

## Solution to problem 77-20 : When is the modified Bessel function equal to its derivative?

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*Editorial note.* The previous proof can be simplified by noting from (4) that  $f(0) = f'(0) = 0 = f''(1) = f'''(1)$ . This leads to a simpler set of equations for (7) and (8). M. L. GLASSER (Clarkson College of Technology) sketches a proof by contour integrals for the related identities:

$$\sum_{j=1}^{\infty} \alpha_j^{-3} \left\{ \frac{\sin \alpha_j - \sinh \alpha_j}{\cos \alpha_j + \cosh \alpha_j} \right\} = 0,$$

$$\sum_{j=1}^{\infty} (\alpha_j^4 - \lambda^4)^{-1} = \left\{ \frac{\sin \lambda \cosh \lambda - \sinh \lambda \cos \lambda}{1 + \cos \lambda \cosh \lambda} \right\} / 4\lambda^3, \quad |\lambda| < \alpha_1.$$

He also indicates that (A) can be derived this way but since the poles involved are second order, the calculations get messy. [M.S.K.]

*Problem 77-20, When is the Modified Bessel Function Equal to its Derivative?*, by I. NASELL (Royal Institute of Technology, Stockholm, Sweden).  
Prove that the equation

$$I_\nu(x) = I'_\nu(x)$$

has exactly one positive solution  $x = \xi(\nu)$  for each  $\nu > 0$ . Investigate the properties of the function  $\xi$ .

Solution by M. L. GLASSER (Clarkson College of Technology).

We note that

$$I_\nu(x) = c_\nu x^\nu \prod_{n=1}^{\infty} \left( 1 + \frac{x^2}{j_{\nu,n}^2} \right)$$

so that

$$\phi_\nu(x) \equiv [\ln I_\nu(x)]' = \frac{\nu}{x} + 2x \sum_{n=1}^{\infty} (x^2 + j_{\nu,n}^2)^{-1} = \frac{I'_\nu(x)}{I_\nu(x)}.$$

It is clear that for  $\nu > 0$ ,  $\phi_\nu(x)$  is positive, continuous and monotonically decreasing. For small  $x$ ,  $\phi_\nu(x) > 1$ ; for large  $x$ , we have from the known asymptotic estimate  $I_\nu(x) = e^x (2\pi x)^{-1/2} \{1 + O(1/x)\}$  that  $\phi_\nu(x) \sim 1 - 1/2x + O(1/x^2) < 1$ . Therefore  $\phi_\nu(x) = 1$  has a unique positive solution  $x = \xi(\nu)$ .

Also solved by the proposer, who shows additionally that

$$\nu + \nu^2 > \xi(\nu) > \max(\nu, \nu^2), \quad \nu > 0$$

$$\xi(\nu) \sim \nu \quad \text{as } \nu \rightarrow 0, \quad \xi(\nu) \sim \nu^2 \quad \text{as } \nu \rightarrow \infty.$$

He also conjectures that  $\xi$  has a positive derivative and refers to his paper, *Rational bounds for ratios of modified Bessel functions*, to appear in SIAM J. Math. Anal.

In a late solution, J. BOERSMA and P. J. DE DOELDER (Technological University, Eindhoven, the Netherlands) also show that

$$\xi(\nu) = \nu + \frac{\nu^2}{2} + \frac{\nu^4}{16} - \frac{\nu^5}{32} + O(\nu^6) \quad \text{for } \nu \rightarrow 0,$$

$$\xi(\nu) = \nu^2 + \frac{1}{2} + \frac{\nu^{-2}}{4} + \frac{\nu^{-4}}{8} + O(\nu^{-6}) \quad \text{for } \nu \rightarrow \infty,$$

$$\xi(\nu) \approx \nu^4 / (\nu^2 - \frac{1}{2}) \quad (\text{correct to four decimal places when } \nu \geq 4),$$

$$\xi'(\nu) > 0.$$