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Abstract

This paper completely solves the discrete time minimum entropy control problem. It is shown that in discrete time the central controller has the additional interpretation as the controller which minimizes a minimum entropy criterion. This is completely analogous to the continuous time. However, although the H_∞ control problems for discrete and continuous time can be connected via the bilinear transform, it is shown that this is not the case for the corresponding minimum entropy control problems and hence the bilinear transform does not connect the central controllers in continuous and discrete time.

Keywords H_∞ control, algebraic Riccati equation, discrete time systems, minimum entropy, bilinear transform.

1 Introduction

For the H_∞ control problem a parametrization was derived of all stabilizing controllers which yielded a closed loop system with H_∞ norm strictly less then some a priori given bound γ. This so-called Q-parametrization centered around a controller which was hence called the central controller. The above was derived for continuous systems [1, 10] and for discrete time systems [2, 5, 10].

For continuous time systems an alternative interpretation to the central controller was given: it was the controller which minimized an entropy criterion (see [6, 8]). The objective of this paper is to show that we can also derive such an interpretation of the central controller in discrete time. Clearly this is not very surprising.

We feel that it is good to have this interpretation because it can be used to derive several properties of the central controllers. In section 5, we for instance show that to design a discrete time controller via a bilinear transform to the continuous time domain might introduce an implicit and undesirable additional weighting function.

The discrete minimum entropy control problem has been studied before in [4] for the special case of a one-block problem and in a much more general setting in [3]. This paper extends these results.

The notation in this paper is quite standard. We will denote by σ the shift operator: $(\sigma x)(k) = Ax(k+1)$. By $\mathcal{D}$ we denote the unit circle. Finally, by $A^\dagger$ we denote the Moore-Penrose inverse of the matrix A and $\rho(A)$ denotes its spectral radius.

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2 Discrete time minimum entropy

Consider the linear time-invariant system:

$$\Sigma : \begin{cases} \sigma x = & Ax + \quad Ew + \quad Bu, \\ z \quad = & C_1 x + \; D_{11} w \quad D_{12} u, \\ y \quad = & C_2 x + D_{21} w, \end{cases} \tag{2.1}$$

Here $A, B, E, C_1, C_2, D_{11}, D_{12}$ and D_{21} are real matrices of suitable dimension. Let G be a proper real-rational matrix which has no poles on the unit circle and which is such that

$$\|G\|_\infty := \sup_{\theta \in [0,2\pi]} \|G\left(e^{i\theta}\right)\| < \gamma.$$

where $\|\cdot\|$ denotes the largest singular value. For such a transfer matrix G, we define the following entropy function:

$$\mathcal{J}(G,\gamma) := -\frac{\gamma^2}{2\pi}\int_0^{2\pi} \ln\det\left[I - \frac{1}{\gamma^2} G^{\sim}\left(e^{i\theta}\right) G\left(e^{i\theta}\right)\right] d\theta \tag{2.2}$$

where $G^{\sim}(s) := G^{\mathrm{T}}(s^{-1})$. The minimum entropy H_∞ control problem for given γ is then defined as:

> infimize $\mathcal{J}(G_{cl},\gamma)$ over all controllers which yield a proper, internally stable closed loop transfer matrix G_{cl} with H_∞ norm strictly less than γ.

We will investigate controllers of the form:

$$\Sigma_F : \begin{cases} \sigma p = & Kp + \; Ly, \\ u \quad = & Mp + Ny. \end{cases} \tag{2.3}$$

In the formulation of our main result we require the concept of *invariant zero* of the system $\Sigma = (A, B, C, D)$. These are all $s \in C$ such that

$$\operatorname{rank}\begin{pmatrix} sI - A & -B \\ C & D \end{pmatrix} < \operatorname{normrank}\begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix}. \tag{2.4}$$

We first formulate the main result from [11]:

Theorem 2.1 : *Consider the system (2.1). Assume that the systems (A, B, C_2, D_{21}) and (A, E, C_1, D_{12}) have no invariant zeros on the unit circle. The following statements are equivalent:*

- *(i) There exists a dynamic compensator Σ_F of the form (2.3) such that the resulting closed loop system is internally stable and the closed loop transfer matrix G_F satisfies $\|G_F\|_\infty < \gamma$.*
- *(ii) There exist symmetric matrices $P \geq 0$ and $Q \geq 0$ such that*

(*a*) *We have*

$$R > 0 \tag{2.5}$$

where

$$\begin{aligned} V &:= B^{\mathrm{T}}PB + D_{12}^{\mathrm{T}}D_{12}, \\ R &:= \gamma^2 I - D_{11}^{\mathrm{T}}D_{11} - E^{\mathrm{T}}PE + \left(E^{\mathrm{T}}PB + D_{11}^{\mathrm{T}}D_{12}\right) V^{\dagger} \left(B^{\mathrm{T}}PE + D_{12}^{\mathrm{T}}D_{11}\right). \end{aligned}$$

(*b*) *P satisfies the discrete algebraic Riccati equation:*

$$P = A^{\mathrm{T}}PA + C_1^{\mathrm{T}}C_1 - \begin{pmatrix} E^{\mathrm{T}}PA + D_{11}^{\mathrm{T}}C_1 \\ B^{\mathrm{T}}PA + D_{12}^{\mathrm{T}}C_1 \end{pmatrix}^{\mathrm{T}} G(P)^{\dagger} \begin{pmatrix} E^{\mathrm{T}}PA + D_{11}^{\mathrm{T}}C_1 \\ B^{\mathrm{T}}PA + D_{12}^{\mathrm{T}}C_1 \end{pmatrix}. \tag{2.6}$$

where

$$G(P) := \begin{pmatrix} D_{11}^{\mathrm{T}}D_{11} - \gamma^2 I & D_{11}^{\mathrm{T}}D_{12} \\ D_{12}^{\mathrm{T}}D_{11} & D_{12}^{\mathrm{T}}D_{12} \end{pmatrix} + \begin{pmatrix} E^{\mathrm{T}} \\ B^{\mathrm{T}} \end{pmatrix} P \begin{pmatrix} E & B \end{pmatrix}. \tag{2.7}$$

(*c*) *For all* $z \in C$ *with* $|z| \geq 1$, *we have*

$$\operatorname{rank}_{\mathcal{R}} \begin{pmatrix} zI - A & -E & -B \\ E^{\mathrm{T}}PA + D_{11}^{\mathrm{T}}C_1 & E^{\mathrm{T}}PE + D_{11}^{\mathrm{T}}D_{11} - I & E^{\mathrm{T}}PB + D_{11}^{\mathrm{T}}D_{12} \\ B^{\mathrm{T}}PA + D_{12}^{\mathrm{T}}C_1 & B^{\mathrm{T}}PE + D_{12}^{\mathrm{T}}D_{11} & B^{\mathrm{T}}PB + D_{21}^{\mathrm{T}}D_{12} \end{pmatrix}$$
$$= n + q + \operatorname{rank}_{\mathcal{R}(z)} C_1(zI - A)^{-1}B + D_{12}$$

(*d*) *We have*

$$S > 0, \tag{2.8}$$

where

$$\begin{aligned} W &:= D_{21}D_{21}^{\mathrm{T}} + C_2QC_2^{\mathrm{T}}, \\ S &:= \gamma^2 I - D_{11}D_{11}^{\mathrm{T}} - C_1QC_1^{\mathrm{T}} + \left(C_1QC_2^{\mathrm{T}} + D_{11}D_{21}^{\mathrm{T}}\right) W^{\dagger} \left(C_2QC_1^{\mathrm{T}} + D_{21}D_{11}^{\mathrm{T}}\right). \end{aligned}$$

(*e*) *Q satisfies the following discrete algebraic Riccati equation:*

$$Q = AQA^{\mathrm{T}} + EE^{\mathrm{T}} - \begin{pmatrix} C_1QA^{\mathrm{T}} + D_{11}E^{\mathrm{T}} \\ C_2QA^{\mathrm{T}} + D_{21}E^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} H(Q)^{\dagger} \begin{pmatrix} C_1QA^{\mathrm{T}} + D_{11}E^{\mathrm{T}} \\ C_2QA^{\mathrm{T}} + D_{21}E^{\mathrm{T}} \end{pmatrix}. \tag{2.9}$$

where

$$H(Y) := \begin{pmatrix} D_{11}D_{11}^{\mathrm{T}} - \gamma^2 I & D_{11}D_{21}^{\mathrm{T}} \\ D_{21}D_{11}^{\mathrm{T}} & D_{21}D_{21}^{\mathrm{T}} \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} Q \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}^{\mathrm{T}}. \tag{2.10}$$

(*f*) *For all* $z \in C$ *with* $|z| \geq 1$, *we have*

$$\operatorname{rank}_{\mathcal{R}} \begin{pmatrix} zI - A & AQC_1^{\mathrm{T}} + ED_{11}^{\mathrm{T}} & AQC_2^{\mathrm{T}} + ED_{21}^{\mathrm{T}} \\ -C_1 & C_1QC_1^{\mathrm{T}} + D_{11}D_{11}^{\mathrm{T}} - I & C_1QC_2^{\mathrm{T}} + D_{11}D_{21}^{\mathrm{T}} \\ -C_2 & C_2QC_1^{\mathrm{T}} + D_{21}D_{11}^{\mathrm{T}} & C_2QC_2^{\mathrm{T}} + D_{21}D_{21}^{\mathrm{T}} \end{pmatrix}$$
$$= n + q + \operatorname{rank}_{\mathcal{R}(z)} C_2(zI - A)^{-1}E + D_{21}$$

(*g*) $\rho(PQ) < \gamma^2$. □

The Riccati equations appearing in the above conditions do not have the classical form due to the Moore-Penrose inverse appearing in the equation. However, it wass shown in [11] that these more general Riccati equations can be connected to classical Riccati equations of lower dimension.
Note that the existence of such P and Q guarantees that the system Σ is detectable from y and stabilizable by u. To present our main result we need one preliminary lemma:

Lemma 2.2 : *Suppose there exist matrices* P *and* Q *such that part (ii) of theorem 2.1 is satisfied. Then there exist matrices* F_0 *and* K_0 *such that the matrices*

$$\begin{aligned} A_{st} &:= A_P - B\left[D_{12,P}^{\dagger}C_{1,P} - (I - V^{\dagger}V)F_0\right] \\ A_{fi} &:= A_P - \left[(A_PYC_{2,P}^{\mathrm{T}} + E_PD_{21,P}^{\mathrm{T}})W_Y^{\dagger} - K_0(I - W_YW_Y^{\dagger})\right]C_{2,P} \end{aligned}$$

are both asymptotically stable. In the above,

$$\begin{aligned} Z &:= E^{\mathrm{T}}PA + D_{11}^{\mathrm{T}}C_1 - [E^{\mathrm{T}}PB + D_{11}^{\mathrm{T}}D_{12}]\,V^{\dagger}\,[B^{\mathrm{T}}PA + D_{12}^{\mathrm{T}}C_1], \\ A_P &:= A + ER^{-1}Z, \\ E_P &:= ER^{-1/2}, \\ C_{1,P} &:= D_{12,P}^{\dagger}(B^{\mathrm{T}}PA + D_{12}^{\mathrm{T}}C_1) + D_{12,P}^{\dagger}[B^{\mathrm{T}}PE + D_{12}^{\mathrm{T}}D_{11}]\,R^{-1}Z, \\ D_{11,P} &:= D_{12,P}^{\dagger}(B^{\mathrm{T}}PE + D_{12}^{\mathrm{T}}D_{11})\,R^{-1/2}, \\ D_{12,P} &:= V^{1/2}, \\ C_{2,P} &:= C_2 + D_{21}R^{-1}Z, \\ D_{21,P} &:= D_{21}R^{-1/2}, \\ Y &:= (\gamma^2 I - QP)^{-1}Q, \\ W_Y &:= D_{21,P}D_{21,P}^{\mathrm{T}} + C_{2,P}YC_{2,P}^{\mathrm{T}}, \\ S_Y &:= I - D_{11,P}D_{11,P}^{\mathrm{T}} - C_{1,P}YC_{1,P}^{\mathrm{T}} + \\ &\qquad (C_{1,P}YC_{2,P}^{\mathrm{T}} + D_{11,P}D_{21,P}^{\mathrm{T}})\,W_Y^{\dagger}\,(C_{2,P}YC_{1,P}^{\mathrm{T}} + D_{21,P}D_{11,P}^{\mathrm{T}})\,. \end{aligned}$$

□

We can now present our main result:

Theorem 2.3 : *Consider the system (2.1). Let $\gamma > 0$ be given. Assume that the systems (A, B, C_1, D_1) and (A, E, C_2, D_2) have no invariant zeros on the imaginary axis and assume that there exists a controller which is such that the closed loop system is internally stable and has H_∞ norm strictly less than γ. The infimum of (2.2), over all internally stabilizing controllers of the form (2.3) which are such that the closed system has H_∞ norm strictly less than γ, is equal to:*

$$\gamma^2 \ln \det \left(\frac{R}{\gamma^2} \right) + \gamma^2 \ln \det S_Y.$$

Let F_0 and K_0 be chosen according to lemma 2.2. The infimum is attained by the controller Σ_F, described by (2.3) where:

$$\begin{aligned}
N &:= -D_{12,P}^{\dagger} \left(C_{1,P} Y C_{2,P}^{\mathrm{T}} + D_{11,P} D_{21,P}^{\mathrm{T}} \right) W_Y^{\dagger}, \\
M &:= -(D_{12,P}^{\dagger} C_{1,P} + N C_{2,P}) + (I - V^{\dagger} V) F_0, \\
L &:= BN + \left(A_P Y C_{2,P}^{\mathrm{T}} + E_P D_{21,P}^{\mathrm{T}} \right) W_Y^{\dagger} - K_0 (I - W_Y W_Y^{\dagger}), \\
K &:= A_P - L C_{2,P} - B[D_{12,P}^{\dagger} C_{1,P} - (I - V^{\dagger} V) F_0].
\end{aligned}$$

□

Remarks:

(*i*) In the system (2.1) we have one direct feedthrough matrix which is identical to zero. It is straightforward to extend the above result to the more general case with all direct feedthrough matrices possibly unequal to zero. (see [7, 10]). The problem is that the infimum might not be attained in this case.

(*ii*) One can also derive that the infimum over all strictly proper compensators is equal to:

$$\gamma^2 \ln \det \left(\frac{R}{\gamma^2} \right) + \gamma^2 \ln \det \left(I - D_{1,P} D_{1,P}^{\mathrm{T}} - C_{1,P} Y C_{1,P}^{\mathrm{T}} \right).$$

(*iii*) We will only prove this result for $\gamma = 1$. The general result can then easily be derived by scaling.

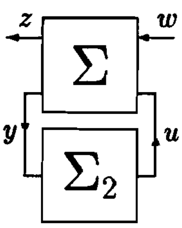
3 Properties of the entropy function

In this section we recall some basic properties of the entropy function. These properties are the discrete-time equivalent of the properties derived in [6, 8].
We first define the property of being inner.

Definition 3.1 : *A proper rational transfer matrix G is called inner if G is a stable rational matrix such that $G^{\sim} G = I$. A system Σ is called inner if the system is internally stable and its transfer matrix is inner.* □

Next, we give two key lemmas. Of the first lemma, the first part stems from [10] while the second part is a discrete-time version of a result in [6, 8].

Lemma 3.2 : *Suppose that two systems Σ and Σ_2, both described by some state space representation, are interconnected in the following way:*



(3.1)

Assume that the system Σ is inner. Moreover, assume that its transfer matrix G is square and has the following decomposition:

$$G\begin{pmatrix} w \\ u \end{pmatrix} =: \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} z \\ y \end{pmatrix} \tag{3.2}$$

such that $G_{21}^{-1} \in H_\infty$ and such that G_{22} is strictly proper. Under the above assumptions the following two statements are equivalent:

(*i*) *The closed loop system (3.1) is internally stable and its closed loop transfer matrix G_{cl} has H_∞ norm less than 1.*

(*ii*) *The system Σ_2 is internally stable and its transfer matrix G_2 has H_∞ norm less than 1.*

Finally if (i), or equivalently (ii), holds, then the following relation between the entropy functions for the different transfer matrices is satisfied:

$$\mathcal{J}(G_{cl}, 1) = \mathcal{J}(G_{11}, 1) + \mathcal{J}(G_2, 1). \tag{3.3}$$

□

Proof : The first claim that the statements (i) and (ii) are equivalent, has been shown in [10]. Remains to show (3.3). The following equality is easily derived using the property that Σ is inner:

$$I - G_{cl}^{\sim}G_{cl} = G_{21}^{\sim}\left(I - G_2^{\sim}G_{22}^{\sim}\right)^{-1}\left(I - G_2^{\sim}G_2\right)\left(I - G_{22}G_2\right)^{-1}G_{21}$$

Therefore, we find that

$$\begin{aligned} \ln\det\left(I - G_{cl}^{\sim}G_{cl}\right) &= \ln\det\left(I - G_{11}^{\sim}G_{11}\right) + \ln\det\left(I - G_2^{\sim}G_2\right) \\ &\quad - \ln\det\left(I - G_{22}G_2\right) - \ln\det\left(I - G_2^{\sim}G_{22}^{\sim}\right) \end{aligned} \tag{3.4}$$

Moreover if statement (i), or equivalently statement (ii), is satisfied then we have

$$\mathcal{J}(G_2,1) = -\frac{1}{2\pi}\int_0^{2\pi} \ln\det\left[I - G_2^{\sim}\left(e^{i\theta}\right) G_2\left(e^{i\theta}\right)\right] d\theta, \tag{3.5}$$

$$\mathcal{J}(G_{11},1) = -\frac{1}{2\pi}\int_0^{2\pi} \ln\det\left[I - G_{11}^{\sim}\left(e^{i\theta}\right) G_{11}\left(e^{i\theta}\right)\right] d\theta. \tag{3.6}$$

It is easily checked that:

$$\int_0^{2\pi} \ln\det\left[I - G_2\left(e^{i\theta}\right) G_{22}\left(e^{i\theta}\right)\right] d\theta = \int_0^{2\pi} \ln\det\left[I - G_{22}^{\sim}\left(e^{i\theta}\right) G_2^{\sim}\left(e^{i\theta}\right)\right] d\theta \tag{3.7}$$

We know that G_2G_{22} is stable and has H_∞ norm strictly less than 1. Therefore the function $\ln\det(I - G_{22}^{\sim}G_2^{\sim})$ is analytic on the unit ball which implies that (using Cauchy's theorem):

$$\int_{\mathcal{D}} \frac{1}{z}\ln\det\left(I - G_{22}^{\sim}(z)G_2^{\sim}(z)\right) dz = \ln\det\left(I - G_{22}^{\sim}(0)G_2^{\sim}(0)\right) = 0$$

The last equality follows because G_{22} is strictly proper. Combining the above, we find (3.3).

■

The following lemma is an essential tool for actually calculating the entropy function of some specific system:

Lemma 3.3 : *Let G be a rational matrix which has a detectable and stabilizable realization (A,B,C,D). Assume that $G, G^{-1} \in H_\infty$. Then we have:*

$$\int_0^{2\pi} \ln|\det G\left(e^{i\theta}\right)|\, d\theta = 2\pi \ln\det D \tag{3.8}$$

□

Proof : We know that $\ln|p| = \mathrm{Re}\, p$. Therefore, we have

$$\int_0^{2\pi} \ln|\det G\left(e^{i\theta}\right)|\, d\theta = \mathrm{Re}\; i \int_{\mathcal{D}} \frac{1}{z}\ln\det G^{\sim}(z) dz$$

We know that $\ln\det G^{\sim}$ is analytic on the unit disc. Hence the integrand on the right-hand side has only one pole in 0. Hence using Cauchy's theorem we find (3.8). ■

Lemma 3.4 : *Let G be a stable transfer matrix with H_∞ norm strictly less than γ and with stabilizable and detectable realization (A,B,C,D). Then we have:*

$$\mathcal{J}(G,\gamma) = -\gamma^2 \ln\det\left(I - \frac{1}{\gamma^2}(B^{\mathrm{T}}XB + D^{\mathrm{T}}D)\right) \tag{3.9}$$

where X is the unique solution of the algebraic Riccati equation:

$$X = A^{\mathrm{T}}XA + C^{\mathrm{T}}C + (A^{\mathrm{T}}XB + C^{\mathrm{T}}D)N^{-1}(B^{\mathrm{T}}XA + D^{\mathrm{T}}C)$$

such that $N := \gamma^2 I - B^{\mathrm{T}}XB - D^{\mathrm{T}}D > 0$ and $A + \gamma^{-2}BB^{\mathrm{T}}X$ is asymptotically stable. □

Proof : The existence and uniqueness of X is a well-known result (see e.g. [10]). It is easily checked that the transfer matrix M with realization

$$(A, B, -N^{-1/2}(B^{\mathrm{T}}PA + D^{\mathrm{T}}C), \gamma^{-1}N^{1/2})$$

satisfies:

$$I - \gamma^{-2}G^{\sim}G = M^{\sim}M$$

Moreover, $M, M^{-1} \in H_{\infty}$, i.e. M is a spectral factor of $I - \gamma^{-2}G^{\sim}G$. We have

$$\mathcal{J}(G,\gamma) = \frac{-\gamma^2}{\pi}\int_0^{2\pi} \ln|\det M\left(e^{i}\theta\right)|\, d\theta$$

and therefore (3.9) is a direct consequence of applying lemma 3.3 to the above equation. ■

Note that lemma 3.4 can be used to show that the entropy function is an upper bound for the H_2 norm:

Corollary 3.5 *:Let G be a stable transfer matrix. For all γ_1 larger than the H_∞ norm of G we have*

$$\mathcal{J}(G,\gamma_1) > \|G\|_2 = \lim_{\gamma\to\infty} \mathcal{J}(G,\gamma).$$

where

$$\|G\|_2 := \frac{1}{2\pi}\int_0^{2\pi} G^{\sim}\left(e^{i}\theta\right) G\left(e^{i}\theta\right)\, d\theta$$

□

Proof : We have $\|G\| = \text{Trace } B^{\mathrm{T}}PB$ where P is defined by

$$P = A^{\mathrm{T}}PA + C^{\mathrm{T}}C.$$

It is easily seen that X, as defined in lemma 3.4, satisfies $X \geq P$ and $\lim_{\gamma\to\infty} X = P$. Moreover,

$$\ln\det(I-S) > \det S, \qquad \lim_{S\to 0}\frac{\ln\det(I-S)}{\det S} = 1.$$

The result then follows by applying the above two properties to:

$$S := \frac{1}{\gamma^2}(B^{\mathrm{T}}XB + D^{\mathrm{T}}D)$$

■

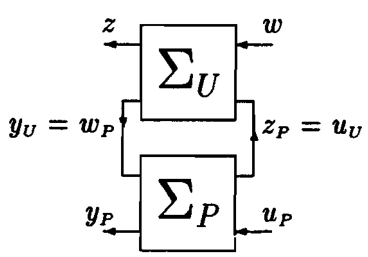
4 A system transformation

Throughout this section we assume that $\gamma = 1$ and that there exist matrices P and Q satisfying the conditions in theorem 2.1 for $\gamma = 1$. Note that this is no restriction when proving theorem 2.3. The assumption $\gamma = 1$ can be easily removed by scaling while the existence of such P and Q is implied by our assumption that there exists an internally stabilizing controller which makes the H_∞ norm strictly less than 1. We use a technique from [10, 9] of transforming the system twice such that the problem of minimizing the entropy function for the original system is equivalent to minimizing the entropy function for the new system we thus obtain. We will show that this new system satisfies some desirable properties which enables us to solve the minimum entropy H_∞ control problem for this new system and hence also for the original system.
We define the following system:

$$\Sigma_P : \begin{cases} \sigma x_P = A_P x_P + E_P w_P + B u_P, \\ z_P = C_{1,P} x_P + D_{11,P} w_P + D_{12,P} u_P, \\ y_P = C_{2,P} x_P + D_{21,P} w_P, \end{cases} \tag{4.1}$$

where the above matrices are defined in theorem 2.3 for $\gamma = 1$.
The following lemma connects Σ_P with the original system Σ and presents some of the properties of the connecting system.

Lemma 4.1 : *Let Σ and Σ_P be defined by (2.1) and (4.1), respectively. For any system Σ_U of suitable dimensions consider the following interconnection:*



(4.2)

and decompose the transfer matrix U of Σ_U as follows:

$$U \begin{pmatrix} w \\ u_U \end{pmatrix} =: \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} w \\ u_U \end{pmatrix} = \begin{pmatrix} z \\ y_U \end{pmatrix},$$

compatible with the sizes of u_U, w, y_U, and z. Then the following holds: there exists a system Σ_U of suitable dimensions such that:

(*i*) *The system Σ_U is inner*

(*ii*) *The transfer matrix U_{21}^{-1} is well-defined and stable*

(*iii*) $\mathcal{J}(U_{11}, 1) = \ln \det R$.

(*iv*) *The system Σ and the interconnection in (4.2) have the same transfer matrix.*

(v) The interconnection in (4.2) is detectable from y_P and stabilizable by u_P. □

where R as defined in theorem (2.3) for $\gamma = 1$.

Proof : All of these properties except part (iii) have been derived in [11] where Σ_U has been explicitly constructed. Note that U_{21} is a spectral factor for $I - U_{11}^{\sim} U_{11}$ which yields (iii) by using the state space realization for U_{21} given in [11] and by applying lemma 3.3. ■

Remark: In case the system (A, B, C_2, D_{21}) is left-invertible we have that V is invertible and we can construct Σ_U directly:

$$\Sigma_U : \begin{cases} \sigma x_U = A_U x_U + E_U w + B_U u_U, \\ z_U = C_{1,U} x_U + D_{11,U} w + D_{12,U} u_U, \\ y_U = C_{2,U} x_U + D_{21,U} w \end{cases} \tag{4.3}$$

where

$$\begin{aligned} A_U &:= A - BV^{-1}(B^{\mathrm{T}}PA + D_{12}^{\mathrm{T}}C_1), \\ E_U &:= E - BV^{-1}(B^{\mathrm{T}}PE + D_{12}^{\mathrm{T}}D_{11}), \\ B_U &:= BV^{-1/2}, \\ C_{1,U} &:= C_1 - D_{12}V^{-1}(B^{\mathrm{T}}PA + D_{12}^{\mathrm{T}}C_1), \\ D_{11,U} &:= D_{11} - D_{12}V^{-1}(B^{\mathrm{T}}PE + D_{12}^{\mathrm{T}}D_{11}), \\ D_{12,U} &:= D_{12}V^{-1/2}, \\ C_{2,U} &:= -R^{-1/2}Z, \\ D_{21,U} &:= R^{1/2}, \end{aligned}$$

and V, R and Z are as defined in theorem 2.3 with $\gamma = 1$.
Combining lemmas 3.2 and 4.1, we find the following theorem:

Theorem 4.2 : *Let the systems (2.1) and (4.1) be given. Moreover, let a compensator Σ_F of the form (2.3) be given. The following two conditions are equivalent:*

- *Σ_F is internally stabilizing for Σ such that the closed loop transfer matrix G_{cl} has H_∞ norm strictly less than 1.*
- *Σ_F is internally stabilizing for Σ_P such that the closed loop transfer matrix $G_{cl,P}$ has H_∞ norm strictly less than 1.*

Moreover, if Σ_F satisfies the above conditions then we have

$$\mathcal{J}(G_{cl}, 1) = \mathcal{J}(G_{cl,P}, 1) + \ln \det R.$$ □

Next, we make another transformation from Σ_P to $\Sigma_{P,Q}$. This transformation is exactly dual to the transformation from Σ to Σ_P. We know there exists a controller which is internally stabilizing for Σ_P which makes the H_∞ norm of the closed loop system strictly less than 1. Therefore if we apply theorem 2.1 to Σ_P we find that that there exists a unique matrix Y such that

(i) We have

$$S > 0, \tag{4.4}$$

where

$$\begin{aligned} W_Y &:= D_{21,P} D_{21,P}^{\mathrm{T}} + C_{2,P} Y C_{2,P}^{\mathrm{T}}, \\ S_Y &:= I - D_{11,P} D_{11,P}^{\mathrm{T}} - C_{1,P} Y C_{1,P}^{\mathrm{T}} + \\ &\quad \left(C_{1,P} Y C_{2,P}^{\mathrm{T}} + D_{11,P} D_{21,P}^{\mathrm{T}}\right) W_Y^{\dagger} \left(C_{2,P} Y C_{1,P}^{\mathrm{T}} + D_{21,P} D_{11,P}^{\mathrm{T}}\right). \end{aligned}$$

(ii) Y satisfies the following discrete algebraic Riccati equation:

$$Y = A_P Y A_P^{\mathrm{T}} + E_P E_P^{\mathrm{T}} - \begin{pmatrix} C_{1,P} Y A_P^{\mathrm{T}} + D_{11,P} E_P^{\mathrm{T}} \\ C_{2,P} Y A_P^{\mathrm{T}} + D_{21,P} E_P^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}} H_P(Y)^{\dagger} \begin{pmatrix} C_{1,P} Y A_P^{\mathrm{T}} + D_{11,P} E^{\mathrm{T}} \\ C_{2,P} Y A_P^{\mathrm{T}} + D_{21,P} E_P^{\mathrm{T}} \end{pmatrix}.$$

where

$$H_P(Y) := \begin{pmatrix} D_{11,P} D_{11,P}^{\mathrm{T}} - I & D_{11,P} D_{21,P}^{\mathrm{T}} \\ D_{21,P} D_{11,P}^{\mathrm{T}} & D_{21,P} D_{21,P}^{\mathrm{T}} \end{pmatrix} + \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix} Y \begin{pmatrix} C_{1,P} \\ C_{2,P} \end{pmatrix}^{\mathrm{T}}. \tag{4.5}$$

(iii) For all $z \in \mathcal{C}$ with $|z| \geq 1$, we have

$$\operatorname{rank}_{\mathcal{R}} \begin{pmatrix} zI - A & C_{1,P} Y A_P^{\mathrm{T}} + D_{11,P} E_P^{\mathrm{T}} & C_{2,P} Y A_P^{\mathrm{T}} + D_{21,P} E_P^{\mathrm{T}} \\ -C_{1,P} & C_{1,P} Y C_{1,P}^{\mathrm{T}} + D_{11,P} D_{11,P}^{\mathrm{T}} - I & C_{1,P} Y C_{2,P}^{\mathrm{T}} + D_{11,P} D_{21,P}^{\mathrm{T}} \\ -C_{2,P} & C_{2,P} Y C_{1,P}^{\mathrm{T}} + D_{21,P} D_{11,P}^{\mathrm{T}} & C_{2,P} Y C_{2,P}^{\mathrm{T}} + D_{21,P} D_{21,P}^{\mathrm{T}} \end{pmatrix}$$

$$= n + q + \operatorname{rank}_{\mathcal{R}(z)} C_1 (zI - A)^{-1} E + D_{12}$$

It can be shown that $Y := (I - QP)^{-1} Q$ satisfies the above conditions.
We define the following system:

$$\Sigma_{P,Q} : \begin{cases} \sigma x_{P,Q} = A_{P,Q} x_{P,Q} + E_{P,Q} w + B_{P,Q} u_{P,Q}, \\ z_{P,Q} = C_{1,P,Y} x_{P,Q} + D_{11,P,Y} w + D_{12,P,Y} u_{P,Q}, \\ y_{P,Q} = C_{2,P} x_{P,Q} + D_{21,P,Y} w, \end{cases} \tag{4.6}$$

where

$$\begin{aligned} \tilde{Z} &:= A_P Y C_{1,P}^{\mathrm{T}} + E_P D_{11,P}^{\mathrm{T}} \\ &\qquad - \left(A_P Y C_{2,P}^{\mathrm{T}} + E_P D_{21,P}^{\mathrm{T}}\right) W_Y^{\dagger} \left(C_{2,P} Y C_{1,P}^{\mathrm{T}} + D_{21,P} D_{11,P}^{\mathrm{T}}\right), \\ A_{P,Q} &:= A_P + \tilde{Z} S_Y^{-1} C_{1,P}, \\ B_{P,Q} &:= B + \tilde{Z} S_Y^{-1} D_{12,P}, \end{aligned}$$

$$
\begin{aligned}
E_{P,Q} &:= \left(A_P Y C_{2,P}^{\mathrm{T}} + E_P D_{21,P}^{\mathrm{T}}\right) D_{21,P,Y}^{\dagger} \\
&\quad + \tilde{Z} S_Y^{-1} \left(C_{1,P} Y C_{2,P}^{\mathrm{T}} + D_{11,P} D_{21,P}^{\mathrm{T}}\right) D_{21,P,Y}^{\dagger}, \\
C_{1,P,Y} &:= S_Y^{-1/2} C_{1,P}, \\
D_{21,P,Y} &:= W_Y^{1/2}, \\
D_{12,P,Y} &:= S_Y^{-1/2} D_{12,P}, \\
D_{11,P,Y} &:= S_Y^{-1/2} \left(C_{1,P} Y C_{2,P}^{\mathrm{T}} + D_{11,P} D_{21,P}^{\mathrm{T}}\right) D_{21,P,Y}^{\dagger},
\end{aligned}
$$

Using theorem 4.2 and a dualized version for the transformation from Σ_P to $\Sigma_{P,Q}$ we can derive the following corollary:

Corollary 4.3 : *Let the systems (2.1) and (4.6) be given. Moreover, let a compensator Σ_F of the form (2.3) be given. The following two conditions are equivalent:*

- *Σ_F is internally stabilizing for Σ such that the closed loop transfer matrix G_{cl} has H_∞ norm strictly less than* 1.
- *Σ_F is internally stabilizing for $\Sigma_{P,Q}$ such that the closed loop transfer matrix $G_{cl,P,Q}$ has H_∞ norm strictly less than* 1.

Moreover, if Σ_F satisfies the above conditions then we have

$$\mathcal{J}(G_{cl}, 1) = \mathcal{J}(G_{cl,P,Q}, 1) + \ln\det R + \ln\det S_Y. \qquad \square$$

From this corollary it is immediate that it is sufficient to investigate $\Sigma_{P,Q}$ to prove the results in our main theorem 2.3. Of course, we would like to know what we gain of our transformation from Σ to $\Sigma_{P,Q}$. This is immediate from the following lemma:

Lemma 4.4 : *Let Σ_F be the compensator described in theorem 2.3. The interconnection of Σ_F and $\Sigma_{P,Y}$ is internally stable and the closed-loop transfer matrix from $w_{P,Y}$ to $z_{P,Y}$ is zero.* $\square$

Combining corollary 4.3 and lemma 4.4 we find that the controller in lemma 4.4 minimizes the entropy over all stabilizing controllers and the minimal achievable entropy is equal to

$$\ln\det R + \ln\det S_Y.$$

The proof of theorem 2.3 is completed by noting that the controller in lemma 4.4 is equal to the controller given in theorem 2.3.

5 Bilinear transform

We basically can derive two different characterizations of all suboptimal controllers for the H_∞ control problem. The characterization as given in [2, 10] for discrete time systems or to apply the bilinear transform. The bilinear transform leaves stability and the H_∞ norm invariant. Hence if we transform our discrete time system to the continuous time via the bilinear transform, then obtain the parametrization for all suboptimal controllers for continuous time systems and finally via the inverse bilinear transform we obtain a parametrization of all suboptimal controllers for our original discrete time system.

We will compare the direct (discrete) derivation with the indirect (continuous) derivation. Clearly, in both cases we obtain a parametrization of all controllers but the central controller obtained via this two techniques is different. The central controller obtained via the discrete design will be a minimum entropy controller. But via the bilinear transform it is not associated with the (standard) continuous time entropy criterion. Suppose we apply the following bilinear transform:

$$z \to \frac{1+s}{1-s}, \qquad s \to \frac{z-1}{z+1}$$

where s denotes the continuous domain while z denotes the discrete domain. If we transform our discrete time plant via the above bilinear transform into a continuous time system, determine the central controller (for an H_∞ norm bound of γ, e.g. using the formulas of [1]), and apply the bilinear transform on the resulting controller then we obtain a discrete time controller which stabilizes the original discrete time system and satisfies the H_∞ norm bound γ. This controller minimizes, over all stabilizing discrete time controllers which yield an H_∞ norm bound less than γ, the following criterion:

$$\mathcal{J}_c(G,1) := -\frac{\gamma^2}{2\pi}\int_0^{2\pi} \frac{2e^{i\theta}}{(e^{i\theta}+1)^2} \ln\det\left[I - \frac{1}{\gamma^2}G^{\sim}\left(e^{i\theta}\right)G\left(e^{i\theta}\right)\right] d\theta$$

If we compare this criterion with (2.2) then we note that an additional weighting function. Because of this additional weighting function the central controller obtained via the bilinear transform will in general put more emphasis on high-frequency behaviour and often result in a worse design unless one compensates for this additional weighting. We feel that this is an additional argument in favour of a direct discrete design compared to a continuous design via the bilinear transform. It is not difficult to check that (2.2) is, via the bilinear transform, related to what in [6] is called entropy at 1. Hence a design via the bilinear transform is possible without creating aditional weighting functions but one should be careful.

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