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Summary

We show that the Fourier-Jacobi transform of index (α,β), $\alpha > -1$, $\beta \in \mathbb{R}$, maps functions of the form

$$t \mapsto \phi(1-2\tanh^2 t)\cosh^{-\nu} t\,,$$

with ϕ an entire analytic function and $\nu \in \mathbb{C}$, such that $\mathrm{Re}(\nu) > \alpha+\beta+1$ and $\frac{1}{2}\nu \notin \{0,-1,-2,\cdots\}$ and $\frac{1}{2}\nu-\beta \notin \{0,-1,-2,\cdots\}$, bijectively onto the functions

$$x \mapsto \Gamma(\tfrac{1}{2}(\nu-\alpha-\beta-1+ix))\,\Gamma(\tfrac{1}{2}(\nu-\alpha-\beta-1-ix))\,\psi(x).$$

Here ψ is an even and entire analytic function of sub-exponential growth, i.e.

$$\forall_{\varepsilon>0} : \sup_{z\in\mathbb{C}} |\psi(z)| \exp(-\varepsilon|z|) < \infty.$$

Our treatment is based on recurrence relations.

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1. Introduction

For $\alpha, \beta \in \mathbb{C}$ we define the function $\Delta_{\alpha,\beta} : \mathbb{R}^+ \to \mathbb{R}$ by

(1) $$\Delta_{\alpha,\beta}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1} \quad , t > 0$$

and the differential operator $D_{\alpha,\beta}$ by

(2) $$D_{\alpha,\beta} = \frac{1}{\Delta_{\alpha,\beta}} \frac{d}{dt} \Delta_{\alpha,\beta} \frac{d}{dt} + (\alpha+\beta+1)^2.$$

Consider the eigenvalue problem

(3) $$\begin{cases} D_{\alpha,\beta}\, u = -\lambda^2 u \\ u'(0) = 0 \ , \ u(0) = 1. \end{cases}$$

By substituting $z = -\sinh^2 t$ a hypergeometric differential equation is obtained with parameters $\frac{1}{2}(\alpha+\beta+1+i\lambda)$, $\frac{1}{2}(\alpha+\beta+1-i\lambda)$, $\alpha+1$ (cf. [E, 2.1(1)]). So if $\alpha \neq -1, -2, -3, \ldots$ the solution of (3) is given by

(4) $$u(t) = \phi_\lambda^{(\alpha,\beta)}(t) = {}_2F_1(\tfrac{1}{2}(\alpha+\beta+1+i\lambda), \tfrac{1}{2}(\alpha+\beta+1-i\lambda); \alpha+1; -\sinh^2 t).$$

Here ${}_2F_1(a,b;c;z)$ denotes the hypergeometric function, which is the unique analytic continuation for $z \notin [1,\infty)$ of the power series

(5) $$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} z^n \quad , |z| < 1.$$

The function $\phi_\lambda^{(\alpha,\beta)}$ is called the Jacobi function of the first kind and of index (α,β).
The Fourier-Jacobi transform of index (α,β), $f \mapsto \hat{f}^{(\alpha,\beta)}$, is formally defined by

(6) $$\hat{f}^{(\alpha,\beta)}(\lambda) = \int_0^\infty f(t)\, \phi_\lambda^{(\alpha,\beta)}(t)\, \Delta_{\alpha,\beta}(t)\, dt.$$

It is well known, see [K], that the Fourier-Jacobi transform of index (α,β) maps the function

(7) $$f(t) = (\cosh t)^{-\alpha-\beta-\delta-i\mu-2}\, P_n^{(\alpha,\delta)}(1-2\tanh^2 t)$$

onto the function

(8) $$\hat{f}^{(\alpha,\beta)}(\lambda) = \frac{2^{2\alpha+2\beta+1}\,\Gamma(\alpha+1)\,(-1)^n}{n!\,\Gamma(\frac{1}{2}(\alpha+\beta+\delta+i\mu+2)+n)\,\Gamma(\frac{1}{2}(\alpha-\beta+\delta+i\mu+2)+n)} \cdot$$
$$\cdot \{\Gamma(\tfrac{1}{2}(\delta+i\mu+1+i\lambda))\,\Gamma(\tfrac{1}{2}(\delta+i\mu+1-i\lambda))\} \cdot$$
$$\cdot W_n(\tfrac{1}{4}\lambda^2; \tfrac{1}{2}(\delta+i\mu+1), \tfrac{1}{2}(\delta-i\mu+1), \tfrac{1}{2}(\alpha+\beta+1), \tfrac{1}{2}(\alpha-\beta+1)),$$

$\beta, \lambda, \mu \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$, $\alpha, \delta > -1$.

Here the $P_n^{(\alpha,\delta)}$ are Jacobi polynomials and the W_n are Wilson polynomials. If we abandon the factor between { } and if we keep the parameters $\alpha, \beta, \delta, \mu$ fixed, then it is clear, that via the Fourier-Jacobi transform the space of polynomials is mapped linearly and bijectively on the space of even polynomials. Let us denote this linear mapping by $\mathbf{F}_{\alpha,\beta,\delta,\mu}$.
We pose ourselves the following problem: Extend $\mathbf{F}_{\alpha,\beta,\delta,\mu}$ bijectively to suitable spaces of analytic functions.
In [BG] we studied the mapping $\mathbf{F}_{-½,-½,\delta,\mu}$. In that special case the Fourier-Jacobi transform reduces to the Fourier-cosine transform. As an extension of the results in the paper [BG] we now study the general mapping $\mathbf{F}_{\alpha,\beta,\delta,\mu}$.
At this point we emphasize that our treatment is inspired by Koornwinder's formula (8) but does not use it.

2. A special infinite upper triangular matrix

In the sequel we take $\alpha > -1$, $\beta \in \mathbb{R}$, $\nu \in \mathbb{C}$ with $\operatorname{Re}\nu > \alpha+\beta+1$ and $\frac{1}{2}\nu, \frac{1}{2}\nu-\beta \neq 0,-1,-2,\ldots$, fixed. We denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Lemma 2.1.

(i) For each $n \in \mathbb{N}_0$ there exist complex numbers $c_{j,n}$, $0 \le j \le n$, such that

$$(2\tanh^2 t - 1)^n \cosh^{-\nu} t = \sum_{j=0}^{n} c_{j,n}\, D^j_{\alpha,\beta}\,(\cosh^{-\nu} t) \quad , t \in \mathbb{R}.$$

(ii) The numbers $c_{j,n}$, $0 \le j \le n$, satisfy the recurrence relation

$$\begin{aligned}(2n+\nu)\,(n+\tfrac{1}{2}\nu-\beta)\,c_{j,n+1} &= c_{j-1,n} + \\ &+ (2n^2+2n(2\alpha+1)+\nu(2\alpha+\beta+2-\tfrac{1}{2}\nu)-(\alpha+\beta+1)^2)\,c_{j,n} + \\ &- 2n(2\alpha+\beta+2-\nu-n)\,c_{j,n-1} - 2n(n-1)\,c_{j,n-2} \qquad , 0 \le j \le n+1\end{aligned}$$

with boundary conditions

$$c_{0,0} = 1 \quad , \quad c_{j,n} = 0 \ \text{ if } \ j<0 \ \text{ or } \ j>n.$$

Proof.
The proof is by induction. Obviously (i) is true for $n=0$, then $c_{0,0}=1$. Next suppose (i) is true for $n = 0,1,\ldots,N$. Applying the differential operator $D_{\alpha,\beta}$ we get

$$D_{\alpha,\beta}\,\{(2\tanh^2 t - 1)^N \cosh^{-\nu} t\} = \sum_{j=1}^{N+1} c_{j-1,N}\, D^j_{\alpha,\beta}\,(\cosh^{-\nu} t).$$

Evaluating the left hand side of this equality, using the induction hypothesis, leads to both assertions (i) and (ii) at once. □

In the following theorem we gather some properties of the numbers $c_{j,n}$, $0 \le j \le n$.

Theorem 2.2.

(i) $$c_{j,j} = \frac{2^{-j}\,\Gamma(\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}\nu-\beta)}{\Gamma(j+\frac{1}{2}\nu)\,\Gamma(j+\frac{1}{2}\nu-\beta)} \quad , j \ge 0.$$

(ii) There exists a positive real number $\xi_{\alpha,\beta,\nu}$ such that

$$\left|\frac{c_{j,n}}{c_{j,j}}\right| \le \xi^n_{\alpha,\beta,\nu} \quad , 0 \le j \le n < \infty.$$

(iii) $$\lim_{j\to\infty} \{c_{j,j}\,(e^{-1}\sqrt{2}\,j)^{2j}\,j^{\nu-\beta-1}\} = \frac{1}{2\pi}\,\Gamma(\tfrac{1}{2}\nu)\,\Gamma(\tfrac{1}{2}\nu-\beta).$$

Proof.

(i) Take $n = j-1$ in the recurrence relation, then $(2j+\nu-2)\,(j+\frac{1}{2}\nu-\beta-1)\,c_{j,j} = c_{j-1,j-1}$. Noting that $c_{0,0} = 1$ the result follows.

(ii) Put $d_{j,n} = \dfrac{c_{j,n}}{c_{j,j}}$, $0 \le j \le n < \infty$. Then from the recurrence relation we obtain,

$$d_{j+1,n+1} = \frac{(2j+\nu)\,(j+\frac{1}{2}\nu-\beta)}{(2n+\nu)\,(n+\frac{1}{2}\nu-\beta)}\,d_{j,n} +$$

$$+\frac{2n^2+2n(2\alpha+1)+\nu(2\alpha+\beta+2-\frac{1}{2}\nu)-(\alpha+\beta+1)^2}{(2n+\nu)\,(n+\frac{1}{2}\nu-\beta)}\,d_{j+1,n} +$$

$$+\frac{2n(2\alpha+\beta+2-\nu-n)}{(2n+\nu)\,(n+\frac{1}{2}\nu-\beta)}\,d_{j+1,n-1} + \frac{2n(n-1)}{(2n+\nu)\,(n+\frac{1}{2}\nu-\beta)}\,d_{j+1,n-2}.$$

Since $\frac{1}{2}\nu, \frac{1}{2}\nu-\beta \ne 0,-1,-2,\ldots,$ there exists $\varepsilon > 0$ such that $|n+\frac{1}{2}\nu| > \varepsilon$ and $|n+\frac{1}{2}\nu-\beta| > \varepsilon$ for all $n \in \mathbb{N}_0$. Applying the triangle inequality we estimate, for instance,

$$\left|\frac{2j+\nu}{2n+\nu}\right| \le 1 + \frac{n}{|n+\frac{1}{2}\nu|} \le 2 + \frac{|\nu|}{2\varepsilon} \quad , 0 \le j \le n.$$

So it easily follows that there exists $\xi_{\alpha,\beta,\nu} > 1$ such that

$$|d_{j+1,n+1}| \le \tfrac{1}{4}\,\xi_{\alpha,\beta,\nu}\,(|d_{j,n}| + |d_{j+1,n}| + |d_{j+1,n-1}| + |d_{j+1,n-2}|),$$

$$, -1 \le j \le n.$$

Now apply induction.

(iii) Follows from (i) and Stirling's formula. □

We gather the numbers $c_{j,n}$ in an upper triangular matrix $C = [c_{j,n}]_{j,n=0}^{\infty}$. The next theorem gives some results on the inverse C^{-1} of C which is also an upper triangular matrix. The proof does not differ much from the preceding proofs.

Theorem 2.3.

(i) The elements $a_{k,j}$, $0 \le k \le j < \infty$ of C^{-1} satisfy

$$D^{j}_{\alpha,\beta} (\cosh^{-\nu} t) = \sum_{k=0}^{j} a_{k,j} \cosh^{-\nu} t \, (2\tanh^2 t - 1)^k \quad , t \in \mathbb{R}.$$

(ii) The numbers $a_{k,j}$, $0 \le k \le j$, satisfy the recurrence relation

$$a_{k,j+1} = (2k+\nu-2)\,(k+\tfrac{1}{2}\nu-\beta-1)\, a_{k-1,j} +$$

$$-\,(2k^2+2k(2\alpha+1)+\nu(2\alpha+\beta+2-\tfrac{1}{2}\nu)-(\alpha+\beta+1)^2)\, a_{k,j} +$$

$$+\,2(k+1)\,(2\alpha+\beta+1-\nu-k)\, a_{k+1,j} + 2(k+2)\,(k+1)\, a_{k+2,j} \quad , 0 \le k \le j+1$$

with boundary conditions

$$a_{0,0} = 1 \quad , \quad a_{k,j} = 0 \ \text{ if } \ k < 0 \ \text{ or } \ k > j.$$

(iii) There exists a positive real number $\eta_{\alpha,\beta,\nu}$ such that

$$|c_{j,j}\, a_{k,j}| \le \eta^{j}_{\alpha,\beta,\nu} \quad , 0 \le k \le j < \infty.$$

Proof.

(i) Follows from Lemma 2.1 (i).

(ii) Applying the differential operator $D_{\alpha,\beta}$ we get

$$D^{j+1}_{\alpha,\beta} (\cosh^{-\nu} t) = \sum_{k=0}^{j} a_{k,j}\, D_{\alpha,\beta}\, \{\cosh^{-\nu} t \, (2\tanh^2 t - 1)^k\}.$$

Evaluating the right hand side of this equality yields the asserted recurrence relation.

(iii) Put $b_{k,j} = c_{j,j}\, a_{k,j}$, $0 \le k \le j < \infty$. Noting that $(2j+\nu)\,(j+\tfrac{1}{2}\nu-\beta)\, c_{j+1,j+1} = c_{j,j}$ it follows from the recurrence relation for the $a_{k,j}$ that

$$b_{k,j+1} = \frac{(2k+\nu-2)\,(k+\tfrac{1}{2}\nu-\beta-1)}{(2j+\nu)\,(j+\tfrac{1}{2}\nu-\beta)}\, b_{k-1,j} +$$

$$-\,\frac{(2k^2+2k(2\alpha+1)+\nu(2\alpha+\beta+2-\tfrac{1}{2}\nu)-(\alpha+\beta+1)^2)}{(2j+\nu)\,(j+\tfrac{1}{2}\nu-\beta)}\, b_{k,j} +$$

$$+\frac{2(k+1)(2\alpha+\beta+1-\nu-k)}{(2j+\nu)(j+\frac{1}{2}\nu-\beta)}\, b_{k+1,j}+\frac{2(k+2)(k+1)}{(2j+\nu)(j+\frac{1}{2}\nu-\beta)}\, b_{k+2,j}.$$

Preceding as in the proof of Theorem 2.2 (ii) yields the result. $\square$

3. The growth behaviour of the Fourier-Jacobi transform of a class of analytic functions.

We start with some auxiliary results. From [E, 2.3 (9)] we extract the following asymptotic formula for the hypergeometric function $z \mapsto {}_2F_1(a,b;c;z)$ for large values of $|z|$. Unless $a-b$ is an integer, there exist λ_1, λ_2 such that

$${}_2F_1(a,b;c;z)=\lambda_1 z^{-a}+\lambda_2 z^{-b}+O(z^{-a-1})+O(z^{-b-1}) \quad , |z| \to \infty.$$

If $a-b$ is an integer, z^{-a} or z^{-b} has to be multiplied by a factor $\log z$. Using these asymptotic formulas noting that $\mathrm{Re}(\nu) > \alpha+\beta+1$ and that the Jacobi function $\phi_\lambda^{(\alpha,\beta)}$ is the solution of the eigenvalue problem (3) it follows by partial integration that

$$\int_0^\infty D_{\alpha,\beta}\,(\cosh^{-\nu} t)\, \phi_\lambda^{(\alpha,\beta)}(t)\, \Delta_{\alpha,\beta}(t)\, dt = -\lambda^2 \int_0^\infty \cosh^{-\nu} t\, \phi_\lambda^{(\alpha,\beta)}(t)\, \Delta_{\alpha,\beta}(t)\, dt.$$

Substituting $x=\sinh^2 t$ in the latter integral, and using the integral formula [P, 2.21.1 (16)] we obtain the following explicit formula for the Fourier-Jacobi transform of $t \mapsto \cosh^{-\nu} t$,

$$\int_0^\infty \cosh^{-\nu} t\, \phi_\lambda^{(\alpha,\beta)}(t)\, \Delta_{\alpha,\beta}(t)\, dt =$$

$$=\frac{2^{2\alpha+2\beta+1}\,\Gamma(\alpha+1)}{\Gamma(\frac{1}{2}\nu)\,\Gamma(\frac{1}{2}\nu-\beta)}\,\Gamma(\tfrac{1}{2}(\nu-\alpha-\beta-1+i\lambda))\,\Gamma(\tfrac{1}{2}(\nu-\alpha-\beta-1-i\lambda)).$$

Let $\phi(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire analytic function and let $f(t)=\phi(1-2\tanh^2 t)\cosh^{-\nu} t$. Consider the following formal computation

$$\hat{f}^{(\alpha,\beta)}(\lambda)=\int_0^\infty \phi(1-2\tanh^2 t)\cosh^{-\nu} t\, \phi_\lambda^{(\alpha,\beta)}(t)\, \Delta_{\alpha,\beta}(t)\, dt =$$

$$=\int_0^\infty [\sum_{n=0}^\infty a_n(1-2\tanh^2 t)^n]\cosh^{-\nu} t\, \phi_\lambda^{(\alpha,\beta)}(t)\, \Delta_{\alpha,\beta}(t)\, dt =$$

$$=\sum_{n=0}^\infty a_n \int_0^\infty (1-2\tanh^2 t)^n \cosh^{-\nu} t\, \phi_\lambda^{(\alpha,\beta)}(t)\, \Delta_{\alpha,\beta}(t)\, dt =$$

$$=\sum_{n=0}^\infty (-1)^n a_n \sum_{j=0}^n c_{j,n} \int_0^\infty D_{\alpha,\beta}^j(\cosh^{-\nu} t)\, \phi_\lambda^{(\alpha,\beta)}(t)\, \Delta_{\alpha,\beta}(t)\, dt =$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^n a_n c_{j,n} (-\lambda^2)^j \int_0^{\infty} \cosh^{-\nu} t \, \phi_\lambda^{(\alpha,\beta)}(t) \, \Delta_{\alpha,\beta}(t) \, dt =$$

$$= \frac{2^{2\alpha+2\beta+1}\, \Gamma(\alpha+1)}{\Gamma(\frac{1}{2}\nu)\, \Gamma(\frac{1}{2}\nu-\beta)} \, \Gamma(\tfrac{1}{2}(\nu-\alpha-\beta-1+i\lambda)) \, \Gamma(\tfrac{1}{2}(\nu-\alpha-\beta-1-i\lambda)) \, \psi(\lambda)$$

with

$$\psi(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^{2j} \quad \text{and} \quad b_j = \sum_{n=j}^{\infty} (-1)^{n+j} c_{j,n} a_n .$$

In order to justify this formal calculation we proceed as follows. Introduce the vectors

$$\underline{a} = \text{column } (a_0, a_1, a_2, \cdots) \quad \text{and} \quad \underline{b} = \text{column } (b_0, b_1, b_2, \cdots)$$

and the infinite diagonal matrix

$$\bar{I} = \text{diag } (1, -1, 1, -1, \ldots, (-1)^n, \cdots).$$

Now the relation between the, supposed, Taylor coefficients of the functions ϕ and ψ can be written as

$$\underline{b} = \bar{I}\, C\, \bar{I}\, \underline{a} \quad \text{and} \quad \underline{a} = \bar{I}\, C^{-1}\, \bar{I}\, \underline{b}.$$

We introduce the following terminology. An entire analytic function $g(z)$ is called sub-exponential if

$$\forall_{\varepsilon>0} : \sup_{z \in \mathbb{C}} |g(z)| \, \exp(-\varepsilon |z|) < \infty.$$

The proof of the following characterization is elementary.

Characterization 3.1.

(i) Consider the Taylor series $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$. The function ϕ is entire analytic if and only if $\forall_{t>0} : (a_n e^{nt})_{n=0}^{\infty} \in l_2$.

(ii) Consider the Taylor series $\psi(z) = \sum_{n=0}^{\infty} b_n z^{2n}$. The function ψ is entire and sub-exponential if and only if $\forall_{t>0} : (b_n n^{2n} e^{nt})_{n=0}^{\infty} \in l_2$. □

In the next theorem we derive some fundamental estimates for the matrices C and C^{-1}.

Theorem 3.2.
For each $t > 0$ there exists $\tau > t$ such that the infinite upper triangular matrices

$$\Theta(t,\tau) := \text{diag } (n^{2n} e^{nt})\, \bar{I}\, C\, \bar{I} \text{ diag } (e^{-n\tau})$$

$$\Xi(t,\tau) := \text{diag } (e^{nt})\, \bar{I}\, C^{-1}\, \bar{I} \text{ diag } (e^{-n\tau} n^{-2n})$$

are bounded as l_2-operators.

Proof.
For $0 \leq j \leq n$ we have

$$|\Theta_{j,n}(t,\tau)| = j^{2j}\, e^{jt}\, |c_{j,j}| \left| \frac{c_{j,n}}{c_{j,j}} \right| e^{-n\tau}$$

$$|\Xi_{k,j}(t,\tau)| = e^{kt}\, |c_{j,j}\, a_{k,j}|\, \frac{1}{|c_{j,j}|}\, j^{-2j}\, e^{-j\tau}.$$

Taking τ sufficiently large the results follow with the aid of Theorems 2.2 and 2.3 and the estimate $\|K\| \leq \sum_{k=-\infty}^{\infty} \sup_{n-j=k} |K_{j,n}|$ for the l_2-operator norm $\|K\|$ of an infinite matrix K. □

Finally, our main result.

Theorem 3.3.
The mapping $F_{\alpha,\beta,\delta,\mu}$ which maps the space of polynomials bijectively on the space of even polynomials can be extended to a bijective continuous linear mapping between the space of entire functions and the space of even entire functions of sub-exponential growth.

Proof.
Let $t > 0$. Consider

$$\operatorname{diag}(n^{2n} e^{nt})\, \underline{b} = \operatorname{diag}(n^{2n} e^{nt})\, \tilde{I}\, C\, \tilde{I}\, \underline{a} =$$

$$= \{\operatorname{diag}(n^{2n} e^{nt})\, \tilde{I}\, C\, \tilde{I}\, \operatorname{diag}(e^{-n\tau})\}\, \operatorname{diag}(e^{n\tau})\, \underline{a}.$$

According to Theorem 3.2 the operator between { } is bounded in l_2 for τ sufficiently large. Furthermore, $\operatorname{diag}(e^{n\tau})\,\underline{a} \in l_2$ for all $\tau > 0$ (see Characterization 3.1(i)). So $\operatorname{diag}(n^{2n} e^{nt})\,\underline{b} \in l_2$. From Characterization 3.1(ii) we conclude that ψ is an entire analytic function of sub-exponential growth.
The inverse $F^{-1}_{\alpha,\beta,\delta,\mu}$, which corresponds to the equality $\underline{a} = \tilde{I}\, C^{-1}\, \tilde{I}\, \underline{b}$ can be dealt with in a similar way.
Thus all formal calculations at the beginning of this section become justified. □

Corollary 3.4.
The Fourier-Jacobi transform of index (α,β) establishes a bijection between the functions

$$\phi(1 - 2\tanh^2 t)\cosh^{-\nu} t \quad , \phi \text{ entire}$$

and the functions

$$\Gamma(\tfrac{1}{2}(\nu-\alpha-\beta-1+iz))\,\Gamma(\tfrac{1}{2}(\nu-\alpha-\beta-1-iz))\;\psi(z)$$

with ψ entire, even, ψ of sub-exponential growth. □

Corollary 3.5.
Comparison with the general formula in Section 1 shows

$$W_N(\tfrac{1}{4}x^2\,;\tfrac{1}{2}(\delta+i\mu+1),\tfrac{1}{2}(\delta-i\mu+1),\tfrac{1}{2}(\alpha+\beta+1),\tfrac{1}{2}(\alpha-\beta+1))=$$
$$=\frac{(-1)^N\,\Gamma(\tfrac{1}{2}(\alpha+\beta+\delta+i\mu+2)+N)\,\Gamma(\tfrac{1}{2}(\alpha-\beta+\delta+i\mu+2)+N)\,N!}{\Gamma(\tfrac{1}{2}(\alpha+\beta+\delta+i\mu+2))\,\Gamma(\tfrac{1}{2}(\alpha-\beta+\delta+i\mu+2))}\cdot$$
$$\cdot\sum_{j=0}^{N}\,[\sum_{n=j}^{N}(-1)^{n+j}\,c_{j,n}\,\alpha_n]\,x^{2j}$$

with α_n such that $P_N^{(\alpha,\delta)}(z)=\sum_{k=0}^{N}\alpha_k\,z^k$. □

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