# Analysing multiprogramming queues by generating functions

TECHNISCHE UNIVERSITEIT EINDHOVEN
Faculteit Wiskunde en Informatica

Memorandum COSOR 91-25

# Analysing Multiprogramming Queues By Generating Functions

I.J.B.F. Adan
J. Wessels
W.H.M. Zijm

Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven
The Netherlands

Eindhoven, October 1991
The Netherlands

# Analysing Multiprogramming Queues By Generating Functions

I.J.B.F. Adan*
J. Wessels†
W.H.M. Zijm‡

**Abstract**

The generating function approach for analysing queueing systems has a longstanding tradition. One of the highlights is the seminal paper by Kingman on the shortest queue problem, where the author shows that the equilibrium probabilities $p_{m,n}$ of the queue lengths can be written as an infinite sum of products of powers. The same approach is used by Hofri to prove that for a multiprogramming model with two queues the boundary probability $p_{0,n}$ can be expressed as an infinite sum of powers. The present paper shows that the latter representation does not always hold, which implies that the multiprogramming problem is essentially more complicated than the shortest queue problem. However, it appears that the generating function approach is very well suited to show when such a representation is available and when not.

## 1 Introduction

There is a long tradition of using generating functions for analysing exponential queueing models. A seminal paper in this area is Kingman's paper [6] on the shortest queue model, in which the author shows that the generating function for the equilibrium probabilities $p_{m,n}$ for the queue lengths is meromorphic. This implies that a partial fraction decomposition is possible which shows that the equilibrium probabilities can be expressed as countable sum of products of powers. Kingman gives the first term of this expansion explicitly and Flatto and McKean [4] give the second. Hofri [5] uses the same approach for a multiprogramming problem with essentially two queues. The equilibrium equations for the multiprogramming problem are quite similar to the equilibrium equations for the shortest queue problem and therefore it seems likely that the same approach works. By concentrating on the boundary

*Eindhoven University of Technology, Department of Mathematics and Computer Science, p.o. Box 513, 5600 MB - Eindhoven, The Netherlands.
†International Institute for Applied Systems Analysis, A-2361 Laxenburg, Austria.
‡University of Twente, Department of Mechanical Engineering, p.o. Box 217, 7500 AE - Enschede, The Netherlands.

probabilities $p_{0,n}$ rather than on the general probabilities, Hofri is able to get the expansion in a more explicit form.

The aim of the present paper is to show that the multiprogramming problem is less similar to the shortest queue problem than it looks at first sight. It appears that the representation of $p_{0,n}$ as an infinite sum of powers does not necessarily hold for every $n$, but only from some $n$ onwards. It also appears that the generating function approach is a good tool to handle this complication.

The complication, mentioned above, stems from a feature, which does not occur in the shortest queue problem, at least not in the symmetric version as treated by Kingman. Recently, another approach has been developed for the shortest queue problem, leading to more explicit representations of the equilibrium probabilities (see [1]). The extension of this new approach to the asymmetric shortest queue problem (see [2]) encounters a similar complication as the one overlooked by Hofri. In fact, also for this new approach, the analogy can be exploited for the analysis of the multiprogramming system (see [3]).

The paper is organised as follows. In section 2 the model is introduced together with a sketch of the analysis of the relevant generating function. Section 3 is devoted to the partial fraction decomposition of this generating function and hence to the conditions for the representation of the boundary probabilities $p_{0,n}$ as countable linear combination of powers. Section 4 contains some concluding remarks.

## 2 The multiprogramming model and its analysis

The multiprogramming system as introduced by Hofri in [5] has the following queueing properties (compare fig. 1). In the queueing model it is supposed that queue III of incoming

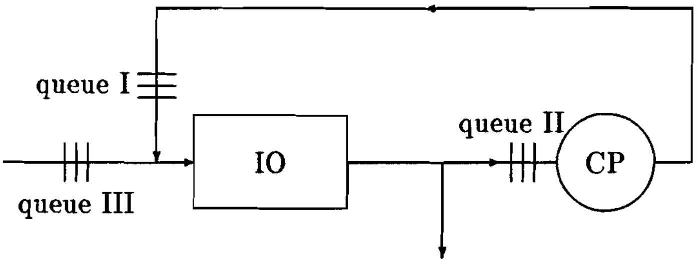


Figure 1: Queueing model for a multiprogramming system

jobs provides an infinite source of ever available jobs. The multiprogramming system consists of an input-output unit (IO) and a central processor (CP). Incoming jobs start at the IO with an exponentially distributed service time with parameter $\mu'$. Subsequently, the job leaves the system with probability $p$ or proceeds to queue II at the CP with probability $1-p$. At the CP a job has an exponentially distributed service time with parameter $\mu$. Next the job is recycled to the IO unit where it joins queue I. The IO unit treats the jobs in queue I with nonpreemtive priority with respect to the new jobs in queue III.

The system may be represented by a continuous time Markov process with states $(i, j)$, $i = 0, 1, \ldots$ and $j = 1, 2, \ldots$ where $i$ and $j$ are the lengths of the queues II and I respectively (including the jobs being served). Let $\{p_{i,j}\}$ be the equilibrium distribution of the Markov chain, which exists if $(1-p)\mu' < \mu$ (see Appendix A in [5]). The determination of this distribution is the main topic of [5]. By the analogy of the Markov process of the multiprogramming system with the Markov process of the shortest queue problem, Hofri can use a similar approach as Kingman used in [6] for the shortest queue problem. In doing so, he shows that the generating function of the boundary probabilities $p_{0,j}$ is a meromorphic function, and he finds the poles and residues. This enables him to express $p_{0,j}$ as an infinite sum of powers.

In this section we proceed by sketching the part of the generating function analysis which is crucial for our discussion. For the other parts and also for more details, the reader is referred to the extensive treatment by Hofri in [5]. Let $G(z, u)$ be the generating function of the equilibrium distribution $\{p_{i,j}\}$:

$$G(z,u) = \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} p_{i,j} z^i u^j.$$

Application of the generating-function approach to the equilibrium equations of the Markov process leads to a functional equation for $G(z, u)$, relating $G(z, u)$ to the boundary values $G(z, 1)$ and $G(0, u)$. Clearly, $G(0, u)$ is the (one-dimensional) generating function of the boundary probabilities $p_{0,j}$, $j = 1, 2, \ldots$. The determination of $G(0, u)$ is crucial. It will be proved that $G(0, u)$ may be continued to a meromorphic function and that the poles and residues can be found. This provides, by partial fraction decomposition of $G(0, u)$, the possibility to express $p_{0,j}$ as an infinite sum of powers. In section 3 it will be shown that this partial fraction decomposition is more complicated than suggested by Hofri and it does not necessarily lead to explicit expressions for all $p_{0,j}$. In the sequel of this section the determination of $G(0, u)$ will first be outlined in more detail. This outline closely follows Hofri's exposition in [5].

Hofri first shows that $G(0, u)$ is regular in $|u| < \max\{1, \mu'/\mu\}$ and introduces the mapping

$$u = h(\zeta) = a + \phi(\zeta + \zeta^{-1}), \tag{1}$$

where $a$ and $\phi$, which are defined by (45) and (57) in [5], are positive constants. $h(\zeta)$ is a conformal mapping from $|\zeta| > 1$ on the whole $u$-plane, excluding the interval $[a-2\phi, a+2\phi]$. The unit circle $|\zeta| = 1$ is mapped two to one on the interval $[a - 2\phi, a + 2\phi]$. Next, the number $r > 1$ is determined as the largest number such that $h(\zeta)$ maps the annular $1/r < |\zeta| < r$ into the disk $|u| < \max\{1, \mu'/\mu\}$. By defining

$$\overline{G}(\zeta) = G(0, h(\zeta)), \quad 1/r < |\zeta| < r,$$

Hofri proves that on the (nonempty) intersection of $1/r < |\zeta| < r$ and $1/r < |\alpha\zeta| < r$, the function $\overline{G}(\zeta)$ satisfies

$$\frac{\overline{G}(\alpha\zeta)}{\overline{G}(\zeta)} = \frac{\alpha}{\beta}\frac{\zeta - \zeta_1/\alpha}{\zeta - 1/\zeta_1}, \tag{2}$$

where $\alpha$ and $1/\zeta_1$, which are defined by (51) and (66) in [5], are strictly larger than unity and $\beta$ is given by

$$\beta = \frac{\mu - \mu'\alpha}{\mu\alpha - \mu'}.$$

Relationship (2) is deduced from the functional equation for $G(z,u)$, and is used to define $\overline{G}(\zeta)$ over $|\zeta| > 1$ recursively as a regular function, except for simple poles at

$$\zeta = \overline{\zeta}_j = \frac{\alpha^{j-1}}{\zeta_1}, \quad j = 2, 3, \ldots \tag{3}$$

with corresponding residues $\overline{g}_j$. Denoting by $h^{\leftarrow}(u)$ the inverse of $h(\zeta)$ from the whole $u$-plane, excluding $[a-2\phi, a+2\phi]$, to $|\zeta| > 1$, it follows that $\overline{G}(h^{\leftarrow}(u))$ is a regular function, except for simple poles at

$$u = u_j = h(\overline{\zeta}_j) = a + \phi\left(\frac{\alpha^{j-1}}{\zeta_1} + \frac{\zeta_1}{\alpha^{j-1}}\right), \quad j = 2, 3, \ldots \tag{4}$$

with corresponding residues

$$g_j = \phi\overline{g}_j\frac{\overline{\zeta}_j^2 - 1}{\overline{\zeta}_j^2}, \quad j = 2, 3, \ldots \tag{5}$$

Since $\overline{G}(h^{\leftarrow}(u))$ and $G(0,u)$ coincide on the interior of the ellipse $|h^{\leftarrow}(u)| = r$, excluding $[a - 2\phi, a + 2\phi]$, it follows that $\overline{G}(h^{\leftarrow}(u))$ is the analytic continuation of $G(0,u)$ over $|u| \geq \max\{1, \mu'/\mu\}$.

So far, it has been proved that $G(0,u)$ can be continued to a meromorphic function over the whole $u$-plane with simple poles at the points $u = u_j$ and corresponding residues $g_j$, $j = 2, 3, \ldots$ To obtain expressions for the boundary probabilities $p_{0,i}$ the meromorphic function $G(0,u)$ is decomposed into partial fractions. This partial fraction decomposition is the main topic of the next section.

# 3 Partial fraction decomposition of the generating function

To decompose $G(0,u)$ into partial fractions, we will use the approach in §7.4 in Whittaker and Watson [7].

Let $E_l$ be the ellipse in the $u$-plane corresponding to $|h^{\leftarrow}(u)| = |\zeta| = (1+\alpha)\overline{\zeta}_l/2$ for

$l = 2, 3, \ldots$. Since

$$\bar{\zeta}_l < \frac{1+\alpha}{2}\bar{\zeta}_l < \bar{\zeta}_{l+1},$$

no ellipse $E_l$ passes through any poles of $G(0, u)$.

If $u$ is not a pole of $G(0, z)$ and if $l$ is sufficiently large such that $E_l$ encloses $u$, then, since the only poles of the integrand are the poles of $G(0, z)$ and the point $z = u$, we have by the Theorem of Residues that

$$\frac{1}{2\pi i}\int_{E_l}\frac{G(0,z)}{z-u}dz = G(0,u) + \sum_{k=2}^{l}\frac{g_k}{u_k - u}. \tag{6}$$

But inserting the expansion

$$\frac{1}{z-u} = \frac{1}{z} + \frac{u}{z^2} + \ldots + \frac{u^{n-1}}{z^n} + \frac{u^n}{z^n(z-u)},$$

where $n$ is some nonnegative integer, yields that

$$\begin{aligned}\frac{1}{2\pi i}\int_{E_l}\frac{G(0,z)}{z-u}dz &= \sum_{k=0}^{n-1}\frac{1}{2\pi i}\int_{E_l}\frac{G(0,z)u^k}{z^{k+1}}dz + \frac{u^n}{2\pi i}\int_{E_l}\frac{G(0,z)}{z^n(z-u)}dz \\ &= \sum_{k=1}^{n-1}p_{0,k}u^k + \sum_{k=0}^{n-1}\sum_{j=2}^{l}\frac{g_j u^k}{u_j^{k+1}} + \frac{u^n}{2\pi i}\int_{E_l}\frac{G(0,z)}{z^n(z-u)}dz.\end{aligned} \tag{7}$$

Hence, by (6) and (7),

$$G(0,u) = \sum_{k=1}^{n-1}p_{0,k}u^k + \sum_{j=2}^{l}\frac{g_j u^n}{u_j^n(u-u_j)} + \frac{u^n}{2\pi i}\int_{E_l}\frac{G(0,z)}{z^n(z-u)}dz. \tag{8}$$

If we can now prove that as $l \to \infty$,

$$\int_{E_l}\frac{G(0,z)}{z^n(z-u)}dz \to 0, \tag{9}$$

then, by letting $l \to \infty$ in (8), we obtain the following decomposition of $G(0, u)$,

$$G(0,u) = \sum_{k=1}^{n-1}p_{0,k}u^k + \sum_{j=2}^{\infty}\frac{g_j u^n}{u_j^n(u-u_j)}. \tag{10}$$

To satisfy condition (9), it is of course desirable to keep $n$ as small as possible.

**Definition:** Let $m$ be the smallest nonnegative integer such that $\frac{1}{|\beta|\alpha^{m-1}} < 1$.

**Lemma** (cf. Lemma 4 in [6]): $G_l = \sup_{u\in E_l}\frac{|G(0,u)|}{|u^m|} \to 0$ as $l \to \infty$.

**Proof:** Since by (1), $u = h(\zeta) \sim \phi\zeta$ as $|\zeta| \to \infty$, it is sufficient to prove that

$$\overline{G}_l = \sup_{|\zeta|=(1+\alpha)\overline{\zeta}_l/2} \frac{|\overline{G}(\zeta)|}{|\zeta^m|} = \sup_{0\le\theta<2\pi} \frac{|\overline{G}((1+\alpha)\overline{\zeta}_l e^{i\theta}/2)|}{|((1+\alpha)\overline{\zeta}_l e^{i\theta}/2)^m|} \to 0 \text{ as } l \to \infty.$$

It holds that

$$\frac{\overline{G}_l}{\overline{G}_{l-1}} \le \frac{\overline{\zeta}_{l-1}^m}{\overline{\zeta}_l^m} \sup_{0\le\theta<2\pi} \frac{|\overline{G}((1+\alpha)\overline{\zeta}_l e^{i\theta}/2)|}{|\overline{G}((1+\alpha)\overline{\zeta}_{l-1} e^{i\theta}/2)|}.$$

Inserting that $\overline{\zeta}_l = \alpha\overline{\zeta}_{l-1}$, by (3), and then applying relation (2), yields

$$\frac{\overline{G}_l}{\overline{G}_{l-1}} \le \frac{1}{|\beta|\alpha^{m-1}} \frac{(1+\alpha)\overline{\zeta}_{l-1}/2 + \zeta_1/\alpha}{(1+\alpha)\overline{\zeta}_{l-1}/2 - 1/\zeta_1}.$$

Hence, since $1/|\beta|\alpha^{m-1} < 1$ and $\overline{\zeta}_{l-1} \to \infty$ as $l \to \infty$, there exists a positive number $R$, strictly less than unity, such that for all $l$ sufficiently large,

$$\frac{\overline{G}_l}{\overline{G}_{l-1}} \le R,$$

which proves that $\overline{G}_l$ tends to zero as $l$ tends to infinity. □

**Remark:** It can be shown that $m$ is always strictly positive, and that $m$ is possibly larger than unity. For instance, for $\mu' = 1$, $\mu = 2$ and $p = 3/25$ we obtain from (51) in [5] that $\alpha = 11/4$ and $\beta = -1/6$, so in that case $m = 3$.

Now as $l \to \infty$,

$$\int_{E_l} \frac{G(0,z)}{z^m(z-u)} dz = \mathcal{O}(G_l),$$

and so by the lemma, this integral tends to zero as $l \to \infty$. Therefore, by inserting $n = m$ in (8) and next letting $l \to \infty$, yields

**Theorem:** $G(0,u) = \sum_{k=1}^{m-1} p_{0,k} u^k + \sum_{j=2}^{\infty} \frac{g_j u^m}{u_j^m (u - u_j)}$.

By investigating the asymptotic behaviour of the poles $u_j$ and the residues $g_j$ as $j \to \infty$, it can be seen that $n = m$ is indeed the smallest nonnegative integer, for which expression (10) is valid. From (4), we obtain that as $j \to \infty$,

$$u_j \sim \frac{\phi}{\zeta_1 \alpha} \alpha^j.$$

The asymptotic behaviour of $g_j$ can be obtained from formula (105) in [5], which should read as

$$\overline{g}_{j+1} = \overline{g}_j \frac{\alpha^2}{\beta} \frac{\alpha^{j-1}/\zeta_1 - \zeta_1/\alpha}{\alpha^{j-1}/\zeta_1 - 1/\zeta_1}, \quad j = 2, 3, \ldots.$$

From that equality and (5) it is easy to show that as $j \to \infty$,

$$g_j \sim C\phi\overline{g}_2 \left(\frac{\alpha^2}{\beta}\right)^j$$

where $C$ is given by

$$C = \prod_{k=2}^{\infty} \frac{\alpha^{k-1}/\zeta_1 - \zeta_1/\alpha}{\alpha^{k-1}/\zeta_1 - 1/\zeta_1} > 0.$$

Hence, for any nonnegative integer $n$, as $j \to \infty$,

$$\frac{g_j}{(u_j)^{n+1}} \sim C\phi\overline{g}_2 \left(\frac{\zeta_1\alpha}{\phi}\right)^{n+1} \left(\frac{1}{\beta\alpha^{n-1}}\right)^j. \tag{11}$$

So, if $n < m$, then the series in (10) is divergent.

We now show how the partial fraction decomposition of $G(0,u)$ leads to the desired expressions for $p_{0,i}$. For $|u| < 1$ we obtain that (notice that $|u_j| \geq 1$ for all $j$)

$$\begin{aligned} \sum_{j=2}^{\infty} \frac{g_j u^m}{u_j^m(u - u_j)} &= -\sum_{j=2}^{\infty} \frac{g_j u^m}{u_j^{m+1}} \sum_{i=0}^{\infty} \frac{u^i}{u_j^i} \\ &= -\sum_{i=0}^{\infty} u^{m+i} \sum_{j=2}^{\infty} \frac{g_j}{u_j^{m+1+i}}, \end{aligned} \tag{12}$$

where interchanging of the summations is allowed since

$$\sum_{j=2}^{\infty}\sum_{i=0}^{\infty} \frac{|g_j u^{m+i}|}{|u_j^{m+1+j}|} \leq \sum_{j=2}^{\infty} \frac{|g_j|}{|u_j|^{m+1}} \frac{1}{1-|u|}$$

and the right-hand side converges by (11) with $n = m$. From (12) and the theorem, it follows that

**Corollary 1:** $p_{0,i} = -\sum_{j=2}^{\infty} \frac{g_j}{u_j^{i+1}}$ for all $i \geq m$.

From (11) follows that this expression for $p_{0,i}$ is not valid for $i < m$.

**Corollary 2:** The series $-\sum_{j=2}^{\infty} \frac{g_j}{u_j^{i+1}}$ is divergent for $i < m$.

# 4 Concluding remarks

Similarly, series expressions can be obtained for the boundary probabilities $p_{i,1}$. We did not pursue here to deduce the partial fraction decomposition for the two-dimensional generating function $G(z,u)$, providing series expressions for $p_{i,j}$. Similarly as for the shortest queue problem, this analysis is much more complicated than the one for the one-dimensional generating functions $G(0, u)$ and $G(z, 1)$, and leads to cumbersome expressions for $p_{i,j}$. However, in [3] it is shown that explicit expressions for $p_{i,j}$ can be obtained by using a compensation approach, which is not based on generating-functions. In particular, the expressions for $p_{i,j}$ are valid for those $i$ and $j$ satisfying $i + j \geq m$, and not for smaller $i$ and $j$.

# References


[1] I.J.B.F. ADAN, J. WESSELS AND W.H.M. ZIJM, "Analysis of the symmetric shortest queue problem", *Stochastic Models,* vol. 6 , pp. 691–713, 1990.

[2] I.J.B.F. ADAN, J. WESSELS AND W.H.M. ZIJM, "Analysis of the asymmetric shortest queue problem", *Queueing Systems,* vol. 8 , pp. 1–58, 1991.

[3] I.J.B.F. ADAN, G.J. VAN HOUTUM, J. WESSELS AND W.H.M. ZIJM, "A Compensation Procedure for Multiprogramming Queues", Memorandum COSOR 91-13, Eindhoven University of Technology, Dep. of Math. and Comp. Sci., 1991, (submitted for publication).

[4] L. FLATTO AND H.P. MCKEAN, "Two queues in parallel", *Comm. Pure Appl. Math.,* vol. 30, pp. 255–263, 1977.

[5] M. HOFRI, "A generating-Function Analysis of Multiprogramming Queues", *International Journal of Computer and Information Sciences,* vol. 7, pp. 121–155, 1978.

[6] J.F.C. KINGMAN, "Two similar queues in parallel", *Ann. Math. Statist.,* vol. 32, pp. 1314–1323, 1961.

[7] E.T WHITTAKER AND G.N. WATSON, *A Course of Modern Analysis,* Cambridge, 1962.

EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science
PROBABILITY THEORY, STATISTICS, OPERATIONS RESEARCH
AND SYSTEMS THEORY
P.O. Box 513
5600 MB Eindhoven, The Netherlands

Secretariate: Dommelbuilding 0.03
Telephone : 040-473130

---

-List of COSOR-memoranda - 1991

| Number | Month | Author | Title |
|---|---|---|---|
| 91-01 | January | M.W.I. van Kraaij<br>W.Z. Venema<br>J. Wessels | The construction of a strategy for manpower planning problems. |
| 91-02 | January | M.W.I. van Kraaij<br>W.Z. Venema<br>J. Wessels | Support for problem formulation and evaluation in manpower planning problems. |
| 91-03 | January | M.W.P. Savelsbergh | The vehicle routing problem with time windows: minimizing route duration. |
| 91-04 | January | M.W.I. van Kraaij | Some considerations concerning the problem interpreter of the new manpower planning system formasy. |
| 91-05 | February | G.L. Nemhauser<br>M.W.P. Savelsbergh | A cutting plane algorithm for the single machine scheduling problem with release times. |
| 91-06 | March | R.J.G. Wilms | Properties of Fourier-Stieltjes sequences of distribution with support in [0,1). |
| 91-07 | March | F. Coolen<br>R. Dekker<br>A. Smit | Analysis of a two-phase inspection model with competing risks. |
| 91-08 | April | P.J. Zwietering<br>E.H.L. Aarts<br>J. Wessels | The Design and Complexity of Exact Multi-Layered Perceptrons. |
| 91-09 | May | P.J. Zwietering<br>E.H.L. Aarts<br>J. Wessels | The Classification Capabilities of Exact Two-Layered Peceptrons. |
| 91-10 | May | P.J. Zwietering<br>E.H.L. Aarts<br>J. Wessels | Sorting With A Neural Net. |
| 91-11 | May | F. Coolen | On some misconceptions about subjective probability and Bayesian inference. |

COSOR-MEMORANDA (2)

| | | | |
|---|---|---|---|
| 91-12 | May | P. van der Laan | Two-stage selection procedures with attention to screening. |
| 91-13 | May | I.J.B.F. Adan<br>G.J. van Houtum<br>J. Wessels<br>W.H.M. Zijm | A compensation procedure for multiprogramming queues. |
| 91-14 | June | J. Korst<br>E. Aarts<br>J.K. Lenstra<br>J. Wessels | Periodic assignment and graph colouring. |
| 91-15 | July | P.J. Zwietering<br>M.J.A.L. van Kraaij<br>E.H.L. Aarts<br>J. Wessels | Neural Networks and Production Planning. |
| 91-16 | July | P. Deheuvels<br>J.H.J. Einmahl | Approximations and Two-Sample Tests Based on P - P and Q - Q Plots of the Kaplan-Meier Estimators of Lifetime Distributions. |
| 91-17 | August | M.W.P. Savelsbergh<br>G.C. Sigismondi<br>G.L. Nemhauser | Functional description of MINTO, a Mixed INTeger Optimizer. |
| 91-18 | August | M.W.P. Savelsbergh<br>G.C. Sigismondi<br>G.L. Nemhauser | MINTO, a Mixed INTeger Optimizer. |
| 91-19 | August | P. van der Laan | The efficiency of subset selection of an almost best treatment. |
| 91-20 | September | P. van der Laan | Subset selection for an -best population: efficiency results. |
| 91-21 | September | E. Levner<br>A.S. Nemirovsky | A network flow algorithm for just-in-time project scheduling. |
| 91-22 | September | R.J.M. Vaessens<br>E.H.L. Aarts<br>J.H. van Lint | Genetic Algorithms in Coding Theory - A Table for $A_3(n,d)$. |
| 91-23 | September | P. van der Laan | Distribution theory for selection from logistic populations. |

COSOR-MEMORANDA (3)

| | | | |
|---|---|---|---|
| 91-24 | October | I.J.B.F. Adan<br>J. Wessels<br>W.H.M. Zijm | Matrix-geometric analysis of the shortest queue problem with threshold jockeying. |
| 91-25 | October | I.J.B.F. Adan<br>J. Wessels<br>W.H.M. Zijm | Analysing Multiprogramming Queues by Generating Functions. |