Solution to Problem 64-1: An asymptotic expansion

Citation for published version (APA):
van Lint, J. H. (1966). Solution to Problem 64-1: An asymptotic expansion. *SIAM Review*, *8*(3), 383-384.
https://doi.org/10.1137/1008079

DOI:
10.1137/1008079

Document status and date:
Published: 01/01/1966

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

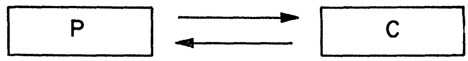

FIG. 2

Problems:

(1) Is there an algorithm to decide whether, for given C and a given function $F: S \to S \cup \{\infty\}$, there exists a control automaton P which has F as its associated mapping?

(2) Is there a general procedure for finding such a P, if it exists?

(3) Characterize the class of all control automata P which have the same associated mapping $F: S \to S \cup \{\infty\}$.

(4) In the class of (3), find those P with the least number of states.

(*Remark*. There is no unique machine with a minimum number of states, in general.)

This model has, among others, the following interpretation: C is a computer with S as set of states, I as set of instructions, and J as set of jump-conditions (conditional transfers). P is a program in flowchart notation, where V is the set of decision vertices (branching points), v_0 the initial vertex.

SOLUTIONS

Problem 64-16 was also solved by J. M. QUONIAM (Saint-Etienne (Loire), France).

Errata:

In the solution of *Problem* 63-14 (April, 1965, p. 290), (13) should read

$$R_{n,m} = \frac{2}{\sqrt{5}} \frac{(p^m - 1)(p^{n-m} - 1)}{p^n - 1}. \tag{13}$$

D. PHILLIPS (Argonne National Laboratory) notes that the solution of the extension of 64-1 (July, 1965) by Trench is incomplete in that he does not establish that the constant of integration, which arises in integrating (8), is equal to zero. Trench, subsequently, sent in a proof that the constant of integration is zero. For completeness here, we restate the original problem and include a different solution which implies that the desired constant of integration is zero.

Problem 64-1, *An Asymptotic Expansion*, by H. O. POLLAK AND L. SHEPP (Bell Telephone Laboratories).

Show that

$$e^{-x} \sum_{n=1}^{\infty} \frac{x^n}{n!} \log n = \log x - \frac{1}{2x} + O(x^{-2}).$$

This problem has arisen in studying the entropy of the Poisson distribution

$$H(x) = \sum_{n=1}^{\infty} p_n(x) \log p_n(x),$$

where

$$p_n(x) = \frac{x^n e^{-x}}{n!}.$$

Solution by J. H. VAN LINT (Technological University, Eindhoven, Netherlands).

We use the known formula for Euler's constant

$$(1) \qquad S_n = \sum_{r=1}^{n} r^{-1} \sim \log n + \gamma + \frac{1}{2(n+1)} + O\left(\frac{1}{n^2}\right).$$

Note that

$$(2) \qquad e^{-x} \sum_{1}^{\infty} \frac{x^n}{n!\, n^2} = O\left\{\frac{e^{-x}}{x^2} \sum_{1}^{\infty} \frac{x^{n+2}}{(n+2)!}\right\} = O(x^{-2}).$$

It now follows from (1) and (2) that

$$\begin{aligned} e^{-x} \sum_{1}^{\infty} \frac{x^n \log n}{n!} & \\ &= e^{-x} \sum_{1}^{\infty} \frac{S_n\, x^n}{n!} - \gamma(1 - e^{-x}) - \frac{e^{-x}}{2} \sum_{n=1}^{\infty} \frac{x^n}{(n+1)!} + O(x^{-2}) \\ &= \gamma + \log x - \mathrm{Ei}\,(-x) - \gamma(1 - e^{-x}) - \frac{e^{-x}}{2x}(e^x - 1 - x) + O(x^{-2}) \\ &= \log x - \frac{1}{2x} + O(x^{-2}). \end{aligned}$$

To obtain more terms of the asymptotic expansion, we just use more terms in the expansion (1).

Problem 65-1, *A Least Squares Estimate of Satellite Attitude*, by GRACE WAHBA (IBM—Federal Systems Division).

Given two sets of n points $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$, and $\{\mathbf{v}_1^*, \mathbf{v}_2^*, \cdots, \mathbf{v}_n^*\}$, where $n \geqq 2$, find the rotation matrix M (i.e., the orthogonal matrix with determinant $+1$) which brings the first set into the best least squares coincidence with the second. That is, find M which minimizes

$$\sum_{j=1}^{n} \| \mathbf{v}_j^* - M\mathbf{v}_j \|^2.$$

This problem has arisen in the estimation of the attitude of a satellite by using direction cosines $\{\mathbf{v}_k^*\}$ of objects as observed in a satellite fixed frame of reference and direction cosines $\{\mathbf{v}_k\}$ of the same objects in a known frame of reference. M is then a least squares estimate of the rotation matrix which carries the known frame of reference into the satellite fixed frame of reference.

Solution by J. L. FARRELL and J. C. STUELPNAGEL (Westinghouse Defense and Space Center).