Stabilizing boundary value problems by perturbing the boundary conditions

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BY PERTURBING THE BOUNDARY CONDITIONS

by

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ABSTRACT

For some ill conditioned BVP the ill-conditioning is due to a lack of a proper dichotomy of the solution space. For such problems we suggest a regularisation technique, by effectively solving a similar problem but subject to an integral BC (being a perturbation of the given one). It is indicated why this improves the conditioning and also how this influences the accuracy of the numerical solution.

AMS (MOS) subject classification 56L10

1. Introduction

Consider the BVP

(1.1) $\frac{dx}{dt} = L(t)x + f(t), \quad -1 \le t \le 1$

(1.2) $M_{-1}\, x(-1) + M_1\, x(1) = b\,,$

where vectors are $\in \mathbb{R}^n$ and matrices $\in \mathbb{R}^{n^2}$

It is well known that the conditioning of (1.1) + (1.2) is closely related to the dichotomy of the ODE (1.1), cf. [1]. By the latter we mean the following: Let there exist a constant κ, a fundamental solution $\Phi(t)$ of (1.1) and an orthogonal projection P such that the following estimates hold

(1.3a) $\| \Phi(t)\, P\, \Phi^{-1}(s) \| \le \kappa\,, \quad t > s$

(1.3b) $\| \Phi(t)\, (I-P)\, \Phi^{-1}(s) \| \le \kappa\,, \quad t < s\,,$

then the ODE (1.1) is said to have a *dichotomy* with *threshold* κ.
Unless the BC (1.2) are chosen such that they do not "control" the modes $\{\Phi(t)\, P\, c \mid c \in \mathbb{R}\}$ at the left point $t=-1$, and the modes $\{\Phi(t)\,(I-P)\, c \mid c \in \mathbb{R}\}$ at the right point $t=1$, we have a "well-conditioned problem", with condition number $\sim \kappa$ (cf. [1], where also a kind of converse was proven: given well-conditioning $O(\kappa)$ we have a dichotomy with threshold $O(\kappa^2)$). Recently, it was also shown (cf. [2]) that if we have a multipoint, or even integral BC, say

(1.4) $\int_{-1}^{1} M(\tau)\, x(\tau)\, dt = b\,,$

then we can indicate a more complex structure of fundamental modes: not only do we have ones that do not grow (in norm) more than a factor κ for increasing t or decreasing t, but also ones that may increase initially and then decrease. Although this more general structure is not necessarily present, it is essential that an appropriate interval condition controls such a mode when it is, in order to have a well-conditioned problem.

Reversing the argument a little, ill-conditioning may be caused by modes, having the above described behaviour (i.e. are increasing on some interval $(-1, T)$ and decreasing on $(T, 1)$). In such cases a different BC might "stabilize" the problem. This idea has induced the present investigation, of which we only give a brief account in this paper: We like to find such an (integral) BC, being a perturbation of (1.2), so that modes, not of either type as described in (1.3a) or (1.3b), are controlled; this may be considered as a *regularization* of (1.1), (1.2). The usual price for such a procedure is that we are solving a nearby problem at best. However, by a judicious choice of this perturbed BC and given moderate accuracy requirements, this may lead to a reasonable strategy. In particular, the proposed method improves error bounds obtained for the original problem. In section 2 we consider a simplified but instructive case. In section 3 we show how we may deal with more general situations.

2. A simple example

In order to demonstrate the regularisation idea let us first examine a simple example, which allows for explicit analytical treatment: Consider the ODE

(2.1) $$\frac{dx}{dt} = \begin{cases} \frac{1}{\varepsilon} x, & -1 \le t < 0 \\ -\frac{1}{\varepsilon} x, & 0 < t \le 1, \end{cases}$$

where x is a scalar function.
A general basis solution (is a fundamental solution here) is given by

(2.2) $$\phi(t) = \begin{cases} e^{t/\varepsilon}, & -1 \le t \le 0 \\ e^{-t/\varepsilon} & 0 \le t \le 1. \end{cases}$$

The graph of $\phi(x)$ is given in Fig. 2.1.

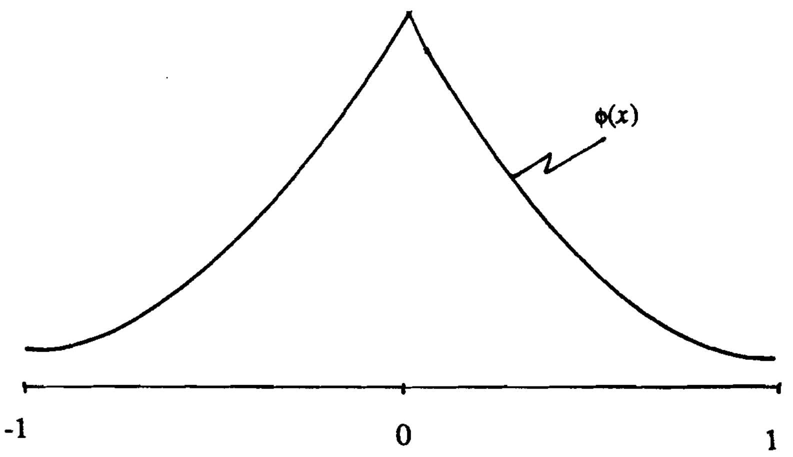

Fig. 1.2

Hence we see that there exists no dichotomic fundamental solution, with a dichotomy constant uniformly bounded in ε. Consequently, no possible choice for a two point BC can make a related BVP well-conditioned.

In order to investigate the effect of integral BC, let us compute $\psi(t) := \int_0^t \phi(\tau)\, d\tau$. We obtain

(2.3a) $$\psi(t) = \int_{-1}^{t} \phi(\tau)\, d\tau = \varepsilon\left[e^{t/\varepsilon} - e^{-1/\varepsilon} \right], \quad t \le 0$$

(2.3b) $$\psi(t) = \int_{-1}^{t} \phi(\tau)\, d\tau = \varepsilon\left[2 - e^{-1/\varepsilon} - e^{-t/\varepsilon} \right], \quad t \ge 0.$$

The graph of $\psi(t)$ is given in Fig. 2.2.

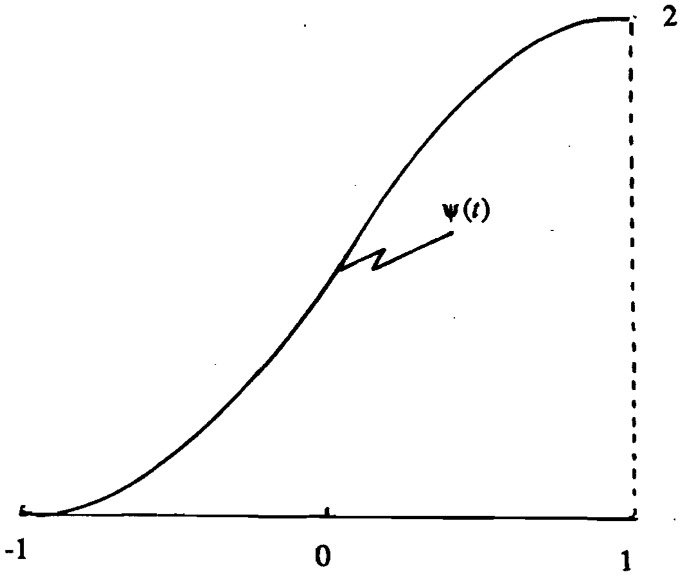

Fig. 2.2

If we consider the *augmented* system

$$(2.4) \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{cases} \begin{bmatrix} -1/\varepsilon & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & -1 \le t < 0 \\ \begin{bmatrix} 1/\varepsilon & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, & 0 < t \le 1, \end{cases}$$

Then, clearly, a fundamental solution, $\Omega(t)$, is given by

$$(2.5) \qquad \Omega(t) = \begin{bmatrix} \phi(t) & 0 \\ \psi(t) & 1 \end{bmatrix}.$$

As can be seen, the first column of $\Omega(t)$ is strongly increasing on (-1, 0) and almost constant on (0, 1) for $\varepsilon \downarrow 0$. However, this choice of fundamental solution is also very *skew*, i.e. $\mathrm{span}((\phi(t), \psi(t))^T) \to \mathrm{span}((0, 1)^T)$.

Therefore a better choice of $\Omega(t)$ (cf. [4,p 88]) is the following

$$(2.6) \qquad \hat{\Omega}(t) = \begin{bmatrix} \phi(t) & -\phi(t) \\ \psi(t) & 2-\psi(t) \end{bmatrix} = [\hat{\Omega}^1(t) \mid \hat{\Omega}^2(t)].$$

A graph of $\|\hat{\Omega}^1(t)\|$ and $\|\Omega^2(t)\|$ is given in Fig. 2.3.

1) In this paper $\| . \|$ means $\| . \|_2$

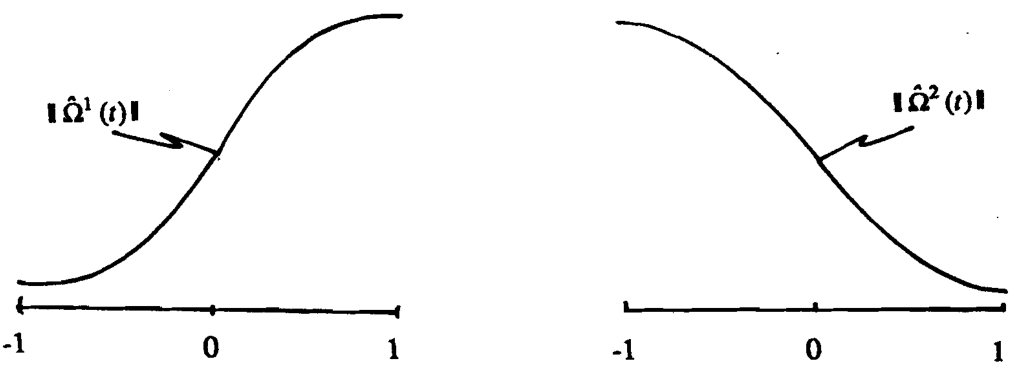

Fig. 2.3

We conclude that the augmented system (2.4) has a dichotomic fundamental solution $\hat{\Omega}(t)$ with a moderate dichotomy constant, uniformly in ε! This consequence lies at the heart of the idea we want to exploit in this paper.
First let us consider a (non biased) BC for (2.1):

(2.7) $$x(-1)+x(1)=b\,,\quad b\in \mathbb{R}.$$

We obtain for the Green's function $G(t,s)$ of (2.1), (2.7).

(2.8a) $$G(t,s)=\begin{cases}\phi(t)\,Q^{-1}\,e^{-1/\varepsilon}\,\phi^{-1}(s)\,, & t>s\\ -\phi(t)\,Q^{-1}\,e^{-1/\varepsilon}\,\phi^{-1}(s)\,, & t<s\,,\end{cases}$$

where we derive Q from the BC (2.7):

(2.8b) $$Q=2\,e^{-1/\varepsilon}.$$

If we take e.g. $t>s$ we find

(2.9) $$G(t,s)=\begin{cases}\frac{1}{2}\,e^{(t-s)/\varepsilon}\,, & t<0,\,s<0\\ \frac{1}{2}\,e^{(-t-s)/\varepsilon}\,, & t>0,\,s<0\\ \frac{1}{2}\,e^{(-t+s)/\varepsilon}\,, & t>0,\,s>0.\end{cases}$$

For $t<s$ a similar type of result can be given, from which we conclude that

(2.10) $$\max_{t,s}\,|\,G(t,s)\,|=\tfrac{1}{2}\,e^{1/\varepsilon}.$$

This bound for $|\,G(t,s)\,|$ of course confirms the lack of suitable dichotomy as was apparent from (2.2), cf. [1].
Let us now consider a regularised BC, cf. (2.7)

$$(2.11) \qquad x(-1) + x(1) + \delta \int_{-1}^{1} x(\tau)\, d\tau = b.$$

Of course δ in (2.11) should be taken fairly small, in order not to perturb the BC too much.
On the other hand we see that (2.1) + (2.11) can be viewed as an augmented system (2.4), satisfying the BC

$$(2.12) \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x(-1) \\ y(-1) \end{bmatrix} + \begin{bmatrix} 1 & \delta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(1) \\ y(1) \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

The crucial point here is that the row vector $(1, \delta)$ in the right BC matrix is "controlling" the non-increasing mode $\hat{\Omega}^1(t)$, and the better the larger δ is. On account of the moderate dichotomy constant of $\hat{\Omega}(t)$ we therefore may expect a conditioning constant for (2.4) + (2.12) $\sim \frac{1}{\delta}$, as compared to $\sim e^{1/\varepsilon}$ for (2.1) + (2.7) (cf. (2.10)).
We shall work this out in a slightly different way now, by considering the Green's function, $\tilde{G}(t, s)$, for (2.1) + (2.11) directly. We obtain

$$(2.13a) \qquad \tilde{G}(t, s) = \phi(t)\, \tilde{Q}^{-1} \left(e^{-1/\varepsilon} + \delta\, \psi(s)\right) \phi^{-1}(s), \quad t > s$$

$$(2.13b) \qquad \tilde{G}(t, s) = -\phi(t)\, \tilde{Q}^{-1} \left(e^{-1/\varepsilon} + \delta\,(2 - \psi(s))\right) \phi^{-1}(s), \quad t < s,$$

where

$$(2.14) \qquad \tilde{Q} \doteq 2\, e^{-1/\varepsilon} + 2\, \varepsilon\, \delta.$$

If we take (e.g.) $t > s$, we find (cf. (2.9))

$$(2.15) \qquad \tilde{G}(t, s) \doteq \left(2 e^{-1/\varepsilon} + 2\, \varepsilon\, \delta\right)^{-1} * \begin{cases} e^{(t-s)/\varepsilon} * (e^{-1/\varepsilon} + \delta\, \varepsilon\, e^{s/\varepsilon} - \delta\, \varepsilon\, e^{-1/\varepsilon}), & t < 0, s < 0 \\ e^{(-t-s)/\varepsilon} * (e^{-1/\varepsilon} + \delta\, \varepsilon\, e^{s/\varepsilon} - \delta\, \varepsilon\, e^{-1/\varepsilon}), & t > 0, s < 0 \\ e^{(-t+s)/\varepsilon} * (e^{-1/\varepsilon} + 2\, \delta \varepsilon - \delta\, \varepsilon\, e^{-1/\varepsilon} - \delta\, \varepsilon\, e^{-s/\varepsilon}), & t > 0, s > 0. \end{cases}$$

Clearly, $|\tilde{G}(t, s)|$ attains its maximum value for $t = 0$, $s = -1$, so (a similar bound follows for $t < s$), so

$$(2.16) \qquad \max |\tilde{G}(t, s)| = \frac{1 + \delta\varepsilon - \delta\varepsilon}{2 e^{-1/\varepsilon} + \delta\varepsilon} \approx \frac{1}{2\, \delta\varepsilon},$$

assuming $e^{-1/\varepsilon} \ll \delta\, \varepsilon$.

Now consider a nonhomogeneous problem,

$$(2.17) \qquad \frac{dx}{dt} = \begin{cases} \frac{1}{\varepsilon}\,(x + f(t)), & -1 \le t < 0 \\ -\frac{1}{\varepsilon}\,(x + f(t)), & 0 < t \le 1, \end{cases}$$

where $f(t)$ is chosen such that (2.17) possesses a (uniformly bounded) particular solution p.
Define the following BC (cf. (2.7), (2.11))

(2.18a) $\quad B\,x := x(-1) + x(1)$

(2.18b) $\quad \tilde{B}_\delta\, x := x(-1) + x(1) + \delta \int_{-1}^{1} x(\tau)\, d\tau.$

The for the exact solution of (2.17), (2.7) there exists a constant c^* such that

(2.19a) $\quad x(t) = \phi(t)\, c^* + p(t),$

where

(2.19b) $\quad 2e^{-1/\varepsilon}\, c^* = b - B\,p.$

By requiring $\max |x(t)| = 0(1)$, we see that

(2.20) $\quad c^* = 0(1).$

(so $| b - B\,p | = 0(e^{-1/\varepsilon})$). Now consider the following two solutions:
Let $x^0(t)$ satisfy (2.1) and (2.11), then there exists a constant c^0 such that

(2.21a) $\quad x^0(t) = \phi(t)\, c^0 + p(t),$

where

(2.21b) $\quad 2\,(e^{-1/\varepsilon} + \varepsilon\,\delta)\, c^0 = b - \tilde{B}_\delta\, p = b - B\,p - \delta \int_{-1}^{1} p(\tau)\, d\tau.$

Let $x^1(t)$ satisfy (2.1) and the BC

(2.22) $\quad \tilde{B}_\delta\, x^1 = b + \delta \int_{-1}^{1} x^0(\tau)\, d\tau.$

Then

(2.23a) $\quad x^1(t) = \phi(t)\, c^1 + p(t),$

with

(2.23b) $\quad 2\,(e^{-1/\varepsilon} + \varepsilon\,\delta)\, c^1 = b - B\,p + 2\varepsilon\,\delta\, c^0.$

Remark 2.24. Given $p(t)$ (which can be computed without employing any (global) BC, we can stably find both c^0 and c^1 by evaluating (2.21a) and (2.23a) at $t = 0$.
From (2.21b) and (2.23b) we can then formally find c^* by

(2.25) $\quad 2\,(e^{-1/\varepsilon} + \varepsilon\,\delta)\, c^1 - 2\varepsilon\,\delta\, c^0 = b - B\,p = 2\, e^{-1/\varepsilon}\, c^*$

The idea here is that c^0, c^1 can be found in a fairly stable way and that, if the cancellation errors in (2.25) are within certain limits, we can use this form for computing c^* and obtain $x(t)$ via (2.19a).
Using (2.19b) in (2.21b) we find

(2.26) $$c^0 = \frac{O(e^{-1/\varepsilon}) + O(\delta)}{2(e^{-1/\varepsilon} + \varepsilon\,\delta)} = 0(\frac{1}{\varepsilon}),$$

assuming again $e^{-1/\varepsilon} \ll \varepsilon\,\delta$.
Similarly we obtain from (2.23b)

(2.27) $$c^1 = 0(\frac{1}{\varepsilon}).$$

Hence, given a machine accuracy ε_M, we find for the numerically computed c^*, $\bar{c}$ say (cf. (2.25))

(2.28) $$|\,\bar{c} - c^*\,| = O(\delta\,\varepsilon_M\,e^{1/\varepsilon}),$$

whence we find for the numerically computed solution $x(t)$, $\bar{x}(t)$ say (cf. (2.19a))

(2.29) $$|\,x(t) - \bar{x}(t)\,| = O(\delta\,\varepsilon_M\,e^{1/\varepsilon}).$$

If we compare this error bound with the one that would result from solving (2.1), (2.7) directly, (or through (2.19b)) viz. $O(\varepsilon_M\,e^{1/\varepsilon})$ (cf. (2.10)) we see that this method gives better results indeed. Of course, we have a natural constraint on the magnitude of δ. For if we solve the "regularized" system we should expect errors $O(\frac{\varepsilon_M}{\delta\,\varepsilon})$, cf. (2.16).

Concluding we obtain that δ must satisfy

(2.30a) $$\frac{\varepsilon_M}{\mathrm{TOL}}\,\frac{1}{\varepsilon} \le \delta \le \frac{\mathrm{TOL}}{\varepsilon_M}\,e^{-1/\varepsilon},$$

which is only meaningful if

(2.30b) $$\frac{\mathrm{TOL}}{\varepsilon_M} \ge \varepsilon^{-1/2}\,e^{1/2\varepsilon}.$$

3. Some generalisations

In this section we shall sketch the more general situation of a BVP where the fundamental solution $\Phi(t)$ is not dichotomic with a fairly moderate dichotomy constant, but instead the following holds

Assumption 3.1 Let there exist orthogonal projections P_1, P_2, P_3, such that $P_1 + P_2 + P_3 = I$, $\mathrm{rank}(P_1) + \mathrm{rank}(P_2) + \mathrm{rank}(P_3) = n$, and a moderate number κ such that

(i) $\qquad \| \Phi(t) P_1 \Phi(s)^{-1} \| \leq \kappa, \quad t \geq s$

(ii) $\qquad \| \Phi(t) P_2 \Phi(s)^{-1} \| \leq \kappa, \quad t \leq s$

(iii) $\qquad \| \Phi(t) P_2 \Phi^{-1}(s) \| \leq \kappa, \quad 0 > t \geq s$

(iv) $\qquad glb\,(\int_{-1}^{1} \Phi(\tau)\, dt) \geq c\,, \quad c$ not small.

Remarks 3.2.

(a) Condition (iii) can be generalized to "turning points" other than 0 and modes within $\Phi(t) P_2$ having different "turning points".

(b) Note that (iv) prevents "directional cancellation" (a necessary and nontrivial extra requirement).

We now employ the same idea as used in the previous section. Rather than (1.2), consider the BC:

(3.2) $$M_{-1}\, x(-1) + M_1\, x(1) + \delta \int_{-1}^{1} x(\tau)\, d\tau = b.$$

Define

(3.3a) $$Q = M_{-1}\, \Phi(-1) + M_1\, \Phi(1)$$

(3.3b) $$E = \int_{-1}^{1} \Phi(\tau)\, d\tau.$$

We shall only investigate the Green's function, of the BVP (1.1), (3.2), $G_\delta(t, s)$ for $t > s$ for $t < s$, the analysis is similar. We obtain

(3.4) $$G_\delta(t, s) = \Phi(t)\ (Q + \delta E)^{-1}\ [M_{-1}\, \Phi(-1) + \delta \int_{-1}^{s} \Phi(\tau)\, d\tau]\, \Phi^{-1}(s).$$

This leads to

Lemma 3.5. Let $C = \max(\| M_{-1} \|, \| M_1 \|$. Then $\| G_\delta(t, s) \| \leq 2\,\kappa (C + 2\delta)\, \| \Phi(t)\, (Q + \delta E)^{-1} \| + \kappa$.

Proof: Let $s < 0$ then

$$G_\delta(t, s) = \Phi(t)\,(Q+\delta E)^{-1}\,[M_{-1}\,\Phi(-1)+\delta\int_{-1}^{s}\Phi(\tau)\,d\tau]\,(P_1+P_2+P_3)I\,\Phi^{-1}(s)$$

$$=-\Phi(t)\,(Q+\delta E)^{-1}\,(M_1\,\Phi(1)+\delta\int_{s}^{1}\Phi(\tau)\,d\tau)\,P_1\,\Phi^{-1}(s)+\Phi(t)\,P_1\,\Phi^{-1}(s)$$

$$+\Phi(t)\,(Q+\delta E)^{-1}\,(M_{-1}\,\Phi(-1)+\delta\int_{-1}^{s}\Phi(\tau)\,d\tau)\,(P_2+P_3)\,\Phi^{-1}(s),$$

from which the assertion can easily be shown. For $s > 0$, we make a similar reformulation, now grouping P_2 with P_1 instead. □

In order to show which effect δE has on the conditioning, we have to make some additional assumptions regarding directional well-conditioning. First, we note that it is *not restrictive* to assume that P_1, P_2 and P_3 are just *diagonal blocks* of the identity matrix. We have

Assumption 3.6.

(i) Let $\Phi(t)$ be normalized such that all columns assume a maximum norm of 1 for some $t[-1, 1]$.

(ii) Let the minimum angle between the columns of Q, and $\eta(\delta)$ the minimum angle between the columns of $Q+\delta E$, be bounded away from zero, say $\zeta := \min(\theta, \eta(\delta)) > 0$.

If we denote the j^{th} column of Q by q_j and the j-th column of $Q+\delta E$ by $q_j(\delta)$, then we can deduce from this assumption (cf. [3, p. 429]

Property 3.7.

$$\|\Phi(t)\,Q^{-1}\| \le \frac{1}{\sin\xi \min\limits_{j=1,\ldots,n}\|q_j\|}$$

and

$$\|\Phi(t)\,(Q+\delta E)^{-1}\| \le \frac{1}{\sin\xi \min\limits_{j=1,\ldots,n}\|q_j(\delta)\|}.$$

□

It can also be shown that the bounds in this property are fairly realistic. Consequently, (as is precisely the case in section 2) we attribute ill-conditioning to $\min\|q_j\|$ being small.
Hence it is realistic to have

Assumption 3.8. Let $\Phi(t)\,(P_1+P_3)$ precisely have the last p columns equal to zero (so $\Phi(t)\,P_2$ precisely has the first $(n-p)$ columns equal to zero). Then assume that $\min\limits_{j=1,\ldots,n-p}\|q_j\| \ge c$, c not small and $\min\limits_{j=0,\ldots,p-1} \| q_{n-j} \| \ll c$

Identifying the constants c in assumptions 3.1 and 3.8, we immediately deduce from Property 3.7.

Corollary 3.9.

Let $\delta \gg \min\limits_{j=0,\ldots,p-1} \| q_{n-j} \|$, then $\| Q(t)\,(Q + \delta E)^{-1} \| \leq \dfrac{1}{\sin\zeta\,\delta c}$

□

Together with Lemma 3.5 this gives us

Corollary 3.10.

Let $\delta \gg \min\limits_{j=0,\ldots,p-1} \| q_{n-j} \|$, then $\| G_\delta(t, s) \| \leq \dfrac{2\kappa(C+\delta)}{\sin\xi\,\delta c} + \kappa$.

□

We conclude that we have regularised our problem, i.e. $\max\limits_{t,s} \| G_\delta(t, s) \| = 0(\frac{1}{\delta}) \gg \max\limits_{t,s} \| G(t, s) \|$.

This fact can now be employed in the same way as it was done in section 2, cf. (2.18) ff. We shall omit further details as the final result is essentially the same.

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