Finite simple subgroups of semisimple complex Lie groups : a survey

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Abstract

We survey recent results regarding embeddings of finite simple groups (and their nonsplit central extensions) in complex Lie groups, especially the Lie groups of exceptional type.

1. Introduction

Throughout this paper, L will be a finite group. Representation theory for L is usually understood to be the study of group morphisms $L \to GL(n,k)$ for distinguished collections of fields k (e.g., all overfields of a fixed field F) and positive integers n. The topic of this survey is motivated by the question as to what happens if $GL(n,\cdot)$ is replaced by another algebraic group $G(\cdot)$.

We shall mainly be concerned with the case where L is a finite simple group (that is, a finite nonabelian simple group) or a central extension thereof, and $G(k)$ is a connected simple algebraic group over a field k. A further restriction of our discussion concerns the field k. It will mostly be taken to be the complex numbers, in which case we will mainly study group morphisms from L to the complex Lie group $G(\mathbb{C})$. (See below for some exceptions in §3 and §5.)

For $G(\cdot)$ of classical type, the theory for representations $L \to G(\mathbb{C})$ differs little from the usual one for $GL(n, \mathbb{C})$. Indeed, a representation $L \to GL(n, \mathbb{C})$ decomposes into irreducible subrepresentations. The decomposition is well controlled by character theory. Given an irreducible representation $\rho : L \to GL(n, \mathbb{C})$, it can be checked whether it is conjugate to a symplectic representation $L \to Sp(n, \mathbb{C})$ or an orthogonal representation $L \to O(n, \mathbb{C})$ by verifying whether its Frobenius-Schur index (that is, $\sum_{g \in L} \rho(g^2)/|L|$) takes the value -1, or 1, respectively (cf. (Isaacs [1976])). Using the criterion for irreducible subrepresentations, it can also be successfully employed for arbitrary (reducible) representations $L \to GL(n, \mathbb{C})$.

Thus, the simple connected complex algebraic groups of exceptional type remain. There are five of them; their universal covers form a chain with respect to group embeddings:

$$G_2(\mathbb{C}) < F_4(\mathbb{C}) < 3 \cdot E_6(\mathbb{C}) < 2 \cdot E_7(\mathbb{C}) < E_8(\mathbb{C}).$$

Here, $3 \cdot E_6(\mathbb{C})$ denotes the universal covering group of type E_6, which has a center of order 3.

In §4 we indicate what is known about the occurrence of finite simple groups in each of these. In §2 we deal with some general theory, and in §3 with correspondences between ordinary (characteristic 0) and modular representations. §5 deals with related embedding problems, mainly focussing on finite maximal subgroups of the same overgroups, and finite simple subgroups of simple algebraic groups in positive characteristic. §6 is concerned with an overview of the calculations needed to establish the harder embeddings. In §7, we end by a discussion of the computational aspects of the constructive proofs outlined in the previous section.

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2. Finiteness results

The theorem below generalises a well-known fact known for $GL(n, \cdot)$. A good reference for a proof is (Slodowy [1993]); it is based on (Weil [1964]).

Consider the set of all maps from L to G, denoted by G^L, as an affine variety by viewing it as the product of $|L|$ copies of the affine variety G. Regard the set of all representations as the subvariety of G^L consisting of all points $\rho : L \to G$ satisfying the polynomial equations $\rho(g)\rho(h) = \rho(gh)$ for all $g, h \in L$. Note that G acts on X by conjugation:

$$(g \cdot \rho)(h) = g\rho(h)g^{-1} \quad (g \in G,\ \rho \in X,\ h \in L).$$

If k is a field, we denote by $\overline{k}$ its algebraic closure. If G is an algebraic group, we denote by G^0 its connected component containing the identity.

2.1. Theorem. *Suppose L is a finite group and G is an algebraic group. If k is a field such that $H^1(L, V) = 0$ for all finite-dimensional kL-modules V, then the number of conjugacy classes of representations $L \to G(\overline{k})$ is finite. In fact, in the variety X of all representations, each G^0-orbit of a representation is an irreducible component of X.*

The vanishing cohomology condition for L on kL-modules is satisfied if $|L|$ and $\operatorname{char} k$ are coprime (this includes the case $\operatorname{char} k = 0$). At least some condition in this direction is necessary as, for any natural number i, the elementary abelian group L of order p^2 embeds into $SL(2, k)$, where $k = (\mathbf{Z}/p)(t)$, via

$$\phi_i : (a, b) \mapsto \begin{pmatrix} 1 & a + bt^i \\ 0 & 1 \end{pmatrix};$$

this gives an infinite set of representations $\{\phi_i\}_i$, no two of which are $G(k)$-conjugate.

There are more detailed results in this direction. For instance, for $p = \operatorname{char} k > 0$, Slodowy (Slodowy [1993]) proves that, when fixing a particular representation of a Sylow p-subgroup of L, the number of conjugacy classes of representations of L extending this representation is finite (thus answering a question of Külshammer).

A very useful consequence of Theorem 2.1 is the following result.

2.2. Corollary. *Let K be an algebraically closed overfield of F and suppose $\operatorname{char} F$ and $|L|$ are coprime. Then any finite subgroup L of $G(K)$ is conjugate to a subgroup of $G(k)$ where k is a finite extension of F inside K.*

PROOF. As before, let X be the subvariety of G^L consisting of all group morphisms $L \to G$. Then X is clearly defined over F. Let $\rho : L \to G(K)$ be the embedding afforded by the hypothesis. The above theorem yields that the $G(K)^0$-orbit of $\rho: L \to G(K)$ is the set of K-points of an irreducible component of X. Therefore, this orbit contains a point defined over a finite extension k of F, that is, there is $g \in G(K)$ such that $g \cdot \rho : L \to G(K)$ satisfies $(g \cdot \rho)(L) < G(k)$. The assertion follows as $(g \cdot \rho)(L) = g\rho(L)g^{-1}$. □

Thus, for any given representation with specified ground field, one may ask for the minimal degree of an extension field of the ground field that realizes it. In particular, it would be interesting to have an analogue of Brauer's result (Brauer [1980]), which states that if F is a finite field, each irreducible representation $\rho : L \to GL(n, F)$ is $GL(n, F)$-conjugate to a representation $\sigma : L \to GL(n, k)$, where k is the smallest subfield of F containing all traces of $\rho(g)$ for $g \in L$.

In order to give a meaning to such an extension from $GL(n, \cdot)$ to arbitrary reductive algebraic groups G, two notions need appropriate generalizations. The first is irreducibility of a representation: a good candidate for simple algebraic groups might be that $\rho(L)$ is not contained in any parabolic subgroup of G. The second is the extension field of the ground field: taking k to be the smallest subfield of F containing all traces of elements of $\rho(L)$ on any of the fundamental weight modules, we would regain Brauer's subfield in case $G = GL(n, \cdot)$.

For example it is shown in (Testerman [1989]) that $G_2(q)$ embeds in $E_6(q)$ as an irreducible group on a 27-dimensional module for $E_6(q)$ if and only if $\sqrt{-7}$ is in $GF(q)$. It is shown in (Cohen & Wales [1993]) that a certain embedding of $L(2, 13)$ into $E_6(q^2)$ is in $E_6(q)$ if and only if $\sqrt{-91}$ is in $GF(q)$.

3. Relation with the finite groups of Lie type

In this section we review some of the folklore on the connection between group embeddings in groups of Lie type defined over the complex numbers and those over a finite field. See also (Griess [1991]), (Cohen, Griess & Lisser [1993]) and (Cohen & Wales [1992]). We are indebted to Prasad, Ramakrishnan, and others, for helpful discussions concerning the contents of this section.

For the duration of this section, let G be a semi-simple algebraic group scheme. Denote by Δ its Dynkin diagram, by r its number of nodes (i.e., the rank of G), and by $\widetilde{\Delta}$ the extended Dynkin diagram of Δ. Furthermore, fix a prime number p. We let $\mathbf{Q}_p$ be the p-adic field and K the p-adic completion of the algebraic closure of $\mathbf{Q}_p$. As is well known, K and $\mathbf{C}$ are isomorphic as fields. Pick an isomorphism which identifies these fields.

For k a finite extension of $\mathbf{Q}_p$, let $\mathfrak{o}$ denote the ring of integers in k and $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$. Then $\mathfrak{o}/\mathfrak{p} \cong \mathbf{F}_q$ where q is a power of p and $\mathbf{F}_q$ is a finite field with q elements. The groups $G(\mathbf{C}) \cong G(K)$, $G(k)$, $G(\mathbf{F}_q)$, and the subgroup $G_\Gamma(\mathfrak{o})$ of $G(K)$, where Γ is a rank r subdiagram of $\widetilde{\Delta}$, are now defined by the group scheme.

Reduction modulo $\mathfrak{p}$ is a homomorphism from $G_\Gamma(\mathfrak{o})$ onto $G_\Gamma(\mathbf{F}_q)$. The kernel of this map is a profinite p-group. The quotient $G_\Gamma(\mathbf{F}_q)$ is a finite group of Lie type Γ. The quotient $G_\Delta(\mathbf{F}_q)$ coincides with $G(\mathbf{F}_q)$. Reduction modulo $\mathfrak{p}$ works because of the following key result.

3.1. Theorem. *Suppose k is a finite extension of $\mathbf{Q}_p$. Then the group $G(k)$ is a locally compact group, when endowed with the p-adic topology. Any finite subgroup of $G(k)$ is compact and is contained in a maximal compact subgroup of $G(k)$. Given a maximal compact subgroup M of $G(k)$ there is a rank r subdiagram Γ of $\tilde{\Delta}$ and an algebraic subgroup G_Γ of G defined over k such that M is conjugate within $G(k)$ to $G_\Gamma(\mathfrak{o})$, where $\mathfrak{o}$ denotes the ring of integers in k.*

PROOF. See (Bruhat & Tits [1972]; Tits [1979]). □

Actually, we have been informed (Serre [1994]) that one can do better: there is a totally ramified extension k_e of k such that $G_\Gamma(\mathfrak{o})$ embeds in $G_\Delta(\mathfrak{o}_e)$, where $\mathfrak{o}_e$ is the ring of integers of k_e. It follows that each finite subgroup of $G(k)$ is conjugate in $G(\overline{k})$ to a subgroup of $G_\Delta(\mathfrak{o}_e)$.

3.2. Theorem. *Suppose L is a finite subgroup of $G(K)$. Then there is a finite extension field k of $\mathbf{Q}_p$ in K, a rank r subdiagram of $\tilde{\Delta}$ and a subgroup L_1 of $G(k)$ conjugate to L such that L_1 is a subgroup of $G_\Gamma(\mathfrak{o})$, where $\mathfrak{o}$ is the ring of integers of k. Reduction modulo the maximal ideal $\mathfrak{p}$ of $\mathfrak{o}$ is a homomorphism from L_1 onto a subgroup of $G_\Gamma(\mathbf{F}_q)$ for some power q of p. The kernel is a p-group.*

PROOF. Choose k as in Corollary 2.2, so that, identifying K and $\mathbf{C}$, the subgroup L of $G(\mathbf{C})$ is conjugate to a subgroup of $G(k)$.

By Theorem 3.1 there is a conjugate L_2 of L_1 in $G(k)$ which is a subgroup of $G_\Gamma(\mathfrak{o})$ for some rank r subdiagram Γ of Δ. The kernel in $G_\Gamma(\mathfrak{o})$ of reduction modulo $\mathfrak{p}$ being a profinite p-group, the kernel of its restriction to L_2 is a finite p-group. The image of L_2 under reduction modulo $\mathfrak{p}$ is a subgroup of $G_\Gamma(\mathbf{F}_q)$. □

In particular, if L has no normal p-subgroup, we find that L embeds in $G(\mathbf{F}_q)$. For, take k as in Theorem 3.2. Then the residue field of $\mathfrak{o}_e$ as above is again $\mathbf{F}_q$, so by the second assertion of the theorem, there is a homomorphism from a conjugate L_1 of L to $G_\Delta(\mathbf{F}_q) = G(\mathbf{F}_q)$. By the last assertion and the hypothesis on normal subgroups of L, the homomorphism is injective.

Now, conversely, given an embedding of L in $G(\mathbf{F}_q)$, we can lift the embedding to one of L in $G(\mathbf{C})$ under a familiar condition, which finds its origin in the following well-known result.

3.3. Lemma. *Suppose H is a finite group which contains a normal subgroup P of order a power of the prime p and H/P is of order prime to p. Then H contains a subgroup isomorphic to H/P, and all such subgroups are conjugate in H.*

PROOF. This is the Schur-Zassenhaus theorem. See, e.g., (Suzuki [1982]), Theorem 8.10 of Chapter 2. □

3.4. Theorem. *Suppose q is a power of p and L is a subgroup of $G(\mathbf{F}_q)$ whose order is not divisible by p. Choose a finite extension k of $\mathbf{Q}_p$ with ring of integers $\mathfrak{o}$ and maximal ideal $\mathfrak{p}$ of $\mathfrak{o}$ such that $\mathfrak{o}/\mathfrak{p} \cong \mathbf{F}_q$. Then there is a subgroup of $G_\Delta(\mathfrak{o})$ which reduces modulo $\mathfrak{p}$ to L. For a fixed isomorphism between the p-adic completion K of the algebraic closure of k and $\mathbf{C}$, there is a unique conjugacy class of subgroups of $G(\mathbf{C})$ for which some conjugate in $G_\Delta(\mathfrak{o})$ reduces to L.*

PROOF. Denote by $\widetilde{L}$ the inverse image in $G_\Delta(\mathfrak{o})$ of L under reduction modulo $\mathfrak{p}$. If N_i is the kernel of reduction modulo $\mathfrak{p}^i$, then N_i is normal in $\widetilde{L}$ and N_1/N_i is a finite p-group. Since the latter quotients are p-groups and p does not divide $|L|$, the lemma above and induction on i yield that there exists a unique (up to conjugacy) subgroup L_i in $\widetilde{L}$ isomorphic to L and mapping onto L_{i-1} under the natural quotient map $\widetilde{L}/N_i \to \widetilde{L}/N_{i-1}$. Thus, we can find a complement of N unique up to conjugacy in $\widetilde{L}$ isomorphic to L. This complement provides an embedding of L in $G_\Delta(\mathfrak{o})$, which by the isomorphism $K \cong \mathbf{C}$ as above leads to an embedding of L in $G(\mathbf{C})$. The conjugacy condition follows from the uniqueness of L in $\widetilde{L}$. □

Caution with the uniqueness statement in the theorem is needed, as, changing the isomorphism between K and $\mathbf{C}$, the conjugacy class of L may change by a Galois conjugation.

4. Established embeddings and open cases

The existence of finite simple subgroups of complex Lie groups has been of interest for some time. Systematic searches for such embeddings received an impetus by *Kostant's conjecture*, formulated in 1983. It asserts that every simple complex Lie group $G(\mathbf{C})$ with a Coxeter number h such that $2h+1$ is a prime power, has a subgroup isomorphic to $L(2, 2h+1)$. For $G(\mathbf{C})$ of classical type, this is readily checked using ordinary representation theory and the Frobenius-Schur index. For $G(\mathbf{C})$ of exceptional type the theorem below and the knowledge that $h = 6, 12, 12, 18, 30$ for the five respective exceptional types give an affirmative case-by-case answer.

A quick overview of the state of the art is supplied by Table 1.

Table 1.
Nonabelian simple groups L a central extension of which embeds in a complex Lie group of exceptional type X_n

X_n	L
G_2	Alt_5, Alt_6, $L(2,7)$, $L(2,8)$, $L(2,13)$, $U(3,3)$
F_4	Alt_7, Alt_8, Alt_9, $L(2,25)$, $L(2,27)$, $L(3,3)$, ${}^3D_4(2)$, $U(4,2)$, $O(7,2)$, $O^+(8,2)$
E_6	Alt_{10}, Alt_{11}, $L(2,11)$, $L(2,17)$, $L(2,19)$, $L(3,4)$, $U(4,3)$, ${}^2F_4(2)'$, M_{11}, J_2
E_7	Alt_{12}, Alt_{13}, $L(2,29)^?$, $L(2,37)$, $U(3,8)$, M_{12}
E_8	Alt_{14}, Alt_{15}, Alt_{16}, Alt_{17}, $L(2,16)$, $L(2,31)$, $L(2,41)^?$, $L(2,32)^?$, $L(2,61)$, $L(3,5)$, $Sp(4,5)$, $G_2(3)$, $Sz(8)^?$

There are two meanings to be attached to this table:

4.1. Theorem. *Let L be a finite simple group and let G be a simple algebraic group of exceptional type X_n.*

(i) *If L occurs on a line corresponding to X_n in Table 1, then a central extension of it embeds in $G(\mathbb{C})$, with a possible exception for the four groups marked with a "?".*

(ii) *If X_n is as in some line of Table 1 and L appears neither in the line corresponding to X_n nor in a line above it, then no central extension of L embeds in $G(\mathbb{C})$.*

Warnings. To simplify the presentation,

a. we have deliberately neglected questions of conjugacy classes of embeddings, and
b. we have not specified the particular nonsplit central extensions of the simple groups involved.

Ad a. An example where the conjugacy class question is more subtle than suggested by the table is provided by $L(2,13)$. By (Cohen & Wales [1993]), it is isomorphic to a subgroup of $F_4(\mathbb{C})$ whose normalizer is a finite maximal closed Lie subgroup of $F_4(\mathbb{C})$, whereas the table only hints at the existence of embeddings via a closed Lie subgroup of $F_4(\mathbb{C})$ of type G_2.

Ad b. For instance, the simple group $L(2,37)$ listed embeds into a group of type E_7 but not in a group of type E_8 because each embedding in an adjoint group of type E_7 lifts to an embedding of $SL(2,37)$ into the universal covering group $2 \cdot E_7(\mathbb{C})$. Of course, the double cover $SL(2,37)$ of $L(2,37)$ embeds in the universal Lie group of type E_7, whence in a Lie group of type E_8.

Another warning concerning Table 1 is perhaps in order: The main theorems in (Cohen & Wales [1992]) and (Cohen & Griess [1987]) only concern subgroups not contained in closed Lie subgroups of positive dimension whereas Table 1 lists all finite simple subgroups (whether in a closed Lie subgroup of positive dimension or not).

Remarks.

i. The choice of central extensions of simple groups rather than just simple groups is important because they are the ones needed for the generalized Fitting subgroup.

ii. The table does not account for all groups that are involved in $E_8(\mathbb{C})$. For instance, no central extension of $L(5,2)$ is embeddable in $E_8(\mathbb{C})$, but a nonsplit extension $2^{\{5+10\}} \cdot L(5,2)$ does embed (cf. (Alekseevskii [1974])).

iii. The group $L(2,29)$ appears in a Lie group of type B_7, whence in one of type E_8. So, if the question whether a central cover of $L(2,29)$ embeds in $E_7(\mathbb{C})$ has a negative answer, the group should appear at the bottom line of Table 1.

iv. Unlike the $GL(n,\cdot)$ case, knowledge of the classes of the individual elements of an embedded group L does not suffice to determine the conjugacy class of L in G. This has been observed by Borovik for the alternating group Alt_6 in $E_8(\mathbb{C})$. The problem of how many conjugacy classes of embeddings of L exist only has a partial solution. See (Griess [1994]) for the full solution concerning G_2.

v. The group $L(2,41)$ does not appear as a possible subgroup of $E_8(\mathbb{C})$ in (Cohen & Griess [1987]), but neither does the argument ruling it out. Also, the group $Sz(8)$ does not appear as a possible subgroup of $E_8(\mathbb{C})$ in [loc. cit.], whereas the argument ruling it out is erroneous.

vi. Another error in [loc. cit.] concerns the character given for $L(2,31)$. The restriction of the adjoint character for $E_8(\mathbb{C})$ to the subgroup isomorphic to $L(2,31)$ constructed by Serre (see below) has a different character.

Proof of Theorem 4.1(i). In some cases where the subgroup to be embedded is "big enough", L can be shown to embed by use of character theoretic arguments, without explicit constructions. Two useful examples are the following two criteria, valid for both finite and algebraically closed fields F:

I. If $\rho : L \to GL(7,F)$ is an irreducible representation and $\rho(L)$ leaves fixed a 3-linear alternating form on F^7, then there is a subgroup of $GL(7,F)$ isomorphic to $G_2(F)$ such that $\rho(L) < G_2(F)$ (cf. (Cohen & Helminck [1988])).

II. If, for some positive integer n, the map $\rho : L \to GL(n,F)$ is an irreducible representation such that $\rho(L)$ fixes a nonzero symmetric bilinear form and a nonzero alternating trilinear form on F^n, but no nonzero alternating quadrilinear form, then $\rho(L)$ preserves a non-trivial Lie algebra product on F^n (cf. (Norton [1988]); earlier, in (Griess [1977]), similar conditions were given).

In both cases, the conditions can be verified using character tables and power maps only.

We now deal with the individual groups occurring in the table.

G_2: Alt_5, Alt_6, $L(2,7)$ have 3-dimensional projective representations, so a central extension occurs in a group of type A_2; the latter diagram occurs in $\widetilde{G}_2$, and so, by (Borel & Siebenthal [1949]), there is a closed Lie subgroup in $G_2(\mathbb{C})$ of type A_2, via which central covers of the three simple groups embed in $G_2(\mathbb{C})$. The above argument I. applies to $L(2,8)$, $L(2,13)$. The group $U(3,3)$ is an index 2 subgroup of the full automorphism group of the Cayley integers, and as such is known to embed in G_2, see (Coxeter [1946]). For more details, see (Cohen & Wales [1983]). Recently, a new approach to this classification appeared in (Griess [1994]).

F_4: $Alt_7 < Alt_8 < Alt_9$, and the latter has an orthogonal representation of degree 8, so embeds in a Lie group of type D_4, whence a central cover embeds in a Lie group of type F_4. Similarly for $O(7,2)$ and $O^+(8,2)$. A central cover of the group $U(4,2)$ has a 4-dimensional orthogonal representation, and so embeds in a group of type A_3, whence in $F_4(\mathbb{C})$. The group $L(2,25)$ occurs as a subgroup of ${}^2F_4(2)$, which can easily be seen to embed in $E_6(\mathbb{C})$ (see below); restriction of the E_6-character on a 27-dimensional high-weight module to the subgroup $L(2,25)$ shows that a vector is left fixed; the stabilizer of this vector must then be a Lie group of type F_4, whence $L(2,25) < F_4(\mathbb{C})$. The group ${}^3D_4(2)$ can be seen to embed in $F_4(\mathbb{C})$ by argument II. above. The group $L(3,3)$ occurs in the split extension $3^3 : L(3,3)$ found by (Alekseevskii [1974]). For an embedding of $L(2,27)$ and more details, see (Cohen & Wales [1992]).

E_6: $Alt_{10} < Alt_{11}$ embeds in a Lie group of type D_5. The groups $L(2,11)$, $L(3,4)$, $U(4,3)$, and J_2 have nontrivial projective representations of dimension less than or equal to 6, and so central extensions embed in $A_5(\mathbb{C})$, whence in $E_6(\mathbb{C})$. Similarly M_{11} has an orthogonal representation of degree

10, so embeds in a group of type D_5, whence in one of type E_6. By arguments II., it can be established that ${}^2F_4(2)'$ embeds in $E_6(\mathbb{C})$. For $L(2,17)$ and $L(2,19)$, and more details, see (Cohen & Wales [1992]).

E_7: $Alt_{12} < Alt_{13}$ is in $D_6(\mathbb{C})$ whence in $E_7(\mathbb{C})$. An embedding for $U(3,8)$ is given in (Griess & Ryba [1991]). Finally, M_{12} embeds in Alt_{12}. For more details on maximality and characters, see (Cohen & Griess [1987]).

E_8: $Alt_{14} < Alt_{15} < Alt_{16} < Alt_{17}$ embed in $D_8(\mathbb{C})$. The group $L(2,16)$ embeds in Alt_{17}. The group $L(3,5)$ occurs in a split extension $5^3 : SL(3,5)$ (cf. (Alekseevskii [1974])). The groups $Sp(4,5)$ and $G_2(3)$ embed in a group of type D_7, whence in $E_8(\mathbb{C})$. Finally, $L(2,61)$ is constructed in (Cohen, Griess & Lisser [1993]). For more details, see (Cohen & Griess [1987]). Using a more elaborate lifting criterion than the one of Theorem 3.4 Serre recently proved (Serre [1994]), at least for groups G of exceptional type, the existence of a subgroup of $G(\mathbb{C})$ isomorphic to $PGL(2,h+1)$ where h is the Coxeter number (starting from the existence of a particular subgroup of type A_1 in $G(\mathbf{F}_{h+1})$). This establishes the existence of a subgroup of $E_8(\mathbb{C})$ isomorphic to $PGL(2,31)$.

Sketch of Proof of Theorem 4.1(ii). This part of the theorem uses the classification of finite simple groups. The proof can be found in (Cohen & Griess [1987]) for E_7 and E_8, (Cohen & Wales [1992]) for F_4 and E_6, and (Cohen & Wales [1983]) for G_2. Some of the main techniques are discussed below.

First, due mainly to (Landazuri & Seitz [1974]), for any given finite simple L there is an explicitly known number r_L such that each nontrivial projective representation of L has degree at least r_L. If a central cover of L embeds in $G(\mathbb{C})$, then the smallest high weight representation has dimension at least r_L. For $G = G_2, F_4, E_6, E_7, E_8$, this gives $r_L \leq 7, 26, 27, 56, 248$. This leads to an explicitly known finite list of simple groups for which the existence of an embedding needs to be checked.

In most cases, the list resulting from the Landazuri-Seitz bound r_L is still too big for a detailed analysis. An extremely useful result that helps to trim down the list further is due to (Borel & Serre [1953]). It states that every supersolvable subgroup of $G(\mathbb{C})$ is embeddable in the normalizer N of a maximal torus T of $G(\mathbb{C})$. Its use lies in the fact that the structure of N is completely determined: $T \cong \mathbb{C}^r$ where r is the rank of G, and N/T is the Weyl group of G. Thus necessary conditions for the existence of an embedding of L in G can be derived in terms of the structure of all supersolvable subgroups of L. For example, the rank of a maximal abelian p-subgroup of L is at most $r+1$ when L is embedded in $G(\mathbb{C})$ (cf. (Cohen & Seitz [1987]; Griess [1991])).

Another useful criterion comes from the limited number of classes of elements of given order in $G(\mathbb{C})$ and knowledge of their centralizers. For instance, the possible traces on small G-modules can be readily computed (it is fully automated in LiE, cf. (Leeuwen, Cohen & Lisser [1992])). This gives rise to necessary conditions on the characters of L for them to be restrictions of characters of the ambient Lie group $G(\mathbb{C})$ on a given small-dimensional high-weight module.

There are other conditions on the characters of L that must hold for them to be restrictions of $G(\mathbb{C})$-characters if L embeds in $G(\mathbb{C})$. For instance, on the adjoint module, L must leave invariant a symmetric bilinear form and an alternating trilinear form; these conditions can be expressed in terms of characters. In (Cohen & Wales [1992]) a more detailed relation between the characters that holds for the Lie group but not for "likely" character restrictions for $G_2(3)$, was used to show that $G_2(3)$ cannot be embedded in $E_6(\mathbb{C})$. More specifically, let ψ, χ be the characters of $3 \cdot E_6(\mathbb{C})$ on high weight modules of dimension 27 and 78, respectively. Then $\psi \otimes \overline{\psi}$ contains χ. Assume now that $G_2(3) < E_6(\mathbb{C})$. Then, by character arguments, we see that $L = 3 \cdot G_2(3) < 3 \cdot E_6(\mathbb{C})$, and that there are unique characters ψ_1, χ_1 of L such that $\psi|L = \psi_1$ and $\chi|L = \chi_1$. Now $\psi_1 \otimes \overline{\psi_1}$ does not contain χ_1, a contradiction with $\chi|L$ occurring in $\psi|L \otimes \overline{\psi}|L$.

If such arguments do not help, an explicit model of the group is useful. This model is usually taken to be the smallest dimensional high-weight module of G. □

In conclusion, in establishing the existence of an embedding of a central extension for finite simple groups, we only encounter computational difficulties for the Suzuki group $Sz(8)$ in $E_8(\mathbb{C})$ and for the groups $L = L(2, s)$ with $s = 17, 19, 27, 29, 32, 37, 41, 61$ in the respective cases $X_n = E_6$, E_6, F_4, E_7, E_8, E_7, E_8, E_8. In view of part (i) of the theorem, we have the following result.

4.2. Theorem. *Kostant's conjecture holds.*

To finish off the question marks of Table 1, the following open questions need to be solved.

4.3. Open problems. *Establish that $L(2, 29)$ embeds in $E_7(\mathbb{C})$, and that $L(2, 41)$, $L(2, 32)$, and $Sz(8)$ embed in $E_8(\mathbb{C})$.*

The first of these, the only case left open for E_7, is probably the most straightforward one. The centralizer of the image of the diagonal of $L(2, 29)$ in $E_7(\mathbb{C})$ is a group of type T_6A_1 (that is, a product of a central torus of dimension 6 and $(P)SL(2, \mathbb{C})$). From §6, it will be clear that this slightly complicates the approach to a construction used for cases where the centralizer of a similar image is minimal, i.e., a maximal torus of G.

5. Related embedding problems

Embedding other groups in simple algebraic groups. In (Borovik [1989]; Borovik [1990]) perhaps the most remarkable finite subgroup of any Lie group appeared: it is a finite maximal closed Lie subgroup of $E_8(\mathbb{C})$, whose socle is $Alt_5 \times Alt_6$. It is the only occurrence of a subgroup of a simple complex Lie group $G(\mathbb{C})$ whose normalizer is a finite maximal closed Lie subgroup of $G(\mathbb{C})$ and whose socle is a product of more than one simple group.

Now let L be a finite maximal closed Lie subgroup of a complex simple Lie group $G(\mathbb{C})$ of exceptional type. If L has a nontrivial abelian normal subgroup, then L is known by (Alekseevskii [1974]; Alekseevskii [1975]). If not, then either L has socle $Alt_5 \times Alt_6$ (and G has type E_8), or L has a simple socle, in which case the results of §4 apply. Thus, finite maximal closed Lie subgroups of $G(\mathbb{C})$ are well understood.

Modular representations of finite simple groups. The analog of Theorem 4.1 for algebraic groups over algebraically closed fields of positive characteristic $p > 0$ is more difficult. One extreme is the situation where $(|L|, p) = 1$; by Theorems 3.2 and 3.4, this can be brought back to the Lie group case. At the other extreme, L may be a group of Lie type of the same characteristic. The study in (Seitz [1991]) shows how intricate this situation is.

On the positive side, many constructions as suggested by Table 1 go through due to the results in §3. To indicate that extra embeddings arise, we mention a few, without attempting to be exhaustive. In (Kleidman & Wilson [1993]), the sporadic simple groups embedding in a finite group of exceptional Lie type are determined. Apart from the sporadic simple groups L that can be found by use of Theorems 4.1(i) and 3.2 above, they found M_{22} (in $E_6(4)$), J_1 (in $G_2(11)$), J_3 (in $E_6(4)$), Ru, HS (both in $E_7(5)$), F_{22} (in $E_6(4)$) and Th (in $E_8(3)$). The reader is warned that here, as opposed to [loc. cit.], no exhaustive list is given of the groups of Lie type in which the sporadic groups occur. Earlier, several of the sporadic groups were hypothesised (notably by Steve Smith for Ru and HS) or proven (e.g. by (Thompson [1976]) for Th and by (Janko [1966]) for J_1) to embed in a group of exceptional Lie type. The added value of [loc. cit.] is that it establishes exactly where these groups occur and that the list is complete.

Modular representations of other groups. In the theory of maximal finite subgroups of algebraic groups over fields of positive characteristic, much progress has been made, especially for finite and algebraically closed fields. The cases where L has a nontrivial normal abelian subgroup have been dealt with in (Cohen et al. [1992]). Borovik's remarkable subgroup remains the sole finite maximal closed subgroup whose socle is a product of more than one simple group (cf. (Borovik [1990]; Liebeck & Seitz [1990])). The remaining case, where the socle is simple and there is no abelian normal subgroup, is very hard. See (Seitz [1992]) and references contained therein, for general results in this direction. The determinations of maximal finite subgroups of groups of Lie type G_2, F_4 and E_6 are fairly satisfactory, due to, among others, (Aschbacher [1991]; Kleidman [1988]; Magaard [1990]). Those for E_7 and E_8 are still unfinished. It should be noted that not all finite subgroups are contained in maximal finite subgroups. Many finite subgroups of a torus are examples.

6. Description of hard embeddings

The earliest description of a method for an embedding of an $L(2,s)$ in an exceptional Lie group was perhaps (Meurman [1982]) (although in this case, according to I. of §4, no explicit construction was necessary). A method along the same lines works in principle for most of the hard cases. The starting point for these constructions is a presentation for $L = L(2,s)$ by generators and relations, together with a model for G.

By way of example, consider the group $L(2,s)$, where s is an odd prime. It has a presentation of the form

$$\begin{aligned} L_1 = \langle u,t,w \mid\ & u^s = t^{(s-1)/2} = 1,\ tut^{-1} = u^{g^2}, \\ & w^2 = 1,\ wtw = t^{-1}, \\ & (uw)^3 = 1, wu^g w = t^a u^i w u^j \rangle \end{aligned}$$

for integers g, a, i, j such that

i. g mod s is a generator of the multiplicative group of Z/s,
ii. $a \equiv (s-3)/2 \pmod{s}$,
iii. $i \equiv g^{-a} \pmod{s}$ and $j \equiv i^{-1} \pmod{s}$.

In particular, the following map on $\{u,t,w\}$ can be extended to an isomorphism $L_1 \to L$.

$$u \mapsto \pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad t \mapsto \pm\begin{pmatrix} \overline{g} & 0 \\ 0 & \overline{g}^{-1} \end{pmatrix}, \quad w \mapsto \pm\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $\overline{g}$ stands for g mod s. The subgroup $\langle u,t\rangle$ of L_1 is clearly isomorphic to a Borel subgroup of L, and hence a Frobenius group of order $s(s-1)/2$, and so a Todd-Coxeter enumeration for the presentation with respect to this subgroup of L_1 (yielding that $\langle u,t\rangle$ has index $s+1$) suffices to show that L and L_1 are isomorphic.

Next, a model for $G(\cdot)$ is needed. In almost all cases, $G(F)$, for a suitable field F, is viewed as a subgroup of the linear group $GL(n, F)$ preserving a form or a multiplication on F^n. In (Cohen, Griess & Lisser [1993]) the Lie algebra product μ is used to define G, that is, $G(F)$ is viewed as the subgroup of $GL(n, F)$ of all matrices preserving μ. In (Griess & Ryba [1991]) $G(F)$ is viewed as the subgroup of $GL(n, F)$ of all matrices which leave invariant (under conjugation) the Lie algebra L corresponding to $G(F)$, where L is presented as a linear space of $n \times n$-matrices over F.

In such a setting, a maximal split torus T is fixed, usually the subgroup of all diagonal matrices in $G(F)$, and its normalizer N in $G(F)$ is a monomial group, which can be described explicitly.

It is easy to embed $\langle u, t \rangle$ in $G(F)$ provided F contains s-th roots of unity. Because this group is supersolvable, the theorem of Borel-Serre mentioned in the proof of 4.1(ii) (a variant for algebraic groups is due to (Springer & Steinberg [1970])) yields that, up to conjugacy, we may assume $\langle u, t \rangle$ is contained in N, the normalizer in G of the standard maximal torus T of G. The structure of N then forces the image of u to lie in T. The T-coset of the image of t in $W = N/T$ belongs to a well-studied conjugacy class inside the Weyl group. In most cases, for instance in the Kostant series, the image of t in W is a regular element in the sense of (Springer [1974]). This implies that all elements in the inverse image $Tt \subset N$ are conjugate in $G(F)$.

Now suppose we are given such an embedding of $\langle u, t \rangle$ into N. We shall denote the images of u and t also by u and t. How do we extend it to an embedding of L_1 into G? In order to find an element in G which is the image of $w \in L_1$ under an embedding, we start with $w_0 \in N$ inducing an involution inverting t in W. Due to the good control over N, such an element w_0 is easy to find. The next stage is to look for $w \in w_0 C$, where C is the centralizer in $G(F)$ of t. If t is a Coxeter element, C is a maximal split torus (in general, the dimension of C is at least the Lie rank r of G). Maximal split tori are conjugate. Suppose now that t is a Coxeter element. In order to be able to compute with elements of C, we need to identify this group with an explicit conjugate of T, that is, we need to find $d \in G$ with $dTd^{-1} = C$. For this operation, it is useful to have $(s-1)/2$-th roots of 1 in F. Thus, an appropriate choice for F would be a field of prime order $1 + ms(s-1)/2$ for a suitable natural number m. (If $L = L(2, 61)$, we can take $m = 1$ so that $|F| = 1831$.) Here, for the first time, we need an explicit model for G. We can take it to be $G(F) = \operatorname{Aut} g$, where g is the corresponding Lie algebra defined over F.

Computationally, finding d is a hard step. An eigenspace decomposition of F^n with respect to d is needed, but is not enough. In (Cohen, Griess & Lisser [1993]), detailed information regarding the behaviour of the eigenspaces under the multiplication μ was exploited to finish this step. But the result is gratifying in that it enables us to explicitly construct C, so that the embedding problem can be transformed into a set of equations, the unkowns of which are the entries of a matrix representing an element $x \in C = dTd^{-1}$. The number of unknown entries in x is $r = \dim T$ if we regard the diagonal entries of $d^{-1}xd$ as monomials in r independent variables, or n, if we regard that diagonal as $n = \dim G$ linear variables.

The final step consists of solving the equations pertaining to the relation $(uw)^3 = 1$ (often the most complicated relation $wu^gw = t^a u^i w u^j$ is not needed). To this end, rewrite this relation as:

$$uw_0xu = w_0xu^{-1}w_0x$$

for $x \in dTd^{-1}$. By letting these matrices act on vectors $y \in d\mathfrak{t}$, where $\mathfrak{t}$ is the Cartan subalgebra related to T, we get the equations

$$uw_0xuy = w_0xu^{-1}w_0y,$$

which are linear in x. Big systems of linear equations are more easily solved than small systems of polynomial equations. Thus, for the case $s = 61$, the linear equations in $n = 248$ variables were quite manageable, whereas the polynomial equations in $r = 8$ variables were extremely difficult to solve. (Recently, A. Reeves, using the software package Macaulay, managed to solve a set of polynomial equations derived in the course of the work described in (Cohen, Griess & Lisser [1993]); it took her Sparc server little over an hour to find the unique solution.) Seeing to it that $|F|$ is coprime with $|L|$, we can conclude by Theorem 3.4 that L embeds in $G(\mathbb{C})$.

In some cases, the lifting argument was not needed; for instance, in (Cohen & Wales [1993]), the group $L(2,13)$ could be explicitly embedded in $3 \cdot E_6(\mathbb{C})$.

An entirely different method of construction, based on computer experiments, is to be found in (Kleidman & Ryba [1993]). The method works with a smaller field F, and has a probabilistic portion (for finding an embedding; the resulting existence proof is not probabilistic).

7. Existence by computer

The kind of proof described in the previous section raises the question about construction of an embedding by means of computer. In this section, we discuss some of the issues regarding an existence proof by computer.

It goes without saying that a computer-free proof, if it did not degenerate into a dull stack of computations accounted for on paper, would be much preferable. But given the fact that no such proof of Kostant's conjecture is in sight, we are faced with the question of what is acceptable as a proof when computations are involved that can no longer be checked by a single person using pad and pencil.

For the sake of exposition, it is convenient to revisit the embedding described in the previous section. So, suppose we are given three square matrices u, t and w of size 248 and we wish to verify that they generate a subgroup of the Lie group of type E_8 isomorphic to the finite group $L = L(2, 61)$. Here, by 'being given' a matrix of this size, we mean that there is a simple routine available for generating them, or that they are on file, because the amount of data is simply too large for visual inspection or typing. We need to be able to multiply two such matrices. These computations are useful since it is possible to verify an identity between products of matrices. In particular, checking whether u, t and w satisfy the defining relations for L_1 is feasible.

For the computations involved it is essential that the entries lie in a field of moderate size, such as $\mathbf{Z}/p$ for p a prime less than 10^6. Multiplication of two matrices of this size would otherwise not be practical. This shows the importance of the lifting results: the computer calculations will only explicitly embed L in $G(\mathbf{Z}/p)$; Theorem 3.4 is subsequently used to derive the existence of an embedding in $G(\mathbb{C})$.

In purpose-dedicated software, multiplication of two 248×248-matrices over such prime order fields takes less than a second. In packages like GAP, MAGMA, and LiE which are specially suited for computations with such matrices, it will take several seconds, which is still acceptable.

As a consequence, it is possible for everyone with access to a workstation with one of the abovementioned packages to perform the necessary matrix multiplications in order to be convinced that the defining relations for L_1 are satisfied. So much for the verification that u, t and w generate a subgroup of $GL(248, \mathbf{Z}/p)$ isomorphic to L.

Another part of the verification that L embeds in $G(\mathbf{Z}/p)$ is the check that u, t and w preserve the E_8 Lie algebra product. To this end, the Lie algebra product μ is given as a vector $\mu(x, y)$ of 248 polynomials in the 2×248 variables x, y (representing vectors of $(\mathbf{Z}/p)^{248}$). Then, for $k = 1, \ldots, 248$ and $g = u, t, w$, it is checked whether, for generic vectors x and y, the k-th component of the vectors $\mu(gx, gy)$ and $g\mu(x, y)$ coincide. By reduction of the check to one component at a time, this computation is feasible in a general purpose package (such as Maple or Mathematica).

In general, it can be argued that, provided the source code and the software used is well documented, widely available and implementable, computations that are independently verifiable (with relative ease) can be accepted as parts of a mathematical proof. The argument in defense of acceptance is that, if the intermediate steps documented in the proof suffice for a monastery of mathematically skilled monks to be able to perform the computations within a reasonable time span, the usual proof check is conceivable (albeit blown out of proportion) in times and places where no computers are available.

Although the proof requires relatively little computer effort, finding such a proof can be much more time consuming. Indeed, this has been the case in the computer search for the right matrices yielding the preceding embedding of L in $G(\mathbf{Z}/p)$, especially w. But, once the 'oracle-like' results are establish, the time-consuming constituents of the computer work need not be repeated (with the minor exception that, in (Cohen, Griess & Lisser [1993]), a uniqueness proof (up to conjugacy) of the embedding of $L(2,61)$ in $E_8(\mathbb{C})$ is given that depends on computer computations).

8. Conclusion

Most embedding questions regarding finite simple groups in complex Lie groups of exceptional type have been solved, except for the four persistent problems of §4.3. More detailed questions are still (partially) open, such as minimal splitting fields, the number of conjugacy classes, a description of integral representations, and a geometric interpretation of the existence of such amazingly small groups as maximal closed Lie subgroups of such huge Lie groups.

References

A.V. Alekseevskii [1974], "Finite commutative Jordan subgroups of complex simple Lie groups," *Funct. Anal. and its Appl.* 8, 277–279.

A.V. Alekseevskii [1975], "Maximal finite subgroups of Lie groups," *Funct. Anal. and its Applications* 9, 248–250.

M. Aschbacher [1991], "The 27-dimensional module for E_6, IV," *J. Algebra* 191, 23–39.

A. Borel & J-P. Serre [1953], "Sur certains sous-groupes des groupes de Lie compacts," *Commentarii Math. Helv.* 27, 128–139.

A. Borel & J. de Siebenthal [1949], "Les sous-groupes fermés de rang maximum des groupes de Lie clos," *Commentarii Math. Helv.* 23, 200–221.

A.V. Borovik [1989], "On the structure of finite subgroups of the simple algebraic groups," *Algebra and Logic* 28, 249–279, (in Russian).

A.V. Borovik [1990], "Finite subgroups of simple algebraic groups," *Soviet Math. Dokl.* 40, 570–573, (transl).

R. Brauer [1980], *Collected papers*, MIT Press, Boston, ed. by P. Fong and W.J. Wong.

F. Bruhat & J. Tits [1972], "Groupes réductifs sur un corps local I. Données radicielles valuées," *Publ. Math. IHES* 41, 5–252.

A.M. Cohen & R.L. Griess, Jr. [1987], "On finite simple subgroups of the complex Lie group of type E_8," *AMS Proceedings of Symposia in Pure Math.* 47, 367–405.

A.M. Cohen, R.L. Griess, Jr. & B. Lisser [1993], "The group $L(2,61)$ embeds in the Lie group of type E_8," *Comm. Algebra* 21, 1889–1907.

A.M. Cohen & A.G. Helminck [1988], "Trilinear alternating forms on a vector space of dimension 7," *Communications in Algebra* 16, 1–25.

A.M. Cohen, M.W. Liebeck, J. Saxl & G.M. Seitz [1992], "The local maximal subgroups of exceptional groups of Lie type, finite and algebraic," *Proc. London Math. Soc.* 64, 21–48.

A.M. Cohen & G.M. Seitz [1987], "The r-rank of groups of exceptional Lie type," *Proceedings of the KNAW* 90, 251–259.

A.M. Cohen & D.B. Wales [1983], "Finite subgroups of $G_2(\mathbb{C})$," *Comm. Algebra* 11, 441–459.

A.M. Cohen & D.B. Wales [1992], "On finite subgroups of $E_6(C)$ and $F_4(C)$," preprint.

A.M. Cohen & D.B. Wales [1993], "Embedding of the group $L(2,13)$ in the groups of Lie type E_6," *Israel J. Math.* 82, 45–86.

H.S.M. Coxeter [1946], "Integral Cayley numbers," *Duke Math. J.* 13, 561–578.

R.L. Griess, Jr. [1977], "Finite groups as automorphisms of Lie algebras," Vortragsbuch, Oberwolfach.

R.L. Griess, Jr. [1994], "Basic conjugacy theorems for G_2," University of Michigan, report, Ann Arbor.

R.L. Griess, Jr. [1991], "Elementary abelian p-subgroups of algebraic groups," *Geom. Dedicata* 39, 253–305.

R.L. Griess, Jr. & A.J.E. Ryba [1991], "Embeddings of $U_3(8)$, $Sz(8)$ and the Rudvalis group in algebraic groups of type E_7," preprint, U. of Michigan.

M. Isaacs [1976], *Character theory of finite groups*, Pure & Appl. Math. #69, Academic Press, New York.

Z. Janko [1966], "A new finite simple group with Abelian Sylow 2-subgroups, and its characterization," *J. Algebra* 3, 147–186.

P.B. Kleidman [1988], "The maximal subgroups of the Chevalley groups $G_2(q)$ with q odd, of the Ree groups ${}^2G_2(q)$ and of their automorphism groups," *J. Algebra* 117, 30–71.

P.B. Kleidman & A.J.E. Ryba [1993], "Kostant's conjecture holds for E_7: $L_2(37) < E_7(\mathbb{C})$," *J. Algebra* 161, 535–540.

P.B. Kleidman & R.A. Wilson [1993], "Sporadic simple subgroups of finite exceptional groups of Lie type," *J. Algebra* 157, 316–330.

V. Landazuri & G. M. Seitz [1974], "On the minimal degrees of projective representations of the finite Chevalley groups," *J. Algebra* 32, 418–443.

M.A.A. van Leeuwen, A.M. Cohen & B. Lisser [1992], *LiE, A package for Lie group computations*, CAN, Amsterdam.

M.W. Liebeck & G. Seitz [1990], "Maximal subgroups of exceptional groups of Lie type, finite and algebraic," *Geom. Dedicata* 35, 353–387.

K. Magaard [1990], "The Maximal Subgroups of the Chevalley Groups $F_4(F)$ where F is a Finite or Algebraically Closed Field of Characteristic $\neq 2, 3$," PhD Thesis, Caltech.

A. Meurman [1982], "An embedding of $PSL(2,13)$ in $G_2(\mathbb{C})$," in *Lie Algebras and Related Topics*, Lecture Notes in Mathematics #933 , Springer Verlag, Berlin, 157–165.

S.P. Norton [1988], "On the group F_{24}," *Geom. Dedicata* 25, 483–501.

G.M. Seitz [1991], "The maximal subgroups of exceptional algebraic groups," *Memoirs of the Amer. Math. Soc.* 90, 1–197.

G.M. Seitz [1992], "Subgroups of finite and algebraic groups," in *Groups, combinatorics and geometry*, M.W. Liebeck & J. Saxl, eds., London Math. Soc. LNS #165, Cambridge University Press, Cambridge, 316–326.

J-P. Serre [1994], Private Communications.

P. Slodowy [1993], "Two notes on a finiteness problem in the representation theory of finite groups," Universität Hamburg, Hamburger Beiträge zur Mathematik.

T.A. Springer [1974], "Regular elements of finite reflection groups," *Inventiones Math.* 25, 159–198.

T.A. Springer & R. Steinberg [1970], "Conjugacy classes," in *Seminar on Algebraic groups and related finite groups*, Springer Lecture Notes in Math. #131, Springer-Verlag, Berlin, 167–266.

M. Suzuki [1982], *Group Theory I*, Grundlehren der math. Wiss. #247, Springer-Verlag.

D. Testerman [1989], "A construction of certain maximal subgroups of the algebraic groups E_6 and F_4," *J. Algebra* 122, 299–322.

J.G. Thompson [1976], "A simple subgroup of $E_8(3)$," in *Finite groups*, N. Iwahori, ed., Tokyo, 113–116.

J. Tits [1979], "Reductive groups over local fields," in *Automorphic forms, representations and L-functions, Oregon State Univ., Corvallis, Ore.*, A. Borel & W. Casselman, eds., Proc. Symp. Pure Math. #XXXIII, Amer. Math. Soc., Providence, R.I., 29–69.

A. Weil [1964], "Remarks on the cohomology of groups," *Annals of Math.* 80, 149–157.