Solution to Problem 93-15 : Two integrals arising from a reaction-diffusion problem

Two Integrals Arising from a Reaction-Diffusion Problem

Problem 93-15, *by* JOHN BILLINGHAM (Schlumberger Cambridge Research, Cambridge, England).

Evaluate the two integrals

$$\text{(i)} \quad \int_0^\infty e^{-s^2} ds / A_n^2(s), \qquad n \text{ even and } \geq 0,$$

$$\text{(ii)} \quad \int_0^\infty \left\{ 1/4s^2 - e^{-s^2}/A_n^2(s) \right\} ds, \quad n \text{ odd and } \geq 1,$$

where

$$A_n(s) = \sum_{r=0}^{n/2} (2s)^{2r} (n/2)!/(2r)!(n/2 - r)!, \qquad n \text{ even and } \geq 0,$$

$$A_n(s) = \sum_{r=0}^{(n-1)/2} (2s)^{2r+1} ((n-1)/2)!/(2r+1)!((n-1)/2 - r)!, \qquad n \text{ odd and } \geq 1.$$

This problem arose as the leading order approximation to the small time solution of a reaction-diffusion problem [1].

REFERENCE

J. BILLINGHAM AND D. J. NEEDHAM, *The development of travelling waves in quadratic and cubic autocatalysis with unequal diffusion rates*. II. *An initial value problem with an immobilized or nearly immobilized autocatalyst*, Philos. Trans. Roy. Soc. London, Ser. A, 336 (1991), pp. 497–539.

Solution by J. BOERSMA AND P. J. DE DOELDER (Eindhoven University of Technology, Eindhoven, the Netherlands).

The two integrals are evaluated by a method due to Belevitch [1, §3], for obtaining indefinite integrals involving independent solutions of a linear differential equation, by means of the Wronskian. First we express $A_n(s)$ in terms of confluent hypergeometric functions Φ [2, §6.1] and of the Hermite polynomial H_n by means of [2, form. 10.13(17), (18)], viz.

$$A_n(s) = \Phi(-\tfrac{1}{2}n, \tfrac{1}{2}; -s^2) = \frac{i^{-n}\left(\frac{1}{2}n\right)!}{n!} H_n(is), \qquad n \text{ even and } \geq 0,$$

$$A_n(s) = 2s\Phi(-\tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{3}{2}; -s^2) = \frac{i^{-n}\left(\frac{1}{2}n - \frac{1}{2}\right)!}{n!} H_n(is), \qquad n \text{ odd and } \geq 1.$$

From the differential equation for Hermite polynomials (see [2, form. 10.13(12)]) we infer that $u = H_n(is)$ satisfies the equation

$$u'' + 2su' - 2nu = 0,$$

where a prime denotes differentiation with respect to s. The latter equation has two independent power-series solutions

$$u_1(s) = \Phi(-\tfrac{1}{2}n, \tfrac{1}{2}; -s^2), \qquad u_2(s) = s\Phi(-\tfrac{1}{2}n + \tfrac{1}{2}, \tfrac{3}{2}; -s^2),$$

constructed in the standard manner. The Wronskian of these solutions is readily found to be

$$W(u_1, u_2) = u_1 u_2' - u_1' u_2 = e^{-s^2}.$$

Furthermore, we establish the expansions

$$\frac{u_1(s)}{u_2(s)} = \frac{1}{s} + O(s) \qquad (s \to 0),$$
$$\frac{u_1(s)}{u_2(s)} = \frac{2\Gamma(\frac{1}{2}n+1)}{\Gamma(\frac{1}{2}n+\frac{1}{2})} + O(s^{-2}) \qquad (s \to \infty),$$

where the latter expansion was obtained from the asymptotics of the Φ-function, as discussed in [1, §6.13.1].

The evaluation of the integrals is now straightforward by setting $e^{-s^2} = u_1 u_2' - u_1' u_2$, $A_n(s) = u_1$ for even $n \geq 0$, and $A_n(s) = 2u_2$ for odd $n \geq 1$. As a result we find

$$\int_0^\infty \frac{e^{-s^2}}{A_n^2(s)} ds = \int_0^\infty \frac{u_1 u_2' - u_1' u_2}{u_1^2} ds = \frac{u_2(s)}{u_1(s)} \Big|_{s=0}^{s=\infty}$$
$$= \frac{\Gamma(\frac{1}{2}n+\frac{1}{2})}{2\Gamma(\frac{1}{2}n+1)}, \qquad n \text{ even and } \geq 0,$$

$$\int_0^\infty \left(\frac{1}{4s^2} - \frac{e^{-s^2}}{A_n^2(s)} \right) ds = \frac{1}{4} \int_0^\infty \left(\frac{1}{s^2} - \frac{u_1 u_2' - u_1' u_2}{u_2^2} \right) ds$$
$$= \frac{1}{4} \left(\frac{u_1(s)}{u_2(s)} - \frac{1}{s} \right) \Big|_{s=0}^{s=\infty} = \frac{\Gamma(\frac{1}{2}n+1)}{2\Gamma(\frac{1}{2}n+\frac{1}{2})}, \qquad n \text{ odd and } \geq 1.$$

Remark. The two results may be combined into the integral

$$\int_0^\infty \frac{e^{-s^2}}{H_n^2(is)} ds = \frac{2^{-n-1}\sqrt{\pi}}{n!}, \qquad n = 0, 1, 2, \ldots$$

where in the case of odd n, the finite part of the divergent integral should be taken.

REFERENCES

[1] V. Belevitch, *The Gauss hypergeometric ratio as a positive real function*, SIAM J. Math. Anal., 13 (1982), pp. 1024–1040.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, Vol. I, II, McGraw-Hill, New York, 1953.

Also solved by C. C. Grosjean (University of Ghent, Ghent, Belgium), W. B. Jordan (Scotia, NY), *and the proposer.*

ERRATA

Problem 92-11*, *by* Malte Henkel (Université de Genève, Switzerland) and R. A. Weston (University of Durham, Durham, UK).