EINDHOVEN UNIVERSITY OF TECHNOLOGY

Department of Mathematics and Computing Science

Memorandum COSOR 83-16

On using discrete random models within decision support systems

by

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Eindhoven, The Netherlands

June 1983

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Abstract:

In this paper we review how models for discrete random systems may be used to support practical decision making. It will be demonstrated how organizational requirements determine to a large extent the type of model to be applied as well as the way in which the model should be applied. This demonstration is given via several practical examples of Markov chain models, cohort models, and Markov decision models. The examples are drawn from various areas ranging from the purely technical to social applications.

It is demonstrated that the models that are needed for supporting the decision making process may vary from purely descriptive models to optimization models. Similarly, the obvious way of application of a model may vary from straightforward numerical analysis to interactive modelling procedures based upon managerial evaluation.

It will also be demonstrated how the numerical methods to be used depend on the structure of the model as well as on applicational aspects. The numerical aspect is strongly related to the aforementioned aspects, since the model choiche heavily determines the computational possibilities.

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1. Introduction

The main goal of the present paper is to demonstrate the practical relevance of discrete mathemetical models for dynamic systems with random aspects. It will be shown that the environment highly affects the model choice as well as the role of the model in the decision making process.

Random dynamic systems can be recognized in a large number of practical fields. Examples will be given from areas like insurance, inventory and production control finance, marketing as well as demographic and manpower processes. We will approach the problems from the point of view of managers who are interested in the analysis of dynamic phenomena because of the information that such an analysis might provide for supporting a decision making process. Intentionally, the phrasing is kept vague, since one of the messages of this paper is that mathematical models should not be designed with the purpose to replace the decision making. Only in rather rare cases whole systems can be formalized in such a way that the decision making can be left over to the mathematical model. Usually, only part of the relevant aspects of a problem can be formalized and the analysis of the corresponding mathematical models might be one of the information sources that can help in reaching a decision. The role played by mathematical models in such situations is one of the subjects of the present paper.
Because of the purposes of the paper, we will focus on the class of practical problems rather than the class of mathematical techniques. Nevertheless, some technicalities are needed to stress our point. It will appear that natural models for the investigated phenomena are dynamic programming models like Markov chains, cohort models and Markov decision models. "Natural" here means that these types of models allow a relatively truthful and credible description of reality, are relatively open to mathematical and numerical analysis, can be fitted naturally into the data collecting, and decision making process.

In the next section the discussion will be initiated by the presentation of a highly simplified example, namely some aspects of the managing of a car insurance company. This example gives some insight in the different types of roles mathematical models can play in decision making processes. It is particularly indicated how the basic mathematical models for client behaviour in this car insurance example might

constitute the kernel of a decision support system for the company.
In section 3 a more general discussion is devoted to the relation between mathematical models for discrete random phenomena and the organizational environment. Since, usually, one cannot convey the role of the decision maker to the analyzer, it becomes clear that models (and computer programs) should be developed in such a way that the model and its analysis can be fitted naturally into the decision making process. This requires, for instance, that sensitivity analyses, parameter variations, and model modifications can be executed in a simple and natural way in order to allow, as one might say, a discussion between the decision makers (management) and the model. In section 3 the nature of this discussion will be explained. Of course, this whole presentation primarily focuses on the class of discrete random systems. This third section also contains a review of the principles of Markov chain models, cohort models, and Markov decision models.
In this third section we propose a partition of the problems in design problems and dynamic control problems. In design problems one has to choose a design for a dynamic system. Once the design has been selected, a descriptive model can be used for the analysis of the process. In dynamic control problems, however, the system can be controlled during its evolution using its past and current behaviour. In fact, the partitioning is not strict, since the steering of a ship is a dynamic control problem and, therefore, the design of a ship's steering equipment would be the design of a dynamic control system.
Section 4 is devoted to the description of some examples of design problems.
Also for another reason the partitioning is not strict, since there are many situations in which redesign is possible during the process. For instance, the determination of a promotion policy in a manpower system would be a typical design problem if it would be done once and for ever, however, that is usually not the case; redesign is quite customary. In this way one obtains a third type of system which we call controlled autonomous systems. Examples of such systems are given in section 5. Several of these examples contain, as the manpower example mentioned, cohort aspects. Like in the pure design problems, part of the relevant information for decision making might be provided analyzing some alternatives via autonomous or descriptive models. So, again, control and optimization need not explicitly be incorporated

in the models. On the contrary, quite often it would not be very sensible to force the decision making into a control or optimization model.
Even the dynamic control models, as described and exemplified in section 6 in the form of Markov decision models, will usually not be applied as strictly action-prescribing models.
In section 7 numerical aspects of solving discrete random systems are treated. As the structure of the model bears a heavy influence on the numerical possibilities, the numerical aspects in their turn have quite some influence on the modelling. Topics like aggregation and decomposition are discussed briefly in their relation to numerical analysis and decision making.
Section 8, finally, is used to reach some conclusions and give some final remarks.

2. An example of a discrete random process

In this section we will introduce an oversimplified version of a vehicle insurance company in order to introduce the type of models as well as the role of these models in the decision process.

In the insurance business the company sets the condition and the (prospective) clients react upon these conditions. So the basic process in the vehicle insurance industry is the behaviour of (prospective) clients in a given environment. Possible actions for these clients are:

signing for an insurance with specific conditions
dropping their insurance
claiming a damage

The decision to claim or not to claim will depend on the size of the damage, but also on the current no-claim discount of the client, the time in the year, and the conditions of the insurance with respect to own risk and no-claim discounts.
For the operations of the insurance company it will be necessary to answer a number of questions concerning these basic processes, for example:

What is for certain conditions the average premium to be paid by a client?
What is the optimal claiming policy for an individual car owner?
How much worse is it for a client to claim as soon as the damage is more than the no-claim discount?
How does the claim limit depend on the time within the (insurance)year?
How sensitive are such results for changes in the accident rate?

How does the accident rate depend on age, area, etc.?

For which (prospective) clients would a specific option be attractive?

In order to analyse such questions, one can try to model the process in which one (prospective) client is involved. Claiming models, for instance, are described in detail by De Leve e.a. [9] and by Hastings [14]. Also for analyzing joining decisions one should consider claiming models and compare the results for claiming models under different conditions.

Let us make such a claiming model. Since it is only used for demonstration purposes, it can be kept very simple. Suppose that the company has the following no-claim discount system. A new client pays the full premium but obtains a 20% discount on the next year's premium if he or she does not claim in the first year. After two consecutive years with no claims even a discount of 30% is granted. However, after a claim the full premium is due for the next year.

It seems sensible for a client to base the claim behaviour upon the time within the insurance year the damage is sustained. We incorporate this aspect into the model by dividing the insurance year in two equal parts (which is not principally different from a division into 12 or 52 parts). Each half year one may now observe the situation of the client with respect to the insurance and record this situation by 2 indices:

the first index indicates the part of the year,
i.e. 0 means that the premium for the next year is due now, whereas 1 means that the client has still half a year to go before the new insurance year starts,

the second index indicates the premium level (100%, 80%, or 70%) for the next insurance year with the proviso for the mid-year observation that this premium might be affected by claims in the coming half year.

So, one obtains 6 situation descriptions - called states - which are depicted in figure 1, in which a possible path of a client time is reproduced

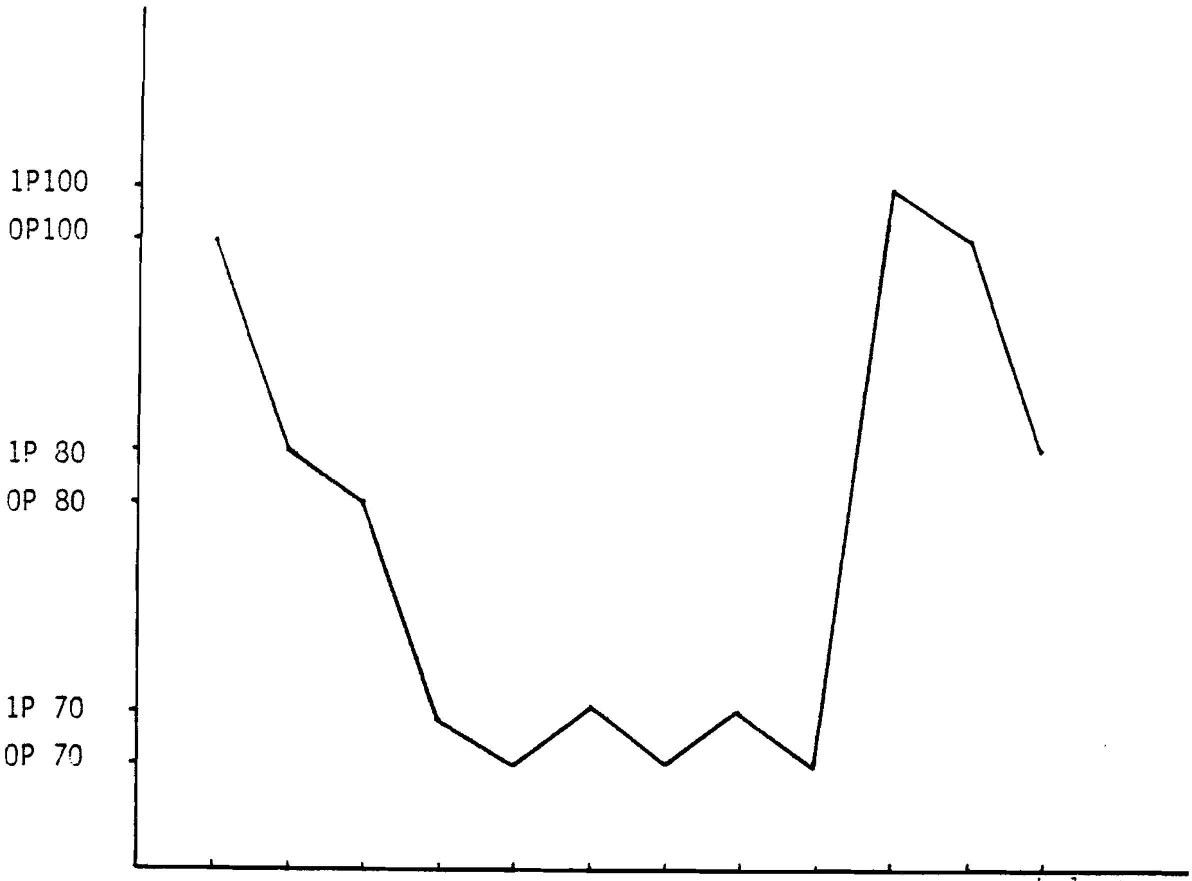

fig. 1: The development of the state over time for the premium situation of a particular car owner

The random element in this process is caused by the occurring or non-occurring of damage for the car owner. But the process, is also influenced by the reaction of the client, namely whether he claims or not.
In this simplified example we use periods of half a year, so let us assume for simplicity that the car holder decides per period whether a certain damage will be claimed or not. In this way no difference is made for the different parts within half a year, but that shortcoming could be repaired by taking shorter periods. Let us also assume that the car owner bases his decision on the state and uses a claim limit for each state, which seems quite sensible. Suppose these possible claim limits belong to a finite set L. Then the accident rates and damage statistics determine for each given state i and for each limit ℓ the probability of a transition to any state j: $P(i,j;\ell)$. Similarly, one can determine the one-stage expected costs $c(1;\ell)$ for a period which starts in state i with claim limit ℓ .

For a given claim policy λ, assigning a claim limit $\lambda(i)$ to each state i, one may compute the total expected discounted costs for an infinite time horizon after start in state i as solution of the set of linear equations in the variables $v(i;\lambda)$ with i from the set of states I, say:

$$(2.1) \quad v(i;\lambda) = c(i;\lambda(i)) + \beta \sum_{j \in I} P(i,j;\lambda(i))\, v(j;\lambda) \quad i \in I$$

where β $(0<\beta<1)$ is the discount factor.
Similar, but slightly more complicated, relations hold for time-dependent policies, finite time horizon, and for time-averages of nondiscounted costs.
Let

$$(2.2) \quad v(i) \equiv \min_{\lambda} v(i;\lambda)$$

the total costs function for the optimal policy, one even has the following elegant relations

$$(2.3) \quad v(i) = \min_{\ell \in L} \{ c(i;\ell) + \beta \sum_{j \in I} P(i,j;\ell)\, v(j) \} \quad i \in I$$

These relations can be used to compute the minimal costs and the optimal claim limits for a given set of parameters $P(i,j;\ell)$, $c(i;\ell)$ (cf.sections 6 and 7).

In this way we have modelled the claim behaviour as a controlled system. Actually, the interest to the insurance company of this model is provided by the fact that it will give a lower bound to its profits, since, most likely, the clients will not behave optimally.

For supporting decision making within the company, it is also relevant to study the behaviour of groups of clients, since the combined efforts of all clients determine the financial well-being of the company. Let us consider a more or less homogeneous group of clients all having the same insurances. Let us suppose that these clients have chosen a claiming policy which results in an average likelihood $P(i,j)$ for a client in state i to make a transition to state j. Using that property, one easily obtains for the expected number $n_t(j)$ of clients in state j at time instant t

$$n_t(j) = \sum_i n_{t-1}(i)\, P(i,j)$$

or in vector-matrix notation

$$n_t = n_{t-1} P$$

with n_t and n_{t-1} as row-vectors.

The $P(i,j)$ may not add up to 1 when summed over the j for fixed i. Namely, there may be a positive probability of dropping the insurance. Also there may be recruits. If the expected number of recruits in state j is $r_t(j)$ for period t, then we obtain

$$(2.4)\quad n_t(j) = r_t(j) + \sum_i n_{t-1}(i)\, P(i,j)$$
or
$$n_t = r_t + n_{t-1} P$$

For the financial consequences of this group of clients in period t we obtain the following forecast

$$\sum_i n_t(i)\, [Pr(i) - c(i)]$$

where $Pr(i)$ denotes the premium of a client in state i and $c(i)$ the expected costs for the company caused by this client. (If recruitment and loosing of clients also gives costs, then they can be incorporated in a simular way).

Also long-term behaviour can be analyzed in this way. But, whatever one analyzes, the most important thing is to compare different policies of the company. Different policies can mean different recruitments, but also different premiums and even different discount systems. In all cases new transition probabilities have to be estimated and then the results can be computed. Also different policies can be used after each other, for instance first some periods a policy to attract new clients and later on a change to a policy for keeping the clients. Therefore, we call this type of system a controlled autonomous system. With completely fixed recruitment and transition probabilities, it would be called an autonomous system.

From the above description it will be clear what kind of function the mathematic models can play in the decision processes within the insurance company. Particularly, the controlled autonomous systems are very useful for the design of good marketing policies as well as for judging financial consequences of different alternatives.

3. Discrete random systems and the organizational environment of decision making.

In the preceding section we introduced three types of models that can be used to support decision making with regard to vehicle insurance. However, the success or failure of an application of such a type of model will heavily depend on the way in which the model and the method of analysis link up with the managerial and organizational circumstances. In recent years, several authorities have re-amphasized the necessity of embedding formal techniques in the integral decision making process. It has been stressed repeatedly that application of formal methods without carefully established relation to the organizational environment are bound to become failures. The necessary approach has been described explicitly in the literature on decision support systems [1] , [19]. This approach, however, requires the combination of expertise in various disciplines and experience with several types of organizational processes. The contribution of mathematical models might indeed be essential, however, use of such models can only be effective if other - usually qualitative-aspects receive the attention which they deserve.

Actually, the examples of the previous section demonstrate that the organizational environment heavily influences the type of model that is required as well as the way in which the model may be applied. For instance, decisions with respect to the introduction of a new type of car insurance are probably made at the highest level within the company. The impact of such decisions lasts for several years. The design and choice of a new type of insurance belong to the category of strategic planning problems and it will be clear that many qualitative aspects play an important role in the final decision. Problems like "how to adapt premiums to increasing repair costs?" are of a different type and might be categorized as control problems.
From the example the differences between strategic planning and control problems will be clear. In general, one usually distinguishes planning and control as being concentrated on different aspects of managing an organization. Such a distinction can never be categorical, however, it

may be helpful to make a conceptual separation between different types of decision making activities. For our purposes, the distinction of Anthony [2] is interesting:

strategic planning,
management control,
operational control.

In this division the control activities are separated in two classes, such that premium adaption belongs to the class of management control problems, whereas claim and client acceptation belong to the operational control problems.
These three categories often coincide with the three levels within an organization that can be characterized as strategic, coordinative and operating. Another closely related division of decisions is obtained by a classification based on the time period of supposed impact:

strategic (or long term) decisions,
tactical (or medium term) decisions,
operational (or short term) decisions.

If, on the other hand, one classifies the decision problems according to their level of fussiness, one might obtain:

unstructured decision problems,
semi-structured decision problems,
structured decision problems.

Premium adaptation here definitely belongs to the class of well-structured decision problems, designing a new car insurance probably belongs to the class of semi-structure decision problems and entering the field of car insurance will belong to the unstructured decision problems.

Moreover, a classification can be made with respect to the decision making process, Keen and Scott Morton [19] distinguish 5 different approaches:

- The rational way of decision making.
  In this classical approach one strives after the "optimal" action out of a set of feasible actions.
- The satisficing process oriented way of decision making of Simon [30].
  The acceptability of the ultimate result for all parties involved gives in this approach the red thread for the decision making process.
- The organizational procedure approach of Cyert and March [7].
  The specific operating procedures are supposed to determine the evolving of the decision making process in this approach.
- The political way of decision making.
  According to this approach the final decisions are made after a

bargaining process between the parties involved.

- The individual differences approach.
  In this approach the problem solving and information processing behaviour of individual decision maker is the determining factor for the decision making process.

In practice usually some mixture of these types of approach for the decision making process will occur. Nevertheless, it will be clear that the type of models that have to be constructed to provide information for supporting decision making processes will vary with the type of the decision making process itself. However, like in the car insurance company, one may observe some tendencies which are related to the other categorization mentioned. Such a tendency may be caused by the fact that higher organizational levels impose side conditions for the lower levels. Moreover, the decision making approach itself is related to the hierarchical level: the political way of decision making is encountered more often at the strategic level, whereas the rational approach for well-structured decison problems is more common at the operational level.
It will be obvious that the classifications given are oversimplified and not at all generally applicable. Nevertheless, this discussion may be of help in studying the different types of decision support that are needed in different situations. The classifications and their relations are given to encourage a sensible choice, taking explicitly into account the organizational environment.
For a reallife situation like the one indicated in the insurance example, three models have been described for supporting decision making in specific situations. The relevance of these models and the way they have to be applied depends on the organizational environment. Even in that simplified example, it was already clear that decision support at the strategic level requires a partly different approach than decision support at the operational level. For instance the decisions with respect to the design of a new insurance might be based on highly aggregate information, moreover, such decisions are not made every day. On the other hand, adaptation to increasing repair costs or accident rates is probably actual at very regular times, for instance, each half year. Advices to clients have to be given on a daily basis. These differences also imply differences in the requirements on the models as well as on allowed response times for delivering information. At the strategic level the data collecting, model building, model analysis and reporting can last a relatively long time period compared with the

operational level. At the lower levels the methods should be more standardized and readily available. The same feature also makes numerical aspects important: only efficient computational procudures can provide short computation times. Therefore numerical aspects will be treated in section 7.

4. Some design problems

In this section some examples of designing autonomous systems will be described. In practice, no system is really autonomous, since there is always a possibility for redesign if there is some need to do that. However, for the exposition it is better to demonstrate first the pure design problems. In section 5 redesign will be taken into account when we consider controlled autonomous systems.

4.1. The vehicle insurances company.

A typical design problem of a strategic character is the design of a completely new type of car insurance. Considering variations of existing insurance conditions is a decesion problem of a tactical nature. Let us consider a simplified version of the example of section 2 and contemplate on some variations. The simplification is made by taking periods of one year. The described insurance problem has a structure characterized by the graph of figure 2 in which each square denotes a premium level.

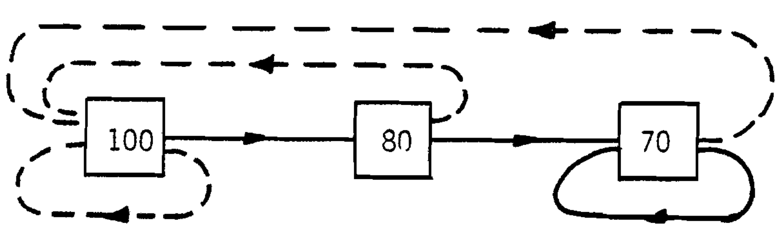

fig.2: Discount structure of the car insurance problem. Squares denote premium levels in percentages, drawn arcs indicate transitions in a year without a claim and dotted arcs indicate transitions in years with a claim.

An alternative could consist of the incorporation of an extra discount level at 60% as indicated in fig.3.
In order to compare the two different premium policies of fig.2 and fig.3 one has to evaluate the consequence for the Markov chains related to the discount structures.

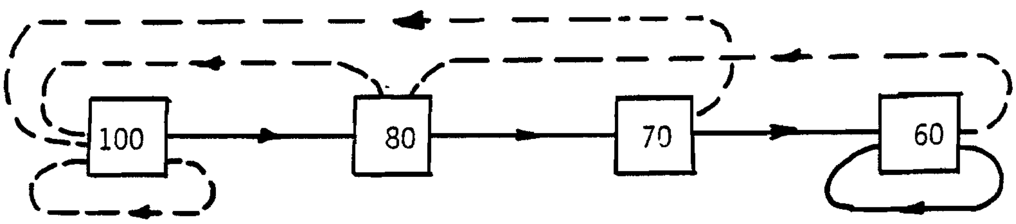

fig.3: Alternative discount structure for the car insurance problem.

The data for the existing discount structure are derived from operational information of the firm. The alternative scheme, however, presents difficulties with respect to claim rates. Particularly for clients on the new level the claim rate is uncertain, but also on the old levels the claim rate might be influenced by the new structure. Attractiveness for new customers of the new scheme is even more difficult to estimate. Nevertheless, for estimations of the entrance numbers of customers and for the claim rates, the new scheme can be analyzed and compared to the existing one. In some steps a new scheme can be developed using a iterative procedure with the following steps per iteration: new structure, estimation of data, analysis, evaluation and comparison.

4.2. The dyeing of leather bags

An essential part of the production process of leather bags is the dyeing of pieces of leather. Before a piece of leather proceeds to the dye-bath it is inspected. As a result it may be decided that before the dyeing a preparatory treatment is necessary. This preparatory treatment increases the likelihood that dyeing results in an even colour and texture of the piece. After dyeing but also after the preparatory treatment a further inspection decides on the further treatment of the piece. The inspection after the dyeing may result in rejection of the piece, acceptation, reducing, but also in a retry with a preparatory treatment. The inspection after the preparatory treatment may result in rejection, or acceptation for the dyeing. The possible steps in this process are depicted in fig.4, in which also percentages of pieces which make that specific transition from the same outgoing position are indicated.

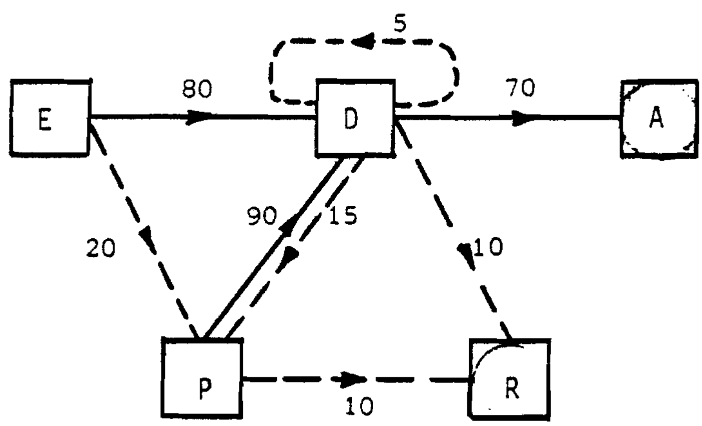

fig.4: Structure of the dyeing process. The stages E(enhance) P(preparation), D(dyeing) include an inspection. The dotted lines indicate some sort of rejection after inspection. Ultimately A(accepted) or R(definitely rejected) are reached.

The design problem to be discussed now has to do with this part of the production process. Two types of alterations are considered, one is organizationally by nature, the other technically. The first possibility consists of shipping the entrance inspection and giving all incoming pieces an initial preparatory treatment. The technical innovation would mean the installation of a new type of dye bath, which improves the dyeing results in such a way that definitive rejection would decrease from 10 to 5 percent and sending from dye-bath to preparatory treatment from 15 to 10 percent.

All changes have costs effects and the question is what to do.

The first approach would be a cost analysis of the existing procedure and all three alternatives. Let us suppose that each of the production phases E , P, D require costs C(E), C(P), C(D) respectively and an accepted piece brings a reward 2 above the entering value, whereas an entering piece has costed already an amount. Then the expected profit on an entering piece of leather can be computed from the following set of equations, where v(E), v(P), v(D) are the expected profits during the rest of the dyeing process for a piece which actually arrived at stage E, P, D respectively:

$$v(E) = -c(E) + .2v(P) + .8\, v(D)$$
$$v(P) = -c(P) - .1c \qquad .9\, v(D)$$
$$v(D) = -c(D) - .1c + .7r + .15v(P) + .05v(D)$$

In a simular way models can be made and analyzed with respect to costs for the two alternatives. Such quantitative information may be of use for deciding which alternative will be chosen. However, not only pure cost motives may be decisive. It may be possible that one of the alternatives has organizational advantages as leading to a more stable process.

These models also offer the possibility of precalculating the required price for accepted pieces such that the loss incurred by the rejects are covered. Also other aspects as production time per piece and throughput can be analyzed. Sensitivity analysis may show how changes in parameters influence the outcomes. This whole range of quantitative information will give support for the decision process that has to result in the selection of one of the three alternatives. In fact, the analysis can easily lead to sensible suggestions for other alternatives both technically and organizationally.

For problems in a highly technological environment one may more often use the intrinsic formal structure of the problem to incorporate the design aspect in the mathematical formulation. Let us consider, as an example, a simple problem from the area of computer performance.

4.3. Data storage on a disk

Suppose that a lot of background data have to be stored on a disk unit. Let us consider the case that the background data belong to 4 large sets each relevant for one type of application. There is a tendency to give subsequent read/write orders for the data sets. When jumping from one data set to the other the read/write head has to be moved radially over the disk and this motion requires time. By minimizing the jumping distance, the goal of working most efficiently would be reached. A disk consists of several disks on one axis with simultaneously moving heads. For our purpose it suffices be consider the disk unit as consisting of one disk each data set occupying a set of circular tracks as indicated in fig.5.

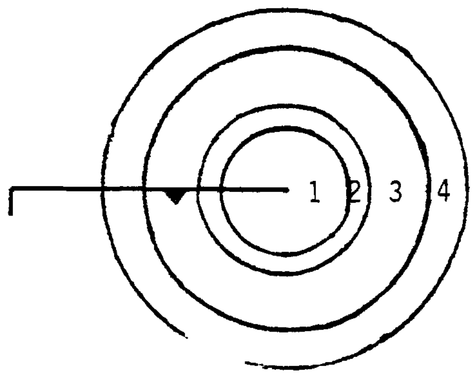

fig.5: Schematic division of a disc in 4 regions each consisting of a number circular tracks. The read/write head moves along the line to find the correct track and the reading or writing can take place since the disk turns around with a constant speed.

The amount of information on a track does not depend on its placing, so each data set consists of a given number of tracks independent of its placing. Now the problem is which placing of the 4 data sets is most efficient. The efficiency of a placing is completely determined by the required moving time between data sets. Naturally, one should place sets near each other if there is heavy traffic between them. Suppose table 1 gives the traffic intensities between the data sets.

from \ to	1	2	3	4
1	0	50	30	20
2	30	0	30	40
3	30	55	0	15
4	40	20	40	0

Table 1: Jumping intensities between data sets. The first line says that from data set 1 the jumps leads in 50% of the cases to data set 2 and in 30% of the cases to data set 3, etc.

To each permutation α of 1,2,3,4 belongs a moving function $f_\alpha(i,j)$ indicating the average time needed to move the head from a track in data set i to the requested track in data set j. For simplicity, one might determine this moving function by computing the time to move the head from the center track of data set i to the center back of data set j. Usually, the moving times are not proportional to the distance. Let p(i,j) indicate the jump probabilities as can be derived from table 1 and let q(i) be the relative visiting frequency to region i, then we obtain the following minimization problem

$$\min_{\alpha} \sum_{i=1}^{4} q(i) \sum_{j=1}^{4} p(i,j)\, f_\alpha(i,j)$$

So here the design problem may be modelled as a formal optimization problem.

For 4 data sets explicit enumeration can be used. However, for large numbers of data sets the computation of the minimal solution has to be approximated by means of a heuristic. We will not treat that topic here further.

We have treated 3 design problems of quite a different nature, but with one feature in common, namely, that they can be treated as pure design problems because of the relative long time between the current design and a futural next design. Note the meaning of "long" is indeed very relative, since in the last example it may be the order of one week. However, that is very long in that type of process in which thousands of jumps per hour are realistic. Many other examples can be given for example for the design of networks of queues [29]. In the next section we will consider problems for which the possibility of redesign is an essential aspect of the analysis.

5. Some examples of controlled autonomous systems

In the preceeding section the choice of an insurance scheme has been considered. Moreover, it has already been noted that it might be sensible to incorporate the possibility of later redesign, for instance if in first instance the scheme should be attractive for prospective clients while later on it should have the feature of keeping clients. In order to analyse these aspects it does not suffice to consider single (prospective) clients however, one needs to consider group behaviour. This is quite often the case. In this section we will illustrate these aspects with two very simple examples from other areas, since these examples require less background knowledge and also less technicalities than the vehicle insurance or the leather dyeing example would require.

5.1. Operations Research Analyst

In a hypothetical country the training for Operations Research Analyst (ORA) consists of two phases each requiring one year. Exams for both phases are only held once a year and a student who fails his exam has to repeat the whole year. Of course, students can leave the program before obtaining the ORA-certificate. A schematic overview of the whole process is given in fig.6, where the squares indicate phases in the curriculum in JA are the students who work for the Junior Analyst-certificate and in ORA are the students who work for the ORA-certificate. The circles indicate possibilities of leaving the program:

N means without any certificate

JAC means with Junior Analyst-certificate

ORAC means with ORA-certificate

JAC can be reached directly from JA and also from ORA after failing and leaving.

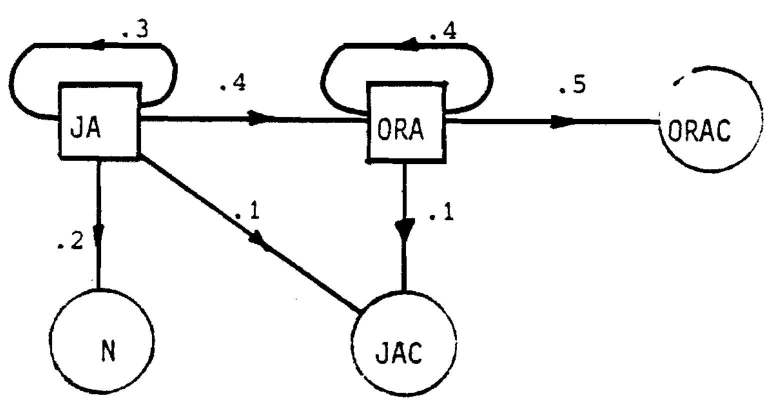

fig.6: The phases for the Operations Research Analyst program.

If the fractions of students for JA and ORA that jump to different possibilities would be known and if also the instream of new students, together with the current occupation of JA and ORA is known , then it will be possible to compute forecasts for the occupations of both phases and for the output in the years to come. Formular (2.1) would provide the tool for such a computation.

Interesting information can be derived by computing the forecasts for occupations and output for example if enrollment increases gradually over 5 years with a total of 50% (enrollment only in JA). Even more interesting would even be the question: how can enrollment increase if capacity in ORA can be increased by only 10% per year up to a total of 50%?

Or: what is the influence if the JA-exam is made more restrictive over a couple of years?

Let us consider the enrollment problem with the capacity restriction. This problem can easily be formalized. However, that is not our intention here. Let us give an informal treatment for the situation with currently 300 students in JA and 200 students in ORA, that means the capacity of ORA may be increased by 20 students a year to 300 students. Moreover, let us suppose that the numbers in figure 6 indicate the transition probabilities of individuals.

Using formula (2.1) we obtain for the occupations in the next period (without recruitment):

$n_o(JA) = 300$ $\qquad$ $n_1(JA) = 90$

$n_o(ORA) = 200$ $\qquad$ $n_1(ORA) = 200$

Recruitment in JA does not increase $n_1(ORA)$, so for the next period no increase in ORA can be envisaged. However, one period further, a recruitment might help.

$$n_1(JA) = 90 + r_1$$
$$n_1(ORA) = 200$$
$$n_2(JA) = .3*(90 + r_1)+r_2$$
$$n_2(ORA) = .4*200 + .4*(90+r_1)$$

Equaling $n_2(ORA)$ to 220 gives

$$r_1 = \frac{220 - .4*200}{.4} - 90 = 260$$

For the next years we obtain analogously

$$r_2 = \frac{240 - .4*220}{.4} - 105 = 275$$

$$r_3 = \frac{260 - .4*240}{.4} - 114 = 296$$

$$r_4 = 317, \; r_5 = 338, \quad r_6 = 309, \quad r_7 = 315, \quad r_8 = 315$$

In this way a simple enrollment policy for the coming years may be developed. However, this policy is based on forecasts, which don't necessarily come true. Therefore, a yearly updating of the policy based on new information is requested.

In this example the only decision variable is the recruitment, although also the transition probabilities can be used as such. Consider, for instance, an alternative set-up for the first phase with higher probabilities of transition to the next phase due to better motivation of the students. Of course such an alternative would be more expensive. Different models of introduction of such an alternative could be analyzed with respect to output and costs.

5.2. Manpower planning

Similar situations as in the proceeding example can be found in regional planning and in manpower planning. We will confine ourselves to the last type of example.

Manpower planning using Markov models has already obtained a lot of attention in the literature, cf. Bartholomew [3], Forbes [11],Vajda [33], Van Nunen/Wessels [27], Verhoeven [35]. Here we will consider an over-simplified example of a computer programming department of a large corporation. This department employs programmers and system analysts. The envisaged requested capacity of both groups is given in table 2.

year	programmers	analysts
1983	200	100
1984	200	100
1985	210	105
1986	220	110
1987	240	120
1988	250	125

Table 2: Envisaged strength for the years to come.

Turnover in both groups is rather high, yearly about 20 and 25% respectively. In the preceding years it appeared that about 20% of the programmers was promoted to system analyst. By making a Markov model based on these data for the behaviour of an individual, one might obtain the means for solving the manpowercapacity problem. See fig.7 for the Markov-model.

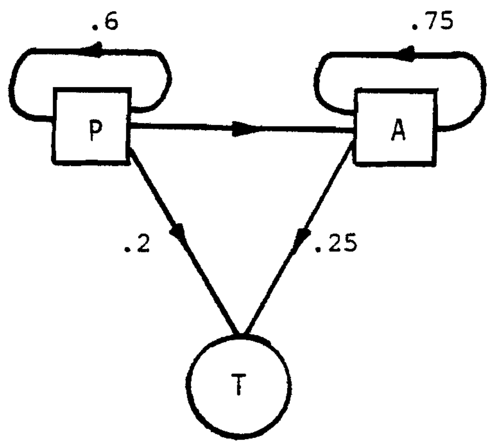

Fig.7: Markov model with states P(programmer), A(analyst), and T(turnover).

In fact the decision variables are rather complicated in this problem, since not only recruitment can be used to control the system, also the promotion policy, educational efforts and stimuli for leaving the organization are available for attaining the goal of the required capacity.

If 1983 is the current year (t=0) and the strength in 1983 is as required in table 2, then the model of fig.7 gives with formula (2.1) without recruitment

$n_o(P) = 200$ $\qquad$ $n_1(P) = 120$

$n_o(A) = 100$ $\qquad$ $n_1(A) = 115$

Considering the numbers of table 2 as minimum requirement, one could satisfy the demands for 1984 (t=1) by recruiting 80 programmers. In table 3 we give the forecasts for the case of no recruitment and for the simple recruitment rule of yearly filling up the requirement by considering them as minima.
With this recruitment policy the group of programmers obtains the right strength. However, in this way the group of system analysts becomes too big. This cannot be remedied by another recruitment policy. The only remedies are to be found in the promotion policy and/or in the turnover.

	no recruitment		with recruitment		
year	P	A	recruitment in P	P	A
1983	200	100	-	200	100
1984	120	115	80	200	115
1985	72	110	90	210	126
1986	43	97	94	220	137
1987	26	81	108	240	147
1988	16	66	106	250	158

Table 3: Expected occupations without and with recruitment

If also in the past the group of system analysts had a size of half the size of the programmers group, then, apparently, the problems arise from a slower growth in the preceding years. A high promotion rate is only possible if growth is fast. Adaptation of the promotion rate to 15% would result in the occupations of table 4, at least, if the turnover is not influenced by this change.

year	recruitment in P	recruitment in A	P	A
1983	-	-	200	100
1984	70	-	200	105
1985	80	-	210	109
1986	83	-	220	113
1987	97	3	240	120
1988	94	-	250	126

Table 4: Expected occupations with recruitment and adapted promotion rate.

Apparently, small additional adaptation can make the scheme of table 4 work perfectly, if necessary. One could also work the other way around and make the promotion rate free (within some bounds), requiring perfect fit of the group sizes. In table 5, the results are given for the situation with only recruitment in the P group.

Table 5 is obtained by the follwoing computations.

$$n_1(A)=100=.75*100+\alpha_1*200 \text{ , so } \alpha_1 = .125$$
$$n_1(P)=200=(.8-\alpha_1)*200 + r_1\text{, so } r_1 = 65$$

$$n_2(A)=105=.75*100 + \alpha_2*200\text{, so } \alpha_2 = .15$$
$$n_2(P)=210= (.8-\alpha_2)*200 + r_2\text{, so } r_2 = 80,$$

etc.

year	recruitment in P	promotion rate	P	A
1983	-	-	200	100
1984	65	.125	200	100
1985	80	. 15	210	105
1986	83	.149	220	210
1987	101	. 17	240	120
1988	93	.146	250	125

Table 5: Occupations obtained by adapting the promotion rate.

The type of analysis given in tables 3-5 will give the responsible managers a lot of information about the behaviour of their system, including the costs. For practical purposes the first aspect to add to the model would be experience, since definitely promotion and possibly turnover will depend on experience. A simple way to incorporate this aspect into the model would be the extension of the state concept by the time spent already in the group. In that way the model as described in fig.8 would be obtained.

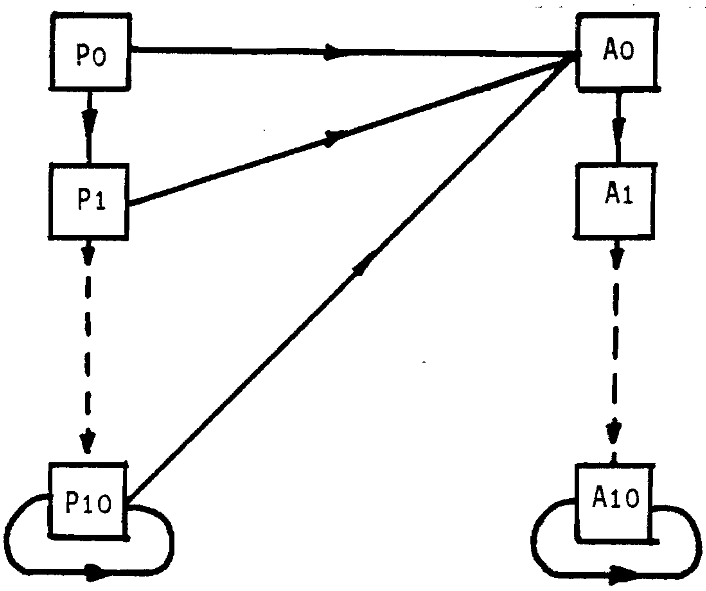

Fig.8: Markov model for individual behaviour with grade age incorporated; for simplicity the turnover possibility is deleted.

This type of model is well-suited to base decision support systems for manpower planning on. For further details on such decision support systems, see Forbes [11], Verhoeven [35].

6. Some examples of controlled systems

In section 2 a Markov decision model has been outlined, which might be used for the determination of the optimal claim limit for a vehicle insurance. Processes that possess a similar control structure occur in various areas ranging from inventory control (cf. Tijms [32], via cash balancing (cf.Hendrikx et.al.[16], Bartmann[4]), fishery (cf. Mendelssohn [23]) to marketing (cf.Van Nunen/Wessels [25]) and ship handling in a sea terminal (cf.Lenssen et.al. [20]).

In the sequel of this section we will describe some types of controlled systems. Particular attention will be given to some modelling aspects and to some implementation aspects as well. Computational aspects are also very important since the models have a tendency to become very large. Computational aspects, however, will be considered in the next section.

6.1. Inventory and production control

Inventory production control problems exist in a large variety. A complete overview of this variety cannot be expected within the framework of this paper. Let us therefore consider some incidental examples.

6.1.1. The classical stochastic inventory problem

A single product is kept in storage. The amount in storage is observed periodically and replenishment orders are fulfilled immediately. The external demand per period for the product is modelled by a random variable and subsequent demands are independent. Shortages in stock are backlogged. Costs involved are restocking costs $c(k) = k + ck$, for $k > 0$, in which k is a fixed and c a proportional ordering cost. Inventory costs are k per unit per period and backlogging costs are b per unit per period.Each period a demand d occurs with probability $p(d)$. Costs are discounted with discount factor β.
The decision process can be modelled by taking the amount of product in stock (positive or negative) as state. As action one can either take the amount ordered, or the new state after the eventual replenishment. The latter choice has some advantages as will appear in section 7, let us therefore take that choice. As a result one obtains analogous to (2.3) the following relations for the costs $v(i)$ belonging to an optimal strategy for an infinite time horizon and starting with inventory i:

$$(6.1)\quad v(i) = \min_{\ell \geq i} \{ c(i;\ell) + \beta \sum_d p(d)\, v(\ell-d) \}$$

where ℓ denotes the action in i and $c(i;\ell)$ is (expected) one period costs if action ℓ is chosen in state i

$$(6.2)\quad c(i;\ell) = \begin{cases} hi + K\delta_{\ell-i} + c(\ell-i) & \text{if } i \geq 0 \\ -bi + K\delta_{\ell-i} + c(\ell-i) & \text{if } i < 0 \end{cases}$$

$$(6.3)\quad \delta_{\ell-i} = \begin{cases} 0 & \text{if } \ell = i \\ 1 & \text{if } \ell > i \end{cases}$$

In principle one might use the set of equations (6.1) to determine an optimal action for each state. However, because of the particular structure of this model, it is known beforehand (cf.Iglehart [17]) that there is an optimal strategy of a particular structure. This type of strategy is determined by two parameters s, S such that replenishment takes only place if stock is below s and the replenishment is such, that it results in the new stock level S. This knowledge might help finding an optimal strategy (cf. section 7), but it is also of implementational relevance. Namely, such a structured strategy is simpler to implement than a general strategy. However, it may be the case that the variety of replenishment order sizes is acceptable but not nice, then it can be computed what extra costs a fixed size replenishment policy would require. This can be done by computing the optimal fixed order size policy, but also by experimenting with an order size of S-s or slightly greater. The fixed-strategy equivalent of (6.1)can be used to compute the costs.

6.1.2 Hard cash inventory

One of the many variants of the preceding classical stochastic inventory problem is the hard cash inventory problem for a branch office of a bank. The main specific feature is that the demand by customers can be positive as well as negative and the same holds for replenishment orders. Backlogging is not allowed, since shortages have to be filled by an emergency replenishment. If one considers this situation and takes as time period half a day, then it will be clear that the demand distribution will depend on the period within

the week (cf. Hendrikx et.al.[16]).Modelling as a Markov decision model gives as natural state concept (i,ℓ) in which i denotes the stocklevel at the beginning of a period and ℓ denotes the period number within the week. This gives a large state space, which even becomes larger if one models a situation in which there is a time-lag between action and effect (cf.Veugen et.al. [37]), then also old actions have to be maintained in the state.
Regrettably, the form of the optimal policy in this situation (consider for simplicity the case without time-lag) is not necessarily as nice as one would hope. One would expect more or less a 3-parameter strategy s_ℓ, S, s_u which requires replenishment up till S if stock is below s_ℓ and deposit of stock until S if stock is above s_u. Because of the time-dependence of demand, one would have dependence of these parameters on the period within the week. This 3-parameter strategy with time-dependence is already inappropriate for implementation. So it is important to find well-structured strategies which are nearly optimal, which appears to be simple in practical situations
by designing 3-parameter strategies which are constant over parts of the week. Such strategies can be found departing from an optimal strategy.
From example 6.1.2, it becomes clear that even single product-single inventory point problems can already lead to excessively large control models. For problems with more products and/or more inventory points therefore, other approaches are necessary. Such approach can be found using aggregation and decomposition. In section 7 some remarks will be made in that direction.

6.2. Replacement and maintenance

Let us characterize the large variety of replacement and maintenance problems by a highly simplified example. For some piece of equipment it has to be decided each quarter of a year whether to replace it by a new one or to keep it and give it the necessary maintenance. The required maintenance depends on the age and on some wastage characteristic. Therefore, the state is defined as (a,w) in which a denotes age and w denotes wastage. The only actions are replace, leading to state (o,o), and keep leaving the state (a,w) untouched. In a period a random transition takes place from state (a,w) to (a+1,w') with $w \geq w'$ and probability $p_{a,w}(w')$ for the new wastage level w'. Using discounting, we obtain the optimal policy from the relations (6.4) which are analogous to (2.3):

(6.4) $v(a,w) = \min\{ -r(a,w) + c + v(o,o),\ m(a,w) + \beta \sum p_{a,w}(w')\, v(a+1,w') \}$

In this minimization the first alternative represents replacing and the second alternative represents keeping. r(a,w) is the next value, c the price of a new piece and m(a,w) the maintenance costs. Again one can observe that the control process possesses a lot of structure. Because of this structure one may expect for sensible forms of the functions r and m that well-structured policies arise (cf. monotone policies). However, practically a policy like: replace when a=A, might be desirable. Again it is simple to find a good A heuristically and evaluate the extra costs of such a policy.

7. Numerical aspects

The numerical problems arising in analyzing discrete random models of the types treated in the preceding sections are partly of a standard nature and partly of an ad hoc nature. Standard are the numerical problems for finding the probability distribution for the state of some Markov chain at time t or in the long run. Also standard are the related problems for costs. Similarly, standard problems arise in cohort models and in the controlled models. Ad hoc problems arise if in a design problem some parameter has to be optimized. In this paper we will only pay attention to numerical problems of the standard type. The main reason that numerical problems tend to be real problems in this area is that models have a tendency to become large or even very large. Of course, a modeller should always be extremely careful in letting models become large, however, there are many reasons for a state space to become multidimensional and state spaces of three or more dimensions are usually rather large. With ten levels in each dimension, one becomes 10^n states in case of an n-dimensional state concept. Extra dimensions can easily be necessary in order to keep track of some sort of cyclic behaviour as in the hard cash inventory example of subsection 6.1.2. Other reasons for extra dimensions may be some time lag in the reaction mechanism (see also the hard cash example) or the importance of some supplementary variable for the reaction mechanism. An example of such a supplementary variable is wastage in the replacement example of subsection 6.2. Another example of such a supplementary variable is the time already spent in a grade in the manpower model of fig.8. In fact, manpower models usually need at least a three-dimensional state consisting of grade (or function), grade age, and age. Quite often more dimensions are needed as indicators for such features as "sex", "time in service", "education".

So, even if the models are kept as simple as possible, the state and/or the action space may be quite large and it becomes essential to develop numerical methods for handling these large models.
In this section a short overview will be given of the state of the art in handling these numerical problems.

7.1. Probabilities in Markov chains

Denoting by P the matrix of transition probabilities and by p(t) the row vector of state probabilities at time instant t, then p(t) can be computed from p(o) via the recursion relation

$$(7.1) \qquad p(t) = p(t-1)\, P \quad \text{for } t=1,2...$$

For equilibrium probabilities, i.e. for $\lim_{t\to\infty} p(t)$ which usually exists, one should be able to solve

$$(7.2) \qquad p = p\, P$$

with respect to p. The form of P is essential for these computations. Its general form is exhibited in fig.9.

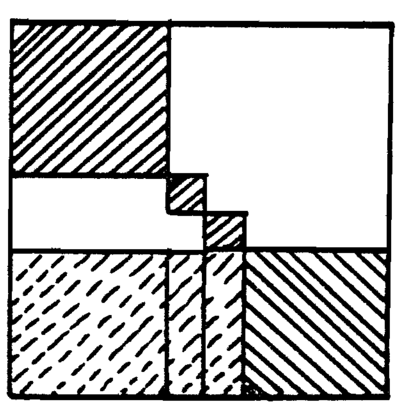

Fig.9: General form of the matrix P of transition probabilities

The figure shows that the solution of (7.1) and of (7.2) can be split-up in smaller problems. For (7.1) we further refer to subsection 7.3, where a similar problem is treated. For (7.2) we start by noting that the part of p corresponds to the last block of rows in P is necessarily equal to zero. For the corresponding states, the equilibrium probabilities can be computed as expected costs if a cost is attached to transition from any state in the last block to a state in one of the other. In each of the other blocks the states can then be aggregated to one superstate. For the solution of such cost problems, the reader is referred to subsection 7.2.

So the only numerical problem to be treated here is the part of (7.2) belonging to one of the upper blocks, i.e. $x = x A$, with A being a matrix of transition probabilities in which all pairs of states are interconnected. Also in this particular case it seems necessary to use the particular structure of the matrix A. Particularly, if the state is multidimensional, then A often has itself again some sort of blockstructure. This structure can be exploited by using the accelerated overrelaxation method of Hadjidimos [13] and block-iteration method of Brandwajn [5]. These methods are quite similar.

Standard successive approximations methods can also be used but have a tendency to converge only slowly, although that is not necessary. In fact, the methods of Jadjidimos and Brandwajn are accelerated versions of standard successive approximations. Also very simple accelerations, like Gauss-Seidel iteration, can already help a lot.
A recent development gives the possibility to compute upper- and lowerbounds of the solution after any iteration, at least, if one of the components of x is already known beforehand. This approach of Van der Wal [38] is very promising.

Also decomposition gives some possibilities, at least if a natural decomposition arises, cf. Courtois [6].

7.2 Costs in a Markov chain

If each visit to state i brings an expected cost c(i), then the total expected costs until time t are denoted by $v_t(j)$ for starting state j. The $v_t(j)$ form a columnvector v_t, which can be computed from

(7.3) $$v_t = c + Pv_{t-1} \qquad \text{for } t=1,2...$$

when v_o is given.

If costs are discounted, (7.3) becomes

(7.4) $v_t = c + \beta P v_{t-1}$, for t=1,2,......

If $o < \beta < 1$, and under some circumstances if $\beta \geq 1$ one may consider $\lim_{t\to\infty} v_t = v$ which satisfies

(7.5) $v = c + \beta P v$

The solution of these equations will not be considered further here, since in subsection 7.4 a more general problem will be treated. It is clear, however, that (7.5) is just a relatively general set of linear equations so that all standard methods from numerical analysis for such sets of equations become available,cf. Young [42]. In [24] 't has been shown that a for the general case the interpretation which leads to (7.5) can be helpful in understanding available numerical techniques.

Many of the specific remarks made in the following subsections are also relevant for this case.

v_t and v give expected costs. For the costs in period t the whole distribution can be obtained, since the state distribution p(t) can be computed from (7.1). For total (discounted) costs, however, it is very difficult to give the distribution because of the dependence between the states at difference times. Nevertheless, the variance can be given quite easily and these variances are practically the most important. For the ∞-horizon discounted problem the variance and second moment of the total expected discounted costs are denoted by V(i) and E(i) if i is the starting state.

Then

(7.6) $V(i) = E(i) - v^2(i)$

(7.7) $E(i) = \mathbb{E}_i [\sum_{t=0}^{\infty} \beta^t c(\chi_t)]^2$

where the random variable χ_t denotes the state at time t and $\mathbb{E}_i$ indicates expectation with given start in state i. Ten one obtains easily for E(i)

$$E(i) = \mathbb{E}_i\, [\sum_{t=o}^{\infty} \beta^t\, c(\chi_t)]^2 =$$

$$= \mathbb{E}_i\, [c(\chi_o) + \sum_{t=1}^{\infty} \beta^t c(\chi_t)]^2 =$$

$$= \mathbb{E}_i\, [c(\chi_o)]^2 + \mathbb{E}_i\, [\sum_{t=1}^{\infty} \beta^t\, c(\chi_t)]^2$$

$$+ 2\, \mathbb{E}_i\, c(\chi_o) [\sum_{t=1}^{\infty} \beta^t\, c(\chi_t)]$$

$$= c^2(i) + \beta^2 \sum_j P(i,j)\, \mathbb{E}_j\, [\sum_{t=o}^{\infty} \beta^t\, c(\chi_t)]^2$$

$$+ 2\beta c(i) \sum_j P(i,j) v(j)$$

$$= c^2(i) + 2\beta c(i) \sum_j P(i,j)\, v(j) + \beta^2 \sum_j P(i,j)\, E(j)$$

With (7.6) $E(i)$ and $E(j)$ can be replaced by $V(i)$ and $V(j)$, which results in

$$(7.8)\quad V(i) = \beta^2 [\sum_j P(i,j)\, v^2(j) - (\sum_j P(i,j) v(j)^2] + \beta^2 \sum_j P(i,j)\, V(j)$$

(7.8) has the same form (7.5) as soon as v is known, viz.

$$(7.9)\quad V = C + \beta^2 PV$$

with

$$(7.10)\quad C = \beta^2\, Pv^2 - (Pv)^2$$

if the square of a vector is defined as the vector with all components squared.

Also the interpretation of (7.8) or (7.9) + (7.10) is simple and interesting. Similar results have been found independently by Sobel [31]. Computationally, the variances of total costs give no new problems.

7.3. Cohorts

For cohort models there are also several types of numerical problems but let us confine attention to (2.4)

$$n_t = r_t + P\, n_{t-1} \qquad t = 1,2,\ldots.$$

for given starting occupation of the states n_o. n_t and r_t are column vectors denoting respectively the expected occupation and the expected recruitment at time t for the different states.

The only real problem arises if the state set is very large and Pn_{t-1} has to be computed quite often. For conversational use it is essential that response times are short in order to facilitate trial-and error exercises.

Let us consider the manpower planning type of models as mentioned in subsection 5.2. Let the state be characterized by grade, grade age, and one or more characteristics like sex, education level, education field, experience, location. These characteristics consist of two types: those that can change over time and that cannot change over time. The latter group is simpler to handel and therefore age (year of birth) is included in that group together with sex and, depending on the situation, some of the other supplementary characteristics. Let us take for simplicity as state characteristics grade, grade age, age, sex, and eduction level with age and sex invariant over time. Then the computation of Pn_{t-1} can be decomposed by computing a similar but much smaller form for any relevant sex-age combination. In these smaller problems one may further use the specific structure which usually shows that grade and education level can only change in one direction. Moreover, grade age just increases by one as long as the grade remains unchanged. The two features can straight forwardly be used to make the computation more efficient.
A further feature can be that age is not really important for state transitions only for retiring. In that case it is important to know whether the forecasts should give age distributions. If not, the retirements can be computed beforehand and one aggregate the ages which gives a substantial saving in computation effort. This latter trick also works if retirement is not at a fixed age.

The forecasts treated so far are the expectations of the actual number. The variances are much more complicated. In fact, variances can only be computed under more assumptions.
If all individuals behave independently, for instance, then there are formulae for the variances (cf. Bartholomew [3]). However, computations are time consuming for large state spaces and the assumption is not always realistic.

7.4. Controlled discrete dynamic systems

Expected total discounted costs in a Markov chain can be computed by solving the set of linear equations (7.5). For a controlled Markov chain, this set of equations can be generalized to the vector-matrix form of (2.3)

$$(7.11)\quad v = \min_{\lambda} \{c_{\lambda} + \beta P_{\lambda} cv\}$$

where λ is an arbitrary policy ascribing an action to each state and c_{λ}, P_{λ} are the cost-vector and transition-matrix belonging to this policy:

$$c_{\lambda}(i) = c\ (i;\ \lambda(i)) \quad \text{and} \quad P_{\lambda}(i,j) = P\ (i,j;\ \lambda(i))$$

Starting from the idea that (7.11) is a generalization of (7.5), there emerge two strongly related approaches, viz. linear programming (see [8] and for a recent survey, Kallenberg [18]) and policy iteration (see Howard [15]).

The linear program giving the minimizing policy in (7.11) becomes

$$\text{minimize} \sum_{i,\ell} x(i;\ell)\ c(i;\ell) \text{ subject to}$$

$$\sum_{\ell} x\ (i;\ell) = \frac{1}{N} + \beta \sum_{j,k} x(j;k)\ \ P(j,i;k) \qquad \text{for } i = 1,\dots,N$$

$$x\ (i;\ell) \geqslant 0$$

(7.12)

For a control problem with 100 states and an average of 30 actions per state the l.p.-problem has 100 constraints and 3000 variables.

As a standard method linear programming is only useful for relatively small problems, since a couple of hundreds of states is characteristic for relatively small problems. A possibility for handling really large problems with linear programming might be found in the recently emerged simplex variants for linear programming problems for optimizing flows in processing networks (for these simplex variants, see Glover/ Klingman [12]; for a flow-in-a-processing-network formulation of a

Markov decision process, see Van der Wal/Wessels [39]).
The policy iteration method exploits the fact that (7.5) gives costs for a given strategy λ:

$$(7.12)\quad v_\lambda = c_\lambda + \beta P_\lambda v_\lambda \qquad \text{(value determination step)}$$

With v_λ, a new policy can be found by

$$(7.13)\quad \min_\mu \{c_\mu + \beta P_\mu v_\lambda\} \qquad \text{(policy improvement step)}$$

By repeating these two steps an optimal solution of (7.11) is found in a finite number of iterations. When the value determination step is executed by solving the set of linear equations, then a couple of hundreds of states is practically the maximum problem size. In some examples, however, the structure of the solution of (7.12) or of the optimal policy can be exploited in such a way that solution of (7.12) is particularly efficient (for a queueing example, see Van Nunen and Puterman [26]) ; for inventory control, see [32]
Also for (7.13) there are possibilities to accelerate the procedure. Such possibilities will be dicussed in the context of the subsequent approach.

A third approach arises by either considering (7.11) as a functional equation and applying the idea of successive approximations to it, or considering the infinite time-horizon problem as the limit of a sequence of finite time-horizon problems. In its simplest form one obtains in this way

$$(7.14)\quad w_t = \min_\lambda \{c_\lambda + \beta P_\lambda w_{t-1}\}$$

where the sequence w_t converges to the solution of (7.11) if t tends to infinity.
Here there are a lot of variants, see Van Nunen/Wessels [28], however no one is overall good (compare Hendrikx et.al [16]), so a good variant

has to be chosen carefully. Usually, problems with a couple of thousands of states can be handled in this way by somebody who has some ability in designing a proper variant and making an efficient program for it. For particularly structured problems, it is sometimes possible to handle problems with tens of thousands of states. For instance, if one of the dimensions of the state has been introduced to keep track of some periodicity in the process, then it can be handled practically as efficient as the same problem without the periodicity (see Veugen et.al [35]).
Not only the variant can be chosen in such a way that convergence is quick, an extra property of these methods is that, usually, extrapolations provide bounds for the solution of (7.11). These bounds are very helpful in speeding up convergence (by the extrapolation) as well as in providing a stop criterion.
As already exhibited in (6.1), the form

$$c_\lambda + \beta P_\lambda v$$

may simplify essentially if P_λ has a particular form. Such a simplication helps considerably in the computation. However, such a structure would disappear if the action in the problem underlying (6.1) would have been modelled as order sizes instead of transitions to a new state. Also many variants of the successive approximations method would destroy this simplification. Actually, in such a problem (cf. Hendrikx et.al [16]) the proper modelling choice and the choice of a variant which can exploit this simplicity is essential for obtaining a solution in a reasonable time.

Concluding one may state:

1. by standard methods only relatively small problems can be handled;
2. medium-sized problems can be handled by properly designed variants of standard methods, however, these variants will be highly problem-dependent;
3. really large problems can only be handled along these lines in rare cases.

For really large problems, with at least some tens of thousands of states, other lines of thought have to be followed. The use of decomposition/composition and aggregation/disaggregation seem to provide solutions. Until now, not much systematic knowledge about those approaches has been obtained.
The systematic knowledge consists primarily of bounds for the distance between the solution of the original problem and an aggregated problem (cf. Whitt [41]). Recently, Mendelssohn [22] has published an iterative aggregation procedure for Markov decision processes. Mendelssohn's approach is based on the iterative aggregation approach for linear programming as developed mostly in the Sovjet-Union, cf. for instance Vakkutinskii et.al [34]. These approaches have mainly a theoretical value. The sparse reports on practical use are concerned with problems in which grid size is a measure for the level of aggregation (Mendelssohn [21], Veugen et.al [37]). However, realistic planning problems, for instance in the area of inventory and production control, are often so complicated that aggregation via the choice of a grid does diminish the problem size sufficiently. Aggregation via grid choice may be considered as a very natural way of aggregation which usually does not destroy the problem structure. If large grid sizes do not help sufficiently, then the next step is to consider forms of decomposition and/or aggregation which do destroy the problem structure more or less.
Hopefully, this destruction is such that it can be repaired somewhat in a subsequent composition and/or disaggregation step. It seems not very likely that a very general approach is possible in this area. Nevertheless, several examples show some progress. All these examples are heavily based on the specific problem structure. We mention here the paper by Federgruen and Zipkin [10] on aggregation in inventory models. Furthermore, we mention the paper by Lenssen et.al [20], on planning in shiphandling, who advocate a iterative aggregation/ disaggregation approach, when the disaggregation is executed via simulation.

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