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ROBUSTNESS OF FEEDBACK STABILIZATION: A TOPOLOGICAL APPROACH

PROEFSCHRIFT

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To my parents

To my motherland

Preface

In September 1985, I came to Eindhoven and started to study systems and control theory. Since then, Prof. Malo Hautus has been supervising me enthusiastically and patiently to learn and to do research leading to this thesis. His stringent scientific style and broad knowledge have influenced me to a great extent. I am grateful to him for his guidance and supervision during the past four years.

Another person who has played a role in my studies is Dr. Kees Praagman. I am indebted to him for always being ready to discuss problems, for his rapid critical reading of my manuscript, and for his support in practical matters. Then, I would like to give my thanks to Drs. Anton Stoorvogel for sharing his many research interests with me, for many stimulating discussions and for the help he offered me; also, thanks goes to my room mate Ir. Ton Geerts for many interesting discussion and for his assistance.

I have been fortunate to learn systems and control theory in a land that holds a concentration of eminent researchers in this field, and from whom I have benefited very much. The subject of my thesis resulted from a suggestion of Prof. Hans Schumacher to compare the gap topology with the graph topology, and I am indebted to him for his constant help and advice. From Prof. Ruth F. Curtain I learnt about the theory of infinite dimensional systems, and her enthusiastic help and support are gratefully acknowledged. Special thanks are given to Prof. Frank M. Callier and Dr. Joseph Winkin from Belgium and Prof. George Zames from Canada for their interest in this research.

I appreciate that Prof. H. Kwakernaak and Prof. J. de Graaf took the trouble to review my thesis and that Dr. Peter Attwood improved the written English of the thesis.

I would like to express my gratitude to the Faculty of Mathematics and Computing Science at Eindhoven University of Technology for its financial support of my research over the last four years. Last but not least I would like to thank Mrs. Harma Koops, the secretary of our group.

In this thesis, a compact and self-contained story is presented on a topological approach to the robustness of feedback stabilization. The investigation was carried out in a general framework including finite and infinite dimensional linear time-invariant systems as well as continuous-time and discrete-time and even 2D-systems. This thesis summarises the extensive work done on this approach including the most recent research. To follow this thesis one needs no more than the background of Hardy class theory, operator theory and the frequency domain approach to control systems.

Contents

Notation, Symbols and Abbreviations

$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of non-negative real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}_+$	the open right half plane
$\mathbb{D}$	the open unit disk
A	a normed integral domain with identity of linear bounded operators
F	a subring of the quotient field of A
$A^{n\times m}$	the set of all nxm matrices with entries in A
$\mathrm{M}(A)$	the set $\bigcup_{n,m} A^{n\times m}$
$U^{m,m}$	the subset of $A^{m\times m}$ consisting of all the unimodular matrices
$B^{n,m}$	the subset of $F^{n\times m}$ consisting of all the matrices having a right Bezout fraction and a left Bezout fraction over $\mathrm{M}(A)$
$C^{n,m}$	the subset of $F^{n\times m}$ consisting of all the systems having stabilizing controllers
X	the space of inputs and outputs
H_∞	a Hardy space consisting of all complex-valued functions f(.) which are analytic in $\mathbb{C}_+$ and satisfy $\|f(.)\| := \sup \{ \|f(s)\| : s \in \mathbb{C}_+ \} < \infty$
$\mathrm{H}_\infty(\mathbb{D})$	a Hardy space consisting of all complex-valued functions f(.) which are analytic in $\mathbb{C}\backslash\bar{\mathbb{D}}$ and satisfy $\|f(.)\| := \sup \{ \|f(s)\| : s \in \mathbb{C}\backslash\bar{\mathbb{D}} \} < \infty$
H_2	a Hardy space consisting of all complex-valued functions f(.) which are analytic in $\mathbb{C}_+$ and satisfy $\|f(.)\| := [\sup \{ \int_0^{2\pi} \|f(\sigma+i\omega)\|^2 d\omega : \sigma > 0 \}]^{1/2} < \infty$
$\mathrm{H}_2(\mathbb{D})$	a Hardy space consisting of all complex-valued functions f(.) which are analytic in $\mathbb{C}\backslash\bar{\mathbb{D}}$ and satisfy $\|f(.)\| := [\sup \{ \int_{-\infty}^{+\infty} \|f(\gamma e^{i\omega})\|^2 d\omega : \gamma > 1 \}]^{1/2} < \infty$
L_∞	a Lebesgue space consisting of all complex-valued functions f(.) satisfying $\|f(.)\| := \mathrm{ess\ sup} \{ \|f(i\omega)\| : \omega \in \mathbb{R} \} < \infty$
L_2	a Lebesgue space consisting of all complex-valued functions f(.) satisfying $\|f(.)\| := [\int_{-\infty}^{+\infty} \|f(i\omega)\|^2 d\omega]^{1/2} < \infty$
RH_∞	the subset of H_∞ consisting of rational functions

$\|P\|$	determinant of the matrix P
P^T	transpose of the matrix P
P^*	adjoint of the operator P
$P^-(s)$	$= \overline{P(-\bar{s})}^T$
$\phi \ominus \psi$	$= \{ x \in \phi : x \perp y \text{ for all } y \in \psi \}$

iff	if and only if
LTI	linear time-invariant
SISO	single-input and single-output
r.b.f.	right Bezout fraction
l.b.f.	left Bezout fraction
g.r.b.f.	generalized right Bezout fraction
g.l.b.f.	generalized left Bezout fraction
resp.	respectively

Robust stabilization

Consider the standard *feedback system* shown in Figure 1.1. It is assumed that P_0 is the *nominal system* which models a natural phenomenon and C_0 is the *ideal controller* designed according to the nominal system P_0 in order to make the closed-loop system achieve some desired purposes, for example, closed-loop stability and/or response improvement. Due to the complicated nature and our limited knowledge, in general, the "real" system is difficult to be identified fully. Moreover, often a model has to be simplified, because it is too complicated to handle. Therefore, the nominal system only describes the "real" system approximately. On the other hand, some errors and/or simplification should be expected when implementing the ideal controller, so that the "real" controller will not necessarily be the same as the ideal controller. Thus, the ideal controller can only be an approximation of the "real" controller. In some sense, nearly all control systems are subject to the uncertainties of both systems and controllers. Consequently, in control system synthesis, it is necessary to study robustness with the uncertainties of both systems and controllers.

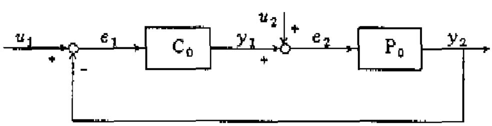

Figure 1.1 Feedback System

Conventionally and conveniently, the "real" system and the "real" controller can be regarded as perturbed versions of the nominal system and the ideal controller, respectively. This thesis is concerned with robustness of feedback stabilization and closed-loop response with respect to uncertainties in systems and controllers. It is supposed that the ideal controller C_0 stabilizes the nominal system P_0 and the closed-loop transfer matrix $H(P_0,C_0)$ from $u := [u_1^T, u_2^T]^T$ to $e := [e_1^T, e_2^T]^T$ achieves the desired response. Then, the central question to be studied here is : *What sort of perturbations can be permitted in P_0 and /or C_0 without destroying the feedback stability and without changing the closed-loop response* $H(P_0,C_0)$ *unacceptably* ? This is called the problem of *robustness of feedback stabilization*, or simply, *robust stabilization*.

Robustness of feedback stabilization is one of the critical problems in control system synthesis, and especially, in applications. In recent years, it has been studied from various points of view and a considerable amount of literature has been devoted to its study. To name some of them (certainly only a few) : The stability radii studied by Hinrichsen and Pritchard [H–P]; structured perturbations studied by Doyle [Do.]; additive system perturbations studied by Chen and Desoer [Ch–D], Vidyasagar and Kimura [V–K], Glover [Gl. 1], and Curtain and Glover [C–G]; multiplicative system perturbations studied by [V–K], stable Bezout factor perturbations studied by [V–K], Glover and McFarlane [G–M], Curtain [Cu. 2]; graph metric approach studied by Vidyasagar [Vi. 1,2] and Zhu [Zh. 4]; gap metric approach studied by Zames and El–Sakkary [Z–E], [El.], Zhu [Zh. 4], Zhu, Hautus and Praagman [Z–H–P 1,2], and Georgiou and Smith [G–S]. This thesis presents recent developments in the gap metric approach to the problem of robust stabilization.

The problem of robust stabilization is concerned with perturbation of a system. First of all, there is a need to measure perturbations, that is, to measure the distance between two systems. This need is typically met by introducing a metric. For stable systems, represented by input–output mappings, the operator norm is a natural measure. However, this norm cannot measure the distance between two unstable systems, and a topology or metric has to be developed for these systems.

Developing a topology or a metric for unstable systems should be related to a special design purpose. A topology which is suitable for one control design purpose might be unsatisfactory for another. More precisely, the characteristics of a topology or metric should match the features of the control design under consideration. The problem of robust stabilization has two basic requirements (for simplicity we will temporarily suppose that there are no perturbations on the controllers) : *i*) the perturbed systems P of the nominal system P_0 should be stabilized by the controller C_0; *ii*) the closed–loop system $H(P,C_0)$, resulting from the perturbation of P_0, should be "close" to $H(P_0,C_0)$. According to these two requirements, a neighborhood of P_0 can be defined as

$$N(P_0,\varepsilon) := \{\ P : P \text{ can be stabilized by } C_0, \text{ and } \|H(P,C_0)-H(P_0,C_0)\| < \epsilon\ \}.$$

By varying ε and P_0, a collection of the neighborhoods can be obtained, which generates a certain topology T. A family $\{P_\lambda\}$ of systems converges to P_0 in the topology T, as $\lambda \longrightarrow 0$ if and only if (iff) P_λ can be stabilized by C_0 when λ is sufficiently close to 0, while $H(P_\lambda,C_0)$ converges to $H(P_0,C_0)$ as $\lambda \longrightarrow 0$. This topology exactly matches the problem of robust stabilization. Unfortunately, this definition doesn't offer a good perspective for carrying out an analysis.

In 1980, Zames and El–Sakkary applied the gap metric to the robustness of feedback stabilization for square finite dimensional linear time–invariant (LTI) systems under unity feedback [Z-El.]. At the same time, Vidyasagar [Vi. 1] proposed the graph metric for finite dimensional LTI systems. A reformulation of these two topologies for a general setting and their comparison are presented in [Zh. 4]. It was shown that both are equal to the topology **T**.

In this thesis, we will report a study on robustness of feedback stabilization using the gap metric approach for a general framework including finite and infinite dimensional as well as continuous-time and discrete-time, and even including 2-D LTI systems. A necessary and sufficient condition for robust stabilization is characterized by the gap topology, and the estimation is given in the gap metric for the influence on the closed-loop transfer matrices by the perturbations the systems and controllers. Moreover, several guaranteed (i.e. sufficient) bounds for robust stabilization are provided in terms of the gap metric. For systems described by the transfer matrices with entries in the quotient field of H_∞, optimally robust controllers and the largest robust stability radius are discussed. Meanwhile, the relationship of the gap metric approach with the graph metric approach and stable Bezout factor perturbation method as well as additive and multiplicative system perturbation methods are presented.

Review of the thesis

This thesis consists of four chapters. Chapter 1 contains preliminaries having of four sections. The framework is outlined in Section 1, which is a general set-up including lumped and distributed as well as continuous-time and discrete-time and 2-D LTI systems. In Section 2, it is proven that the operators induced by systems are closed. And this property will be used twice in this thesis : first, to apply the gap topology; secondly, to apply the theorem of Lax. We will discuss the relationship between Bezout fractions and stabilizing controllers in Section 3. It is shown there, in general, that the existence of a right (or left) Bezout fraction ensures the existence of some stabilizing controllers. A useful fact in Lemma 1.3.3, hidden in the parameterization of all stabilizing controllers, will be revealed in this section too. Finally, a mathematical formulation for robustness of feedback stabilization will be given in Section 4.

Chapter 2 is devoted to a qualitative description of the robustness of feedback stabilization, and a necessary and sufficient condition for robust stabilization is characterized in terms of the gap topology. There are five sections in Chapter 2. The first describes a preparatory stage before the gap topology being discussed, in which the gap between two closed subspaces of a Banach space is introduced. The definition and some basic properties of the gap topology are introduced in Section 2, in which we will also prove the diagonal product property of the gap topology. In Section 3, a necessary and sufficient condition for robustness of feedback stabilization is given in the gap topology. Moreover, a lower and an upper bound are obtained for estimating the influence upon the closed–loop transfer matrix of perturbations of the system and controller. The graph topology is generalized to our framework in Section 4, in which a proof is provided for the diagonal product property without using spectral factorization. In the last section (Section 5), the gap topology is compared with the graph topology.

Chapter 3 gives a quantitative description of robust stabilization for systems in the general framework. In this chapter, several bounds are given which guarantee robust stabilization, and some useful techniques are developed. In Section 1, the definition of the gap metric is discussed, then, the concept of generalized Bezout fractions is developed and it is used to define the graph metric. A relationship between the gap metric and generalized Bezout fractions is presented in Section 2. This is one of the key techniques that are used in Chapters 3 and 4. In Section 3, the main results are provided, namely, the guaranteed bounds for robust stabilization.

In the first three chapters, the transfer matrices of the systems under consideration are supposed to have their entries in the quotient field of an arbitrary normed integral domain consisting of linear bounded operators. In Chapter 4, a special case is examined, that is, transfer matrices with entries in the quotient field of H_∞. Since H_∞ has a rich mathematical background and more structure, many results in the first three chapters can be deepened. Especially, one of the bounds obtained in Chapter 3 is shown to be the sharpest in this special case. In Section 1 of this chapter, the relationship of Bezout fractions with stabilizing controllers is again discussed. It is shown that H_∞ is a Hermite ring i.e. a transfer matrix has a right Bezout fraction iff it has a left Bezout fraction. The existence of normalized Bezout fractions is presented in Section 2, it will be a cornerstone for later developments. In Section 3, it is shown that the neighborhoods of a system in the gap metric are exactly the neighborhoods obtained by perturbing the right normalized Bezout fractions of the system. It follows that one of the guaranteed bounds given in Chapter 3 is the sharpest. Optimally robust controllers and the largest robust stability radius of a system are discussed in Section 4, where several related problems such as the influence of the uncertainties in optimally robust controllers and the variation of the closed–loop systems etc. are discussed. Section 5 is devoted to the discussion of the computation of the gap metric, and in this section a computable formula of the gap metric found by Georgiou and a lower and an upper bound of the gap metric obtained by Zhu, Hautus and Praagman are presented. In Section 6, we discuss the design of finite dimensional controllers for infinite dimensional systems via the largest robust stability radius and optimally robust controllers. Finally, in Section 7, a procedure for computing optimally robust controllers and the largest robust stability radius is presented. Some numerical examples are also provided there.

Preliminaries

1.1 Framework

A framework will now be formulated, which is a unifying approach for dealing with both lumped and distributed, as well as continuous-time and discrete-time LTI systems. This framework provides a connection of systems with operators which makes it possible to apply operator theory to control system synthesis.

SET-UP Let $\boldsymbol{A}$ be a commutative normed integral domain with identity of linear bounded operators mapping a Banach space X into X, and $\boldsymbol{F}$ be a subring of the quotient field of $\boldsymbol{A}$.

ASSUMPTION 1.1.1 It is assumed that any nonzero element f of $\boldsymbol{A}$ maps X into X injectively, and if f maps X onto X surjectively, then f^{-1} is in $\boldsymbol{A}$ also. Moreover, each element $P \in \boldsymbol{F}$ is supposed to have a coprime fractions over $\boldsymbol{A}$, which is unique up to multiplications by the units of $\boldsymbol{A}$.

Note that the coprime fractions do not necessarily have to be Bezout fractions, whose definition will be given in Section 3.

X is regarded as the space of (single) inputs and outputs. $\boldsymbol{A}$ is interpreted as the set of all single-input and single-output (SISO) stable systems, while $\boldsymbol{F}$ is the universe of all the SISO systems under consideration.

Since each nonzero element f of $\boldsymbol{A}$ is an injective linear bounded operator mapping X into X, the inverse f^{-1} exists as a linear (possibly, unbounded) operator mapping the range R(f) ($\subseteq$ X) of f onto X. It follows that for each system $P = h/f \in \boldsymbol{F}$, a linear (possibly, unbounded) operator **P** can be defined as follows.

DEFINITION 1.1.2 Let $P \in \boldsymbol{F}$ and $f,h \in \boldsymbol{A}$ be a coprime fraction of P. A linear operator **P** can be defined : The Dom(**P**) of **P** is defined as R(f) and the action of **P** on $x \in \mathrm{Dom}(\mathbf{P})$ is defined as $\mathbf{P}x := hf^{-1}x$. The operator **P** is called the *operator induced* by the system P.

Because the coprime fractions of $P \in \boldsymbol{F}$ are unique up to multiplications by the units of $\boldsymbol{A}$, it is easy to check that the induced operator $\mathbf{P}$ by $P \in \boldsymbol{F}$ does not depend on a special coprime fraction of P.

LEMMA 1.1.3 Suppose that $f,h \in \boldsymbol{A}$ and $f \neq 0$. Then, the operator $f^{-1}h$ is equal to hf^{-1} on $\mathrm{Dom}(hf^{-1})$.

PROOF Let $x \in \mathrm{Dom}(hf^{-1})$ and $hf^{-1}x = y$. Then, $fhf^{-1}x = fy$. Since f and h are commutative, it follows that $hx = fy$. Therefore, $hx \in \mathrm{Dom}(f^{-1})$ and $f^{-1}hx = y$. This implies that $f^{-1}h$ is equal to hf^{-1} on $\mathrm{Dom}(hf^{-1})$. ∎

LEMMA 1.1.4 Let $P_1,P_2 \in \boldsymbol{F}$. Then $\mathbf{P}_1 = \mathbf{P}_2$ if and only if (iff) $P_1 = P_2$.

PROOF "$\Leftarrow$" This is trivial.

"$\Rightarrow$" Suppose that $(f_i,h_i) \in \boldsymbol{A}$ is a coprime fraction of P_i $(i=1,2)$. For any $x\ (\neq 0) \in \mathrm{Dom}(\mathbf{P}_1)$ $(= \mathrm{Dom}(\mathbf{P}_2))$, we have that $h_1f_1^{-1}x = h_2f_2^{-1}x$. It follows from Lemma 1.1.3 that $f_1^{-1}h_1x = f_2^{-1}h_2x$, i.e., $f_2h_1x = f_1h_2x$. Define $g := f_2h_1 - f_1h_2$ $(\in \boldsymbol{A})$. Since g is not injective, it is zero. Thus, $P_1 = P_2$. ∎

Because of this lemma, $\mathbf{P}$ can be identified with P, and, for simplicity, $\mathbf{P}$ is denoted also by P.

LEMMA 1.1.5 The induced operator by $P \in \boldsymbol{F}$ is a bounded mapping of X into X iff $P \in \boldsymbol{A}$.

PROOF "$\Leftarrow$" It is trivial.

"$\Rightarrow$" Suppose that $(f,h) \in \boldsymbol{A}$ is a coprime fraction of P. Since $\mathrm{Dom}(P) = \mathrm{Dom}(f^{-1}) = R(f) = X$, f^{-1} is bounded, i.e., $f^{-1} \in \boldsymbol{A}$. Hence, $P \in \boldsymbol{A}$. ∎

The sum of two systems in $\boldsymbol{F}$ is their parallel connection, and the product of two systems is their cascade connection.

ASSUMPTION 1.1.6 The algebraic properties of operators induced by systems in $\boldsymbol{F}$ are defined by those of systems, i.e., the sum of two operators is the operator induced by the sum of the relevant systems and the product of two operators is the operator induced by the product of the relevant systems.

The following examples show that the above framework is reasonable and includes many important situations.

EXAMPLE 1.1.7 Assume that A is the set of all rational functions without poles in the closed right half plane including infinity and F is the set of all rational functions. Let X be the Hardy space H_2. It is well known that each system $P \in A$ induces a so-called Laurent operator [Fr. $p48$], a linear bounded mapping from H_2 to H_2, which is injective if $P \neq 0$. Identify each system P in A with its Laurent operator, then, A will be a normed integral domain. It is a routine to check that A, F and H_2 satisfy Assumption 1.1.1. This case represents continuous-time lumped LTI systems.

EXAMPLE 1.1.8 Assume that A is the set of all rational functions without poles in $\{ z \in \mathbb{C} : |z| \geq 1 \}$. Let F be the set of all rational functions and X be the Hardy space $H_2(\mathbb{D})$. As in Example 1.1.7, each system of A induces a Laurent operator mapping $H_2(\mathbb{D})$ into $H_2(\mathbb{D})$. If we identify the systems in A with their Laurent operators, then A becomes a normed integral domain. It is easy to check that A, F and $H_2(\mathbb{D})$ satisfy Assumption 1.1.1. This case stands for discrete-time lumped LTI systems.

Note that Examples 1.1.7 and 1.1.8 also include the so-called singular (or generalized) finite dimensional continuous-time and discrete-time lumped LTI systems, respectively.

EXAMPLE 1.1.9 Let $(\mathrm{LTI})_+$ denote the set of all real-valued Laplace transformable distributions with support on $\mathbb{R}_+$. Define

$$\mathcal{L}_{1,\sigma_0}(\mathbb{R}_+) := \{ f : f:\mathbb{R}_+ \longrightarrow \mathbb{R};\ \int_0^\infty |f(t)|e^{-\sigma_0 t}\, dt < \infty \},$$

and

$$A(\sigma_0) := \left\{ f \in (\mathrm{LTI})_+ : f(t) = \begin{cases} 0 & t<0 \\ f_\alpha(t) + \sum_{i=0}^{\infty} f_i \delta(t-t_i) & t\geq 0 \end{cases},\right.$$

$$\left. \text{where, } f_\alpha(\cdot) \in \mathcal{L}_{1,\sigma_0}(\mathbb{R}_+);\ t_i \in \mathbb{R}_+ \text{ and } \sum_{i=0}^{\infty} |f_i| e^{-\sigma_0 t_i} < \infty \right\}.$$

Moreover, define

$$A_-(\sigma_0) := \{ f \in A(\sigma_0) : \exists \sigma_1 < \sigma_0 \text{ such that } f \in A(\sigma_1) \}.$$

If "$\hat{\ }$" denote the Laplace transform, the interpretation of $\hat{A}(\sigma_0)$ and $\hat{A}_-(\sigma_0)$ is fairly obvious. Furthermore, define

$$\hat{A}_-^{\infty}(\sigma_0) := \{\ f \in \hat{A}_-(\sigma_0)\ :\ \exists\ \rho>0,\ \text{such that}\ \inf_{|s|>\rho} |f(s)| > 0\ \},$$

and

$$\hat{B}(\sigma_0) := [\hat{A}_-(\sigma_0)][\hat{A}_-^{\infty}(\sigma_0)]^{-1}.$$

Let A be $\hat{A}_-(0)$ and F be $\hat{B}(0)$, and assume that X is the Hardy space H_2. This is the transfer-function algebra introduced by Callier and Desoer [C-D 1,2], which describes a class of continuous-time distributed LTI systems. It is a routine to check that A, F and H_2 satisfy Assumption 1.1.1.

EXAMPLE 1.1.10 Let F be the set of all rational functions of two variables; i.e., F consists of all functions P(s,t) that are rational with respect to s and t, respectively. The poles of P(s,t) are defined as the pairs (s,t) so that the denominator of P(s,t) is zero. And P(s,t) is said to be stable if all of its poles are in $\mathbb{O} := \{ (s,t) : \text{Re } s < 0; \text{Re } t < 0.\}$. Let A be the subset of F, which consists of all stable elements. Define X to be the space consisting of all the functions of two variables g(s,t) which are analytic in both variables everywhere outside $\bar{\mathbb{O}}$ and satisfy

$$\|g(\cdot,\cdot)\| := [\ \sup_{\alpha>0;\,\beta>0} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |g(\alpha+i\omega\ ,\ \beta+i\gamma)|^2\ d\omega\ d\gamma\ ]^{1/2} < \infty.$$

It can be easily check that A, F and X defined here also satisfy Assumption 1.1.1. This case describes a class of 2D- LTI systems.

Note that, actually, there are various definitions of stability for 2-D systems and the definition given above is only one of the possibilities.

Denote the set of all matrices with entries in F (resp. A) by M(F) (resp. M(A)), the subset of M(F) (resp. M(A)) consisting of all nxm matrices by $F^{n\times m}$ (resp. $A^{n\times m}$), and the subset of $A^{m\times m}$ consisting of all unimodular matrices in $A^{m\times m}$ by $U^{m,m}$. Note that $U \in A^{m\times m}$ is unimodular neither implies nor is implied by the situation that its entries are unimodular. The norm of X^m is defined as $\|x\| := [\sum_{i=1}^{m} \|x_i\|^2]^{1/2}$. It follows from [Ka. p153] that

LEMMA 1.1.11 The mapping $U \longrightarrow U^{-1}$ defined on $U^{m,m}$ is continuous and $U^{m,m}$ is an open subset of $A^{m,m}$.

Each element $P \in F^{n\times m}$ induces an operator mapping a subspace of X^m into X^n in an obvious way. In the next section, we will prove that this operator is closed.

1.2 Closedness of $P \in F^{n \times m}$

In this section we show that each operator induced by system $P \in F^{n \times m}$ is a closed operator mapping a subspace of X^m into X^n. This property is essential for the application of the gap topology, because the gap topology is only defined for closed linear operators. Moreover, it is also a crucial property for applying Lax's theorem in order to prove the existence of normalized Bezout fractions in Chapter 4.

Suppose that T is a linear operator mapping a subspace of a Banach space Y into another Banach space Z. T is said to be *closed* if its graph

$$G(T) := \{ (x , Tx) : x \in Dom(T) \subseteq Y \}$$

is a closed subspace of $Y \times Z$.

THEOREM 1.2.1 Under Assumption 1.1.1, each system $P \in F^{n \times m}$ is a closed linear operator mapping a subspace of X^m into X^n

The proof of this theorem is based on the following lemma.

LEMMA 1.2.2 Assume that W, Y and Z are Banach spaces, S a closed linear operator mapping a subspace of W into Y and that T is a linear bounded injective operator mapping Z into Y. Then, the combined operator $T^{-1}S$ mapping a subspace of W into Z is closed.

Note that the injectivity of T implies the existence of T^{-1}, which is defined on the range R(T) of T.

PROOF We apply the well-known fact that a linear operator K mapping a subspace of Y into Z is closed iff Kx=y whenever $x_n \longrightarrow x$ and $Kx_n \longrightarrow y$ for $n \longrightarrow \infty$. Let $w_n, w \in W$, $w_n \longrightarrow w$ and $T^{-1}Sw_n \longrightarrow z$. Since T is continuous, $Sw_n \longrightarrow Tz$. By the closedness of S, we have $Sw = Tz$. Hence, $T^{-1}Sw = z$, and this implies that $T^{-1}S$ is closed. ■

PROOF OF THEOREM 1.2.1 Since $P \in F^{n \times m}$, there is an element $d \in A$ and a matrix $N \in A^{n \times m}$ such that $P = d^{-1}N$. Therefore, if we take $S = N$ and $T = d\,I$, where I is the nxn identity matrix, then, according to Lemma 1.2.2, P must be closed. ■

1.3 Bezout fractions and stabilizing controllers

For each $P \in F^{n\times m}$, $(D,N) \in M(A)$ is said to be a *right Bezout fraction* (r.b.f.) of P over $M(A)$ if

1) $D \in A^{m\times m}$, $N \in A^{n\times m}$ and $|D| \neq 0$;

2) there are two matrices Y and Z in $M(A)$ such that

(1.3.1) $$YD + ZN = I$$

3) $P = ND^{-1}$

Left Bezout fractions (l.b.f.) are defined analogously. It is easy to check that an r.b.f. (resp. l.b.f.) of $P \in F^{n\times m}$ is unique up to right (resp. left) multiplication by matrices in $U^{m,m}$ [Vi. 2 p75].

Equation (1.3.1) is called a Bezout identity. It plays an important role in control system synthesis. Later on we will show that a stabilizing controller can be obtained by solving a Bezout identity and that all stabilizing controllers can be parameterized by solving two Bezout identities (one is related to an r.b.f., another to an l.b.f.).

In general, not every matrix in $F^{n\times m}$ has an r.b.f. (resp. l.b.f.), and the fact that a system having an r.b.f. neither implies nor is implied by the fact that it has an l.b.f.. That each matrix P with entries in the quotient field of A has an r.b.f. iff it has an l.b.f. is equivalent to the fact that A is a *Hermite ring* [Vi. 2 p347]. In Chapter 4 we will prove that H_∞ is a Hermite ring.

Denote by $B^{n,m}$ the subset of $F^{n\times m}$ consisting of all elements which have both an r.b.f. and an l.b.f. over $M(A)$, and by $M(B)$ the set $\cup_{n,m}B^{n,m}$. Note the fact that a matrix $P \in F^{n\times m}$ has a right (resp. left) Bezout fraction does not imply that each of its entries has one; for instance, the matrix

$$P = \begin{bmatrix} se^{-s} & 0 \\ \frac{-1}{s+1} & 1 \end{bmatrix} = \begin{bmatrix} \frac{se^{-s}}{s+1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 1 & 1 \end{bmatrix}^{-1} = ND^{-1}$$

has a right Bezout fraction (D,N) over $M(H_\infty)$. But se^{-s} does not have a Bezout fraction (see Chapter 4).

Now we will introduce the *feedback system* shown in Figure 1.3.1, where $P \in M(F)$ represents a system and $C \in M(F)$ a controller; u_1,u_2 denote external inputs, e_1,e_2 inputs to the controller and system respectively, and y_1, y_2 outputs of the compensator and system, respectively. This model is versatile enough to accommodate several control problems, for instance, the problem of tracking or disturbance rejection or desensitization to noise or feedback compensation or cascade compensation etc.. For convenience, we will refer to such a set-up as a feedback system.

Suppose that P,C $\in$ M($\boldsymbol{F}$). The transfer matrix from $u := [u_1^T, u_2^T]^T$ to $e := [e_1^T, e_2^T]^T$ is

$$(1.3.2) \qquad H(P,C) := \begin{bmatrix} (I+PC)^{-1} & -P(I+CP)^{-1} \\ C(I+PC)^{-1} & (I+CP)^{-1} \end{bmatrix}.$$

Throughout this thesis it is assumed that P and C have compatible dimensions, also that the well-posedness condition $|I+PC| \neq 0$ is satisfied so that H(P,C) makes sense.

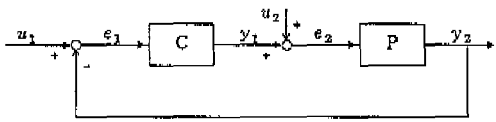

Figure 1.3.1 Feedback System

A transfer matrix is said to be stable if it is in M($\boldsymbol{A}$). The feedback system shown in Figure 1.3.1 is said to be stable if the transfer matrix W(P,C) from u to $y := [y_1^T, y_2^T]^T$ is stable. But it turns out that W(P,C) is stable iff H(P,C) is stable, because

$$W(P,C) = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} (H(P,C)-I).$$

Since H(P,C) has a slightly simpler form than W(P,C), we always deal with H(P,C) when studying stability of feedback system.

A system P $\in$ M($\boldsymbol{F}$) is said to be stabilizable if there is an element C in M($\boldsymbol{F}$) such that H(P,C) is stable. If H(P,C) is stable, then C is called a *stabilizing controller* of P. One can verify that the conditions for stability are symmetric in P and C, i.e., H(P,C) is stable iff H(C,P) is stable. The set of all stabilizing controllers of P is denoted by S(P).

LEMMA 1.3.1 If P $\in \boldsymbol{F}^{n \times m}$ has an r.b.f., then P has a stabilizing controller.

PROOF Assume that (D,N) $\in$ M($\boldsymbol{A}$) is an r.b.f. of P, and (Y,Z) $\in$ M($\boldsymbol{A}$) such that

$$YD + ZN = I.$$

If $|Y| \neq 0$, define $C := Y^{-1}Z$. It follows from

$$H(P,C) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -N \\ D \end{bmatrix} (YD + ZN)^{-1} [Z, Y]$$

that C is a stabilizing controller of P. Now suppose that $|Y| = 0$. Choosing $K \in A^{m\times m}$ such that $[Y^T,K^T]^T$ has full column rank, and define

$$V := \{ R \in A^{m\times m} : |Y + RK| \neq 0 \}.$$

It is shown [Vi. 2 *p*111] that V is an open dense subset of $A^{m\times m}$ (Note that although [Vi. 2 *p*111] states this property for principal ideal domains, the proof suits for the general case). Now, take $R \in V$ such that $\|RKD\| < 1$. Thus, $I + RKD$ is unimodular and it follows from

$$(Y + RK)D + ZN = I + RKD$$

that $C := (Y + RK)^{-1}Z$ is a stabilizing controller. ∎

Analogously, it can be shown that, if $P \in F^{n\times m}$ has an l.b.f., then it has stabilizing controllers too. Moreover, assume $P \in F^{n\times m}$ has stabilizing controllers, then, it can be easily proved that P has an r.b.f. iff all of its stabilizing controllers have an l.b.f., and that P has l.b.f. iff all of its stabilizing controllers have an r.b.f. [Vi. 2 *p*363]. Hence, if P is in $M(B)$, then the stabilizing controllers of P always exist and all its stabilizing controllers are in $M(B)$ too. Furthermore, we can parameterize all of the stabilizing controllers of $P \in M(B)$.

Assume that (D,N) and $(\tilde{D},\tilde{N})$ are an r.b.f. and an l.b.f., respectively, of P. Let $C_0 \in S(P)$ and (Y,Z) and $(\bar{Y},\bar{Z})$ be an r.b.f. and an l.b.f., respectively, of C_0, such that

$$(1.3.3) \qquad \begin{bmatrix} -Z & Y \\ \tilde{D} & \tilde{N} \end{bmatrix} \begin{bmatrix} -N & \bar{Y} \\ D & \bar{Z} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

It is readily shown that

$$(1.3.4) \qquad \begin{bmatrix} -Z-R\tilde{D} & Y-R\tilde{N} \\ \tilde{D} & \tilde{N} \end{bmatrix} \begin{bmatrix} -N & \bar{Y}-NR \\ D & \bar{Z}+DR \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

where R is an arbitrary element of $A^{m\times n}$.

LEMMA 1.3.2 [Vi. 2 *p*108]

$$S(P) = \{ (Y-R\tilde{N})^{-1}(Z+R\tilde{D}) : R \in A^{m\times n},\ |Y-R\tilde{N}| \neq 0 \}$$

$$= \{ (\bar{Z}+DR)(\bar{Y}-NR)^{-1} : R \in A^{m\times n}\ \ |\bar{Y}-NR| \neq 0 \}.$$

LEMMA 1.3.3 If $|Y-R\tilde{N}|\neq 0$, then $|\tilde{Y}-NR|\neq 0$, and vice versa.

PROOF Assume $|Y-R\tilde{N}|\neq 0$. According to Lemma 1.3.2, $C := (Y-R\tilde{N})^{-1}(Z+R\tilde{D}) \in S(P)$. Let (Y_c,Z_c) be an r.b.f. of C. Again by Lemma 1.3.2, there exists an $R_c \in A^{n\times m}$ such that $|\tilde{Y}-NR_c|\neq 0$ and $(Y_c,Z_c) = (\tilde{Y}-NR_c,\tilde{Z}+DR_c)$. It follows from

$$(Y-R\tilde{N})^{-1}(Z+R\tilde{D}) = Z_cY_c^{-1}$$

that $R_c = R$. Hence $|\tilde{Y}-NR|\neq 0$. The converse can be proved similarly. ∎

Now the next theorem follows readily from the above arguments.

THEOREM 1.3.4 Let $P \in B^{n\times m}$. Then $C \in S(P)$ iff an $R \in F^{n\times m}$ exists such that

$$|Y-R\tilde{N}|\neq 0, \qquad C = (Y-R\tilde{N})^{-1}(Z+R\tilde{D}) = (\tilde{Z}+DR)(\tilde{Y}-NR)^{-1}.$$

If P has neither an r.b.f. nor an l.b.f., a stabilizing controller of P may exist (see an example given by Anantharam [An.]) or may not (for example: $P(s) = se^{-s}$). If A is H_∞, then $P \in M(F)$ has an r.b.f. iff it has an l.b.f. iff it has a stabilizing controller. A detailed discussion will be given in Chapter 4 for the case $A = H_\infty$.

1.4 Robustness of feedback stabilization

In this section, we will formulate the central problem studied in this thesis. Suppose that we have a sequence of systems $\{P_\lambda\}$ and a sequence of controllers $\{C_\lambda\}$ parameterized by λ taking values in a metric space Λ. Also suppose that $H(P_0,C_0)$ is stable. *The question is : when will* $H(P_\lambda,C_\lambda)$ *be stable as* λ *is sufficiently close to* 0, *and* $H(P_\lambda,C_\lambda) \longrightarrow H(P_0,C_0)$ *as* $\lambda \longrightarrow 0$.

The space, Λ, of the parameters λ could occur as a result of perturbations, disturbances, approximations, measurement errors, modelling errors and parameter uncertainties, or could correspond to the physical characteristics that are intrinsic to the problem at hand.

Roughly speaking, P_0 is the *nominal system* or a mathematical model, which approximately describes the unknown real physical system; while C_0 is the *ideal controller* which is designed according to the nominal system. In theory, the ideal controller C_0 stabilizes the nominal system, P_0, i.e., $H(P_0,C_0)$ is stable, besides, $H(P_0,C_0)$ is the expected response. In practice, it is hoped that both real physical system P_λ, which is close to P_0, and real controller C_λ, which is close to C_0, will form a stable pair too, i.e., $H(P_\lambda,C_\lambda)$ is stable, and, in addition, $H(P_\lambda,C_\lambda)$ is close to $H(P_0,C_0)$. This problem is referred to *robustness of feedback stabilization*, or simply, *robust stabilization*.

For the study of robustness of feedback stabilization, we need a topology or a metric in order to describe the distances from P_0 to P_λ and from C_0 to C_λ. This topology should be compatible with the robustness of feedback stabilization in the sense that the perturbation (P_λ,C_λ) from (P_0,C_0) is a stable pair and $H(P_\lambda,C_\lambda)$ is close to $H(P_0,C_0)$ when P_λ is close to P_0 and C_λ is close to C_0 in the topology. According to these requirements, the following topology can be defined and it will be compatible with the problem of robust stabilization.

Let $\mathcal{C}^{n,m}$ be the subset of $\mathcal{F}^{n\times m}$ consisting of all the elements which possess stabilizing controllers. We define a basic neighborhood $\mathcal{N}$ of $P_0 \in \mathcal{C}^{n,m}$ as

$$\mathcal{N} := \mathcal{N}(P_0,C_0,\varepsilon) := \{\ P : H(P,C_0) \text{ is stable and } \|H(P,C_0) - H(P_0,C_0)\| < \varepsilon\ \},$$

where C_0 is a stabilizing controller of P_0. By varying ε over $\mathbb{R}_+$, varying C_0 over the set $S(P_0)$ of all stabilizing controllers of P_0 and varying P_0 over $\mathcal{C}^{n,m}$, we will obtain a collection of basic neighborhoods, which forms a basis for a topology (denoted by T) over $\mathcal{C}^{n\times m}$.

Unfortunately, although this topology perfectly describes the robustness of feedback stabilization, this definition as given has little structures and doesn't offer a good perspective for analysis. In the next chapter, we will introduce the gap topology and show that it is equal to topology T on $\mathcal{C}^{n,m}$.

The Gap Topology and the Graph Topology

2.1 The gap between two closed subspaces

Let Y be a Banach space and ϕ,ψ be two linear closed subspaces of Y. The gap is a measure of the "distance" between two linear closed subspaces. It is given in terms of two directed gaps, and the *directed gap* from ϕ to ψ is defined as

$$(2.1.1) \qquad \overrightarrow{\delta}(\phi,\psi) := \sup_{x \in S_\phi} \inf_{y \in \psi} \| x - y \|,$$

where

$$S_\phi := \{ x \in \phi : \|x\| = 1 \}.$$

If $\phi = 0$, then define $\overrightarrow{\delta}(\phi,\psi) := 0$. The *gap* between ϕ and ψ is defined as

$$(2.1.2) \qquad \delta(\phi,\psi) := \max \{ \overrightarrow{\delta}(\phi,\psi) , \overrightarrow{\delta}(\psi,\phi) \}.$$

The following relations are direct consequences from the definition.

$$\overrightarrow{\delta}(\phi,\psi) = 0 \text{ iff } \phi \subseteq \psi ; \qquad \delta(\phi,\psi) = 0 \text{ iff } \phi = \psi;$$

$$\delta(\phi,\psi) = \delta(\psi,\phi); \qquad 0 \le \delta(\phi,\psi) \le 1.$$

In general, $\delta(.,.)$ is not a metric for the space of all linear closed subspaces of Y, because it may not satisfy the triangle inequality. But the function $\gamma(.,.)$ defined by

$$(2.1.3) \qquad \overrightarrow{\gamma}(\phi,\psi) := \sup_{x \in S_\phi} \inf_{y \in S_\psi} \|x - y\| \ ; \ \gamma(\phi,\psi) := \max\{ \overrightarrow{\gamma}(\phi,\psi), \overrightarrow{\gamma}(\psi,\phi)\}$$

$$\overrightarrow{\gamma}(0,\psi) = 0 \qquad \overrightarrow{\gamma}(\phi,0) = 2 \quad (\text{if } \phi \neq 0)$$

is a metric and $\delta(\phi,\psi) \le \gamma(\phi,\psi) \le 2\delta(\phi,\psi)$. Although the gap function $\delta(.,.)$ is not a metric, it is more convenient than the proper metric function $\gamma(.,.)$ for applications, since its definition is slightly simpler.

We will end this section by giving an intuitive illustration of the gap function. First, we consider the case of ϕ and ψ being two lines on the plane shown in Figure 2.1.1. In this case we have $\overrightarrow{\delta}(\phi,\psi) = \overrightarrow{\delta}(\psi,\phi) = \sin(\theta)$. Next, let ϕ be a line and ψ be a plane, and their relationship is shown in Figure 2.1.2. Then, we have $\overrightarrow{\delta}(\phi,\psi) = \sin(\gamma)$ and $\overrightarrow{\delta}(\psi,\phi) = 1$.

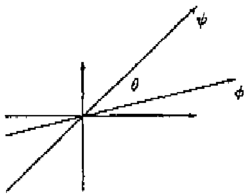

Figure 2.1.1

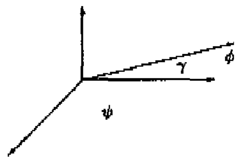

Figure 2.1.2

2.2 The gap topology for $F^{n\times m}$

According to Definition 1.1.2, each system $P \in F^{n\times m}$ is a linear operator mapping a subspace of X^m into X^n and by Theorem 1.2.1, this operator is closed i.e. the graph of P,

$$G(P) := \{ (x , Px) : x \in Dom(P) \}$$

is a closed subspace of $X^m \times X^n$. The *directed gap* and the *gap* between two systems in $F^{n\times m}$ are defined as the directed gap and the gap between their graphs, respectively, that is, for P_1,P_2 in $F^{n\times m}$

$$\overrightarrow{\delta}(P_1,P_2) := \overrightarrow{\delta}(G(P_1) , G(P_2)) \; ; \qquad \delta(P_1,P_2) := \delta(G(P_1) , G(P_2))$$

It is easy to see that $\delta(P_1,P_2) = 0$ iff $P_1 = P_2$. We will define a basic neighborhood of $P_0 \in F^{n\times m}$ as

$$N(P_0,\epsilon) := \{ P \in F^{n\times m} : \delta(P_0,P) < \epsilon \}.$$

Now, by varying ϵ over $(0,1]$ and varying P_0 over $F^{n\times m}$, we can obtain a collection of basic neighborhoods. This collection forms a base for a topology on $F^{n\times m}$ which is called the *gap topology*.

The following properties, Theorem 2.2.1–2.2.4 are quoted from [Ka. p197–206], and they will be used later,

THEOREM 2.2.1 If $P_0 \in A^{nxm}$ and $P \in F^{nxm}$ satisfy

$$\delta(P,P_0) < (1+\|P_0\|^2)^{-1/2},$$

then, P is in A^{nxm}

A consequence of this theorem is that A^{nxm} is an open subset of F^{nxm} in the gap topology. Thus, any system is stable, if it is sufficiently close to a given stable system.

THEOREM 2.2.2 On A^{nxm}, the gap topology is equal to the topology induced by the operator norm.

THEOREM 2.2.3 Let $P_i \in F^{nxm}$ (i=1,2) and $P_0 \in A^{nxm}$. Then,

$$\text{(2.2.1)} \qquad \delta(P_1+P_0,P_2+P_0) \leq 2(1+\|P_0\|^2)\ \delta(P_1,P_2).$$

Another way of writing (2.2.1) is

$$\text{(2.2.2)} \qquad \delta(P_1,P_2) \leq 2(1+\|P_0\|^2)\ \delta(P_1+P_0\ ,\ P_2+P_0).$$

THEOREM 2.2.4 If $P_i \in F^{mxm}$ (i=1,2) are invertible, then

$$\delta(P_1^{-1}\ ,\ P_2^{-1}) = \delta(P_1,P_2).$$

Note that according to Theorem 2.2.4 the gap between two SISO systems, whose transfer functions are polynomials, can be obtained by computing the gap between their inverses, whose transfer functions are strictly proper rational functions.

It is well-known that the norm topology in A^{nxm} is a *product topology*, i.e., a family $\{P_\lambda\}$ of matrices in A^{nxm} converges to P_0 iff each entry family $\{p_\lambda^{(i,j)}\}$ converges to $p_0^{(i,j)}$ for all i,j. In Section 5, we will show that the gap topology is equal to the graph topology on $B^{n,m}$. But, Vidyasagar [Vi. 2 p246] showed that the graph topology is not a product topology. Hence, *the gap topology is not a product topology on F^{nxm}*.

Below, we will prove that the gap topology is a *diagonal product topology*. This property plays an important role in dealing with feedback systems. Suppose that $P_i \in F^{nxm}$

$(i=1,2)$ have the following diagonal form

$$(2.2.3) \qquad P_i = \begin{bmatrix} P_i^1 & 0 \\ 0 & P_i^2 \end{bmatrix},$$

where $P_i^j \in F^{n_j \times m_j}$ $(j = 1,2;\ i = 1,2)$ and $n_1+n_2 = n;\ m_1+m_2 = m$.

THEOREM 2.2.5 Let $P_i \in F^{n\times m}$ have the diagonal form (2.2.3) (i=1,2). Then,

$$(2.2.4) \qquad \max\{\delta(P_1^1,P_2^1),\delta(P_1^2,P_2^2)\} \le \delta(P_1,P_2) \le \delta(P_1^1,P_2^1) + \delta(P_1^2,P_2^2).$$

PROOF By definition

$$\overrightarrow{\delta}(P_1,P_2) := \sup_{x\in SG(P_1)} \ \inf_{y\in G(P_2)} \| x - y \|,$$

where x and y are in $X^m \times X^n$ i.e.

$$x = (x^1,x^2) \in G(P_1^1)\times G(P_1^2), \qquad\qquad y = (y^1,y^2) \in G(P_2^1)\times G(P_2^2).$$

Therefore, we can write $\overrightarrow{\delta}(P_1,P_2)$ as:

$$(2.2.5) \qquad \overrightarrow{\delta}(P_1,P_2) := \sup_{x\in SG(P_1)} \ \inf_{y\in G(P_2)} [\|x^1 - y^1\|^2 + \|x^2 - y^2\|^2]^{1/2}$$

Now, we can prove the first inequality of (2.2.4). From (2.2.5), we get

$$\begin{aligned}(2.2.6) \qquad \overrightarrow{\delta}(P_1,P_2) &\ge \sup_{\substack{x^1\in SG(P_1^1)\\ x^2=0}} \ \inf_{y\in G(P_2)} [\|x^1 - y^1\|^2 + \|0 - y^2\|^2]^{1/2} \\ &= \sup_{\substack{x^1\in SG(P_1^1)\\ x^2=0}} \ \inf_{\substack{y^1\in G(P_2^1)\\ y^2=0}} [\|x^1 - y^1\|^2 + \|0 - 0\|^2]^{1/2} \\ &= \sup_{x^1\in SG(P_1^1)} \ \inf_{y^1\in G(P_2^1)} \|x^1 - y^1\| \\ &= \overrightarrow{\delta}(P_1^1,P_2^1),\end{aligned}$$

i.e., $\qquad \overrightarrow{\delta}(P_1,P_2) \ge \overrightarrow{\delta}(P_1^1,P_2^1)$.

From (2.2.5) to (2.2.6), if we take $x^1 = 0$ instead of $x^2 = 0$, we can get

$$\overrightarrow{\delta}(P_1,P_2) \geq \overrightarrow{\delta}(P_1^2,P_2^2).$$

Hence

$$\overrightarrow{\delta}(P_1,P_2) \geq \max\ \{\overrightarrow{\delta}(P_1^1,P_2^1)\ ,\ \overrightarrow{\delta}(P_1^2,P_2^2)\}.$$

By symmetry,

$$\overrightarrow{\delta}(P_2,P_1) \geq \max\ \{\overrightarrow{\delta}(P_2^1,P_1^1)\ ,\ \overrightarrow{\delta}(P_2^2,P_1^2)\}.$$

Consequently, we get the first inequality of (2.2.4). To prove the second inequality, we apply

$$(\alpha + \beta)^{1/2} \leq \alpha^{1/2} + \beta^{1/2} \qquad\qquad \forall\ \alpha\ ,\ \beta\ \geq 0$$

to (2.2.5) and obtain

$$\begin{aligned}\overrightarrow{\delta}(P_1,P_2) &\leq \sup_{x \in SG(P_1)}\ \inf_{y \in G(P_2)}\ [\|x^1 - y^1\| + \|x^2 - y^2\|] \\ &= \sup_{x \in SG(P_1)}\ [\inf_{y^1 \in G(P_2^1)} \|x^1 - y^1\| + \inf_{y^2 \in G(P_2^2)} \|x^2 - y^2\|] \\ &\leq \sup_{x \in SG(P_1)}\ \inf_{y^1 \in G(P_2^1)} \|x^1 - y^1\| + \sup_{x \in SG(P_1)}\ \inf_{y^2 \in G(P_2^2)} \|x^2 - y^2\| \\ &= \sup_{x^1 \in SG(P_1^1)}\ \inf_{y^1 \in G(P_2^1)} \|x^1 - y^1\| + \sup_{x^2 \in SG(P_1^2)}\ \inf_{y^2 \in G(P_2^2)} \|x^2 - y^2\| \\ &= \overrightarrow{\delta}(P_1^1,P_2^1) + \overrightarrow{\delta}(P_1^2,P_2^2),\end{aligned}$$

i.e., $\overrightarrow{\delta}(P_1,P_2) \leq \overrightarrow{\delta}(P_1^1,P_2^1) + \overrightarrow{\delta}(P_1^2,P_2^2).$

By symmetry

$$\overrightarrow{\delta}(P_2,P_1) \leq \overrightarrow{\delta}(P_2^1,P_1^1) + \overrightarrow{\delta}(P_2^2,P_1^2).$$

As a result, the second inequality of (2.2.4) is true. ∎

COROLLARY 2.2.6 Let $\{P_\lambda\}$ be a family of systems in $F^{n \times m}$ which has the following diagonal form $P_\lambda = \begin{bmatrix} P_\lambda^1 & 0 \\ 0 & P_\lambda^2 \end{bmatrix}$, where $P_\lambda^j \in F^{n_j \times m_j}$ ($j = 1,2$) and $n_1+n_2 = n$; $m_1+m_2 = m$. Then,

$\delta(P_\lambda,P_0) \longrightarrow 0$ as $\lambda \longrightarrow 0$ iff $\delta(P^1_\lambda,P^1_0) \longrightarrow 0$ and $\delta(P^2_\lambda,P^2_0) \longrightarrow 0$ simultaneously as $\lambda \longrightarrow 0$.

This property is called the diagonal product property and it will be used in the next section.

REMARK 2.2.7 In a completely analogous way, it can be proved that (2.2.4) will still hold if P_i is defined by $P_i = \begin{bmatrix} 0 & P^1_i \\ P^2_i & 0 \end{bmatrix}$ instead of (2.2.3).

2.3 A necessary and sufficient condition for robust stabilization

In this section we apply the gap topology to the problem of robust stabilization. It is shown that on $C^{n\times m}$ the gap topology is compatible with this problem and coincides with the topology T defined in Section 1.4.

THEOREM 2.3.1. Assume that $\{P_\lambda\} \subset F^{n\times m}$ and $\{C_\lambda\} \subset F^{m\times n}$. Then

(2.3.1) $$\frac{1}{4}\max\{\delta(P_\lambda,P_0)\ ,\ \delta(C_\lambda,C_0)\} \le \delta(H_\lambda,H_0) \le 4[\delta(P_\lambda,P_0) + \delta(C_\lambda,C_0)].$$

PROOF It is easy to check that $H_\lambda := H(P_\lambda,C_\lambda)$ can be written as

$$H_\lambda = (I + FG_\lambda)^{-1},$$

where

$$F := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \qquad G_\lambda := \begin{bmatrix} C_\lambda & 0 \\ 0 & P_\lambda \end{bmatrix}.$$

According to Theorem 2.2.4, we know

$$\delta(H_\lambda,H_0) = \delta((I + FG_\lambda)^{-1}\ ,\ (I + FG_0)^{-1})$$

$$= \delta((I + FG_\lambda)\ ,\ (I + FG_0)).$$

From (2.2.1) and (2.2.2), we have

$$\frac{1}{4}\ \delta(FG_\lambda\ ,\ FG_0) \le \delta(H_\lambda,H_0) \le 4\delta(FG_\lambda\ ,\ FG_0).$$

But by Remark 2.2.7,

$$\max\{\delta(P_\lambda,P_0)\ ,\ \delta(C_\lambda,C_0)\} \le \delta(FG_\lambda,FG_0) \le \delta(P_\lambda,P_0) + \delta(C_\lambda,C_0).$$

Hence

$$\frac{1}{4}\max\{\delta(P_\lambda,P_0)\ ,\ \delta(C_\lambda,C_0)\} \le \delta(H_\lambda,H_0) \le 4[\delta(P_\lambda,P_0) + \delta(C_\lambda,C_0)]. \qquad \blacksquare$$

The following corollary gives a necessary and sufficient condition for robustness of feedback stabilization

COROLLARY 2.3.2 Suppose $\lambda \longrightarrow P_\lambda$ and $\lambda \longrightarrow C_\lambda$ are functions mapping Λ into the set $F^{n\times m}$ of systems and the set $F^{m\times n}$ of controllers, respectively. Moreover, assume that the corresponding closed-loop transfer matrix $H(P_\lambda,C_\lambda)$ is stable at $\lambda = 0$, i.e., $H(P_0,C_0) \in M(A)$. Then, the following two statements are equivalent.

i) $H(P_\lambda,C_\lambda)$ is stable when λ is sufficiently close to 0 and satisfies:

(2.3.2) $\qquad \|H(P_\lambda,C_\lambda) - H(P_0,C_0)\| \longrightarrow 0 \quad (\lambda \longrightarrow 0);$

ii) $\delta(P_\lambda,P_0) \longrightarrow 0 \quad (\lambda \longrightarrow 0)$ and $\delta(C_\lambda,C_0) \longrightarrow 0 \quad (\lambda \longrightarrow 0)$.

PROOF "*i*) $\Rightarrow$ *ii*)" According to Theorem 2.2.2, the gap topology is identical to the topology induced by the operator norm. Since $H(P_\lambda,C_\lambda)$ is stable, (2.3.2) is equivalent to $\delta(H(P_\lambda,C_\lambda)\ ,\ H(P_0,C_0)) \longrightarrow 0$. Using Theorem 2.3.1, we know that *i*) implies *ii*).

"*ii*) $\Rightarrow$ *i*)" From Theorem 2.3.1, $\delta(H(P_\lambda,C_\lambda)\ ,\ H(P_0,C_0)) \longrightarrow 0$. According to Theorem 2.2.1, $H(P_\lambda,C_\lambda)$ is stable as λ is sufficiently close to 0. Again by Theorem 2.2.2, $\delta(H(P_\lambda,C_\lambda)\ ,\ H(P_0,C_0)) \longrightarrow 0$ implies (2.3.2). $\blacksquare$

Recall that $C^{n,m}$ is a subset of $F^{n\times m}$ consisting of all the systems which possess stabilizing controllers in $M(F)$. The following result is a simple outcome of the above corollary.

COROLLARY 2.3.3. In the gap topology, $C^{n,m}$ is an open subset of $F^{n\times m}$.

Finally, we will show that the restriction of the gap topology to $C^{n,m}$ is equal to the topology T defined in Section 1.4. For a system $P \in C^{n,m}$, a basic neighborhood of P in the topology T is defined as

$$\mathcal{N}(P_0,C_0,\varepsilon) := \{\ P : H(P,C_0) \text{ is stable and } \|H(P,C_0) - H(P_0,C_0)\| < \varepsilon\ \},$$

where C_0 is a stabilizing controller of P_0.

Suppose that $\{P_\lambda\} \subset \mathcal{C}^{n,m}$ converges to $P_0 \in \mathcal{C}^{n,m}$ in the topology T. Then, we see that $H(P_\lambda,C_0)$ is stable when λ is sufficiently close to 0 and $\|H(P_\lambda,C_0) - H(P_0,C_0)\| \longrightarrow 0$. According to Corollary 2.3.2, we know that $\{P_\lambda\}$ converges to P_0 in the gap topology. Conversely, suppose that $\{P_\lambda\} \in \mathcal{C}^{n,m}$ converges to $P_0 \in \mathcal{C}^{n,m}$ in the gap topology. By Theorem 2.3.1 and 2.2.1, we know that $H(P_\lambda,C_0)$ is stable when λ is sufficiently close to 0 and $H(P_\lambda,C_0)$ converges to $H(P_0,C_0)$, which means that $\{P_\lambda\}$ converges to P_0 in the topology T.

2.4 The graph topology for $\mathcal{B}^{n,m}$

The definition of the graph topology and its essential properties are presented in this section. The graph topology was proposed by Vidyasagar and thoroughly studied in his monograph [Vi. 2]. There are two distinguishing features in the present formulation :

i) The definition and theorems are carried out for a general setting;

ii) In [Vi. 2] spectral factorization of rational matrices is used to prove the *diagonal product property* of the graph topology. However, the spectral factorization problem has not yet been solved satisfactorily for a general matrix ring. So, we provide a proof, which is independent of spectral factorization.

The only proof given in this section is for the diagonal product property. The proofs of all the other results are simple translations of [Vi. 2], hence we will omit them here.

LEMMA 2.4.1 Suppose that $P_0 \in \mathcal{B}^{n,m}$ and (D_0,N_0) is an r.b.f. of P_0. Then, there exists a constant $\mu = \mu(D_0,N_0) > 0$ such that: if a pair $(D,N) \in M(\mathcal{A})$ satisfies

$$\|(D,N) - (D_0,N_0)\| < \mu,$$

then $|D| \neq 0$ and (D,N) is an r.b.f. of $P := ND^{-1}$.

Let ε be any positive number less than $\mu(D_0,N_0)$, then

$$(2.4.1) \qquad \mathcal{N}(D_0,N_0,\varepsilon) := \{\ P = ND^{-1} : \|(D,N) - (D_0,N_0)\| < \varepsilon\ \}$$

is a basic neighborhood of P_0.

Now by varying ε over $(0\ ,\ \mu(D_0,N_0))$, varying (D_0,N_0) over the set of the r.b.f.'s of P_0, and varying P_0 over $\mathcal{B}^{n,m}$ we can obtain a collection of basic neighborhoods.

LEMMA 2.4.2 The collection of the basic neighborhoods defines a topology on $B^{n,m}$.

We call this topology *graph topology*. In this topology two systems P_1 and P_2 are "close" if for each r.b.f. (D_1,N_1) of P_1 there exists an r.b.f. (D_2,N_2) of P_2 such that $\|(D_1,N_1)-(D_2,N_2)\|$ is small. A family $\{P_\lambda\}$ converges to P_0 in the graph topology, if for each r.b.f. (D_0,N_0) of P_0 there exist r.b.f.'s (D_λ,N_λ) of P_λ such that $\|(D_\lambda,N_\lambda)-(D_0,N_0)\| \longrightarrow 0$ $(\lambda \longrightarrow 0)$.

THEOREM 2.4.3 $A^{n\times m}$ is an open subset of $B^{n,m}$ in the graph topology and on $A^{n\times m}$ the graph topology is equal to the topology induced by the operator norm.

THEOREM 2.4.4 Assume that $P_\lambda \in B^{n,m}$ has a diagonal form $P_\lambda = \begin{bmatrix} P_\lambda^1 & 0 \\ 0 & P_\lambda^2 \end{bmatrix}$, where $P_\lambda^j \in F^{n_j\times m_j}$ $(j = 1,2)$ and $n_1+n_2 = n$; $m_1+m_2 = m$. Then, $P_\lambda \longrightarrow P_0$ (as $\lambda \longrightarrow 0$) iff $P_\lambda^1 \longrightarrow P_0^1$ and $P_\lambda^2 \longrightarrow P_0^2$ (as $\lambda \longrightarrow 0$) simultaneously.

PROOF "$\Leftarrow$" Assume that (D_0^i,N_0^i) is an r.b.f. of P_0^i (i=1,2). Since $\{P_\lambda^i\}$ converges to P_0^i, there are r.b.f.'s $(D_\lambda^i,N_\lambda^i)$ of P_λ^i such that

$$(D_\lambda^i,N_\lambda^i) \longrightarrow (D_0^i,N_0^i) \quad (\text{as } \lambda \longrightarrow 0) \quad (i= 1,2).$$

Let

$$N_\lambda = \begin{bmatrix} N_\lambda^1 & 0 \\ 0 & N_\lambda^2 \end{bmatrix}, \qquad D_\lambda = \begin{bmatrix} D_\lambda^1 & 0 \\ 0 & D_\lambda^2 \end{bmatrix}. \tag{2.4.2}$$

Then (D_λ,N_λ) is clearly an r.b.f. of P_λ and (D_0,N_0) defined by (2.4.2) with $\lambda = 0$ is an r.b.f. of P_0. Since the topology induced by the norm is a product topology, we have

$$(D_\lambda,N_\lambda) \longrightarrow (D_0,N_0) \quad (\text{as } \lambda \longrightarrow 0).$$

Therefore, $\{P_\lambda\}$ converges to P_0 in the graph topology.

"$\Rightarrow$" Suppose $(D_\lambda^i,N_\lambda^i)$ is an r.b.f. of P_λ^i; then (D_λ,N_λ) defined by (2.4.2) is an r.b.f. of P_λ. Since $\{P_\lambda\}$ converges to P_0, there exists a family $\{U_\lambda\}$ of unimodular matrices such that

$$\begin{bmatrix} D_\lambda^1 & 0 \\ 0 & D_\lambda^2 \\ N_\lambda^1 & 0 \\ 0 & N_\lambda^2 \end{bmatrix} \begin{bmatrix} U_{1\lambda} & U_{3\lambda} \\ U_{4\lambda} & U_{2\lambda} \end{bmatrix} \longrightarrow \begin{bmatrix} D_0^1 & 0 \\ 0 & D_0^2 \\ N_0^1 & 0 \\ 0 & N_0^2 \end{bmatrix} \quad (\lambda \longrightarrow 0),$$

where U_λ is partitioned in a obvious way. Hence

$$(D^i_\lambda,N^i_\lambda)U_{i\lambda} \longrightarrow (D^i_0,N^i_0) \quad (\lambda \longrightarrow 0) \quad (i=1,2).$$

Since (D^i_0,N^i_0) is an r.b.f of P^i_0, there exist Y^i and Z^i in $M(A)$ such that

$$[Y^i,Z^i]\begin{bmatrix} D^i_0 \\ N^i_0 \end{bmatrix} = I \quad (i=1,2).$$

Thus,

$$[Y^i,Z^i]\begin{bmatrix} D^i_\lambda \\ N^i_\lambda \end{bmatrix} U_{i\lambda} \longrightarrow I \quad (i=1,2).$$

Hence, when λ is sufficiently close to 0, $[Y^i,Z^i]\begin{bmatrix} D^i_\lambda \\ N^i_\lambda \end{bmatrix}U_{i\lambda}$ is unimodular. Consequently, $U_{i\lambda}$ is unimodular. As a result, P^i_λ converges to P^i_0 in the graph topology (i=1,2). ∎

THEOREM 2.4.5 Suppose $\lambda \longrightarrow P_\lambda$ and $\lambda \longrightarrow C_\lambda$ are functions mapping Λ into the set $B^{n,m}$ of systems and the set $B^{m,n}$ of controllers, respectively. Moreover, assume that the corresponding closed-loop transfer matrix $H(P_\lambda,C_\lambda)$ is stable at $\lambda = 0$, i.e., $H(P_0,C_0) \in M(A)$. Then, the following two statements are equivalent.

i) $H(P_\lambda,C_\lambda)$ is stable when λ is sufficiently close to 0 and satisfies:

$$\text{(2.4.3)} \qquad \|H(P_\lambda,C_\lambda) - H(P_0,C_0)\| \longrightarrow 0 \quad (\lambda \longrightarrow 0);$$

ii) P_λ converges to P_0 and C_λ to C_0 in the graph topology simultaneously.

2.5 Comparing the gap topology with the graph topology

It is obvious that the gap topology is defined for a larger set of systems than the graph topology. In this section, we aim to prove that the gap topology is equal to the graph topology, if it is restricted to $B^{n,m}$.

THEOREM 2.5.1 Let $\{P_\lambda\} \subset B^{n,m}$. Then, $\{P_\lambda\}$ converges to $P_0 \in B^{n,m}$ in the gap topology iff it converges in the graph topology.

PROOF Since P_0 is in $B^{n,m}$, it can be stabilized, i.e., there is a controller $C \in B^{n,m}$ such that $H(P_0,C)$ is in $M(A)$. Suppose that $\{P_\lambda\}$ converges to $P_0 \in B^{n,m}$ in the gap topology, then, according to Theorems 2.3.1 and 2.2.2, $H(P_\lambda,C)$ is in $M(A)$ when λ is sufficiently close to 0 and

$$\|H(P_\lambda,C) - H(P_0,C)\| \longrightarrow 0 \quad (\lambda \longrightarrow 0).$$

Because of Theorem 2.4.5, this implies that $\{P_\lambda\}$ converges to P_0 in the graph topology. The converse implication follows by reversing the steps above. ∎

REMARK 2.5.2. It follows from the proof of the above theorem that if a topology K defined on a subset M of $M(F)$ possesses following two properties:

i) $M(A)\cap M$ is an open subset of M in the topology K, and restricted to $M(A)\cap M$, topology K is equivalent to the topology generated by the norm;

ii) $H(P_\lambda,C_\lambda)$ converges to $H(P_0,C_0)$ in the topology K iff P_λ converges to P_0 and C_λ converges to C_0 in the topology K, simultaneously,

then, the topology K is the restriction of the gap topology to M.

Sufficient Conditions for Robustness of Feedback Stabilization

3.1 The gap metric and the graph metric

First, we discuss the gap metric. As said before, in general, the function $\delta(.,.)$ is not a metric. But the function $\gamma(.,.)$ as defined by (2.1.3) is a metric and it induces the same topology as $\delta(.,.)$. This implies that the gap topology can be metrized. If the space X of inputs and outputs is a Hilbert space, then $\delta(.,.)$ is a metric. *In this chapter, if without specification, it is assumed that* X *is a Hilbert space.* For $P \in F^{n\times m}$, according to Theorem 1.2.1, the graph G(P) of P is a closed subspace of $X^m \times X^n$. Let $\Pi(P)$ denote the orthogonal projection from $X^m \times X^n$ onto the graph G(P). If $P_i\,(i=1,2) \in F^{n\times m}$, then, it is easy to see that

$$\overrightarrow{\delta}(P_1,P_2) := \sup_{x\in SG(P_1)} \inf_{y\in G(P_2)} \| x - y \| = \sup_{x\in SG(P_1)} \|(I-\Pi(P_2))x\|$$

$$= \sup_{x\in X,\, \|x\|=1} \|(I-\Pi(P_2))\, \Pi(P_1)x\|$$

$$= \|(I-\Pi(P_2))\, \Pi(P_1)\| \qquad (3.1.1)$$

From this formula, it is shown [K–V–Z. *p*205] that

$$\delta(P_1,P_2) = \|\Pi(P_1) - \Pi(P_2)\| \qquad (3.1.2)$$

For densely defined closed operators P, [C–L] gave a representation of $\Pi(P)$. But, in general, operators induced by systems are not densely defined. For instance, in the case described by Example 1.1.7 we take $P(s) = \frac{1}{s-1}$ and it is easy to check that $[\mathrm{Dom}(P)]^{\perp}$ is the subspace spanned by $\frac{1}{s+1}$. Fortunately, we can find a representation of $\Pi(P)$ for $P \in B^{n,m}$. In order to do this, we need to prove

LEMMA 3.1.1 Suppose that (D,N) is an r.b.f. of $P \in B^{n,m}$, then $S := D^*D+N^*N$ is bijective, where D^* stands for the *adjoint operator* of D.

PROOF First, note that S is a bounded operator mapping X^m into X^m. $Sx=0$ implies that $[D^T,N^T]^T x=0$. Because $[D^T,N^T]^T$ has a left inverse, it is injective. So we obtain the injectivity of S.

To prove that S is surjective, we recall the following equation

$$\text{(3.1.3)} \qquad \mathrm{Image}\begin{bmatrix} D \\ N \end{bmatrix} \oplus \mathrm{Ker}[D^*,N^*] = X^m \times X^n.$$

Since (D,N) is an r.b.f. of P, there are Y and Z in $M(A)$ such that $[Y,Z]\begin{bmatrix} D \\ N \end{bmatrix} = I$. Hence $[D^*,N^*]\begin{bmatrix} Y^* \\ Z^* \end{bmatrix} = I$. For each $y \in X^m$, $\begin{bmatrix} Y^* \\ Z^* \end{bmatrix} y \in X^m \times X^n$. By (3.1.3), there are $x \in X^m$ and $z \in \mathrm{Ker}[D^*,N^*]$ such that $\begin{bmatrix} Y^* \\ Z^* \end{bmatrix} y = \begin{bmatrix} D \\ N \end{bmatrix} x + z$. Thus $y = [D^*,N^*]\begin{bmatrix} D \\ N \end{bmatrix} x$. Hence, S is surjective. This completes the proof. ■

Analogously, it can be proved that $S := \tilde{D}\tilde{D}^* + \tilde{N}\tilde{N}^*$ is bijective, if $(\tilde{D},\tilde{N})$ is an l.b.f. of P.

LEMMA 3.1.2 If (D,N) and $(\tilde{D},\tilde{N})$ are an r.b.f. and an l.b.f. of $P \in B^{n,m}$, respectively, then

$$\text{(3.1.4)} \qquad \Pi(P) = \begin{bmatrix} D \\ N \end{bmatrix} [D^*D + N^*N]^{-1} [D^*,N^*]$$

$$\text{(3.1.5)} \qquad = I - \begin{bmatrix} -\tilde{D}^* \\ \tilde{N}^* \end{bmatrix} [\tilde{D}\tilde{D}^* + \tilde{N}\tilde{N}^*]^{-1} [-\tilde{D},\tilde{N}]$$

To prove Lemma 3.1.2, it suffices to check that: *i*) the right hand side of (3.1.4) (resp. (3.1.5)) is self-adjoint and idempotent; *ii*) its image is G(P).

In order to define the graph metric for $B^{n,m}$, we need to generalize the definitions of right and left Bezout fractions.

DEFINITION 3.1.3 Suppose that $D \in B(X^m)$ and $N \in B(X^m,X^n)$. (D,N) is said to be a *generalized right Bezout fraction* (g.r.b.f.) of $P \in B^{n,m}$ if

i) D is invertible;

ii) $Y \in B(X^m)$ and $Z \in B(X^n,X^m)$ exist such that

$YD + ZN = I$;

iii) $P = ND^{-1}$.

Note that the condition *iii*) holds in the operator sense, i.e., ND^{-1} is the operator induced by the system P.

It is easy to see that the g.r.b.f. is unique up to right multiplications by the units of $B(X^m)$ [Vi. 2 p75]. An r.b.f. is certainly a g.r.b.f., but not conversely. *Generalized left Bezout fractions* (g.l.b.f.) are defined similarly.

In the above definition the generalized Bezout factors D and N of the system $P \in B^{n,m}$ as well as the solutions Y and Z of the Bezout identity are just bounded operators and are not necessarily in $M(A)$. But, this concept is necessary for defining the graph metric in a general framework. Moreover, the generalized Bezout fraction is a useful tool for obtaining some guaranteed bounds for robustness of feedback stabilization. We emphasize that the concept of generalized Bezout fraction is only a tool or a bridge and our final results are not expressed in terms of generalized Bezout fractions.

REMARK 3.1.4 If g.r.b.f.'s are used instead of r.b.f.'s, Lemmas 3.1.1 and 3.1.2 are still valid.

Suppose that T is a linear operator mapping a Hilbert space Y into another Hilbert space Z. T is said to be *isometric* on Y, if $\|Tx\| = \|x\|$ for all $x \in Y$, or equivalently, $T^*T = I$. And T is said to be *unitary* from Y to Z, if it is isometric and surjective. It can be easily checked that a necessary and sufficient condition for T to be unitary is $T^{-1} = T^*$.

An (resp. a generalized) r.b.f. (D,N) of $P \in B^{n,m}$ is said to be *normalized* if $[D^T,N^T]^T$ is isometric on X^m, i.e., $\|[D^T,N^T]^T x\| = \|x\|$ for all $x \in X^m$, or equivalently,

$$(3.1.6) \qquad D^*D + N^*N = I.$$

The reason why we call it normalized instead of normalized is that we normalize the (generalized) r.b.f.'s using their adjoint operators and it is different from what is called normalized conventionally, which is only defined for H_∞-matrices. In Chapter 4 we will give the definition of normalized r.b.f.'s and compare it with normalized r.b.f.'s. It can be easily checked that normalized (resp. generalized) r.b.f.'s are unique up to right multiplications by the elements in $U^{m,m}$ (resp. in the set of units of $B(X^m)$), which are unitary on X^m.

LEMMA 3.1.5 $P \in B^{n,m}$ always has a normalized g.r.b.f. and a normalized g.l.b.f..

PROOF Suppose that (D,N) is an r.b.f., it is known from Lemma 3.1.2 that $S := D^*D + N^*N$ is bijective. Hence, S and S^{-1} are positive operators and there is a square root $S^{1/2}$, which is also positive. It is trivial to check that $(DS^{-1/2},NS^{-1/2})$ is a normalized g.r.b.f.. Similarly, a normalized g.l.b.f. can be obtained. ∎

Now we are in the position to define the graph metric. Let (D_i,N_i) be a normalized g.r.b.f. of $P_i \in B^{n,m}$ (i=1,2), and define

$$\overrightarrow{d}^*(P_1,P_2) = \inf_{U\in B(X^m),\|U\|\le 1} \left\| \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} - \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} U \right\|,$$

$$d(P_1,P_2) = \max\{\overrightarrow{d}^*(P_1,P_2)\ ,\ \overrightarrow{d}^*(P_2,P_1)\}.$$

Then, by an analogous procedure as in [Vi.2 *p*262–265], it can be proved that $d(.,.)$ is a metric (we call it the graph metric), which induces the graph topology.

3.2 The gap metric and generalized Bezout fractions

The main purpose of this section is to find the relationship between the gap metric and generalized Bezout fractions. This is of interest on its own, and in addition is one of the key techniques required in the sequel.

LEMMA 3.2.1 Assume that $P \in B^{n,m}$, $D \in B(X^m)$ and $N \in B(X^m,X^n)$. Then, (D,N) is a g.r.b.f. of P iff $[D^T,N^T]^T$ maps X^m bijectively onto the graph G(P) of P.

PROOF "⇒" We can easily check that $[D^T,N^T]^T$ maps X^m injectively into G(P). We show that $[D^T,N^T]^T$ is also surjective. For each $w = [x^T,(Px)^T]^T \in G(P)$, define $z := D^{-1}x$. Then we have $[D^T,N^T]^T\ z = w$. Hence, $[D^T,N^T]^T$ must be surjective.

"⇐" Suppose that $(\bar{D},\bar{N})$ is an r.b.f. of P. According to the necessity part, $[\bar{D}^T,\bar{N}^T]^T$ maps X^m bijectively onto G(P). By assumption, $[D^T,N^T]^T$ also maps X^m bijectively onto G(P). Hence, for each $x \in X^m$, there is a unique $y \in X^m$ such that

$$(3.2.1)\qquad \begin{bmatrix} D \\ N \end{bmatrix} x = \begin{bmatrix} \bar{D} \\ \bar{N} \end{bmatrix} y$$

and vice versa. Since $(\bar{D},\bar{N})$ is an r.b.f. of P, there exist $Y,Z \in M(A)$ such that

$$Y\bar{D} + Z\bar{N} = I$$

Therefore,

$$(YD + ZN)x = y.$$

Hence $U := YD + ZN$ maps X^m to X^m bijectively. Since $[D^T,N^T]^T = [\bar{D}^T,\bar{N}^T]^T U$, (D,N) must be a generalized r.b.f. P. ∎

The next lemma is an alternative version of a result in [K–V–Z p206].

LEMMA 3.2.2 Let $P_i \in F^{n,m}$ (i=1,2). Then

i) $\delta(P_1,P_2) < 1$ iff $\Pi(P_1)$ maps $G(P_2)$ bijectively onto $G(P_1)$;

ii) If $\delta(P_1,P_2) < 1$, then $\overrightarrow{\delta}(P_1,P_2) = \overrightarrow{\delta}(P_2,P_1) = \delta(P_1,P_2)$.

Part *i*) of the theorem is proved in [K–V–Z p206] and the proof given there also establishes part *ii*). Using Lemmas 3.2.1 and 3.2.2, we can prove

THEOREM 3.2.3 Let $P_1 \in B^{n,m}$, (D_1,N_1) be a g.r.b.f. of P_1 and $P_2 \in F^{n\times m}$. Define

$$\begin{bmatrix} D_2 \\ N_2 \end{bmatrix} := \Pi(P_2) \begin{bmatrix} D_1 \\ N_1 \end{bmatrix}.$$

Then, (D_2,N_2) is a g.r.b.f. of P_2 iff $\delta(P_1,P_2) < 1$.

PROOF "$\Rightarrow$" That $[D_2^T,N_2^T]^T$ is a g.r.b.f. of P_2 implies that $\Pi(P_2)$ maps $G(P_1)$ bijectively onto $G(P_2)$. It follows from Lemma 3.2.2 that $\delta(P_1,P_2) < 1$.

"$\Leftarrow$" According to Lemma 3.2.2, $\delta(P_1,P_2) < 1$ implies that $\Pi(P_2)$ maps $G(P_1)$ bijectively onto $G(P_2)$. Hence, $[D_2^T,N_2^T]^T$ maps X^m bijectively onto $G(P_2)$. By Lemma 3.3.1, (D_2,N_2) is a g.r.b.f. of P_2. ∎

Now we will consider the relationship between the gap metric and generalized left bezout fractions in order to to get an analogous result to Theorem 3.2.3.

Suppose that (D,N) is a g.l.b.f. of $P \in B^{n,m}$, and define $T_P:=N^*(-D^*)^{-1}$. Then T_P is uniquely determined by P and independent of the g.l.b.f.'s of P. Moreover, (D,N) is a g.l.b.f. of P iff $(-D^*,N^*)$ is a g.r.b.f. of T_P.

LEMMA 3.2.4 Let $P \in B^{n,m}$, $D \in B(X^n)$ and $N \in B(X^m,X^n)$. Then, (D,N) is a g.l.b.f. of P iff

$$\mathrm{Ker}[N,-D] = G(P) \ ; \ \overline{\mathrm{Image}[N,-D]} = X^n$$

PROOF "$\Rightarrow$" For all $(x,y) \in G(P)$, $y=Px$ i.e. $y = D^{-1}Nx$. Hence, $Nx - Dy = 0$, that is $(x,y) \in \mathrm{Ker}[N,-D] := \{(x,y) \in X^m \times X^n : Nx - Dy = 0\}$. It is obvious that $(x,y) \in G(P)$ whenever $(x,y) \in \mathrm{Ker}[N,-D]$. Hence $\mathrm{Ker}[N,-D] = G(P)$. Since (D,N) is a g.l.b.f. of P, $(-D^*,N^*)$ is a g.r.b.f. of T_P. By Theorem 3.2.1, $[-D^{*T},N^{*T}]^T$ is injective, and hence, $\mathrm{Ker}[-D^{*T},N^{*T}]^T = 0$. But

$$\mathrm{Ker}[-D^{*T},N^{*T}]^T = 0 \quad \Leftrightarrow \quad \overline{\mathrm{Image}[N,-D]} = X^n.$$

"$\Leftarrow$" Take a g.l.b.f. $(\hat{D},\hat{N})$ of P. According to the necessity part, we can obtain

$$\mathrm{Ker}[\hat{N},-\hat{D}] = \mathrm{Ker}[N,-D] = G(P)$$

Hence,

$$(\mathrm{Ker}[\hat{N},-\hat{D}])^{\perp} = (\mathrm{Ker}[N,-D])^{\perp}.$$

So that

$$\mathrm{Image}[-\hat{D}^{*T},\hat{N}^{*T}]^T = \mathrm{Image}[-D^{*T},N^{*T}]^T = G(T_P).$$

By assumption, $[-D^{*T},N^{*T}]^T$ is injective. Consequently, $[-D^{*T},N^{*T}]^T$ maps X^n bijectively onto $G(T_P)$. It follows from Theorem 3.2.1 that $(-D^*,N^*)$ is a g.r.b.f. of T_P. Thus, we have shown that (D,N) is a g.l.b.f. of P. ■

THEOREM 3.2.5 Suppose (D_1,N_1) is a g.l.b.f. of $P_1 \in \mathcal{B}^{n,m}$ and $P_2 \in \mathcal{F}^{n\times m}$. Define

$$\begin{bmatrix} D_2 \\ N_2 \end{bmatrix} := [\Pi(P_2)]^{\perp} \begin{bmatrix} -D_1^* \\ N_1^* \end{bmatrix},$$

then, $(-D_2^*, N_2^*)$ is a g.l.b.f. of P_2 iff $\delta(P_1,P_2) < 1$.

PROOF The following facts can be checked easily

$$1)\ [\Pi(P_2)]^{\perp} = \Pi(T_{P_2});\ 2)\ \delta(P_1,P_2) = \delta(T_{P_1},T_{P_2}).$$

So, it is sufficient to prove that (D_2,N_2) is a g.r.b.f. of T_{P_2} iff $\delta(T_{P_1},T_{P_2}) < 1$, which follows from Theorem 3.2.3 ■

This section is concluded by presenting a corollary of Theorem 3.2.3, which will be used to discuss optimally robust controllers.

COROLLARY 3.2.6 Suppose that (D_i,N_i) is a g.r.b.f. of $P_i \in \mathcal{F}^{n\times m}$ (i=1,2). Then, $\delta(P_1,P_2) < 1$ iff $N_1^* N_2 + D_1^* D_2$ is bijective.

PROOF According to Theorem 3.2.3, $\delta(P_1,P_2) < 1$ iff $\Pi(P_2) \begin{bmatrix} D_1 \\ N_1 \end{bmatrix}$ is a g.r.b.f. of P_2. By (3.1.4), $\Pi(P_2) = \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} (D_2^*D_2 + N_2^*N_2)^{-1} (D_2^*,N_2^*)$. Hence, $\delta(P_1,P_2) < 1$ iff $(D_2^*D_2+N_2^*N_2)^{-1}$ $(D_2^*,N_2^*)\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}$ is bijective. Since $(D_2^*D_2 + N_2^*N_2)^{-1}$ is bijective, $\delta(P_1,P_2) < 1$ iff $(D_2^*,N_2^*)\begin{bmatrix} D_1 \\ N_1 \end{bmatrix}$ is bijective. This completes the proof. ■

3.3 Guaranteed bounds for robust stabilization

In this section we will present various bounds which can guarantee the stability of a perturbed feedback system, if the perturbations of the system and the controller are within these bounds.

Throughout this section, we suppose that $\lambda \longrightarrow P_\lambda$ and $\lambda \longrightarrow C_\lambda$ are functions mapping the metric space Λ into the set $F^{n\times m}$ of systems and the set $F^{m\times n}$ of controllers, respectively. Moreover, we assume that the corresponding closed-loop transfer matrix $H(P_\lambda,C_\lambda)$ is stable at $\lambda = 0$, i.e., $H(P_0,C_0) \in M(A)$.

THEOREM 3.3.1 Suppose that X is a Banach space. If

$$\delta(P_\lambda,P_0) + \delta(C_\lambda,C_0) < \frac{1}{4}\,(1 + \|H(P_0,C_0)\|^2)^{-1/2}, \tag{3.3.1}$$

then $H(P_\lambda,C_\lambda)$ is stable.

PROOF It follows from (2.3.1) that

$$\delta(H(P_\lambda,C_\lambda),H(P_0,C_0)) \le 4\,[\delta(P_\lambda,P_0) + \delta(C_\lambda,C_0)].$$

Hence we have

$$\delta(H(P_\lambda,C_\lambda),H(P_0,C_0)) < (1 + \|H(P_0,C_0)\|^2)^{-1/2},$$

According to Theorem 2.2.1, $H(P_\lambda,C_\lambda)$ is stable. ■

The next bound given in the graph metric is quoted from [Vi. 2 *p*290].

THEOREM 3.3.2 If

$$d(P_\lambda,P_0)\|T(P_0,C_0)\| + d(C_\lambda,C_0)\|T(C_0,P_0)\| < 1, \tag{3.3.2}$$

where

$$T(P,C) := H(P,C) - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},$$

then $H(P_\lambda,C_\lambda)$ is stable.

THEOREM 3.3.3 Let (D_0,N_0) be an r.b.f. of P_0 and $(\dot{D}_0,\dot{N}_0)$ be an l.b.f. of C_0. Denote

$$(3.3.3) \qquad A_0 := \begin{bmatrix} D_0 \\ N_0 \end{bmatrix} \; ; \; B_0 := [\dot{D}_0,\hat{N}_0] \; ; \; w := \|A_0\| \; \|B_0\| \; \|[B_0A_0]^{-1}\|.$$

If

$$(3.3.4) \qquad \delta(P_\lambda,P_0) + \delta(C_\lambda,C_0) < w^{-1},$$

then $H(P_\lambda,C_\lambda)$ is stable.

It is easy to check that, if (D,N) is a generalized r.b.f. of $P \in \mathcal{F}^{nxm}$ and (Y,Z) is a generalized l.b.f. of $C \in \mathcal{F}^{mxn}$, then $H(P,C)$ is stable iff $[Y,Z][D^T,N^T]^T$ is bijective.

PROOF First, it is easy to show that the right hand side of (3.3.4) is not larger than 1. According to Theorem 3.2.3 and 3.2.5, (D_λ,N_λ) and $(\tilde{D}_\lambda,\hat{N}_\lambda)$ defined by

$$A_\lambda := \begin{bmatrix} D_\lambda \\ N_\lambda \end{bmatrix} := \Pi(P_\lambda) \begin{bmatrix} D_0 \\ N_0 \end{bmatrix} \; ; \; B_\lambda := \begin{bmatrix} -\dot{D}_\lambda^* \\ \dot{N}_\lambda^* \end{bmatrix} := [\Pi(C_\lambda)]^\perp \begin{bmatrix} -\dot{D}_0^* \\ \dot{N}_0^* \end{bmatrix}$$

are a g.r.b.f. of P_λ and a g.l.b.f. of C_λ, respectively.

$$\|B_\lambda A_\lambda - B_0A_0\|$$

$$= \|(B_\lambda - B_0)A_\lambda + B_0(A_\lambda - A_0)\| \leq \|B_\lambda - B_0\|\|A_\lambda\| + \|B_0\|\|A_\lambda - A_0\|$$

$$= \|[\Pi(C_\lambda)]^\perp B_0 - [\Pi(C_0)]^\perp B_0\| \; \|A_\lambda\| + \|B_0\| \; \|\Pi(P_\lambda)A_0 - \Pi(P_0)A_0\|$$

$$\leq \|[\Pi(C_\lambda)]^\perp - [\Pi(C_0)]^\perp\| \; \|B_0\| \; \|A_\lambda\| + \|B_0\| \; \|\Pi(P_\lambda) - \Pi(P_0)\| \; \|A_0\|$$

$$\leq [\|[\Pi(C_\lambda)]^\perp - [\Pi(C_0)]^\perp\| + \|\Pi(P_\lambda) - \Pi(P_0)\|] \; \|A_0\|\|B_0\| \qquad (\text{since } A_\lambda = \Pi(P_\lambda)A_0)$$

$$=(\delta(P_\lambda,P_0) + \delta(C_\lambda,C_0)) \; \|A_0\|\|B_0\| < \|[B_0A_0]^{-1}\|^{-1}.$$

Therefore, $B_\lambda A_\lambda$ is bijective and $H(P_\lambda,C_\lambda)$ is stable. ∎

Now, we give a bound which is similar to (3.3.4), but it depends only upon the right Bezout fractions of P_0 and C_0.

THEOREM 3.3.4 Let (D_0,N_0) be an r.b.f. of P_0 and $(\hat{D}_0,\hat{N}_0)$ an r.b.f. of C_0. Denote

$$A_0 := \begin{bmatrix} D_0 \\ -N_0 \end{bmatrix} ; \; B_0 := \begin{bmatrix} \hat{N}_0 \\ \hat{D}_0 \end{bmatrix} ; \; w := \|[B_0,A_0]^{-1}\|.\max\{\|A_0\|,\|B_0\|\};.$$

If

$$(3.3.5) \qquad \delta^2(P_\lambda,P_0) + \delta^2(C_\lambda,C_0) < w^{-2},$$

then $H(P_\lambda,C_\lambda)$ is stable.

Note that, if (D,N) is a generalized r.b.f. of $P \in \mathcal{F}^{n\times m}$ and (Y,Z) is a generalized r.b.f. of $C \in \mathcal{F}^{m\times n}$, then $H(P,C)$ is stable iff $\begin{bmatrix} D & Z \\ -N & Y \end{bmatrix}$ is bijective.

PROOF First, it is easy to check that the right hand side of (3.3.5) is smaller than 1. According to Theorem 3.2.3, (D_λ,N_λ) and $(\hat{D}_\lambda,\hat{N}_\lambda)$ defined by

$$A_\lambda := \begin{bmatrix} D_\lambda \\ -N_\lambda \end{bmatrix} := \Pi(-P_\lambda) \begin{bmatrix} D_0 \\ -N_0 \end{bmatrix}; \; B_\lambda := \begin{bmatrix} \hat{D}_\lambda \\ \hat{N}_\lambda \end{bmatrix} := \Pi(C_\lambda) \begin{bmatrix} \hat{D}_0 \\ \hat{N}_0 \end{bmatrix}$$

are g.r.b.f.'s of P_λ and of C_λ, respectively.

$$\|[B_\lambda,A_\lambda] - [B_0,A_0]\|^2 = \|[B_\lambda - B_0 \, , \, A_\lambda - A_0]\|^2$$

$$\leq \|B_\lambda - B_0\|^2 + \|A_\lambda - A_0\|^2 = \|\Pi(C_\lambda)B_0 - \Pi(C_0)B_0\|^2 + \|\Pi(P_\lambda)A_0 - \Pi(P_0)A_0\|^2$$

$$\leq \delta^2(P_\lambda,P_0)\|A_0\|^2 + \delta^2(C_\lambda,C_0))\|B_0\|^2 < \|[B_0,A_0]^{-1}\|^{-2}.$$

Hence, $[B_\lambda,A_\lambda]$ is bijective. Consequently, $H(P_\lambda,C_\lambda)$ is stable. ∎

In the same way, we can also find another bound by using only the l.b.f.'s. Since the techniques are the same, we omit it.

Transfer Matrices with Entries in The Quotient Field of H_∞

4.1 Basic properties

In the last three chapters we studied robustness of feedback stabilization for an arbitrary normed integral domain A consisting of linear bounded operators. In this chapter we will discuss a special situation, in which A is H_∞ and F is the quotient field of H_∞. It is shown by Smith [Sm.] that H_∞ is a *pseudo-Bezout domain*, i.e., every two elements of H_∞ have a greatest common divisor. If the space X of inputs and outputs is chosen to be H_p $(1 \leq p \leq \infty)$, then A, F and H_p will satisfy Assumption 1.1.1 and the results from Chapters 1 and 2 as well as Theorem 3.3.1 can be applied to this framework. But, in this chapter a further study will be made for the case when X is H_2, a Hilbert space. First, we point out that the class of transfer matrices with entries in the quotient field of H_∞ includes many cases of interest in theory and in applications. For example, it covers :

i) Finite dimensional LTI systems, i.e. systems described by rational matrices (see Example 1.1.7);

ii) Semigroup systems i.e. systems governed by :

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad\qquad x(0) = x_0$$
$$y(t) = Cx(t) + Du(t),$$

where A is the infinitesimal generator of a strongly continuous semigroup T(t) on a Hilbert space H, B is an operator mapping $\mathbb{R}^m$ into H, C mapping H into $\mathbb{R}^n$ and D mapping $\mathbb{R}^m$ into $\mathbb{R}^n$, (A,B) is supposed to be stabilizable and/or (C,A) to be detectable (for details, see [Cu. 1] by Curtain).

iii) The Callier–Desoer class (see Example 1.1.9).

It is well known that H_∞ is not a Bezout domain i.e. not every matrix in $M(F)$ has a Bezout fraction over $M(H_\infty)$. For instance, $P(s) = se^{-s}$ has a coprime fraction $(\frac{se^{-s}}{s+1}, \frac{1}{s+1})$, but does not have a Bezout fraction.

Recall that $C^{n,m}$ is a subset of $F^{n\times m}$ consisting of all systems possessing stabilizing controllers and $B^{n,m}$ a subset of $F^{n\times m}$ consisting of all systems possessing right and left Bezout fractions. It follows from Section 1.3 that $B^{n,m} \subset C^{n,m}$. The following theorem was proved in two different ways by Inouye [In.] and Smith [Sm.], respectively.

THEOREM 4.1.1. Assume F is the quotient field of H_∞. Then, $B^{n,m} = C^{n,m}$.

Note that an example was given by Anantharam [An.] which showed that, in general, $B^{n,m}$ and $C^{n,m}$ are not equal. Using Theorem 4.1.1, we can prove that H_∞ is a *Hermite ring* i.e. a system $P \in F^{n\times m}$ has an r.b.f. iff it has an l.b.f. (for original mathematical definition of Hermite ring we refer to [Vi. 2 *p*345]).

THEOREM 4.1.2. H_∞ is a Hermite ring i.e. a system $P \in F^{n\times m}$ has an r.b.f. iff it has an l.b.f..

PROOF Let $(D,N) \in M(H_\infty)$ be an r.b.f. of $P \in F^{n\times m}$. According to Lemma 1.3.1, $P \in C^{n,m}$. By Theorem 4.1.1, $P \in B^{n,m}$. Hence, P has an l.b.f.. The inverse part can be proved in a similar way. ■

It is known that each system $P \in F^{n\times m}$ induces an operator (denoted by P also) mapping a subspace of H_2^m into H_2^n. For a system $P \in F^{n\times m}$ we can also define another operator P_l mapping a subspace of L_2^m into L_2^n : the domain $Dom(P_l)$ is defined as

$$Dom(P_l) := \{\ x(.) \in L_2^m : P(.)x(.) \in L_2^n\ \};$$

and P_l acting on $x(.) \in Dom(P_l)$ is defined as the product $P(.)x(.)$, i.e., $(P_l x)(.) = P(.)x(.)$. It can be readily checked that when $P \in H_\infty^{n\times m}$, the adjoint operator of P_l is $\overline{P(-\bar{s})}^T (=:P^{\sim}(s))$, and the adjoint operator P^* of P is equal to the restriction of $T_+^n P^{\sim}$ to H_2^n, i.e., $(P^* x)(s) = T_+^n[P^{\sim}(s)x(s)]$ $(\forall\ x(.) \in H_2^n)$ where T_+^n is the orthogonal projection from L_2^n to H_2^n.

Suppose that $P \in L_\infty^{n\times m}$, the *Toeplitz operator* $\mathcal{T}_P$ mapping H_2^m into H_2^n with symbol P is defined as

$$(\mathcal{T}_P x)(s) := T_+^n P(s)x(s) \qquad\qquad \forall\ x \in H_2^m.$$

It is known [Ha.] and [Z-S] that the norm of a Toeplitz operator is equal to the norm of its symbol, i.e., $\|\mathcal{T}_P\| = \|P\|$ (for $\forall\ P \in L_\infty^{n\times m}$). For any $P \in H_\infty^{n\times m}$, since the adjoint operator P^* of P is equal to the restriction of $T_+^n P^{\sim}$ to H_2^n, P^* is just the Toeplitz operator with symbol $P^{\sim}$.

A matrix $V(.) \in H_\infty^{n\times m}$ is said to be *inner* if $V^{\sim}(i\omega)V(i\omega) = I$ $(\forall\ \omega \in R)$, or equivalently, it is an isometric on L_2^m. A matrix $V(.) \in H_\infty^{nxm}$ is said to be *outer* if it is surjective, or equivalently, it has full row rank or it has a right-inverse in H_∞^{mxn}. It is readily shown that an matrix $V(.) \in H_\infty^{nxm}$ is inner and outer iff it is unitary (in this case, n = m).

Recall that (D,N) is said to be *normalized* if $[D^T,N^T]^T$ is isometric on H_2^m, i.e., $\|[D^T,N^T]^Tx\| = \|x\|$ for all $x \in H_2^m$, which is equivalent to

$$D^*(i\omega)D(i\omega) + N^*(i\omega)N(i\omega) = I \qquad \forall\ \omega \in \mathbb{R}. \tag{4.1.1}$$

Then, we have

$$\|T_+^m[D^*D + N^*N - I]\| = 0.$$

As said before, the norm of a Toeplitz operator is equal to the norm of its symbol. Hence

$$\|[D^*D + N^*N - I]\| = 0,$$

i.e.

$$D^*(i\omega)D(i\omega) + N^*(i\omega)N(i\omega) = I \qquad \forall\ \omega \in \mathbb{R}. \tag{4.1.2}$$

This is equivalent to say that (D,N) is isometric on L_2^m.

REMARK The author thanks Drs. Anton A. Stoorvogel for his suggestion of the proof from (4.1.1) to (4.1.2).

Assume that $[D^T,N^T]^T$ is a normalized r.b.f. of $P \in B^{n,m}$, it can be checked that if $[D^T,N^T]^TU$ is also a normalized r.b.f. of P for a $U \in U^{m,m}$, then $U^*(i\omega)U(i\omega) = I$ for all $\omega \in \mathbb{R}$, i.e., U is unitary on L_2^m. Recall that a necessary and sufficient condition for U to be unitary is that U^*, the adjoint operator of U, is equal to U^{-1}. Thus, U has to be a constant matrix. Hence, U is a unitary constant matrix i.e. $U \in \mathbb{C}^{mxm}$, $\bar{U}^T = U^{-1}$.

In the next section, we will prove the existence of normalized Bezout fractions, which is a cornerstone for later developments.

4.2 Existence of normalized Bezout fractions

In Section 3.1 it was shown that normalized g.r.b.f.'s and g.l.b.f.'s always exist for $P \in B^{n,m}$. But the existence of normalized Bezout fractions of $P \in B^{nxm}$ is not trivial. Callier and Winkin proved the existence of normalized r.b.f. for SISO systems in the Callier–Desoer class (see Example 1.1.9) in [C–W 1], and in [C–W 2] they proved the existence for semigroup systems with bounded input and output operators. The existence for semigroup systems with unbounded input and output operators was shown by Curtain [Cu. 2] and [Zh. 3], while the existence of normalized Bezout fractions of transfer matrices with entries in the quotient field of H_∞ was obtained in [Zh. 2]. In this section, we will quote the main results from [Zh. 2].

The proof of the existence of normalized r.b.f.'s given below depends on Lax's theorem [La.]. Before presenting this theorem, we have to introduce the concept of shift invariant subspaces. A subspace $\phi \subset H_2^k$ is said to be *shift invariant* if $e^{-\alpha s}\phi \subseteq \phi$ for all positive α.

LEMMA 4.2.1 For each $P \in F^{n\times m}$, the graph G(P) of P is a shift invariant subspace of H_2^{n+m}.

PROOF Because Dom(P) consists of all the elements $x(.)$ in H_2^m such that $P(\cdot)x(\cdot) \in H_2^n$ and $(Px)(s) = P(s)x(s)$, for any $w(.) \in G(P)$ there exists an $x(.) \in H_2^m$ such that $w(s) = \begin{bmatrix} x(s) \\ P(s)x(s) \end{bmatrix}$. It follows from $e^{-\alpha s}x(s) \in H_2^m$ and $e^{-\alpha s}P(s)x(s) = P(s)e^{-\alpha s}x(s) \in H_2^n$ that $e^{-\alpha s}w(s) = \begin{bmatrix} e^{-\alpha s}x(s) \\ P(s)e^{-\alpha s}x(s) \end{bmatrix} \in G(P)$ $(\forall\, \alpha > 0)$. This completes the proof. ■

Lax's theorem [La.] can be stated as follows.

THEOREM 4.2.2 Suppose that ϕ is a closed shift invariant subspace of H_2^k. Then, there is an integer $p > 0$ and an inner matrix $A \in H_\infty^{k\times p}$, such that A maps H_2^p bijectively onto ϕ.

Now we are able to prove the main theorem of this section.

THEOREM 4.2.3 If $P \in F^{n\times m}$ has an r.b.f., then it has a normalized r.b.f..

PROOF According to Theorem 1.2.1 and Lemma 4.2.1, G(P) is a closed shift invariant subspace of H_2^{n+m}. By Lax's theorem, there is an integer $p > 0$ and an inner matrix $A \in H_\infty^{(n+m)\times p}$, which maps H_2^p bijectively onto G(P). Assume that (D,N) is an r.b.f. of P. From Lemma 3.2.1, $[D^T,N^T]^T$ maps H_2^m bijectively onto G(P). Suppose that $(Y,Z) \in M(H_\infty)$ satisfies $YD + ZN = I$. Then, of course, $[Y,Z][D^T,N^T]^T$ maps H_2^m bijectively onto H_2^m. Consequently $[Y,Z]$ maps G(P) onto H_2^m bijectively. Hence, $[Y,Z]A$ maps H_2^p onto H_2^m bijectively. Since $[Y,Z]A$ is a H_∞–matrix, $[Y,Z]A$ is bijective iff $[Y,Z]A$ is unimodular. Thus, we have $p = m$. If we partition A as $[\bar{D}^T,\bar{N}^T]^T$ with $\bar{D} \in H_\infty^{m\times m}$ and $\bar{N} \in H_\infty^{n\times m}$, then, according to Lemma 3.2.1, $(\bar{D},\bar{N})$ is an r.b.f. of P. Since it is normalized, this completes the proof. ■

The existence of normalized l.b.f.'s and the corresponding results of discrete-time LTI systems can be found in [Zh. 2].

4.3 Optimally robust controllers (1)

In Theorem 3.3.3, a guaranteed bound w^{-1} ($w := \|A_0\|\|B_0\|\|B_0A_0\|$) was obtained for robustness of feedback stabilization. For the case of $\mathcal{A} = H_\infty$, we can maximize w^{-1} by choosing an appropriate stabilizing controller. Moreover, it can be shown that the maximum w_g^{-1} is the *sharpest* bound. In this section, we will assume that there are no perturbations on the controllers.

Suppose that (D_0,N_0) and $(\tilde{D}_0,\tilde{N}_0)$ are a normalized r.b.f. and l.b.f. of P_0, respectively. Let C_0 be a stabilizing controller of P_0 and $(\hat{Y}_0,\hat{Z}_0)$ and (Y_0,Z_0) be an r.b.f. and l.b.f. of C_0, respectively, such that

$$(4.3.1) \qquad \begin{bmatrix} -Z_0 & Y_0 \\ \tilde{D}_0 & \tilde{N}_0 \end{bmatrix} \begin{bmatrix} -N_0 & \hat{Y}_0 \\ D_0 & \hat{Z}_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

It follows from Lemma 1.3.4 that the set of all stabilizing controllers is

$$S(P) = \{\ (Y_0-R\tilde{N}_0)^{-1}(Z_0+R\tilde{D}_0) = (\hat{Z}_0+D_0R)(\hat{Y}_0-N_0R)^{-1} : R \in H_\infty^{m\times n},\ |Y_0-R\tilde{N}_0| \neq 0\ \}.$$

Recall from (1.3.4) that

$$(4.3.2) \qquad \begin{bmatrix} -Z_0-R\tilde{D}_0 & Y_0-R\tilde{N}_0 \\ \tilde{D}_0 & \tilde{N}_0 \end{bmatrix} \begin{bmatrix} -N_0 & \hat{Y}_0-N_0R \\ D_0 & \hat{Z}_0+D_0R \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Let $C := (Y_0-R\tilde{N}_0)^{-1}(Z_0+R\tilde{D}_0)$ be any controller in $S(P_0)$ and assume that P_λ is a perturbed version of P_0. According to Theorem 3.3.3, if

$$\delta(P_\lambda,P_0) < w^{-1},$$

where

$$w := \left\| \begin{bmatrix} D_0 \\ N_0 \end{bmatrix} \right\| \ \|[(Y_0-R\tilde{N}_0),(Z_0+R\tilde{D}_0)]\| \ \left\|[(Y_0-R\tilde{N}_0),(Z_0+R\tilde{D}_0)] \begin{bmatrix} D_0 \\ N_0 \end{bmatrix}\right\|$$

$$= \|[(Y_0-R\tilde{N}_0),(Z_0+R\tilde{D}_0)]\|,$$

then C also stabilizes P_λ. Now, we minimize w by choosing the controllers in $S(P_0)$, i.e., we solve

$$(4.3.3) \qquad \inf_{R\in H_\infty^{m\times n}} \|[(Y_0-R\tilde{N}_0),(Z_0+R\tilde{D}_0)]\| \quad (=: w_g).$$

In the next section, we will discuss how to achieve this infimum. Now let us just suppose that it can be achieved for some $R \in H^{m\times n}$. Define

(4.3.4) $\quad K(P_0, \varepsilon) := \{ P \in F^{n\times m} : \delta(P_0,P) < \varepsilon \} \qquad \varepsilon > 0.$

According to Theorem 3.3.3, if a solution R_g of (4.3.3) is found, then $C_g := (Y_0-R_g\tilde{N}_0)^{-1}(Z_0+R_g\tilde{D}_0)$ stabilizes $K(P_0,w_g^{-1})$. We will show that the bound w_g^{-1} is the sharpest in the sense that there are no controllers which can stabilize $K(P_0,\varepsilon)$ if $\varepsilon > w_g^{-1}$. In other words, the largest number ε such that $K(P_0,\varepsilon)$ can be stabilized by one single controller is w_g^{-1}.

Recall that (D_0,N_0) is a normalized r.b.f. of $P_0 \in B^{n,m}$. Define

(4.3.5) $\quad R(P_0,\varepsilon) := \{ P = (N_0+\Delta_n)(D_0+\Delta_d)^{-1} \in F^{n\times m} : \| \begin{bmatrix} \Delta_d \\ \Delta_n \end{bmatrix} \| < \varepsilon \} \quad \varepsilon > 0.$

Note that (Δ_d,Δ_n) does not have to be in $M(H_\infty)$ but only that $P \in F^{n\times m}$, because $(D_0+\Delta_d\ ,\ N_0+\Delta_n)$ may be a generalized right bezout fraction of P. In fact, it is more reasonable and more natural to assume that the perturbations (Δ_d,Δ_n) are in a wider class than just in $M(H_\infty)$ as long as it can be handled. Since the normalized r.b.f.'s are unique up to the multiplication by unitary matrices in $\mathbb{C}^{m\times m}$, $R(P_0,\varepsilon)$ is independent of the normalized r.b.f.'s of P_0. The following theorem was proved by Vidyasagar and Kimura in [V–K].

THEOREM 4.3.1 The controller $C = (Y_0-R\tilde{N}_0)^{-1}(Z_0+R\tilde{D}_0)$ stabilizes $R(P_0,\varepsilon)$ iff

$$\|[(Y_0-R_0\tilde{N}_0),(Z_0+R_0\tilde{D}_0)]\| \le \varepsilon^{-1}.$$

COROLLARY 4.3.2 The largest number ε such that $R(P_0,\varepsilon)$ can be stabilized by one single controller is w_g^{-1}, and if R_g is a solution of (4.3.3), then $C_g := (Y_0-R_g\tilde{N}_0)^{-1}(Z_0+R_g\tilde{D}_0)$ is a controller stabilizing $R(P_0,w_g^{-1})$.

Making use of Corollary 3.2.6, Theorem 3.2.3 and Lemma 3.2.2, we can prove

THEOREM 4.3.3 If $0 < \varepsilon \le 1$, then

$$K(P_0,\varepsilon) = R(P_0,\varepsilon).$$

PROOF "$\supseteq$" Taking any $P = (N_0+\Delta_n)(D_0+\Delta_d)^{-1} \in R(P_0,\varepsilon)$. Since $\| \begin{bmatrix} D_0 \\ N_0 \end{bmatrix} \| = 1$, we know that $\|[D_0^*,N_0^*]\| = 1$. Because

$$\|[D_0^*,N_0^*] \begin{bmatrix} \Delta_d \\ \Delta_n \end{bmatrix}\| \le \|[D_0^*,N_0^*]\| \ \| \begin{bmatrix} \Delta_d \\ \Delta_n \end{bmatrix}\| < \varepsilon < 1,$$

$$[D_0^*, N_0^*]\begin{bmatrix} D_0+\Delta_d \\ N_0+\Delta_n \end{bmatrix} = I + [D_0^*, N_0^*]\begin{bmatrix} \Delta_d \\ \Delta_n \end{bmatrix}$$

is a bijective mapping. According to Corollary 3.2.6, $\delta(P_0,P) < 1$. By Lemma 3.2.2, $\overrightarrow{\delta}^*(P_0,P) = \overrightarrow{\delta}^*(P,P_0) = \delta(P_0,P)$. Now we can check

$$\begin{aligned}
\overrightarrow{\delta}^*(P_0,P) &= \sup_{x \in SG(P_0)} \inf_{y \in G(P)} \|x - y\| \\
&= \sup_{x \in H_2^m, \|x\|=1} \inf_{y \in G(P)} \left\|\begin{bmatrix} D_0 \\ N_0 \end{bmatrix} x - y\right\|.
\end{aligned}$$

Because of $\begin{bmatrix} D_0+\Delta_d \\ N_0+\Delta_n \end{bmatrix} H_2^m \subseteq G(P)$, we have

$$\begin{aligned}
\overrightarrow{\delta}^*(P_0,P) &\le \sup_{x \in H_2^m, \|x\|=1} \inf_{y \in H_2^m} \left\|\begin{bmatrix} D_0 \\ N_0 \end{bmatrix} x - \begin{bmatrix} D_0+\Delta_d \\ N_0+\Delta_n \end{bmatrix} y\right\| \\
&\le \sup_{x \in H_2^m, \|x\|=1} \left\|\begin{bmatrix} D_0 \\ N_0 \end{bmatrix} x - \begin{bmatrix} D_0+\Delta_d \\ N_0+\Delta_n \end{bmatrix} x\right\| \\
&= \left\|\begin{bmatrix} \Delta_d \\ \Delta_n \end{bmatrix}\right\| < \varepsilon.
\end{aligned}$$

Hence, $P \in K(P_0,\varepsilon)$.

"$\subseteq$" Take $P \in K(P_0,\varepsilon)$. Since $\delta(P_0,P) < 1$, by Theorem 3.2.3, (D,N) given by

$$\begin{bmatrix} D \\ N \end{bmatrix} := \Pi(P)\begin{bmatrix} D_0 \\ N_0 \end{bmatrix}$$

is a generalized r.b.f. of P. Because

$$\left\|\begin{bmatrix} D \\ N \end{bmatrix} - \begin{bmatrix} D_0 \\ N_0 \end{bmatrix}\right\| = \left\|\Pi(P)\begin{bmatrix} D_0 \\ N_0 \end{bmatrix} - \Pi(P_0)\begin{bmatrix} D_0 \\ N_0 \end{bmatrix}\right\| \le \delta(P_0,P) < \varepsilon,$$

$P \in R(P_0,\varepsilon)$. This completes the proof. ■

As a consequence of Corollary 4.3.2 and Theorem 4.3.3, we can see that the largest number, ε, such that $K(P_0,\varepsilon)$ can be stabilized by one controller is w_g^{-1}, and if R_g is a solution of (4.3.3), then $C_g := (Y_0 - R_g\tilde{N}_0)^{-1}(Z_0 + R_g\tilde{D}_0)$ is a controller stabilizing $K(P_0, w_g^{-1})$. It follows that we call w_g^{-1} as the *largest robust stability radius* of P_0 and C_g as an *optimally robust controller* of P_0. We emphasize that the largest robust stability radius is a intrinsic value of each system, and this value can be used as index to describe robustness of feedback stability of a given system.

Note that a result similar to Theorem 4.3.3 was proved by Georgiou and Smith [G–S]. There are two different features between Georgiou and Smith's work and Theorem 4.3.3 :

i) The result was proved in [G–S] under the assumption

$$0 < \varepsilon < \lambda(P_0) := \inf_{s \in \mathbb{C}_+} \sigma_{min} \begin{bmatrix} D_0(s) \\ N_0(s) \end{bmatrix} \leq 1,$$

whereas, in Theorem 4.3.3, the ε can be any number in $(0,1]$;

ii) [G–S] showed that $R(P_0,\varepsilon)\bigcap B^{n,m} = K(P_0,\varepsilon)\bigcap B^{n,m}$, whereas, Theorem 4.3.3 provided a slightly more general result, i.e., $R(P_0,\varepsilon) = K(P_0,\varepsilon)$.

Moreover, the techniques used in Theorem 4.3.3 are different from that used in [G–S]. This section is ended by an example to show that, sometimes, $\lambda(P_0)$ can be very small. Let $P_0(s) = \alpha/[s-\beta]$ $(\alpha,\beta > 0)$. It is easy to prove that $(\alpha/[s+\gamma], [s-\beta]/[s+\gamma])$ is a normalized r.b.f. of P_0, where $\gamma^2 = \alpha^2 + \beta^2$. It is easy to see that

$$\lambda(P_0) = \inf_{s \in C_+} [|\alpha/[s+\gamma]|^2 + |[s-\beta]/[s+\gamma]|^2]^{1/2}$$

$$\leq \alpha/[\beta+\gamma] \longrightarrow 0 \qquad (\alpha \longrightarrow 0 \text{ and /or } \beta \longrightarrow \infty).$$

4.4 Optimally robust controllers (2)

In this section, first, the "dual" versions of Theorem 3.3.3 and Theorem 4.3.2 will be presented before we discuss their relations with the original versions. Then, we deduce that the infimum (4.3.3) is achievable for some $R \in H_\infty^{mxn}$ and present three formulas for computing w_g. Afterwards, the problems of additive and multiplicative perturbations, uncertainties in optimally robust controllers, the structure of the neighborhoods and the variation of the closed-loop systems as well as the dual problem of optimally robust controllers are discussed successively.

Suppose that (D_0,N_0) and $(\hat{D}_0,\hat{N}_0)$ are a normalized r.b.f. and l.b.f. of $P_0 \in B^{n,m}$, respectively. Let C_0 be a stabilizing controller of P_0 and $(\tilde{Y}_0,\tilde{Z}_0)$ and (Y_0,Z_0) be an r.b.f. and an l.b.f. of C_0, respectively, such that (4.3.1) holds.

THEOREM 4.4.1 (Dual with Theorem 3.3.3) Let $C = (\hat{Z}_0+D_0R)(\hat{Y}_0-N_0R)^{-1}$ with $R \in H_\infty^{mxn}$. Assume that $P_\lambda \in F^{nxm}$ and $C_\lambda \in F^{mxn}$ are perturbed version of P_0 and C, respectively. If

$$(4.4.1) \qquad \delta(P_\lambda,P_0) + \delta(C_\lambda,C) < \left\| \begin{bmatrix} \hat{Y}_0-N_0R \\ \hat{Z}_0+D_0R \end{bmatrix} \right\|^{-1}.$$

then $H(P_\lambda,C_\lambda)$ is stable.

THEOREM 4.4.2 (Dual with Theorem 4.3.2) Define

$$(4.4.2) \qquad L(P_0,\varepsilon) := \{ P = (\hat{D}_0+\hat{\Delta}_d)^{-1}(\hat{N}_0+\hat{\Delta}_n) \in F^{n\times m} : \|[\hat{\Delta}_d,\hat{\Delta}_n]\| < \varepsilon \} \qquad \varepsilon > 0.$$

Then $C = (\hat{Z}_0+D_0R)(\hat{Y}_0-N_0R)^{-1}$ with $R \in H^{m\times n}$ stabilizes $L(P_0,\varepsilon)$ iff

$$\left\| \begin{bmatrix} \hat{Y}_0-N_0R \\ \hat{Z}_0+D_0R \end{bmatrix} \right\| \le \varepsilon^{-1}.$$

It is obvious that, in order to get a maximal robust stability radius for Theorem 4.4.1 and 4.4.2, we have to compute

$$(4.4.3) \qquad \inf_{R\in H_\infty^{m\times n}} \left\| \begin{bmatrix} \hat{Y}_0-N_0R \\ \hat{Z}_0+D_0R \end{bmatrix} \right\|.$$

To this extent we will use

THEOREM 4.4.3

$$\left\| \begin{bmatrix} \hat{Y}_0-N_0R \\ \hat{Z}_0+D_0R \end{bmatrix} \right\| = \|[(Y_0-R\hat{N}_0),(Z_0+R\hat{D}_0)]\| = \sqrt{1 + \|V + R\|^2} \quad \forall\, R \in H_\infty^{m\times n},$$

where $V := \tilde{D}_0\hat{Z}_0-\tilde{N}_0\hat{Y}_0 = Z_0\tilde{\hat{D}}_0 - Y_0\tilde{\hat{N}}_0$.

PROOF Define $Q := \begin{bmatrix} -\tilde{N}_0 & \tilde{D}_0 \\ \hat{D}_0 & \hat{N}_0 \end{bmatrix}$. Then $Q^{\sim} = \begin{bmatrix} -N_0 & \tilde{\hat{D}}_0 \\ D_0 & \tilde{\hat{N}}_0 \end{bmatrix}$ is unitary on L_2^{m+n}. This proof is divided into two parts.

i) Since multiplication by unitary matrices does not change the norms, we have

$$(4.4.4) \qquad \left\| \begin{bmatrix} \hat{Y}_0-N_0R \\ \hat{Z}_0+D_0R \end{bmatrix} \right\| = \left\| Q \begin{bmatrix} \hat{Y}_0-N_0R \\ \hat{Z}_0+D_0R \end{bmatrix} \right\| = \left\| \begin{bmatrix} \tilde{D}_0\hat{Z}_0- \tilde{N}_0\hat{Y}_0 \\ I \end{bmatrix} + \begin{bmatrix} I \\ 0 \end{bmatrix} R \right\|,$$

and

$$(4.4.5)\qquad \|[(Y_0-R\dot{N}_0),(Z_0+R\bar{D}_0)]\| = \|[(Z_0+R\bar{D}_0),(-Y_0+R\hat{N}_0)]\|$$

$$=\|[(Z_0+R\dot{D}_0),(-Y_0+R\dot{N}_0)]\; Q^{-}\| = \|[-I\;,\; Z_0\dot{\bar{D}}_0 - Y_0\hat{\bar{N}}_0] + R[0\;,\;I]\|.$$

Define

$$(4.4.6)\qquad V := \bar{D}_0\hat{Z}_0 - \bar{N}_0\hat{Y}_0.$$

We will show $V = Z_0\bar{D}_0 - Y_0\dot{N}_0$ also. Since

$$Q\begin{bmatrix}\hat{Y}_0\\ \dot{Z}_0\end{bmatrix} = \begin{bmatrix}-\tilde{N}_0 & \tilde{D}_0\\ \dot{D}_0 & \dot{N}_0\end{bmatrix}\begin{bmatrix}\dot{Y}_0\\ \hat{Z}_0\end{bmatrix} = \begin{bmatrix}V\\ I\end{bmatrix}$$

and

$$\begin{bmatrix}\dot{Y}_0\\ \hat{Z}_0\end{bmatrix} = Q^{-}\begin{bmatrix}V\\ I\end{bmatrix} = \begin{bmatrix}-N_0 & \bar{D}_0\\ D_0 & \bar{N}_0\end{bmatrix}\begin{bmatrix}V\\ I\end{bmatrix} = \begin{bmatrix}-N_0\\ D_0\end{bmatrix}V + \begin{bmatrix}\hat{D}_0\\ \bar{N}_0\end{bmatrix},$$

we have

$$[-Z_0,Y_0]\begin{bmatrix}\dot{Y}_0\\ \dot{Z}_0\end{bmatrix} = [-Z_0,Y_0]\begin{bmatrix}-N_0\\ D_0\end{bmatrix}V + [-Z_0,Y_0]\begin{bmatrix}\bar{D}_0\\ \dot{N}_0\end{bmatrix},$$

i.e.,

$$0 = V + [-Z_0,Y_0]\begin{bmatrix}\hat{D}_0\\ \bar{N}_0\end{bmatrix}.$$

It follows that $V = \bar{D}_0\dot{Z}_0 - \bar{N}_0\dot{Y}_0 = Z_0\bar{D}_0 - Y_0\bar{N}_0$. As a result of this part, we have

$$(4.4.4)\qquad \left\|\begin{bmatrix}\dot{Y}_0-N_0R\\ \dot{Z}_0+D_0R\end{bmatrix}\right\| = \left\|\begin{bmatrix}V\\ I\end{bmatrix} + \begin{bmatrix}I\\ 0\end{bmatrix}R\right\|,$$

$$(4.4.5)\qquad \|[(Y_0-R\dot{N}_0),(Z_0+R\bar{D}_0)]\| = \|[-I\;,\;V] + R[0\;,\;I]\|.$$

ii)

$$(4.4.7)\qquad \left\|\begin{bmatrix}V\\ I\end{bmatrix} + \begin{bmatrix}I\\ 0\end{bmatrix}R\right\|^2 = \left\|\begin{bmatrix}V+R\\ I\end{bmatrix}\right\|^2 = \sup_{x\in H_2^n,\,\|x\|=1}\left\|\begin{bmatrix}V+R\\ I\end{bmatrix}x\right\|^2$$

$$= \sup_{x\in H_2^n, \|x\|=1} [\ \|x\|^2 + \|[V + R]x\|^2] = 1 + \sup_{x\in H_2^n, \|x\|=1} \|[V + R]x\|^2$$

$$= 1 + \|V + R\|$$

$$= \sup_{x\in H_2^n, \|x\|=1} (\|x\|^2 + \|(V + R)x\|^2) = \|[-I, (V + R)]\|^2.$$

Our claim follows from (4.4.4), (4.4.5) and (4.4.7). ■

Note that the above theorem was also proved implicitly by Georgiou and Smith [G–S] in a different way. It follows from Theorem 4.4.3 and (4.4.7) that

$$w_g^2 = \inf_{R\in H_\infty^{m\times n}} \|[(\tilde{Y}_0-R\tilde{N}_0),(\tilde{Z}_0+R\tilde{D}_0)]\|^2 = \inf_{R\in H_\infty^{m\times n}} \left\| \begin{bmatrix} \hat{Y}_0-N_0R \\ \hat{Z}_0+D_0R \end{bmatrix} \right\|^2$$

$$= 1 + \inf_{R\in H_\infty^{m\times n}} \|V + R\|^2 \tag{4.4.8}$$

where $V = \bar{D}_0\bar{Z}_0-\tilde{N}_0\tilde{Y}_0 = Z_0\bar{D}_0 - Y_0\hat{N}_0$.

REMARK *i*) It was shown by Georgiou and Smith that $R(P_0,\varepsilon)\cap B^{n,m}$ and $L(P_0,\varepsilon)\cap B^{n,m}$ are indeed different sets [G–S]. Consequently, $R(P_0,\varepsilon)$ and $L(P_0,\varepsilon)$ are different.

ii) Using state space representation, Habets [Ha.] showed that V is antistable and its McMillan degree is the same as P_0.

Finding an R such that (4.4.8) is minimal is a standard *Nehari problem* and it is known to have solutions [Ne.] and [Gl. 2]. For a detailed discussion of the Nehari problem we refer to Francis [Fr.] and Glover [Gl. 2], where solutions for this problem have been constructed. The infimum value, w_g, is related to the norm of a *Hankel operator*. For a matrix $Q \in L_\infty^{n\times m}$, the Hankel operator $\mathcal{H}_Q : H_2^m \longrightarrow (H_2^n)^\perp$ with symbol Q is defined as

$$\mathcal{H}_Q\, x = (I-T_+^n)Qx, \qquad \forall\ x \in H_2^m,$$

where T_+^n is the orthogonal projection from L_2^n onto H_2^n. It is explained in [Fr.] and [Gl. 2] that

$$\inf_{R\in H_\infty^{m\times n}} \|V + R\| = \|\mathcal{H}_V\|.$$

Hence

$$(4.4.9) \qquad w_g = [1 + \|\mathcal{H}_V\|^2]^{1/2}.$$

The next theorem gives notable formulas for w_g. Originally, they were proved by Glover and McFarlane [G–M 2] using state space representations, later, Georgiou and Smith [G–S] gave an operator theoretic proof.

THEOREM 4.4.4 [G–M] and [G–S]

$$(4.4.10) \qquad w_g^{-1} = (1 - \|\mathcal{H}_{(D^-,N^-)}\|^2)^{1/2} = (1 - \|\mathcal{H}_{(\bar{D},\bar{N})^-}\|^2)^{1/2}.$$

As said before, if R_g is a solution of (4.4.8) (or 4.3.3), then $C_g = (Y_0-R_g\hat{N}_0)^{-1}(Z_0+R_g\hat{D}_0) = (\hat{Z}_0+D_0R_g)(\hat{Y}_0-N_0R_g)^{-1}$ is a controller stabilizing $K(P_0,w_g^{-1})$ (= $R(P_0,w_g^{-1})$) and $L(P_0,w_g^{-1})$. Now, we show that C_g can also stabilize other kinds of perturbations. Assume that $P_0 \in B^{n,m}\cap M(L_\infty)$ and $\varepsilon > 0$, define

$$(4.4.11) \qquad A(P_0,\varepsilon) := \{\, P \in B^{n,m}\cap M(L_\infty) : \|P - P_0\| < \varepsilon, \text{ P has the same number of open right half plane poles as } P_0 \text{ including multiplicities } \},$$

$$(4.4.12) \qquad M(P_0,\varepsilon) := \{\, P = (I+M)P_0 \in B^{n,m} : M \in M(L_\infty) \text{ with } \|M\| < \varepsilon, \text{ P has the same number of open right half plane poles as } P_0 \text{ including multiplicities } \}.$$

The following theorem was proved by Vidyasagar and Kimura [V–K].

THEOREM 4.4.5 *i*) A controller C stabilizes $A(P_0,\varepsilon)$ iff

$$(4.4.13) \qquad \|C(I+P_0C)^{-1}\| \le \varepsilon^{-1}.$$

ii) A controller C stabilizes $M(P_0,\varepsilon)$ iff

$$(4.4.14) \qquad \|P_0(I+CP_0)^{-1}C\| \le \varepsilon^{-1}.$$

Since each stabilizing controller C of P_0 can be written as $C = (Y_0-R\hat{N}_0)^{-1}(Z_0+R\hat{D}_0) = (\hat{Z}_0+D_0R)(\hat{Y}_0-N_0R)^{-1}$ with $R \in H^{m\times n}$, (4.4.13) is equivalent to

$$(4.4.15) \qquad \|(\hat{Z}_0+D_0R)\hat{D}_0\| \le \varepsilon^{-1},$$

and (4.4.14) is equivalent to

(4.4.16) $\quad \|N_0(Z_0+R\tilde{D}_0)\| \leq \varepsilon^{-1}$.

Because $C_g = (Y_0-R_g\tilde{N}_0)^{-1}(Z_0+R_g\tilde{D}_0) = (\tilde{Z}_0+D_0R_g)(\tilde{Y}_0-N_0R_g)^{-1}$ with

$$\|[(Y_0-R_g\tilde{N}_0),(Z_0+R_g\tilde{D}_0)]\| = \left\|\begin{bmatrix} \tilde{Y}_0-N_0R_g \\ \tilde{Z}_0+D_0R_g \end{bmatrix}\right\| = w_g,$$

C_g stabilizes $A(P_0,\alpha)$ with

(4.4.17) $\quad \alpha^{-1} := \|(\tilde{Z}_0+D_0R_g)\tilde{D}_0\|$.

Similarly, C_g also stabilizes $M(P_0,\mu)$ with

(4.4.18) $\quad \mu^{-1} := \|N_0(Z_0+R_g\tilde{D}_0)\|$.

Note that both α and μ are larger than w_g^{-1}, because of $\|\tilde{D}_0\| \leq 1$ and $\|N_0\| \leq 1$. It is obvious that C_g may not be a optimally robust controller with respect to $A(P_0,\varepsilon)$ or $M(P_0,\varepsilon)$. It should be kept in mind that if R_g is a solution to the Nehari problem (4.4.8), then the controller $C_g = (Y_0-R_g\tilde{N}_0)^{-1}(Z_0+R_g\tilde{D}_0) = (\tilde{Z}_0+D_0R_g)(\tilde{Y}_0-N_0R_g)^{-1}$ not only stabilizes $K(P_0,w_g^{-1})$ ($= R(P_0,w_g^{-1})$) and $L(P_0,w_g^{-1})$ but also stabilizes $A(P_0,\alpha)$ and $M(P_0,\mu)$.

Next, let us discuss the influence of an approximate solution of (4.4.8). Let $N(P_0)$ denote the set of all solutions of (4.4.8). Suppose that $R_f \in H_\infty^{m\times n}$ is an ε-approximate solution of (4.4.8), that is :

$$\mathrm{dist}(R_f,N(P_0)) = \inf\{\ \|R_f - R_g\| : R_g \in N(P_0)\ \} < \varepsilon \qquad\qquad \varepsilon > 0.$$

Then, the controller $C_f = (Y_0-R_f\tilde{N}_0)^{-1}(Z_0+R_f\tilde{D}_0) = (\tilde{Z}_0+D_0R_f)(\tilde{Y}_0-N_0R_f)^{-1}$ stabilizes $K(P_0,(w_g+\varepsilon)^{-1})$ ($= R(P_0,(w_g+\varepsilon)^{-1})$) and $L(P_0,(w_g+\varepsilon)^{-1})$, because

$$\left\|\begin{bmatrix} \tilde{Y}_0-N_0R_f \\ \tilde{Z}_0+D_0R_f \end{bmatrix}\right\| = \left\|\begin{bmatrix} \tilde{Y}_0-N_0R_g \\ \tilde{Z}_0+D_0R_g \end{bmatrix} + \begin{bmatrix} -N_0 \\ D_0 \end{bmatrix}(R_f - R_g)\right\|$$

$$\leq \left\|\begin{bmatrix} \tilde{Y}_0-N_0R_g \\ \tilde{Z}_0+D_0R_g \end{bmatrix}\right\| + \left\|\begin{bmatrix} -N_0 \\ D_0 \end{bmatrix}(R_f - R_g)\right\|$$

$< w_g + \varepsilon.$

Similarly, we can show that C_f also stabilizes $A(P_0,(\alpha^{-1}+\varepsilon)^{-1})$ and $M(P_0,(\mu^{-1}+\varepsilon)^{-1})$.

Now, let's consider the total set of systems which can be stabilized by the optimal robust controller C_g. Start by writing

$$Y_g := Y_0-R_g\hat{N}_0 \ , \ Z_g := Z_0+R_g\hat{D}_0 \ , \ \hat{Y}_g := \hat{Y}_0-N_0R_g \ , \ \hat{Z}_g := \hat{Z}_0+D_0R_g.$$

Then we have

$$(4.4.19) \quad \begin{bmatrix} -Z_g & Y_g \\ \hat{D}_0 & \hat{N}_0 \end{bmatrix} \begin{bmatrix} -N_0 & \hat{Y}_g \\ D_0 & \hat{Z}_g \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

and

$$(4.4.20) \quad \begin{bmatrix} -Z_g & Y_g \\ \hat{D}_0+SZ_g & \hat{N}_0-SY_g \end{bmatrix} \begin{bmatrix} -N_0+\hat{Y}_gS & \hat{Y}_g \\ D_0+\hat{Z}_gS & \hat{Z}_g \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

The set of all systems which can be stabilized by C_g is

$$(4.4.21) \quad S(C_g) := \{ \ (\hat{D}_0+SZ_g)^{-1}(\hat{N}-SY_g) \ : \ S \in H_\infty^{nxm}, \ |\hat{D}_0+SZ_g| \neq 0 \ \},$$

where

$$(\hat{D}_0+SZ_g)^{-1}(\hat{N}_0-SY_g) = (N_0-\hat{Y}_gS)(D_0+\hat{Z}_gS)^{-1} \qquad \forall \ S \in H_\infty^{nxm}, \ |\hat{D}_0+SZ_g| \neq 0.$$

The most important point is

LEMMA 4.4.6 $P \in \mathbf{F}^{nxm}$ is stabilized by C_g iff there exists an $S \in H_\infty^{nxm}$ such that

$$|\hat{D}_0+SZ_g| \neq 0 \text{ and } P = (\hat{D}_0+SZ_g)^{-1}(\hat{N}_0-SY_g) = (N_0-\hat{Y}_gS)(D_0+\hat{Z}_gS)^{-1}.$$

Using this lemma we can look into the structure of the neighborhoods $R(P_0,w_g^{-1})$ ($=$ $K(P_0,w_g^{-1})$ and $L(P_0,w_g^{-1})$.

COROLLARY 4.4.7 *i*) $P \in R(P_0,w_g^{-1})$ iff there exists an $S \in H_\infty^{nxm}$ such that

$$|\dot{D}_0+\dot{Z}_gS| \neq 0, \ \left\|\begin{bmatrix} \dot{Z}_g \\ \dot{Y}_g \end{bmatrix} S\right\| < w_g^{-1} \text{ and } P = (N_0-\dot{Y}_gS)(D_0+\dot{Z}_gS)^{-1};$$

ii) $P \in L(P_0,w_g^{-1})$ iff there exists an $S \in H_\infty^{n\times m}$ such that

$$|\hat{D}_0+SZ_g| \neq 0, \ \|S[Z_g,-Y_g]\| < w_g^{-1} \text{ and } P = (\hat{D}_0+SZ_g)^{-1}(\hat{N}_0-SY_g).$$

The following theorem is about the variation of the closed-loop transfer matrices.

THEOREM 4.4.8 Let $P \in K(P_0,w_g^{-1})$. Then

$$\|H(P,C_g) - H(P_0,C_g)\| \leq w_g\ \delta(P,P_0) < 1. \tag{4.4.22}$$

PROOF Recall that (D_0,N_0) and $(\hat{D}_0,\hat{N}_0)$ are a normalized r.b.f. and a normalized l.b.f. of P_0, respectively, and that $(\dot{Y}_g,\dot{Z}_g)$ and (Y_g,Z_g) are an r.b.f. and an l.b.f. of C_g, respectively, such that, (4.4.19) holds. Moreover, $\|[Y_g,Z_g]\| = \|[\dot{Y}_g^T,\dot{Z}_g^T]^T\| = w_g$. Suppose that (D,N) is an r.b.f. of P such that $Y_gD + Z_gN = I$. Then, it is not difficult to check that

$$H(P_0,C_g) = \begin{bmatrix} I-N_0Z_g & -N_0Y_g \\ D_0Z_g & D_0Y_g \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -N_0 \\ D_0 \end{bmatrix} [Z_g,Y_g]$$

and

$$H(P,C_g) = \begin{bmatrix} I-NZ_g & -NY_g \\ DZ_g & DY_g \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -N \\ D \end{bmatrix} [Z_g,Y_g]$$

hold. Hence

$$\|H(P,C_g) - H(P_0,C_g)\| = \left\|\begin{bmatrix} -N \\ D \end{bmatrix} [Z_g,Y_g] - \begin{bmatrix} -N_0 \\ D_0 \end{bmatrix} [Z_g,Y_g]\right\|$$

$$\leq \left\|\begin{bmatrix} -N \\ D \end{bmatrix} - \begin{bmatrix} -N_0 \\ D_0 \end{bmatrix}\right\| \|[Z_g,Y_g]\| = w_g \left\|\begin{bmatrix} N \\ D \end{bmatrix} - \begin{bmatrix} N_0 \\ D_0 \end{bmatrix}\right\|.$$

But

$$\vec{\delta}(P_0,P) = \sup_{x\in SG(P_0)} \inf_{y\in G(P)} \|x - y\| = \sup_{x\in H_2^m, \|x\|=1} \inf_{y\in G(P)} \left\|\begin{bmatrix} D_0 \\ N_0 \end{bmatrix}x - y\right\|.$$

Note $\begin{bmatrix} D \\ N \end{bmatrix} H_2^m = G(P)$, we have

$$\vec{\delta}(P_0,P) = \sup_{x\in H_2^m, \|x\|=1} \inf_{y\in H_2^m} \left\|\begin{bmatrix} D_0 \\ N_0 \end{bmatrix}x - \begin{bmatrix} D \\ N \end{bmatrix}y\right\|$$

$$\leq \sup_{x \in \mathbf{H}_2^m, \|x\|=1} \left\| \begin{bmatrix} D_0 \\ N_0 \end{bmatrix} x - \begin{bmatrix} D \\ N \end{bmatrix} x \right\| = \left\| \begin{bmatrix} N \\ D \end{bmatrix} - \begin{bmatrix} N_0 \\ D_0 \end{bmatrix} \right\|.$$

Since $P \in K(P_0, w_g^{-1})$, $\overrightarrow{\delta}(P,P_0) = \overrightarrow{\delta}(P_0,P) = \delta(P_0,P) < w_g^{-1}$. Consequently, we have obtained

$$\|H(P,C_g) - H(P_0,C_g)\| \leq w_g\ \delta(P_0,P) < 1. \qquad \blacksquare$$

We will conclude this section by discussing dual problem of optimally robust controllers. According to Theorem 3.3.3, if $P_\lambda \in \boldsymbol{F}^{n\times m}$ and $C_\lambda \in \boldsymbol{F}^{m\times n}$ satisfy

$$(4.4.23) \qquad \delta(P_\lambda,P_0) + \delta(C_\lambda,C_g) < w_g^{-1},$$

then $H(P_\lambda,C_\lambda)$ is stable. Now we suppose there are no perturbations on the system P_0. Then (4.4.23) implies that whenever $\delta(C_\lambda,C_g) < w_g^{-1}$, $H(P_0,C_\lambda)$ is stable. In other words, P_0 stabilizes $K(C_g,w_g^{-1}) = \{\ C \in \boldsymbol{F}^{m\times n} : \delta(C,C_g) < w_g^{-1}\ \}$. Consequently, the largest robust stability radius w_{cg}^{-1} of C_g is not smaller than w_g^{-1}, the largest robust stability radius of P_0, i.e., the robustness of C_g with respect to feedback stabilization is better than that of P_0. We will carry out a little further study about the relation between w_{cg}^{-1} and w_g^{-1}.

Assume that $U_l \in \boldsymbol{U}^{m,m}$ and $U_r \in \boldsymbol{U}^{n,n}$ such that $U_l[Y_g,Z_g]$ and $[\dot{Y}_g^T,\dot{Z}_g^T]^T U_r$ are normalized, respectively. It follows from (4.4.20) that

$$(4.4.24) \quad \begin{bmatrix} -U_l Z_g & U_l Y_g \\ U_r^{-1}\dot{D}_0+SU_lZ_g & U_r^{-1}\dot{N}_0-SU_lY_g \end{bmatrix} \begin{bmatrix} -N_0U_l^{-1}+\dot{Y}_gU_rS & \dot{Y}_gU_r \\ D_0U_l^{-1}+\dot{Z}_gU_rS & \dot{Z}_gU_r \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Now the largest robust stability radius w_{cg}^{-1} is

$$w_{cg} = \inf_{S \in \mathbf{H}_\infty^{n\times m}} \left\| \begin{bmatrix} -N_0U_l^{-1}+\dot{Y}_gU_rS \\ D_0U_l^{-1}+\dot{Z}_gU_rS \end{bmatrix} \right\| = \inf_{S \in \mathbf{H}_\infty^{n\times m}} \|[U_r^{-1}\dot{D}_0+SU_lZ_g\ ,\ U_r^{-1}\dot{N}_0-SU_lY_g]\|.$$

It is easy to check that $E := \begin{bmatrix} (\hat{Y}_gU_r)^* & (\hat{Z}_gU_r)^* \\ -U_lZ_g & U_lY_g \end{bmatrix}$ is unitary on $\mathbf{L}_2^{m+n}$. Hence

$$w_{cg} = \inf_{S \in \mathbf{H}_\infty^{n\times m}} \left\| E \begin{bmatrix} -N_0U_l^{-1}+\dot{Y}_gU_rS \\ D_0U_l^{-1}+\dot{Z}_gU_rS \end{bmatrix} \right\|$$

$$= \inf_{S \in \mathbf{H}_\infty^{n\times m}} \left\| \begin{bmatrix} [-(\dot{Y}_gU_r)^*N_0U_l^{-1}+(\dot{Z}_gU_r)^*D_0U_l^{-1}]-S \\ I \end{bmatrix} \right\|.$$

Let $V_c = [-(\tilde{Y}_g U_r)^{\sim} N_0 U_l^{-1} + (\tilde{Z}_g U_r)^{\sim} D_0 U_l^{-1}]$, then $w_{cg} = [1 + \|\mathcal{H}_{V_c}\|^2]^{1/2}$. Recall from (4.4.6) that $V := \tilde{D}_0 \tilde{Z}_0 - \tilde{N}_0 \tilde{Y}_0$. Therefore, we have

$$V_c = [-(\tilde{Y}_g U_r)^{\sim} N_0 U_l^{-1} + (\tilde{Z}_g U_r)^{\sim} D_0 U_l^{-1}] = U_r^{\sim}[-\tilde{Y}_g^{\sim} N_0 + \tilde{Z}_g^{\sim} D_0] U_l^{-1} \;=\; U_r^{\sim} V^{\sim} U_l^{-1}.$$

We wish that a further study on the relation between $\|\mathcal{H}_V\|$ and $\|\mathcal{H}_{V_0}\|$ will be made elsewhere.

4.5 A formula, a lower and an upper bound for the gap metric

Georgiou gave a formula for the gap metric for rational matrices in [Ge.], which is a consequence of the *Commutant Lifting Theorem* (for one version see Young [Yo.]). In this section, using the result of Section 4.2, we show that this formula is also valid for distributed LTI systems. Then a lower and an upper bound for the gap metric given by Zhu, Hautus and Praagman [Z–H–P 2] will be presented here.

A proof of the following theorem is given in [Yo.].

THEOREM 4.5.1 Let $B \in H_\infty^{q\times p}$ and $C \in H_\infty^{p\times p}$ be inner matrices and $F \in H_\infty^{q\times p}$. Let ϕ be a closed subspace of L_2^q containing $FC^{\sim}H_2^p + BH_2^p$ and define

$$\text{(4.5.1)} \qquad T : C^{\sim}H_2^p \longrightarrow \phi \ominus BH_2^p$$

as

$$\text{(4.5.2)} \qquad T := \Pi\, F|_{C^{\sim}H_2^p}\,,$$

where Π is the orthogonal projection from L_2^q to $\phi \ominus BH_2^p$. Then

$$\text{(4.5.3)} \qquad \inf_{Q \in H_\infty^{p\times p}} \|F + BQC\| = \|T\|.$$

Let $P_i \in \mathcal{B}^{n,m}$ (i=1,2). From Section 3.1, we have

$$\vec{\delta}^*(P_2,P_1) = \|(I-\Pi(P_1))\Pi(P_2)\|.$$

According to Section 4.2, there will exists a normalized r.b.f. (D,N) for each $P \in \mathcal{B}^{n,m}$. Suppose that (D_i,N_i) is a normalized r.b.f. of $P_i \in \mathcal{B}^{n,m}$ (i=1,2). Define

$$A_i := \begin{bmatrix} D_i \\ N_i \end{bmatrix}.$$

Then

$$\overrightarrow{\delta}(P_2,P_1) = \|(I-A_1A_1^*)A_2A_2^*\|$$

$$(4.5.4) \qquad = \|(I-A_1A_1^*)A_2\| \qquad\qquad (A_2^*A_2 = I)$$

Now we apply Theorem 4.5.1 to (4.5.4). Let $B = A_1$, $C = I$, $F = A_2$, $\phi = H_2^{(n+m)}$, and Π be the orthogonal projection from $L_2^{(n+m)}$ to $H_2^{(n+m)} \ominus A_1H_2^m$, then, by Theorem 4.5.1, we have

$$\|\Pi A_2|_{H_2^m}\| = \inf_{Q\in H_\infty^{m\times m}} \|A_2 - A_1Q\|.$$

But

$$\Pi A_2|_{H_2^m} = (I-A_1A_1^*)A_2|_{H_2^m}.$$

Hence, by (4.5.4)

$$(4.5.5) \qquad \overrightarrow{\delta}(P_2,P_1) = \inf_{Q\in H_\infty^{m\times m}} \|A_2 - A_1Q\|.$$

REMARK Suppose that $\{\psi_n\}$ is a sequence of closed subspaces in $\mathbb{R}^3$ and ϕ is a closed subspace of $\mathbb{R}^3$. It follows from the example shown by Figure 2.1.2 that $\overrightarrow{\delta}(\phi,\psi_n)$ may converge to zero while $\delta(\psi_n,\phi) = \overrightarrow{\delta}(\psi_n,\phi) = 1$. But this situation can't happen in the case of the transfer matrices with entries in the quotient field of H_∞. That is, if $\{P_k\}$ is a sequence of $B^{n,m}$ and $P_0 \in B^{n,m}$. Then, $\overrightarrow{\delta}(P_0,P_k) \longrightarrow 0$ implies $\delta(P_0,P_k) \longrightarrow 0$.

PROOF Assume that (D_k,N_k) is a normalized r.b.f. of P_k and $A_k := [D_k^T,N_k^T]^T$. Moreover, suppose that $C_g = Y_g^{-1}Z_g$ (Y_g,Z_g satisfy (4.4.19)) is an optimally robust controller of P_0. Since $\overrightarrow{\delta}(P_0,P_k) \longrightarrow 0$, by (4.5.5) there exists a sequence $\{Q_k\}$ with $Q_k \in H_\infty^{m\times m}$ such that

$$\|A_0 - A_kQ_k\| \longrightarrow 0 \qquad (k \longrightarrow \infty).$$

Hence,

$$[Y_g,Z_g](A_0 - A_kQ_k) = (I - [Y_g,Z_g]A_kQ_k) \longrightarrow 0 \qquad (k \longrightarrow \infty).$$

Therefore, $[Y_g,Z_g]A_k$ and Q_k have to be unimodular for sufficiently large k. Consequently, $\{P_k\}$ converges to P_0 in the graph topology (hence, also in the gap topology). Thus, $\delta(P_0,P_k) \longrightarrow 0$ $(k \longrightarrow 0)$. ■

Note that $\overrightarrow{\delta}(P_k,P_0) \longrightarrow 0$, in general, doesn't imply $\delta(P_k,P_0) \longrightarrow 0$. For example, take $P_0 = 0$ and $P_{\alpha,\beta}(s) = \alpha/[s-\beta]$. It is known that $\delta(P_{\alpha,\beta},P_0) = 1$ for all $\alpha,\beta > 0$, but

$$\overrightarrow{\delta}^*(P_k,P_0) = \inf_{Q\in H_\infty} \left\| \begin{bmatrix} \frac{s-\beta}{s+\gamma} \\ \frac{\alpha}{s+\gamma} \end{bmatrix} - \begin{bmatrix} I \\ 0 \end{bmatrix} Q \right\| \qquad \gamma = (\alpha^2+\beta^2)^{1/2}.$$

$$\leq \left\| \begin{bmatrix} \frac{s-\beta}{s+\gamma} \\ \frac{\alpha}{s+\gamma} \end{bmatrix} - \begin{bmatrix} I \\ 0 \end{bmatrix} \frac{s-\beta}{s+\gamma} \right\| = \left\| \begin{bmatrix} 0 \\ \frac{\alpha}{s+\gamma} \end{bmatrix} \right\| \longrightarrow 0 \ (\alpha \longrightarrow 0).$$

Since $K := \begin{bmatrix} D_1^- & N_1^- \\ -\tilde{N}_1 & \tilde{D}_1 \end{bmatrix}$ is unitary, i.e., $K^-K = KK^- = I$, it follows that

$$\overrightarrow{\delta}^*(P_2,P_1) = \inf_{Q\in H_\infty^{m\times m}} \|K[A_2 - A_1Q]\|$$

$$= \inf_{Q\in H_\infty^{m\times m}} \left\| \begin{bmatrix} D_1^-D_2+N_1^-N_2 - Q \\ -\tilde{N}_1D_2+\tilde{D}_1N_2 \end{bmatrix} \right\|. \tag{4.5.6}$$

This is a standard "2–block" H_∞ optimization problem, and can be solved by standard techniques from H_∞ theory [Fr.].

A lower and an upper bound is presented in [Z–H–P 2], their computations are certainly simpler than the computation of (4.5.6). Moreover, the lower bound is of interest on its own. Suppose that (D_i,N_i) is a normalized r.b.f. of $P_i \in B^{n,m}$ (i=1,2). Define $A_i := [D_i^T,N_i^T]^T$. It is known from (3.1.4) and (3.1.2) that

$$\delta(P_1,P_2) = \|A_1A_1^* - A_2A_2^*\|.$$

THEOREM 4.5.2 [Z–H–P 2] *i)* The following inequality holds

$$\|A_1A_1^- - A_2A_2^-\| \leq \|A_1A_1^* - A_2A_2^*\|. \tag{4.5.7}$$

If $\delta(P_1,P_2) < 1$, then

$$\|A_1A_1^* - A_2A_2^*\| \leq \|A_1A_1^- - A_2A_2^-\| + \min\{\|\mathcal{H}_{A_1}\|, \|\mathcal{H}_{A_2}\|\}$$

holds also.

Since A_i^* is the restriction of $T_+^{n+m}A_i^-$ to H_2^m, $\|A_1A_1^- - A_2A_2^-\|$ is computed much more easily than $\|A_1A_1^* - A_2A_2^*\|$. For instance, in the case of $P_i(.)$ being rational, $A_iA_i^-$ is just a rational matrix, whereas A_i^* is a Toeplitz operator. The norm of a rational matrix can be computed by a program designed by Bruinsma and Steinbuch [B–S]. We will show that to get the lower bound, it is not necessary to find a normalized r.b.f. of P over $M(H_\infty)$.

$(D(.),N(.)) \in M(L_\infty)$ is said to be a right Bezout fraction of $P \in B^{n,m}$ over $M(L_\infty)$ if

i) $D(.)$, $N(.) \in M(L_\infty)$ and $|D(.)| \neq 0$;

ii) $Y(.)$, $Z(.) \in M(L_\infty)$ exist such that

$$Y(.)D(.) + Z(.)N(.) = I;$$

iii) $P(.) = N(.)D(.)^{-1}$.

According to the definition of the operator N_l in Section 4.1.1, in the same way as Lemma 3.1.1 we can prove that $\tilde{D}_l D_l + \tilde{N}_l N_l$ maps $\mathbf{L}_2^m$ onto $\mathbf{L}_2^m$ bijectively . And analogously to Lemma 3.1.2, we can also show that the orthogonal projection $\Pi_l(P)$ onto the graph of P_l is

$$\Pi_l(P) = \begin{bmatrix} D_l \\ N_l \end{bmatrix} [\tilde{D}_l D_l + \tilde{N}_l N_l]^{-1} [\tilde{D}_l, \tilde{N}_l]. \tag{4.5.8}$$

It is obvious that an r.b.f. $[D^T,N^T]^T$ of P over $M(H_\infty)$ is always an r.b.f. of P over $M(L_\infty)$. Consequently,

$$\Pi_l(P) = \begin{bmatrix} D_l \\ N_l \end{bmatrix} [\tilde{D}_l D_l + \tilde{N}_l N_l]^{-1} [\tilde{D}_l, \tilde{N}_l] = \begin{bmatrix} \mathrm{D}_l \\ \mathrm{N}_l \end{bmatrix} [\tilde{\mathrm{D}}_l \mathrm{D}_l + \tilde{\mathrm{N}}_l \mathrm{N}_l]^{-1} [\tilde{\mathrm{D}}_l, \tilde{\mathrm{N}}_l].$$

If $[D^T,N^T]^T$ is normalized, we have

$$\Pi_l(P) = \begin{bmatrix} D_l \\ N_l \end{bmatrix} [\tilde{D}_l D_l + \tilde{N}_l N_l]^{-1} [\tilde{D}_l, \tilde{N}_l] = \begin{bmatrix} \mathrm{D}_l \\ \mathrm{N}_l \end{bmatrix} [\tilde{\mathrm{D}}_l, \tilde{\mathrm{N}}_l].$$

Suppose that (D_i,N_i) is a right Bezout fraction of $P_i \in B^{n,m}$ over $M(L_\infty)$ (i=1,2), and that $(\mathrm{D}_i,\mathrm{N}_i)$ is a normalized r.b.f. of P_i (i=1,2). Then, we have

$$\begin{aligned}
&\|\Pi_l(P_1) - \Pi_l(P_2)\| \\
&= \left\| \begin{bmatrix} D_{1l} \\ N_{1l} \end{bmatrix} [\tilde{D}_{1l} D_{1l} + \tilde{N}_{1l} N_{1l}]^{-1} [\tilde{D}_{1l}, \tilde{N}_{1l}] - \begin{bmatrix} D_{2l} \\ N_{2l} \end{bmatrix} [\tilde{D}_{2l} D_{2l} + \tilde{N}_{2l} N_{2l}]^{-1} [\tilde{D}_{2l}, \tilde{N}_{2l}] \right\| \\
&= \left\| \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} [\tilde{D}_1 D_1 + \tilde{N}_1 N_1]^{-1} [\tilde{D}_1, \tilde{N}_1] - \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} [\tilde{D}_2 D_2 + \tilde{N}_2 N_2]^{-1} [\tilde{D}_2, \tilde{N}_2] \right\| \\
&= \left\| \begin{bmatrix} \mathrm{D}_{1l} \\ \mathrm{N}_{1l} \end{bmatrix} [\tilde{\mathrm{D}}_{1l}, \tilde{\mathrm{N}}_{1l}] - \begin{bmatrix} \mathrm{D}_{2l} \\ \mathrm{N}_{2l} \end{bmatrix} [\tilde{\mathrm{D}}_{2l}, \tilde{\mathrm{N}}_{2l}] \right\| \\
&= \left\| \begin{bmatrix} \mathrm{D}_1 \\ \mathrm{N}_1 \end{bmatrix} [\tilde{\mathrm{D}}_1, \tilde{\mathrm{N}}_1] - \begin{bmatrix} \mathrm{D}_2 \\ \mathrm{N}_2 \end{bmatrix} [\tilde{\mathrm{D}}_2, \tilde{\mathrm{N}}_2] \right\|.
\end{aligned}$$

Hence, we have shown that

THEOREM 4.5.3 The lower bound $\left\| \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} [\tilde{D}_1, \tilde{N}_1] - \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} [\tilde{D}_2, \tilde{N}_2] \right\|$ in (4.5.7) is equal to

$$(4.5.9) \qquad \left\| \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} [\tilde{D}_1 D_1 + \tilde{N}_1 N_1]^{-1} [\tilde{D}_1, \tilde{N}_1] - \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} [\tilde{D}_2 D_2 + \tilde{N}_2 N_2]^{-1} [\tilde{D}_2, \tilde{N}_2] \right\|.$$

If (D_i, N_i) is normalized, i.e., $\tilde{D}_i(i\omega)D_i(i\omega) + \tilde{N}_i(i\omega)N_i(i\omega) = I$ ($\forall\ \omega \in \mathbb{R}$, $i=1,2$), then (4.5.9) is

$$\left\| \begin{bmatrix} D_1 \\ N_1 \end{bmatrix} (\tilde{D}_1, \tilde{N}_1) - \begin{bmatrix} D_2 \\ N_2 \end{bmatrix} (\tilde{D}_2, \tilde{N}_2) \right\|.$$

The advantage of the using an r.b.f. over $M(L_\infty)$ is that it is much more easily obtained than an r.b.f. over $M(H_\infty)$.

4.6 Finite dimensional controller design

In this section we will discuss the design of finite dimensional controllers for infinite dimensional systems via the largest robust stability radius and optimally robust controllers.

In general, there are two ways to design a finite dimensional controller for an infinite dimensional system : 1) first to design an infinite dimensional controller and then approximate this controller by a finite one; 2) approximate the infinite dimensional system by a finite one, then, design a finite dimensional controller according to this finite dimensional system. According to these two general principles, we propose the following two schemes.

Let $P_0 \in B^{n,m}$ be the system for which a finite dimensional controller will be designed. Denote by $w_g^{-1}(P_0)$ the largest robust stability radius of P_0.

SCHEME 4.6.1

Step 1 Find $w_g^{-1}(P_0)$ and an optimally robust controller C_g;

Step 2 Find a finite approximation C_f of C_g such that $\delta(C_f, C_g) < w_g^{-1}(P_0)$.

THEOREM 4.6.2 The finite dimensional controller C_f obtained in Scheme 4.6.1 stabilizes P_0. Moreover,

$$(4.6.1) \qquad \|H(P_0, C_g) - H(P_0, C_f)\| \le w_g(P_0)\delta(C_f, C_g) < 1.$$

PROOF According to the definition of w_g^{-1} (4.4.8) and Theorem 3.3.3, we know that C_f stabilizes P_0. By (4.4.22) we have

$$\|H(P_0,C_g) - H(P_0,C_f)\| \le w_g(C_g)\delta(C_f,C_g).$$

It follows from the last topic of Section 4.1 that $w_g(C_g) \le w_g(P_0)$. Hence, (4.6.1) is true. ■

SCHEME 4.6.3

Step 1 Find a finite approximation P_f of P_0 such that $\delta(P_f,P_0) \le w_g^{-1}(P_f)$;

Step 2 Compute an optimally robust controller C_f of P_f (the algorithms will be presented in the next section).

THEOREM 4.6.4 The finite dimensional controller C_f obtained in Scheme 4.6.3 stabilizes P_0. Moreover,

(4.6.2) $$\|H(P_0,C_f) - H(P_f,C_f)\| \le w_g(P_f)\delta(P_f,P_0) < 1.$$

PROOF That C_f stabilizes P_0 follows from the definition of $w_g(P_f)$ and Theorem 3.3.3, and (4.6.2) follows from (4.4.22). ■

4.7 Computation of optimally robust controllers

For a given finite dimensional LTI system i.e. a real rational matrix P_0, the following algorithms are presented for computing its largest robust stability radius and an approximate optimally robust controllers.

First, we remark that using state space representation Glover and McFarlane [G–M 2] gave some nice formulas for computing the largest robust stability radius and optimally robust controllers for proper rational transfer matrices. Algorithm 4.7.1 depends on Glover and Mcfarlane's work [G–M 2].

ALGORITHM 4.7.1 (For proper transfer matrices)

Step 1 Find a minimal realization (A,B,C,D) of P_0

Step 2 Solve the following two Algebraic Riccati Equations

$$(A-BH^{-1}D^TC)^TX + X(A-BH^{-1}D^TC) - XBH^{-1}B^TX + C^TL^{-1}C = 0,$$

$$(A-BD^TL^{-1}C)Y + Y(A-BD^TL^{-1}C)^T - YC^TL^{-1}CY + BH^{-1}B^T = 0,$$

where $\quad H := I+D^TD \qquad\qquad L := I+DD^T.$

Step 3 Set $\quad A_c := A - BF \qquad F := H^{-1}(D^TC+B^TX)$

$$A_o := A - KC \qquad K := (BD^T+YC^T)L^{-1}$$

Then $$\begin{bmatrix} -Z_0 & Y_0 \\ \tilde{D}_0 & \tilde{N}_0 \end{bmatrix}\begin{bmatrix} -N_0 & \hat{Y}_0 \\ D_0 & \hat{Z}_0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},$$

and (D_0,N_0), $(\tilde{D}_0,\tilde{N}_0)$ are a normalized r.b.f. and a normalized l.b.f. of $P_0(\cdot)$ respectively, where

$$\begin{bmatrix} -Z_0 & Y_0 \\ \tilde{D}_0 & \tilde{N}_0 \end{bmatrix} := \begin{bmatrix} -H^{1/2}F(sI-A_o)^{-1}K & H^{1/2}[I+F(sI-A_o)^{-1}(B-KD)] \\ L^{-1/2}[I-C(sI-A_o)^{-1}K] & L^{-1/2}[C(sI-A_o)^{-1}(B-KD)+D] \end{bmatrix},$$

$$\begin{bmatrix} -N_0 & \hat{Y}_0 \\ D_0 & \hat{Z}_0 \end{bmatrix} := \begin{bmatrix} -[(C-DF)(sI-A_c)^{-1}B+D]H^{-1/2} & [I+(C-DF)(sI-A_c)^{-1}K]L^{1/2} \\ [I-F(sI-A_c)^{-1}B]H^{-1/2} & F(sI-A_c)^{-1}KL^{1/2} \end{bmatrix}.$$

Step 4 Compute $\lambda_{max}(YX)$, the largest eigenvalue of YX; $w_g = \sqrt{(1 + \lambda_{max}(YX))}$.

Step 5 Take $\gamma > w_g$, and set $W_1 = I + XY - \gamma^2 I$. Then

$$[A_c+\gamma^2 W_1^{T-1}YC^T(C-DF) : \gamma^2 W_1^{T-1}YC^T : B^TX : -D^T]$$

is a state space representation of an approximate optimally robust control controller.

ALGORITHM 4.7.2 (For singular systems, i.e., non-proper transfer matrices)

Step 1 Find a right and a left Bezout fractions of P_0.

Step 2 Using spectral factorization, find a normalized right and left bezout fraction (D_0,N_0), $(\tilde{D}_0,\tilde{N}_0)$.

Step 3 Solve the following two Bezout identities

$$Y_0D_0 + Z_0N_0 = I \qquad \tilde{D}_0\tilde{Y}_0 + \tilde{N}_0\tilde{Z}_0 = I.$$

Step 4 Set $V = D_0\tilde{Z}_0 - N_0\tilde{Y}_0$ ($= Z_0\tilde{D}_0 - Y_0\tilde{N}_0$), find the strictly proper antistable part V_- of V, which is strictly proper, and compute a minimal realization $[A_2,B_2,C_2,0]$ of V_-.

Step 5 Solve the following two Lyapunov equations

$$A_2L_c + L_cA_2^T = B_2B_2^T, \qquad A_2^TL_o + L_oA_2 = C_2^TC_2.$$

Compute the largest eigenvalue λ_{max} of L_cL_o. Then

$$\|\mathcal{H}_V\| = \sqrt{\lambda_{max}}\ (=:\alpha) \qquad w_g = \sqrt{(1 + \lambda_{max})}$$

Step 6 Fix an approximate margin $\varepsilon > 0$ for the Nehari problem

$$\|V_- - R\| < \alpha(1+\varepsilon) \tag{4.7.1}$$

Set $L_q := L_o[\alpha(1+\varepsilon)]^{-2}$, $N := (I-L_qL_c)^{-1}$ and $C_3 := (\alpha+\alpha\varepsilon)^{-1}C_2$. Define

$$L_1(.) := [A_2,\ L_cNC_3^T,\ C_3,\ I] \qquad L_2(.) := [A_2,\ N^TB_2,\ C_3,\ 0]$$

$$L_3(.) := [-A_2^T,\ NC_3^T,\ -B_2^T,\ 0] \qquad L_4(.) := [-A_2^T,\ NL_qB_2,\ B_2^T,\ I].$$

Then, the set of all solutions to problem (*) is

$$\mathcal{N}(V,\varepsilon) := \{\ V-\alpha(1+\varepsilon)(L_1Y+L_2)(L_3Y+L_4)^{-1} \in M(RH_\infty) : Y \in M(RH_\infty),\ \|Y\|\le 1\ \}. \tag{4.7.2}$$

The set of all approximate optimally robust controllers is

$$S_{opt}(P_0,\varepsilon) := \{\ C = (Y_0-R\tilde{N}_0)^{-1}(Z_0+R\tilde{D}_0) = (\tilde{Z}_0+D_0R)(\tilde{Y}_0-N_0R)^{-1} : \\ R \in \mathcal{N}(V,\varepsilon),\ |\tilde{Y}_0-N_0R|\neq 0\ \} \tag{4.7.3}$$ ■

Note that if a Y in (4.7.2) is fixed, the mapping from $\varepsilon \in [0, 1]$ to $C \in S_{opt}(P_0,\varepsilon)$ is continuous at 0. Hence, any element in (4.7.3) is indeed an approximate optimally robust controller in the sense discussed in Section 4.4.

Algorithm 4.7.1 is programmed by Habets [Ha.] in PC–Matlab m.-file, and we will use the program to give some numerical examples. Our first numerical example is to show the variation of the largest robust stability radius with the pole of the plant.

EXAMPLE 4.7.3 Let $P_\beta(s) = 1/[s-\beta]$. The following diagram shows the values of the largest robust stability radius w_g^{-1} corresponding to different β.

β	−10	−5	−3	−1	−0.5	−0.2	−0.01	−0.005	−0.0001	0
w_g^{-1}	0.9988	0.9951	0.9871	0.9239	0.8507	0.7733	0.7106	0.7089	0.7075	0.7071
β	10	5	3	1	0.5	0.2	0.01	0.005	0.0001	0
w_g^{-1}	0.0498	0.0985	0.1602	0.3827	0.5257	0.6340	0.7036	0.7053	0.7068	0.7071

From the above table one can see that the larger the unstable pole is, the smaller the w_g^{-1} is, and w_g^{-1} may approach to zero as the unstable pole goes to infinity.

Next, we fix the pole of and change the coefficient.

EXAMPLE 4.7.4 Let $P_\alpha(s) = \alpha/[s-1]$. The following diagram shows the values of the largest robust stability radius w_g^{-1} corresponding to different α.

$\pm\alpha$	0	1e−7	1e−5	1e−3	0.5e−2	0.01	0.2	0.35	0.5	1
w_g^{-1}	1	4.9e−8	5.0e−6	5.0e−4	2.5e−3	0.0050	0.0985	0.16750	0.22985	0.3827
$\pm\alpha$	1e+8	1e+4	1000	500	50	10	5	3	2.28	1.67
w_g^{-1}	0.7071	0.7071	0.7068	0.7064	0.7000	0.6710	0.6340	0.5847	0.5470	0.4931

This table tells us that when the coefficient is very small it influences w_g^{-1} significantly, but after certain large number it lost its influence on w_g^{-1}.

Note that, if $P_0(s) = \alpha/s$ ($\alpha \neq 0$), then w_g^{-1} is independent of α.

Our last example is to find the largest robust stability radius and an approximate optimally robust controller for an approximation of a homogeneous beam with viscous damping.

EXAMPLE 4.7.5 The transfer matrix of homogeneous beam with viscous damping

[Bo. $p96$] is

$$P(s) = 1/(\rho\alpha)\begin{bmatrix} 1/(2s^2) & 0 \\ 0 & 3/(2s^2) \end{bmatrix} + 1/(\rho\alpha)\sum_{i=3}^{\infty}\frac{G_i}{s^2 + \alpha_1\lambda_i^4 s + \alpha_2\lambda_i^4},$$

where $G_i = \begin{bmatrix} v_i^2(0) & 0 \\ 0 & (v_i'(0))^2 \end{bmatrix}$, $\lambda_i \in \mathbb{R}$ is a spectrum point of the homogeneous beam and $v_i(t)$ is the eigenfunctions corresponding to the λ_i, α_1 is the damping coefficient and α_2 is the stiffness coefficient and $\rho\alpha$ is the mass per unit length. We use the parameters given in [Bo. $p122$]

$$\alpha_1 = 3.89\times10^{-4}; \qquad \alpha_2 = 1.129; \qquad \rho\alpha = 47.2.$$

We will compute the largest robust stability radius and an approximate (in the sense that at the Step 5 of Algorithm 4.7.1, γ is chosen to be w_g+10^{-5}) optimally robust controller for the 20th order approximation P_0 of P.

The largest robust stability radius of $P_0(s)$ is $w_g^{-1} = 0.3828$. P_0 and an approximate optimally robust controller is C_g given by

$$P_0(s) = \begin{bmatrix} \frac{n1}{d1} & 0 \\ 0 & \frac{n2}{d2} \end{bmatrix}, \qquad C_g(s) = \begin{bmatrix} \frac{cn11}{cd11} & \frac{cn12}{cd12} \\ \frac{cn21}{cd21} & \frac{cn22}{cd22} \end{bmatrix},$$

where

$$n1 = 0.0008s^4 + 0.0004s^3 + 0.3890s^2 + 0.0073s + 5.2790;$$
$$d1 = 0.0167s^6 + 0.0152s^5 + 14.6946s^4 + 0.6868s^3 + 498.3416s^2;$$

$$n2 = 0.0001s^4 + 0.0001s^3 + 0.0648s^2 + 0.0015s + 1.0583;$$
$$d2 = 0.0001s^6 + 0.0001s^5 + 0.1396s^4 + 0.0459s^3 + 33.3013s^2;$$

$$cn11 = -0.0003s^5 - 0.0002s^4 - 0.2203s^3 - 0.0197s^2 - 7.4722s - 0.3186;$$
$$cd11 = 0.0001s^5 + 0.0001s^4 + 0.0913s^3 + 0.0275s^2 + 3.0930s + 0.7689;$$

$$\frac{cn21}{cd21} \approx \frac{cn12}{cd12} \approx 0;$$

$$cn22 = -0.0442s^3 - 0.0178s^2 - 10.5444s - 0.7787;$$
$$cd22 = 0.0002s^4 + 0.0184s^3 + 0.0473s^2 + 4.3632s + 1.8794.$$

The author would like to thank Dr. J. Bontsema and Prof. R.F. Curtain for their generous help when he worked on this example.

In a general framework including finite and infinite dimensional LTI systems as well as discrete–time and continuous–time and 2D – systems, the gap topology approach to robust stabilization was studied in the present thesis. A necessary and sufficient condition was given in the gap topology (Corollary 2.3.2). And the estimation was also presented for the variation of the closed–loop system according to perturbations of system and controller (Theorem 2.3.1). Several guaranteed (sufficient) bounds were found for robust stabilization in the gap metric (or in the graph metric). A thorough study was made for the case when the transfer matrices have their entries in the quotient field of H_∞. Especially, the largest robust stability radius and optimally robust controllers were investigated. Moreover, the following related problems were also discussed such as : the existence of normalized Bezout fractions, the variation of the closed–loop systems, the design of finite dimensional controllers, the computation of the gap metric, optimally robust controllers and the largest robust stability radius and so on. The section : **Review of the thesis** in page 3 and 4, gives additional information about contents.

Now, we present the following related topics, which are worth further investigation.

1) In Section 3.3, several bound were presented and one of them was maximized in Chapter 4 for a special case. The first problem is to compare these bounds and to maximize them for the general framework.

2) Generalization of the whole theory from Chapter 4 to 2–D systems described in Example 1.1.10. To do this, one needs to generalized Lax's theorem, the Commutant Lifting Theorem and some other H_∞ theories etc. to the two variables case.

3) Approximation of an infinite dimensional system by a family of finite dimensional systems in the gap topology. First, it can easily be checked that an infinite dimensional $P_0 \in B^{n,m}$ (in the notations of Chapter 4) can be approximated by a family of rational matrices iff P_0 has an r.b.f. or l.b.f., which are continuous on $i\mathbb{R}$. Many related problems such as its relation with L_∞–approximation, computation of error bounds and state space version etc should be studied.

4) Application of the theory in Section 4.6 to some concrete distributed LTI systems such as delay systems, neutral systems and flexible beams. Note that we still do not know whether the controller C_g stabilizes P or not in Example 5.7.5.

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Curriculum Vitae

The author of this thesis was born in Xi'an, China on March 24, 1960. From September 1978 to December 1984, he studied at the Department of Mathematics of Xi'an Jiaotong University, China, where he received his Bachelor's and Master's degree in computation of mathematics in 1982 and 1984, respectively. Afterwards, he taught at Xi'an Jiaotong University until August 1985.

From September 1985 to August 1989, he worked as a Research Assistant in Systems and Control Theory at the Department of Mathematics and Computing Science of Eindhoven University of Technology, the Netherlands. His research at Eindhoven led to the present thesis. From June 1988 to August 1988, he joined Young Scientists' Summer Program (YSSP) at International Institute for Applied Systems Analysis at Laxenburg, Austria. Currently, he holds a position as a Visiting Researcher Associate at the College of Sciences of Clemson University, U.S.A..

PROPOSITIONS

accompanying the dissertation

ROBUSTNESS OF FEEDBACK STABILIZATION : A TOPOLOGICAL APPROACH

by
S.Q. Zhu
Eindhoven, The Netherlands August 1989

1. Given a set of points $\{ \overline{x}_1, \overline{x}_2, \dots \overline{x}_m \}$ in $\mathbb{R}^n$, a closed convex set $C \subset \mathbb{R}^n$ and a set of numbers $\{ \alpha_1, \alpha_2, \dots \alpha_m \}$ in $\mathbb{R}_+$, find an x_* such that

$$(1.1) \qquad L(x_*) = \min_{x \in C} L(x),$$

where

$$(1.2) \qquad L(x) := \sum_{i=1}^{m} \alpha_i \|x - \overline{x}_i\|.$$

This is so–called the *optimal location problem*. Consider the following iteration

$$x_0 \in C, \; x_0 \neq \overline{x}_i \; (i=1,\dots m), \; \varepsilon > 0,$$

$$z_{k+1} = \text{solution of} \min_{x \in C} \|y - (x_k - \lambda_k \frac{d}{dx} L(x)|_{x=\overline{x}_k})\|,$$

$$\text{where } \lambda_k^{-1} := \sum_{i=1}^{m} \frac{\alpha_i}{\|x_k - \overline{x}_i\|},$$

y_{k+1} satisfies $\|y_{k+1} - z_{k+1}\| \le \delta_{k+1}$
where $\delta_{k+1} \longrightarrow 0$,

$$x_{k+1} = \gamma_k x_k + (1-\gamma_k) y_k,$$

where $\gamma_k \in [\varepsilon, 1]$ such that $x_{k+1} \neq \overline{x}_i$ for $\forall$ i.

Then, $\{x_k\}$ converges to x_*.

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2. Let $A \in \mathbb{R}^{n \times n}$ be positive symmetric. A norm $\|\cdot\|_A$ can be defined on $\mathbb{R}^n$

$$(2.1) \qquad \|x\|_A := \langle Ax, x \rangle^{1/2} \quad \forall x \in \mathbb{R}^n.$$

Suppose that C is a closed convex subset of $\mathbb{R}^n$ and $b \in \mathbb{R}^n$. Now consider the following linear variational inequality problem : find an $x_* \in C$ such that

(2.2) $\quad <x - x_*, Ax_* - b> \geq 0 \qquad$ for $\forall\, x \in C$.

An interesting fact is that the solution x_* of (2.2) is the projection of the solution of the equation $Ax = b$ to C in the norm $\|\cdot\|_A$, i.e.

$$x_* = \text{solution} \min_{x \in C} \|x - A^{-1}b\|_A.$$

3. Let $G \in R^{n\times n}$ be symmetric, C be a closed convex subset of R^n and $b \in R^n$. Denote $f(x) = Gx - b$. The linear variational inequality problem VI(C,f) is to find an $x_* \in C$ such that

(3.1) $\quad <x - x_*, Gx_* - b> \geq 0 \qquad$ for $\forall\, x \in C$.

Many iterations have been designed for solving VI(R^n_+,f) [1] and as far as we are aware there are little numerical methods for solving VI(C,f) for an arbitrary closed convex subset C. We propose the following iteration which solves VI(C,f).

Step 1 $\quad x_0 \in C$;

Step 2 $\quad y_{k+1}$ is the solution of VI(C,f_k),
where $f_k(x) = f(x_k) + D_k(x-x_k) + Q_k(y_{k+1} - x_k)$,
and D_k and Q_k are arbitrary matrices;

Step 3 $\quad x_{k+1} = (1 - \alpha_k)x_k + \alpha_k y_{k+1}$,
where $\alpha_k \in [\alpha, 1]$ and $\alpha > 0$.

Under certain conditions, $\{x_k\}$ converges to a solution of VI(C,f).

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4. Consider the problem of linear quadratic optimal control with stability related to the parameterized finite dimensional linear time-invariant systems $\{A_\lambda, B_\lambda, C_\lambda, D_\lambda\}$:

(4.1) $$J^*_\lambda(x) := \min J_\lambda(x_0) := \{\int_0^{+\infty} \|C_\lambda x + D_\lambda u\|^2 dt : u \in \mathfrak{L}_2[0, \infty) \text{ such that } x(+\infty) = 0\}.$$

If (A_λ, B_λ) is supposed to be stabilizable and (C_λ, A_λ) detectable, then

(4.2) $$\overline{\text{limit}}_{\lambda \to \lambda_0} J^*_\lambda(x_0) \leq J^*_{\lambda_0}(x_0)$$

holds.

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5. For a given system P there are many "indices" to describe it. For example, McMillan degree, number of poles, number of unstable poles, the distance from the set of unstable systems and the minimal quadratic cost (with or without stability) etc.. Now, we defined another index for the system P, that is, its largest robust stability radius w_g^{-1}, which is the largest radius of the ball $K(P,\varepsilon)$ such that $K(P,\varepsilon)$ can be stabilized by one single controller.

6. "Tao" has many meanings in chinese, mainly it means : a) "Taoism" (Taoism and Buddhism are the two dominant religions in China. It is known that Buddhism is imported and Taoism is self–created two thousand years ago); b) "search for excellence" (or "to be excellent") in your career; c) "high moral standards"; d) "to be perfect" (this is "Siquan" in chinese). etc..

Nowadays, many people are still interested in "Tao". But they have paid too much attention to the meaning b) "Search for excellence" and forget about c), especially businessmen.

7. At Pluto : babies are produced in factories, cars are born in families, judges are the prisoners and Parliament consists of thieves, cows milk people and only wolves are allowed to enter McDonald's.