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A curious implication of Spitzers identity

F.W. Steutel, T.U. Eindhoven

Abstract. Spitzer's identity can be read as follows: Let W_n denote the waiting time of the n -th customer in a $G|G|1$ -queue, and let N be geometrically distributed on $(0, 1, \dots)$ and independent of $(W_n)_1^\infty$. Then W_{N+1} is infinitely divisible.

0. Introduction and summary

It is well known that random sums of the form

$$S_n = X_1 + \dots + X_N$$

are infinitely divisible (inf div) if X_1, X_2, \dots are i.i.d. and N is geometrically distributed on $\{0, 1, \dots\}$ and independent of $(X_n)_1^\infty$.

In this note it will appear that from Spitzer's identity it follows that

$$W_{N+1} = W_1 + W_2 - W_1 + \dots + W_{N+1} - W_N$$

is inf div, where W_n denotes the waiting time of the n -th customer in a $G|G|1$ -queue, and N is independent of (W_n) ; here, however the $W_{n+1} - W_n$ are *dependent*, and do *not* have the same distribution. In Section 1, Spitzer's identity is given with some necessary context, Section 2 contains basic facts on inf div distributions, and in Section 3 the two ingredients are combined. Section 4 gives some additional remarks.

In what follows F , with or without suffix, will denote a distribution function, and \hat{F} its Laplace-Stieltjes transform (LSt).

1. Spitzer's identity

In the well-known $G|G|1$ -queueing system customers arrive at times $0, A_1, A_1 + A_2, \dots$ and are served during periods B_1, B_2, \dots ; all A 's and B 's are independent. We write $S_0 = 0$,

$$S_n = \sum_{k=1}^n (B_k - A_k), \quad n = 1, 2, \dots,$$

and $S_k^+ = \max(0, S_k)$, $k = 1, 2, \dots$. If it is assumed that the first customer finds the server free, then the waiting time W_n of the n -th customer is given by $W_1 = 0$, and

$$W_{n+1} \stackrel{d}{=} \max(S_0, S_1, \dots, S_n), \quad n = 1, 2, \dots \quad (1.1)$$

Now Spitzer's identity (Loève (1977)) reads, for $|z| < 1$ and $\operatorname{Re} s \geq 0$,

$$\sum_{n=0}^{\infty} E e^{-sW_{n+1}} z^n = \exp \left\{ \sum_{k=1}^{\infty} \frac{1}{k} E e^{-sS_k^+} z^k \right\}. \quad (1.2)$$

2. Infinite divisibility

A random variable X is called inf div if for every $n \in \mathbb{N}$ one has

$$X \stackrel{d}{=} X_{1,n} + \dots + X_{n,n},$$

where the $X_{j,n}$ are iid. We only need the following results (Feller (1971), Steutel (1970)).

Lemma 1. A nonnegative random variable is inf div if and only if it has a LSt of the form

$$\hat{F}(s) = E e^{-sX} = \exp \left\{ \int_0^{\infty} \frac{e^{-sx} - 1}{x} dK(x) \right\}, \quad (2.1)$$

where K is a nondecreasing function, which, necessarily, has the property $\int_1^{\infty} x^{-1} dK(x) < \infty$.

Lemma 2. A nonnegative, integer-valued random variable M with $P(M = n) = p_n$, and $p_0 > 0$ is inf div if and only if its probability generating function has the form,

$$P(z) := \sum_{n=0}^{\infty} p_n z^n = \exp \left\{ \sum_{n=0}^{\infty} \frac{r_n}{n+1} (z^{n+1} - 1) \right\}, \quad (2.2)$$

with $r_n \geq 0$, $n = 0, 1, 2, \dots$, and, necessarily, $\sum_0^{\infty} r_n / (n+1) < \infty$.

3. W_{N+1} is infinitely divisible

We rewrite (1.2) as follows. Put $z = p \in (0, 1)$ and multiply by $(1-p)$; this yields (use $-\sum_1^{\infty} p^k/k = \log(1-p)$),

$$\sum_{n=0}^{\infty} (1-p)p^n E e^{-sW_{n+1}} = \exp \left\{ \sum_{k=1}^{\infty} \frac{p^k}{k} (E e^{-sS_k^+} - 1) \right\}. \quad (3.1)$$

The left-hand side of the equation above is equal to the LSt of W_{N+1} , where N is independent of $(W_n)_1^{\infty}$, and

$$P(N = n) = (1-p)p^n \quad (n = 0, 1, \dots, \infty). \quad (3.2)$$

The right-hand side can be rewritten as

$$\exp \left\{ \int_0^{\infty} \frac{e^{-sx} - 1}{x} dK(x) \right\},$$

with K given by

$$K(x) = \sum_{k=1}^{\infty} \frac{p^k}{k} \int_0^x y dF_{S_k^+}(y) \quad (3.3)$$

Combining the results above we obtain the main result of this note.

Theorem. If W_n is the waiting time of the n -th customer in a $G|G|1$ -queue, started empty, and N is a rv independent of $(W_n)_1^\infty$ satisfying (3.2), then the rv W_{N+1} is infinitely divisible.

It should be pointed out that for fixed n the W_n are in general not inf div, since they are bounded if the B 's are bounded; if $W_n \xrightarrow{d} W$ as $n \rightarrow \infty$, then W is inf div (see end of Section 4).

4. Further remarks

Spitzer's identity (1.2) can also be related to Lemma 2. Apart from a multiplicative constant in the right-hand side, (1.2) is of the form (2.2) with

$$p_n = p_n(s) = Ee^{-sW_{n+1}}; \quad r_n = Ee^{-sS_{n+1}^+} .$$

So we see that, for every s , the sequence $(Ee^{-sW_{n+1}})_0^\infty$ is an infinitely divisible sequence (not necessarily summing to 1).

Another result follows if we take logarithms in (1.2) and (2.2), and differentiate. Equation (2.2) then yields

$$np_n = \sum_{k=0}^{n-1} p_k r_{n-k-1} \quad (n = 1, 2, \dots) .$$

Similarly, (1.2) leads to

$$nEe^{-sW_{n+1}} = \sum_{k=0}^{n-1} Ee^{-sW_{k+1}} \cdot Ee^{-sS_{n-k}^+} ,$$

or in terms of distribution functions

$$F_{W_{n+1}}(w) = \frac{1}{n} \sum_{k=1}^{n-1} (F_{W_{k+1}} * F_{S_{n-k}^+}) \quad (n = 1, 2, \dots) , \quad (4.1)$$

which is not easily verified directly for $n > 1$. In fact, one can derive (4.1) from (1.1) and obtain a proof of Spitzer's identity. This was done by Heinrich (1985), who proves (4.1) by induction; in Kotz et al. (1985), where I found the reference to Heinrich, the identity is misquoted: the factor $1/k$ in the right-hand side of (1.2) is missing. Finally, letting $p \uparrow 1$ in (3.1) we obtain the LSt \widehat{F}_W of the limiting waiting time (assume $ES_1 < 0$):

$$\widehat{F}_W(s) = \exp \left\{ \int_0^\infty (e^{-sx} - 1) d \sum_1^\infty \frac{1}{k} F_{S_k^+}(x) \right\},$$

which shows that W is inf div, as is well known.

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