Solution to problem 77-20 : When is the modified Bessel function equal to its derivative?

Citation for published version (APA):
Boersma, J., & Doelder, de, P. J. (1978). Solution to problem 77-20 : When is the modified Bessel function equal to its derivative? *SIAM Review*, *20*(4), 862-862. https://doi.org/10.1137/1020114

DOI:
10.1137/1020114

Document status and date:
Published: 01/01/1978

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Editorial note. The previous proof can be simplified by noting from (4) that $f(0)=f'(0)=0=f''(1)=f'''(1)$. This leads to a simpler set of equations for (7) and (8). M. L. GLASSER (Clarkson College of Technology) sketches a proof by contour integrals for the related identities:

$$\sum_{j=1}^{\infty} \alpha_j^{-3}\left\{\frac{\sin \alpha_j - \sinh \alpha_j}{\cos \alpha_j + \cosh \alpha_j}\right\}=0,$$

$$\sum_{j=1}^{\infty} (\alpha_j^4-\lambda^4)^{-1}=\left\{\frac{\sin \lambda \cosh \lambda - \sinh \lambda \cos \lambda}{1+\cos \lambda \cosh \lambda}\right\}\Big/ 4\lambda^3, \qquad |\lambda|<\alpha_1.$$

He also indicates that (A) can be derived this way but since the poles involved are second order, the calculations get messy. [M.S.K.]

Problem 77-20, *When is the Modified Bessel Function Equal to its Derivative?*, by I. NASELL (Royal Institute of Technology, Stockholm, Sweden).

Prove that the equation

$$I_\nu(x)=I'_\nu(x)$$

has exactly one positive solution $x=\xi(\nu)$ for each $\nu>0$. Investigate the properties of the function ξ.

Solution by M. L. GLASSER (Clarkson College of Technology).

We note that

$$I_\nu(x)=c_\nu x^\nu \prod_{n=1}^{\infty}\left(1+\frac{x^2}{j_{\nu,n}^2}\right)$$

so that

$$\phi_\nu(x)\equiv[\ln I_\nu(x)]'=\frac{\nu}{x}+2x\sum_{n=1}^{\infty}(x^2+j_{\nu,n}^2)^{-1}=\frac{I'_\nu(x)}{I_\nu(x)}.$$

It is clear that for $\nu>0$, $\phi_\nu(x)$ is positive, continuous and monotonically decreasing. For small x, $\phi_\nu(x)>1$; for large x, we have from the known asymptotic estimate $I_\nu(x)=e^x(2\pi x)^{-1}\{1+O(1/x)\}$ that $\phi_\nu(x)\sim 1-1/2x+O(1/x^2)<1$. Therefore $\phi_\nu(x)=1$ has a unique positive solution $x=\xi(\nu)$.

Also solved by the proposer, who shows additionally that

$$\nu+\nu^2>\xi(\nu)>\max(\nu,\nu^2), \qquad \nu>0$$

$$\xi(\nu)\sim\nu \quad \text{as } \nu\to 0, \qquad \xi(\nu)\sim\nu^2 \quad \text{as} \quad \nu\to\infty.$$

He also conjectures that ξ has a positive derivative and refers to his paper, *Rational bounds for ratios of modified Bessel functions*, to appear in SIAM J. Math. Anal.

In a late solution, J. BOERSMA and P. J. DE DOELDER (Technological University, Eindhoven, the Netherlands) also show that

$$\xi(\nu)=\nu+\frac{\nu^2}{2}+\frac{\nu^4}{16}-\frac{\nu^5}{32}+O(\nu^6) \qquad \text{for } \nu\to 0,$$

$$\xi(\nu)=\nu^2+\frac{1}{2}+\frac{\nu^{-2}}{4}+\frac{\nu^{-4}}{8}+O(\nu^{-6}) \quad \text{for } \nu\to\infty,$$

$\xi(\nu)\approx\nu^4/(\nu^2-\frac{1}{2})$ (correct to four decimal places when $\nu\geqq 4$),

$\xi'(\nu)>0.$