in u and uniformly in x, with $x \in [\delta, 1-\delta]$, for any $\delta > 0$. By lemma 8.2 the integrals in (8.8) converge uniformly in n, j and $x \in [\delta, 1-\delta]$. It follows that

$$E(|U_n|) \to 2\int_0^{\infty} (1 - \Phi(u))du = 2\int_0^{\infty} u\varphi(u)du = \frac{2}{\sqrt{2\pi}}$$

uniformly in $x \in [\delta, 1-\delta]$, and that

$$E(|U_n| - \frac{j}{\sqrt{x}})_+ \to 2\int_{j/\sqrt{x}}^{\infty} (1 - \Phi(u))du = 2\int_{j/\sqrt{x}}^{\infty} (u - \frac{j}{\sqrt{x}})\varphi(u)du ,$$

uniformly in j and $x \in [\delta, 1-\delta]$.

As, also by lemma 8.2, the sums in (8.7) converge uniformly in n and $x \in [\delta, 1-\delta]$, it follows that $c_n(x) \to c(x)$ for all x, and uniformly for $x \in [\delta, 1-\delta]$ for any $\delta > 0$. This proves (8.1).

In order to prove (8.2), we note that from $c_n(x) \le \frac{1}{2}\sqrt{x(1-x)}$ (cf. (6.5)) it easily follows that $\max_x c_n(x) = c_n(x_n)$, with x_n bounded away from 0 and 1. As $c_n(x) \to c(x)$, uniformly in x, and as $\max_x c(x) = c(\frac{1}{2})$, it follows that $\lim_{n\to\infty} \max_x c_n(x) = c(\frac{1}{2})$, because for large n and arbitrary $\varepsilon > 0$ we have

$$c_n(x_n) \ge c_n(\tfrac{1}{2}) \ge c(\tfrac{1}{2}) - \varepsilon ,$$

whereas on the other hand

$$c_n(x_n) = c_n(x_n) - c(x_n) + c(x_n) - c(\tfrac{1}{2}) + c(\tfrac{1}{2}) \le \varepsilon + c(\tfrac{1}{2}) . \qquad \blacksquare$$

Remark 8.2. The expression for $c(\frac{1}{2})$ occuring in (8.2) can be rewritten as

$$c(\tfrac{1}{2}) = \frac{1}{2\sqrt{2\pi}} + \sum_{j=1}^{\infty} e^{-2j^2} - 2\sum_{j=1}^{\infty} j\{1 - \Phi(2j)\} ,$$

where Φ is defined by (8.9). This formula has been used to compute $c(\frac{1}{2})$.

n	xmax	max $c_n(x)$	n	xmax	max $c_n(x)$	n	xmax	max $c_n(x)$
1	0.5000	0.250000	35	0.5000	0.209205	69	0.5000	0.208501
2	0.3333	0.209513	36	0.4865	0.209125	70	0.4929	0.208404
3	0.5000	0.216506	37	0.5000	0.209193	71	0.5000	0.208344
4	0.4000	0.212480	38	0.4872	0.209040	72	0.4946	0.208289
5	0.5000	0.217008	39	0.5000	0.209016	73	0.4985	0.208289
6	0.4403	0.210300	40	0.4878	0.208793	74	0.4933	0.208311
7	0.5000	0.211744	41	0.5000	0.208685	75	0.5000	0.208386
8	0.4452	0.210940	42	0.4908	0.208501	76	0.4935	0.208408
9	0.5000	0.212891	43	0.5000	0.208506	77	0.5000	0.208465
10	0.4541	0.211364	44	0.4889	0.208562	78	0.4937	0.208472
11	0.5000	0.211496	45	0.5000	0.208723	79	0.5000	0.208512
12	0.4689	0.209518	46	0.4894	0.208740	80	0.4938	0.208504
13	0.5000	0.209928	47	0.5000	0.208851	81	0.5000	0.208526
14	0.4670	0.209821	48	0.4898	0.208827	82	0.4940	0.208502
15	0.5000	0.210687	49	0.5000	0.208889	83	0.5000	0.208507
16	0.4706	0.210304	50	0.4902	0.208825	84	0.4941	0.208468
17	0.5000	0.210635	51	0.5000	0.208837	85	0.5000	0.208455
18	0.4735	0.209934	52	0.4906	0.208732	86	0.4942	0.208401
19	0.5000	0.209778	53	0.5000	0.208698	87	0.5000	0.208372
20	0.4810	0.209001	54	0.4909	0.208554	88	0.4944	0.208303
21	0.5000	0.209136	55	0.5000	0.208475	89	0.5000	0.208258
22	0.4784	0.209173	56	0.4931	0.208375	90	0.4957	0.208227
23	0.5000	0.209619	57	0.4981	0.208376	91	0.4988	0.208227
24	0.4800	0.209526	58	0.4915	0.208412	92	0.4946	0.208239
25	0.5000	0.209766	59	0.5000	0.208519	93	0.5000	0.208294
26	0.4815	0.209532	60	0.4918	0.208542	94	0.4947	0.208314
27	0.5000	0.209572	61	0.5000	0.208620	95	0.5000	0.208357
28	0.4827	0.209196	62	0.4921	0.208619	96	0.4948	0.208367
29	0.5000	0.209053	63	0.5000	0.208669	97	0.5000	0.208399
30	0.4872	0.208693	64	0.4923	0.208643	98	0.4950	0.208398
31	0.5000	0.208734	65	0.5000	0.208665	99	0.5000	0.208419
32	0.4849	0.208795	66	0.4925	0.208615	100	0.4950	0.208408
33	0.5000	0.209051	67	0.5000	0.208609	1000	0.4995	0.207998
34	0.4857	0.209043	68	0.4927	0.208534	1001	0.5000	0.208000

Table 8.1.

We conclude this section with table 8.1, containing the numerical values of the coefficients $c_n = \max_{x\in[0,1]} c_n(x)$, and the points where these maxima are attained, for n = 1,2,...,100,1000,1001. These data were computed on the Burroughs 6700 of the Computing Centre of the Eindhoven University of Technology. In computing these numbers use was made of formulae (4.12) and (5.3). Taking into account theorem 5.1 (where it was proved that $c_1 = c_1(\frac{1}{2})$, $c_3 = c_3(\frac{1}{2})$, $c_5 = c_5(\frac{1}{2})$), remark 7.2 (containing the assertion that $c_7 = c_7(\frac{1}{2})$, $c_9 = c_9(\frac{1}{2})$, $c_{11} = c_{11}(\frac{1}{2})$), and examining the first part of the table, one is led to the conjecture that if n is odd, $c_n = c_n(\frac{1}{2})$. For n = 57, however, the computer indicates that $c_{57} > c_{57}(\frac{1}{2})$. A similar phenomenon takes place for n = 73 and n = 91.

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