A curious implication of Spitzers identity

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Abstract. Spitzer's identity can be read as follows: Let W_n denote the waiting time of the n-th customer in a $G|G|1$-queue, and let N be geometrically distributed on $(0, 1, ...)$ and independent of $W_n)_1^\infty$. Then W_{N+1} is infinitely divisible.

0. Introduction and summary

It is well known that random sums of the form

$$S_n = X_1 + ... + X_N$$

are infinitely divisible (inf div) if $X_1, X_2, ...$ are i.i.d. and N is geometrically distributed on $\{0, 1, ...\}$ and independent of $(X_n)_1^\infty$.
In this note it will appear that from Spitzer's identity it follows that

$$W_{N+1} = W_1 + W_2 - W_1 + ... + W_{N+1} - W_N$$

is inf div, where W_n denotes the waiting time of the n-th customer in a $G|G|1$-queue, and N is independent of (W_n); here, however the $W_{n+1} - W_n$ are *dependent*, and do *not* have the same distribution. In Section 1, Spitzer's identity is given with some necessary context, Section 2 contains basic facts on inf div distributions, and in Section 3 the two ingredients are combined. Section 4 gives some additional remarks.
In what follows F, with or without suffix, will denote a distribution function, and $\widehat{F}$ its Laplace-Stieltjes transform (LSt).

1. Spitzer's identity

In the well-known $G|G|1$-queueing system customers arrrive at times $0, A_1, A_1 + A_2, ...$ and are served during periods $B_1, B_2, ...$; all A's and B's are independent. We write $S_0 = 0$,

$$S_n = \sum_{k=1}^{n} (B_k - A_k),\ n = 1, 2, ... ,$$

and $S_k^+ = \max\,(0, S_k)$, $k = 1, 2, ...$ If it is assumed that the first customer finds the server free, then the waiting time W_n of the n-th customer is given by $W_1 = 0$, and

$$W_{n+1} \overset{d}{=} \max(S_0, S_1, ..., S_n),\ n = 1, 2, ... \tag{1.1}$$

Now Spitzer's identity (Loève (1977)) reads, for $|z| < 1$ and $Re\ s \geq 0$,

$$\sum_{n=0}^{\infty} E e^{-sW_{n+1}} z^n = \exp\left\{\sum_{k=1}^{\infty} \frac{1}{k} E e^{-sS_k^+} z^k\right\} . \tag{1.2}$$

2. Infinite divisibility

A random variable X is called inf div if for every $n \in I\!N$ one has

$$X \stackrel{d}{=} X_{1,n} + ... + X_{n,n} ,$$

where the $X_{j,n}$ are iid. We only need the following results (Feller (1971), Steutel (1970)).

Lemma 1. A nonnegative random variable is inf div if and only if it has a LSt of the form

$$\widehat{F}(s) = Ee^{-sX} = \exp\left\{\int_0^{\infty} \frac{e^{-sx}-1}{x} dK(x)\right\} , \tag{2.1}$$

where K is a nondecreasing function, which, necessarily, has the property $\int_1^{\infty} x^{-1} dK(x) < \infty$.

Lemma 2. A nonnegative, integer-valued random variable M with $P(M = n) = p_n$, and $p_0 > 0$ is inf div if and only if its probability generating function has the form,

$$P(z) := \sum_{n=0}^{\infty} p_n z^n = \exp\left\{\sum_{n=0}^{\infty} \frac{r_n}{n+1}(z^{n+1}-1)\right\} , \tag{2.2}$$

with $r_n \geq 0$, $n = 0, 1, 2, ...$, and, necessarily, $\sum_0^{\infty} r_n/(n+1) < \infty$.

3. W_{N+1} is infinitely divisible

We rewrite (1.2) as follows. Put $z = p \in (0,1)$ and multiply by $(1-p)$; this yields (use $-\sum_1^{\infty} p^k/k = \log(1-p)$),

$$\sum_{n=0}^{\infty} (1-p)p^n E e^{-sW_{n+1}} = \exp\left\{\sum_{k=1}^{\infty} \frac{p^k}{k}(Ee^{-sS_k^+} - 1)\right\} . \tag{3.1}$$

The left-hand side of the equation above is equal to the LSt of W_{N+1}, where N is independent of $(W_n)_1^{\infty}$, and

$$P(N = n) = (1-p)p^n \quad (n = 0, 1, ..., n) . \tag{3.2}$$

The right-hand side can be rewritten as

$$\exp\left\{\int_0^{\infty} \frac{e^{-sx}-1}{x} dK(x)\right\} ,$$

with K given by

$$K(x) = \sum_{k=1}^{\infty} \frac{p^k}{k} \int_0^x y \; dF_{S_k^+}(y) \tag{3.3}$$

Combining the results above we obtain the main result of this note.

Theorem. If W_n is the waiting time of the n-th customer in a $G|G|1$-queue, started empty, and N is a rv independent of $(W_n)_1^\infty$ satisfying (3.2), then the rv W_{N+1} is infinitely divisible.

It should be pointed out that for fixed n the W_n are in general not inf div, since they are bounded if the B's are bounded; if $W_n \xrightarrow{d} W$ as $n \to \infty$, then W is inf div (see end of Section 4).

4. Further remarks

Spitzer's identity (1.2) can also be related to Lemma 2. Apart from a multiplicative constant in the right-hand side, (1.2) is of the form (2.2) with

$$p_n = p_n(s) = Ee^{-sW_{n+1}}; \; r_n = Ee^{-S_{n+1}^+} \; .$$

So we see that, for every s, the sequence $(Ee^{-sW_{n+1}})_0^\infty$ is an infinitely divisible sequence (not necessarily summing to 1).
Another result follows if we take logarithms in (1.2) and (2.2), and differentiate. Equation (2.2) then yields

$$np_n = \sum_{k=0}^{n-1} p_k r_{n-k-1} \quad (n = 1, 2, ...) \; .$$

Similarly, (1.2) leads to

$$nEe^{-sW_{n+1}} = \sum_{k=0}^{n-1} Ee^{-sW_{k+1}} \cdot Ee^{-sS_{n-k}^+} \; ,$$

or in terms of distribution functions

$$F_{W_{n+1}}(w) = \frac{1}{n} \sum_{k=1}^{n-1} (F_{W_{k+1}} * F_{S_{n-k}^+} \quad (n = 1, 2, ...) \; , \tag{4.1}$$

which is not easily verified directly for $n > 1$. In fact, one can derive (4.1) from (1.1) and obtain a proof of Spitzer's identiy. This was done by Heinrich (1985), who proves (4.1) by induction; in Kotz et al. (1985), where I found the reference to Heinrich, the identity is misquoted: the factor $1/k$ in the right-hand side of (1.2) is missing. Finally, letting $p \uparrow 1$ in (3.1) we obtain the LSt $\widehat{F}_W$ of the limiting waiting time (assume $ES_1 < 0$):

$$\widehat{F}_W(s) = \exp\left\{\int_0^\infty (e^{-sx} - 1) d \sum_1^\infty \frac{1}{k} F_{S_k^+}(x)\right\} ,$$

which shows that W is inf div, as is well known.

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