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Abstract

This paper is concerned with the discrete time, full-information H_∞ control problem. It turns out that, as in the continuous time case, the existence of an internally stabilizing controller which makes the H_∞ norm strictly less than 1 is related to the existence of a stabilizing solution to an algebraic Riccati equation. However the solution of this algebraic Riccati equation has to satisfy an extra condition. Moreover it is interesting to note that in general state feedbacks do not suffice and we have to include the disturbance in our feedback.

Keywords : Discrete time, Algebraic Riccati equation, H_∞ control, Full information, Static feedback.

1 Introduction

In recent years a considerable amount of papers appeared about the, by now, well-known H_∞ optimal control problem (e.g. [1], [2], [3], [6], [7], [10], [11], [12], [13]. However all these papers discuss the continuous time case. In this paper we will in contrast with the above papers discuss the discrete time case.
In the above papers several methods were used to solve the H_∞ control problem, e.g. frequency domain approach, polynomial aproach and time domain approach. Recently there appeared a paper solving the discrete time H_∞ control problem using frequency domain techniques ([5]). In contrast with that paper this paper will use time-domain techniques which have a lot of familiarities with the paper [12] which deals with the continuous time case.
We make the assumption that we deal with the special case that both disturbance and state are available for feedback. The other assumptions we have to make are weaker than the assumptions in [5]. We do not assume that the system matrix A is invertible. Moreover we replace the assumption that the direct feedthrough matrix from control input to output is injective by the assumption that the transfer matrix from control input to output is left invertible as a rational matrix which is weaker. The only other assumption we have to make is, that a subsystem has no invariant zeros on the unit circle.
As in the continuous time case the necessary and sufficient conditions for the existence of an internally stabilizing controller which makes the closed loop transfer matrix have norm less than 1 involve a positive semi-definite stabilizing solution of an algebraic Riccati equation. However, compared to the continuous time case, P has to satisfy another assumption: a matrix depending on P should be positive definite.
Another difference with the continuous time is, that in the discrete time, even if $D_2 = 0$, we can not always achieve our goal with a static state feedback. In general, we also need a static feedback depending on the disturbance.
This paper gives the general outline of the proof. Some of the details however are not given.

The outline of the paper is as follows. In section 2 we will formulate the problem and give the main results. In section 3 we will derive necessary conditions under which there exists an internally stabilizing feedback which makes the H_∞ norm less than 1. In section 4 we will show that these conditions are also sufficient. We will end with some concluding remarks in section 5.

2 Problem formulation and main results

We consider the following system:

$$\Sigma : \begin{cases} x(k+1) & = & Ax(k) & + & Bu(k) & + & Ew(k) \\ z(k) & = & Cx(k) & + & D_1u(k) & + & D_2w(k) \end{cases} \tag{2.1}$$

where $x(k) \in \mathcal{R}^n$ is the state, $u(k) \in \mathcal{R}^m$ is the control input, $w(k) \in \mathcal{R}^l$ the unknown disturbance and $z(k) \in \mathcal{R}^p$ the, to be controlled, output. Moreover A, B, E, C, D_1 and D_2 are matrices of appropriate dimensions. Our final objective is to find a static feedback $u(k) = F_1x(k)+F_2w(k)$ such that the closed loop system is internally stable and for the closed loop system the ℓ_2-induced norm from disturbance w to the output z is minimized over all internally stabilizing static feedbacks. Here internally stable means that $A + BF_1$ is asymptotically stable, i.e. all eigenvalues lie inside the open unit disc. Denote by G_F the closed loop transfer matrix:

$$G_F(s) := (C + D_1F_1)(sI - A - BF_1)^{-1}(E + BF_2) + (D_2 + D_1F_2). \tag{2.2}$$

The ℓ_2-induced norm is given by:

$$\begin{aligned}\|G_F\|_\infty &= \sup_{\substack{w\in\ell_2^l\\ w\neq 0}} \frac{\|z\|_2}{\|w\|_2}\\ &= \sup_{\theta\in[0,2\pi]} \|G_F(e^{i\theta})\|\end{aligned}$$

where the ℓ_2-norm is given by:

$$\|p\|_2 := \left(\sum_{k=0}^{\infty} p^T(k)p(k)\right)^{1/2}$$

and $\|.\|$ denotes the Euclidian norm. In this paper we will derive necessary and sufficient conditions for the existence of a feedback $F=(F_1,F_2)$ which is internally stabilizing and which is such that the closed loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$. In the formulation of our main result we will need the concept of invariant zero. z is called an *invariant zero* of a system (A,B,C,D) if

$$\operatorname{rank}_{\mathcal{R}} \begin{pmatrix} zI-A & -B\\ C & D\end{pmatrix} < \operatorname{rank}_{\mathcal{R}(s)} \begin{pmatrix} sI-A & -B\\ C & D\end{pmatrix}.$$

A system (A,B,C,D) is called *left invertible* if the transfer matrix $C(sI-A)^{-1}B+D$ is left invertible as a matrix with entries in the field of rational functions. We can now formulate our main result:

Theorem 2.1 : *Let the system (2.1) be given with zero initial state. Assume (A, B, C, D_1) has no invariant zeros on the unit circle and is left invertible. The following statements are equivalent:*

(i) *There exists a feedback $F=(F_1,F_2)$ such that $A+BF_1$ is asymptotically stable and the resulting closed loop transfer matrix G_F satisfies $\|G_F\|_\infty < 1$.*

(ii) *There exists a symmetric matrix $P\geq 0$ such that*

1. *The matrix G(P) is invertible, where:*

$$G(P) := \left[\begin{pmatrix} D_1^T D_1 & D_1^T D_2\\ D_2^T D_1 & D_2^T D_2 - I\end{pmatrix} + \begin{pmatrix} B^T\\ E^T\end{pmatrix} P \begin{pmatrix} B & E\end{pmatrix}\right] \tag{2.3}$$

2. *P satisfies the following discrete algebraic Riccati equation:*

$$P = A^T PA + C^T C - \begin{pmatrix} B^T PA + D_1^T C\\ E^T PA + D_2^T C\end{pmatrix}^T G(P)^{-1} \begin{pmatrix} B^T PA + D_1^T C\\ E^T PA + D_2^T C\end{pmatrix} \tag{2.4}$$

3. *The matrix A_{cl} is asymptotically stable, where:*

$$A_{cl} := A - \begin{pmatrix} B^T\\ E^T\end{pmatrix}^T G(P)^{-1} \begin{pmatrix} B^T PA + D_1^T C\\ E^T PA + D_2^T C\end{pmatrix} \tag{2.5}$$

4. *We have*

$$R > 0 \tag{2.6}$$

where

$$R := I - D_2^T D_2 - E^T PE + \left(E^T PB + D_2^T D_1\right)\left(D_1^T D_1 + B^T PB\right)^{-1}\left(B^T PE + D_1^T D_2\right)$$

The inverse in the above matrix always exists.

Moreover, in case a P satisfies (ii), then the feedback $F = (F_1, F_2)$ given by

$$F_1 := -(D_1^T D_1 + B^T PB)^{-1}(B^T PA + D_1^T C) \tag{2.7}$$

$$F_2 := -(D_1^T D_1 + B^T PB)^{-1}(B^T PE + D_1^T D_2) \tag{2.8}$$

satisfies (i).

Remark :

(i) Necessary and sufficient conditions whether we can find an internally stabilizing feedback which makes the H_∞ norm less than some a priori given upper bound γ can be easily derived from theorem 2.1 by scaling.

(ii) If we compare these conditions with the conditions for the continuous time case we note that condition (2.6) is added. A simple example showing that this assumption is not superfluous is given by the system:

$$\begin{cases} x(k+1) &= & & u(k) + 2w(k) \\ z(k) &= & \begin{pmatrix} 1 \\ 0 \end{pmatrix} x(k) &+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \end{cases} \tag{2.9}$$

There doesn't exist a feedback F satisfying part (i) of theorem 2.1 but there does exist a positive semidefinite matrix P satisfying (2.4) and such that $A_{cl} = 0$ and hence asymptotically stable, namely $P = 1$. However for this P we have $R = -1$.

The general outline of the proof will be reminiscent of the proof given in [12] for the continuous time case. The extra condition (2.6), the invertibility of (2.3) and the requirement of left invertibility instead of assuming that D_1 is injective will give rise to a substantial increase in the amount of intricacies in the proof. This paper will however only give the general outline of the proof. The detailed proof will appear in a future paper.

3 Necessary conditions for the existence of suboptimal controllers

In this section we assume that part (i) of theorem 2.1 is satisfied. We will show that the conditions in (ii) are necessary. Consider system (2.1). For given disturbance w and control input u let $x_{u,w,\xi}$ and $z_{u,w,\xi}$ denote the resulting state and output respectively for initial value $x(0) = \xi$. If $\xi = 0$ we will simply write $x_{u,w}$ and $z_{u,w}$. Note that it is easily seen that the following statement is a direct result from theorem 2.1 part (i):

Assumption 3.1 : *(A,B) is stabilizable. Moreover, for initial state zero, there exists a $\delta > 0$ such that for all $w \in \ell_2^l$ there exists $u \in \ell_2^m$ for which $x_{u,w} \in \ell_2^n$ and $\|z_{u,w}\|_2^2 \leq (1 - \delta^2)\|w\|_2^2$.*

We will show that assumption 3.1 already implies that the conditions in part (ii) of theorem (2.1) are satisfied. This implies that even if we allow more general feedbacks, e.g. dynamic feedbacks, we cannot achieve more. We will assume $D_1^T[\begin{array}{cc} C & D_2 \end{array}] = 0$ for the time being and we will derive the more general statement later. In order to prove the conditions (ii) of theorem 2.1 we will solve the following sup-inf problem:

$$\sup_{w \in \ell_2^l} \inf_u \ \{ \ \|z_{u,w,\xi}\|_2^2 - \|w\|_2^2 \ \mid \ u \in \ell_2^m \text{ such that } x_{u,w,\xi} \in \ell_2^n \ \} \tag{3.1}$$

for arbitrary initial value ξ. Let L be such that $D_1^T D_1 + B^T L B$ is invertible and let it be the positive semi-definite solution of the following discrete algebraic Riccati equation:

$$L = A^T L A + C^T C - A^T L B \left(D_1^T D_1 + B^T L B\right)^{-1} B^T L A \tag{3.2}$$

such that

$$A_L := A - B\left(D_1^T D_1 + B^T L B\right)^{-1} B^T L A \tag{3.3}$$

is asymptotically stable. The existence of such an L is guaranteed if (A, B) is stabilizable and moreover (A, B, C, D_1) has no invariant zeros on the unit circle and is left invertible (see [9]). The assumption that (A, B) is stabilizable is made in assumption 3.1. Moreover (A, B, C, D_1) has no invariant zeros on the unit circle and is left invertible by the original assumptions of theorem 2.1. We define

$$r(k) := -\sum_{i=k}^{\infty} \left[X_1 A^T\right]^{i-k} X_1 \left(L E w(i) + C^T D_2 w(i+1)\right) \tag{3.4}$$

where

$$X_1 := I - L B \left(D_1^T D_1 + B^T L B\right)^{-1} B^T \tag{3.5}$$

Note that r is well-defined since $A_L = X_1^T A$ asymptotically stable implies that $X_1 A^T$ is asymptotically stable. Next we define

$$\begin{aligned} y(k) &= \left(D_1^T D_1 + B^T L B\right)^{-1} B^T \left[A^T r(k+1) - L E w(i) - C^T D_2 w(i+1)\right] && (3.6)\\ \tilde{x}(k+1) &= A_L \tilde{x}(k) + B y(k) + E w(k), \qquad \tilde{x}(0) = \xi && (3.7)\\ \eta(k) &= -X_1 L A \tilde{x}(k) + r(k) && (3.8) \end{aligned}$$

for $k = 0, 1, \ldots$. It can be checked straightforwardly that $r, \tilde{x}, \eta \in \ell_2$. Moreover η satisfies the following backwards difference equation:

$$\eta(k-1) = A^T \eta(k) - C^T C \tilde{x}(k) - C^T D_2 w(k). \tag{3.9}$$

This can be checked by deriving a backwards difference equation for r and some calculations.

Lemma 3.2 : *Let the system (2.1) be given. Moreover let w and ξ be fixed. Then*

$$-\left(D_1^T D_1 + B^T L B\right)^{-1} B^T L A \tilde{x} + y = \arg \inf_u \ \{\|z_{u,w,\xi}\|_2 \mid u \in \ell_2^m \text{ such that } x \in \ell_2^n\}$$

Proof : This can be shown using the sufficient conditions for optimality in [8, Section 5.2]. It has to be adapted for the infinite horizon case but it still works. In [12] a similar method is used. Uniqueness of the optimizing u can be shown using the left invertibility of (A, B, C, D_1). ■

Define $\mathcal{F}(\xi, w) = (\tilde{x}, \tilde{u}, \eta)$ and $\mathcal{G}(\xi, w) = C\tilde{x} + D_1\tilde{u} + D_2 w$. It is clear from the previous lemma that $\mathcal{F}$ and $\mathcal{G}$ are bounded linear operators. Define

$$\mathcal{C}(\xi, w) := \|\mathcal{G}(\xi, w)\|_2^2 - \|w\|_2^2 \tag{3.10}$$

$$\|w\|_C := (-\mathcal{C}(0, w))^{1/2} \tag{3.11}$$

It can be easily shown that $\|.\|_C$ defines a norm. Using our assumption 3.1 it can be shown straightforwardly that

$$\|w\|_2 \geq \|w\|_C \geq \delta \|w\|_2 \tag{3.12}$$

where δ is such that assumption 3.1 is satisfied. Hence $\|.\|_C$ and $\|.\|_2$ are equivalent norms. Define

$$\mathcal{C}^*(\xi) = \sup_{w \in \ell_2^l} \mathcal{C}(\xi, w) \tag{3.13}$$

We can derive the following properties of $\mathcal{C}^*$:

Lemma 3.3

(i) *For all $\xi \in \mathcal{R}^n$ we have*

$$0 \leq \xi^T L \xi \leq \mathcal{C}^*(\xi) \leq \frac{\xi^T L \xi}{\delta^2} \tag{3.14}$$

where δ is such that (3.12) is satisfied.

(ii) *For all $\xi \in \mathcal{R}^n$ there exists a unique $w_* \in \ell_2^l$ such that $\mathcal{C}^*(\xi) = \mathcal{C}(\xi, w_*)$*

Proof : Part (i) is shown by using that the cost of the discrete time linear quadratic problem with internal stability (which is $\xi^T L \xi$, see [9]) is an underbound for $\mathcal{C}^*(\xi)$ and we can make some estimations, using assumption 3.1, to obtain an upper bound for $\mathcal{C}^*(\xi)$.
Part (ii) can be proven in the same way as in [12]. It strongly depends on the formula:

$$\|w_\alpha - w_\beta\|_C^2 = 2.\mathcal{C}(\xi, w_\alpha) + 2.\mathcal{C}(\xi, w_\beta) - 4.\mathcal{C}(\xi, 1/2\,(w_\alpha + w_\beta)) \tag{3.15}$$

which is true for arbitrary $\xi \in \mathcal{R}^n$. ■

Define $\mathcal{H} : \mathcal{R}^n \to \ell_2^l$, $\xi \to w_*$.

Lemma 3.4 : *Let $\xi \in \mathcal{R}^n$ be given. $w_* = \mathcal{H}\xi$ is the unique ℓ_2-function w satisfying:*

$$(I - D_2^T D_2)\, w = -E^T \eta_* + D_2^T C x_* \tag{3.16}$$

where $(x_*, u_*, \eta_*) = \mathcal{F}(\xi, w)$.

Proof : Define $(x_*, u_*, \eta_*) = \mathcal{F}(\xi, w_*)$. Moreover define $w_0 := -E^T\eta(w_*) + D_2^T D_2 w_* + D_2^T C x_*$ and $(x_0, u_0, \eta_0) := \mathcal{F}(\xi, w_0)$. It can be shown that:

$$\mathcal{C}(\xi, w_*) = \mathcal{C}(\xi, w_0) - \|w_0 - w_*\|_2^2 - \|z_{u_0,w_0,\xi} - z_{u_*,w_*,\xi}\|_2^2 \tag{3.17}$$

Since w_* was maximizing $\mathcal{C}(\xi, w)$ over all w, this implies $w_0 = w_*$. That w_* is the unique solution of the equation (3.16) can be shown in a similar way. Assume that besides w_* also w_1 satisfies (3.16). Let $(x_1, u_1, \eta_1) := \mathcal{F}(\xi, w_1)$. It can be shown that:

$$\mathcal{C}(\xi, w_*) = \mathcal{C}(\xi, w_1) - \|w_* - w_1\|_{\mathcal{C}}^2 \tag{3.18}$$

Since w_* was maximizing, we find $\|w_* - w_1\|_{\mathcal{C}} = 0$ and hence $w_* = w_1$. q.e.d. ■

Lemma 3.5 *There exist constant matrices K_1, K_2 and K_3 such that*

$$u_* = K_1 x_*, \qquad \eta_* = K_2 x_*, \qquad w_* = K_3 x_*.$$

Proof : This can be shown by first looking at time zero and deriving the existence of K_1, K_2 and K_3 for time zero. Then using time-invariance it can be shown that K_1, K_2 and K_3 satisfy lemma 3.5 for all $t \geq 0$. ■

Lemma 3.6 : *There exists a $P \geq 0$ such that $\eta_*(k) = -Px_*(k+1)$ $k = -1, 0, 1, \ldots$. where $\eta(-1)$ is defined by (3.9). Moreover for this P we find*

$$\mathcal{C}^*(\xi) = \xi^T P \xi. \tag{3.19}$$

Proof : The existence of a P satisfying $\eta_*(k) = -Px_*(k+1)$ $k = -1, 0, 1, \ldots$ can be derived straightforwardly from the backwards difference equation 3.9 and lemma 3.5. Here (3.19) is then proven by deriving the equation:

$$\mathcal{C}(\xi, w_*) + 2\eta_*^T(-1)x_*(0) = -\mathcal{C}(\xi, w_*)$$

Since $\mathcal{C}(\xi, w_*) = \mathcal{C}^*(\xi)$ and $\eta_*(-1) = -P\xi$ we find (3.19). ■

Lemma 3.7 : *Assume (A, B, C, D_1) has no invariant zeros on the unit circle and is left invertible. Moreover assume that $D_1^T[C \quad D_2] = 0$. If part (i) of theorem 2.1 is satisfied then there exists a symmetric matrix $P \geq 0$ satisfying part (ii) of theorem 2.1.*

Proof : By using lemma 3.4 it can be shown that the matrix $Z := I - D_2^T D_2 - E^T X_1 L E$ is invertible. Using this we find after some tedious calculations that

$$\left\{I + \left[B\left(D_1^T D_1 + B^T L B\right)^{-1} B^T - X_1^T E Z^{-1} E^T X_1\right](P - L)\right\} x_*(k+1) =$$

$$X_1^T \left\{A + EZ^{-1}E^T X_1 LA + EZ^{-1}D_2^T C\right\} x_*(k) \qquad (3.20)$$

Since $u_*(k)$ and $w_*(k)$ are uniquely determined by $x_*(k)$ also $x_*(k+1)$ is uniquely determined by $x_*(k)$. This is the main reasoning to show that the matrix on the left is invertible. This is equivalent to the invertibility of the matrix (2.3). Moreover if we define A_{cl} by (2.5) then we find $x_*(k+1) = A_{cl}x_*(k)$. Since $x_* \in \ell_2^n$ for all initial values ξ we know that A_{cl} is asymptotically stable. Next we show that P satisfies the discrete algebraic Riccati equation (2.4). From (3.9) combined with lemma 3.6 it can be derived that:

$$P = A^T P A_{cl} + C^T C + C^T D_2 Z^{-1} \left\{E^T X_1 (P - L) A_{cl} + D_2^T C + E^T X_1 LA\right\} \qquad (3.21)$$

By some extensive calculations this turns out to be equivalent to the discrete algebraic Riccati equation (2.4). Next we show that P is symmetric. Note that both P and P^T satisfy the DARE. Using this we find that:

$$(P - P^T) = A_{cl}^T (P - P^T) A_{cl}.$$

Since A_{cl} is asymptotically stable this implies that $P = P^T$. P can be shown to be positive semi definite by combining lemma 3.3 and (3.19). Remains to be shown (2.6). Since the matrix $G(P)$ defined by (2.3) is invertible it can be shown using the Schur complement that R is invertible. We will use a homotopy argument to prove that in fact we have $R > 0$. Assume we replace E by $E(\alpha) = \alpha E$ and D_2 by $D_2(\alpha) = \alpha D_2$. It can be easily checked that for all $\alpha \in [0, 1]$ assumption 3.1 is satisfied. Moreover it can be shown that $R(\alpha)$ is a continuous function in α. Since $R(0) > 0$ and $R(\alpha)$ is invertible for all $\alpha \in [0, 1]$ by a homotopy argument we find $R = R(1) > 0$. This is exactly (2.6) and hence the proof is completed. ■

Corollary 3.8 : *Assume (A, B, C, D_1) has no invariant zeros on the unit circle and is left invertible. If part (i) of theorem 2.1 is satisfied then there exists a symmetric matrix $P \geq 0$ satisfying part (ii) of theorem 2.1.*

Proof : We first apply a preliminary feedback $u = \tilde{F}_1 x + \tilde{F}_2 w + v$ such that

$$D_1^T \left(C + D_1 \tilde{F}_1\right) = 0, \qquad D_1^T \left(D_2 + D_1 \tilde{F}_2\right) = 0.$$

Denote the new A, C, D_2 and E by $\tilde{A}, \tilde{C}, \tilde{D}_2$ and $\tilde{E}$. For this new system part (i) of theorem 2.1 is satisfied. Hence since for this new system $D_1^T[\tilde{C} \quad \tilde{D}_2] = 0$ we find conditions in terms of the new parameters. Rewriting in terms of the original parameters gives the desired conditions as given in part (ii) of theorem 2.1. ■

4 Sufficient conditions for the existence of suboptimal controllers

In this section we will show that if there exists a P satisfying the conditions of theorem 2.1 then the feedback as suggested by theorem 2.1 satisfies condition (i). In order to do this we first need a number of preliminary results.

A system is called inner if the transfer matrix of the system, denoted by G satisfies:

$$G(z)G^T(z^{-1}) = I \tag{4.1}$$

We now formulate a generalization of [5, lemma 5]. The proof is a slightly more complicated since if G has a pole in zero then $G^T(z^{-1})$ is not proper any more. Nevertheless it can be shown by simply writing out (4.1).

Lemma 4.1 : *Assume we have a system*

$$\Sigma_{st} : \left\{ \begin{array}{lllll} x(k+1) & = & Ax(k) & + & Bu(k) \\ z(k) & = & Cx(k) & + & Du(k) \end{array} \right. \tag{4.2}$$

Assume A is stable. The system Σ_{st} is inner if there exists a matrix X satisfying:

1. $X = A^T X A + C^T C$
2. $D^T C + B^T X A = 0$
3. $D^T D + B^T X B = I$

We define the following system:

$$\Sigma_U : \left\{ \begin{array}{lllllll} x_U(k+1) & = & A_U x_U(k) & + & B_U u_U(k) & + & E_U w(k), \\ y_U(k) & = & C_{1,U} x_U(k) & + & & + & D_{12,U} w(k), \\ z_U(k) & = & C_{2,U} x_U(k) & + & D_{21,U} u_U(k) & + & D_{22,U} w(k), \end{array} \right. \tag{4.3}$$

where

$$\begin{array}{rcl} A_U & := & A - BW^{-1}(B^T PA + D_1^T C) \\ B_U & := & BW^{-1/2} \\ E_U & := & E - BW^{-1}(B^T PE + D_1^T D_2) \\ C_{1,U} & := & -R^{-1/2}\left(E^T PA + D_2^T C - [E^T PB + D_2^T D_1]\, W^{-1}\, [B^T PA + D_1^T C]\right) \\ C_{2,U} & := & C - D_1 W^{-1}(B^T PA + D_1^T C) \\ D_{12,U} & := & R^{1/2} \\ D_{21,U} & := & D_1 W^{-1/2} \\ D_{22,U} & := & D_2 - D_1 W^{-1}(B^T PE + D_1^T D_2) \end{array}$$

Lemma 4.2 : *The system Σ_U as defined by (4.3) is internally stable and inner. Denote the transfer matrix of Σ_U by U. We decompose U:*

$$U \begin{pmatrix} w \\ u_U \end{pmatrix} =: \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} w \\ u_U \end{pmatrix} = \begin{pmatrix} z_U \\ y_U \end{pmatrix}$$

compatible with the sizes of w, u_U, z_U and y_U. Then U_{21} is invertible and its inverse is in H_∞ .

Proof : It can be easily checked that P as defined by theorem 2.1 (a)-(d) satisfies the conditions (a)-(c) of lemma 4.1. (a) of lemma 4.1 turns out to be equal to the discrete algebraic Riccati equation (3.2). (b) and (c) follow by simply writing out the equations in the original system parameters of system (2.1).
Next we note that $P \geq 0$ and

$$P = A_U^T P A_U + \begin{pmatrix} C_{1,U}^T & C_{2,U}^T \end{pmatrix} \begin{pmatrix} C_{1,U} \\ C_{2,U} \end{pmatrix} \tag{4.4}$$

Using standard Lyapunov theory it can then be shown that A_U is asymptotically stable.
To show that U_{21}^{-1} is an H_∞ function we write down a realization for U_{21}^{-1} and note that $A_{cl} = A_U - E_U D_{12,U}^{-1} C_{1,U}$. The proof is then trivial. ■

Lemma 4.3 : *Assume there exists a P satisfying the conditions in (ii) of theorem 2.1. In that case the feedback $u = F_1 x + F_2 w$ where F_1, F_2 are given by (2.7) and (2.8) satisfies condition (i) of theorem 2.1.*

Proof : First note that G_F as given by (2.2) for this particular F is equal to U_{11} and moreover $A + BF_1$ is equal to A_U. This implies that $F = (F_1, F_2)$ is internally stabilizing and G_F as a submatrix of an inner matrix satisfies $\|G_F\| \leq 1$. Using the fact that U_{21} is invertible in H_∞ it can be shown that the inequality is strict. ■

Note that theorem 2.1 is simply a combination of corollary 3.8 and lemma 4.3. Therefore the main result has been proven.

5 Concluding remarks

In this paper the *discrete time* full information case H_∞ control problem has been investigated. As in the continuous time case the solvability is related to an algebraic Riccati equation. However, in contrast to the continuous time case, it turns out that, even in case $D_2 = 0$ the feedback we find is in general not a state feedback but also an disturbance feedback. Another interesting feature is the extra condition $R > 0$.
The assumptions made in this paper are exactly the discrete time versions of the two main assumptions which are often made in the continuous time.
This paper is naturally a preliminary step towards the measurement feedback case which will be elaborated in an future paper. Another interesting item for future research is finding algorithms to calculate stabilizing solutions of the discrete algebraic Riccati equation (2.4) and discuss issues like uniqueness of stabilizing solutions. I have only been able to reduce this problem to a generalized eigenvalue problem and prove uniqueness in case D_1 and D_2 satisfy certain prerequisites.

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