

# The Fourier-Jacobi transform of analytic functions which are (almost) periodic in the imaginary direction

**Citation for published version (APA):**

van Berkel, C. A. M., & Graaf, de, J. (1989). *The Fourier-Jacobi transform of analytic functions which are (almost) periodic in the imaginary direction*. (RANA : reports on applied and numerical analysis; Vol. 8924). Technische Universiteit Eindhoven.

**Document status and date:**

Published: 01/01/1989

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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RANA 89-24

December 1989

**THE FOURIER-JACOBI TRANSFORM  
OF ANALYTIC FUNCTIONS WHICH  
ARE (ALMOST) PERIODIC IN  
THE IMAGINARY DIRECTION**

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**Summary**

We show that the Fourier-Jacobi transform of index  $(\alpha, \beta)$ ,  $\alpha > -1$ ,  $\beta \in \mathbb{R}$ , maps functions of the form

$$t \mapsto \phi(1 - 2 \tanh^2 t) \cosh^{-\nu} t,$$

with  $\phi$  an entire analytic function and  $\nu \in \mathbb{C}$ , such that  $\operatorname{Re}(\nu) > \alpha + \beta + 1$  and  $\frac{1}{2}\nu \notin \{0, -1, -2, \dots\}$  and  $\frac{1}{2}\nu - \beta \notin \{0, -1, -2, \dots\}$ , bijectively onto the functions

$$x \mapsto \Gamma\left(\frac{1}{2}(\nu - \alpha - \beta - 1 + ix)\right) \Gamma\left(\frac{1}{2}(\nu - \alpha - \beta - 1 - ix)\right) \psi(x).$$

Here  $\psi$  is an even and entire analytic function of sub-exponential growth, i.e.

$$\forall \varepsilon > 0 : \sup_{z \in \mathcal{L}} |\psi(z)| \exp(-\varepsilon |z|) < \infty.$$

Our treatment is based on recurrence relations.

A.M.S. Classifications: 33A65, 30D15, 42A38.

Key Words: Fourier-Jacobi transform, analytic functions.

### 1. Introduction

For  $\alpha, \beta \in \mathbb{C}$  we define the function  $\Delta_{\alpha, \beta} : \mathbb{R}^+ \rightarrow \mathbb{R}$  by

$$(1) \quad \Delta_{\alpha, \beta}(t) = (2 \sinh t)^{2\alpha+1} (2 \cosh t)^{2\beta+1}, \quad t > 0$$

and the differential operator  $D_{\alpha, \beta}$  by

$$(2) \quad D_{\alpha, \beta} = \frac{1}{\Delta_{\alpha, \beta}} \frac{d}{dt} \Delta_{\alpha, \beta} \frac{d}{dt} + (\alpha + \beta + 1)^2.$$

Consider the eigenvalue problem

$$(3) \quad \begin{cases} D_{\alpha, \beta} u = -\lambda^2 u \\ u'(0) = 0, \quad u(0) = 1. \end{cases}$$

By substituting  $z = -\sinh^2 t$  a hypergeometric differential equation is obtained with parameters  $\frac{1}{2}(\alpha + \beta + 1 + i\lambda), \frac{1}{2}(\alpha + \beta + 1 - i\lambda), \alpha + 1$  (cf. [E, 2.1(1)]). So if  $\alpha \neq -1, -2, -3, \dots$  the solution of (3) is given by

$$(4) \quad u(t) = \phi_{\lambda}^{(\alpha, \beta)}(t) = {}_2F_1\left(\frac{1}{2}(\alpha + \beta + 1 + i\lambda), \frac{1}{2}(\alpha + \beta + 1 - i\lambda); \alpha + 1; -\sinh^2 t\right).$$

Here  ${}_2F_1(a, b; c; z)$  denotes the hypergeometric function, which is the unique analytic continuation for  $z \notin [1, \infty)$  of the power series

$$(5) \quad \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1.$$

The function  $\phi_{\lambda}^{(\alpha, \beta)}$  is called the Jacobi function of the first kind and of index  $(\alpha, \beta)$ .

The Fourier-Jacobi transform of index  $(\alpha, \beta)$ ,  $f \mapsto \tilde{f}^{(\alpha, \beta)}$ , is formally defined by

$$(6) \quad \tilde{f}^{(\alpha, \beta)}(\lambda) = \int_0^{\infty} f(t) \phi_{\lambda}^{(\alpha, \beta)}(t) \Delta_{\alpha, \beta}(t) dt.$$

It is well known, see [K], that the Fourier-Jacobi transform of index  $(\alpha, \beta)$  maps the function

$$(7) \quad f(t) = (\cosh t)^{-\alpha - \beta - \delta - i\mu - 2} P_n^{(\alpha, \delta)}(1 - 2 \tanh^2 t)$$

onto the function

$$(8) \quad \tilde{f}^{(\alpha, \beta)}(\lambda) = \frac{2^{2\alpha+2\beta+1} \Gamma(\alpha+1) (-1)^n}{n! \Gamma(\frac{1}{2}(\alpha + \beta + \delta + i\mu + 2) + n) \Gamma(\frac{1}{2}(\alpha - \beta + \delta + i\mu + 2) + n)} \cdot \\ \cdot \{ \Gamma(\frac{1}{2}(\delta + i\mu + 1 + i\lambda)) \Gamma(\frac{1}{2}(\delta + i\mu + 1 - i\lambda)) \} \cdot \\ \cdot W_n\left(\frac{1}{4} \lambda^2; \frac{1}{2}(\delta + i\mu + 1), \frac{1}{2}(\delta - i\mu + 1), \frac{1}{2}(\alpha + \beta + 1), \frac{1}{2}(\alpha - \beta + 1)\right),$$

$$\beta, \lambda, \mu \in \mathbb{R}, \quad n \in \mathbb{N} \cup \{0\}, \quad \alpha, \delta > -1.$$

Here the  $P_n^{(\alpha, \delta)}$  are Jacobi polynomials and the  $W_n$  are Wilson polynomials. If we abandon the factor between  $\{ \}$  and if we keep the parameters  $\alpha, \beta, \delta, \mu$  fixed, then it is clear, that via the Fourier-Jacobi transform the space of polynomials is mapped linearly and bijectively on the space of even polynomials. Let us denote this linear mapping by  $F_{\alpha, \beta, \delta, \mu}$ .

We pose ourselves the following problem: Extend  $F_{\alpha, \beta, \delta, \mu}$  bijectively to suitable spaces of analytic functions.

In [BG] we studied the mapping  $F_{-\frac{1}{2}, -\frac{1}{2}, \delta, \mu}$ . In that special case the Fourier-Jacobi transform reduces to the Fourier-cosine transform. As an extension of the results in the paper [BG] we now study the general mapping  $F_{\alpha, \beta, \delta, \mu}$ .

At this point we emphasize that our treatment is inspired by Koornwinder's formula (8) but does not use it.

## 2. A special infinite upper triangular matrix

In the sequel we take  $\alpha > -1$ ,  $\beta \in \mathbb{R}$ ,  $\nu \in \mathbb{C}$  with  $\operatorname{Re} \nu > \alpha + \beta + 1$  and  $\frac{1}{2} \nu, \frac{1}{2} \nu - \beta \neq 0, -1, -2, \dots$  fixed. We denote  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

### Lemma 2.1.

(i) For each  $n \in \mathbb{N}_0$  there exist complex numbers  $c_{j,n}$ ,  $0 \leq j \leq n$ , such that

$$(2 \tanh^2 t - 1)^n \cosh^{-\nu} t = \sum_{j=0}^n c_{j,n} D_{\alpha, \beta}^j (\cosh^{-\nu} t), \quad t \in \mathbb{R}.$$

(ii) The numbers  $c_{j,n}$ ,  $0 \leq j \leq n$ , satisfy the recurrence relation

$$\begin{aligned} (2n + \nu) \left(n + \frac{1}{2} \nu - \beta\right) c_{j,n+1} &= c_{j-1,n} + \\ &+ (2n^2 + 2n(2\alpha + 1) + \nu(2\alpha + \beta + 2 - \frac{1}{2} \nu) - (\alpha + \beta + 1)^2) c_{j,n} + \\ &- 2n(2\alpha + \beta + 2 - \nu - n) c_{j,n-1} - 2n(n-1) c_{j,n-2} \quad , \quad 0 \leq j \leq n+1 \end{aligned}$$

with boundary conditions

$$c_{0,0} = 1 \quad , \quad c_{j,n} = 0 \quad \text{if } j < 0 \quad \text{or } j > n.$$

### Proof.

The proof is by induction. Obviously (i) is true for  $n = 0$ , then  $c_{0,0} = 1$ . Next suppose (i) is true for  $n = 0, 1, \dots, N$ . Applying the differential operator  $D_{\alpha, \beta}$  we get

$$D_{\alpha, \beta} \left\{ (2 \tanh^2 t - 1)^N \cosh^{-\nu} t \right\} = \sum_{j=1}^{N+1} c_{j-1,N} D_{\alpha, \beta}^j (\cosh^{-\nu} t).$$

Evaluating the left hand side of this equality, using the induction hypothesis, leads to both assertions (i) and (ii) at once. □

In the following theorem we gather some properties of the numbers  $c_{j,n}$ ,  $0 \leq j \leq n$ .

**Theorem 2.2.**

(i) 
$$c_{j,j} = \frac{2^{-j} \Gamma(\frac{1}{2} \nu) \Gamma(\frac{1}{2} \nu - \beta)}{\Gamma(j + \frac{1}{2} \nu) \Gamma(j + \frac{1}{2} \nu - \beta)}, \quad j \geq 0.$$

(ii) There exists a positive real number  $\xi_{\alpha, \beta, \nu}$  such that

$$\left| \frac{c_{j,n}}{c_{j,j}} \right| \leq \xi_{\alpha, \beta, \nu}^n, \quad 0 \leq j \leq n < \infty.$$

(iii) 
$$\lim_{j \rightarrow \infty} \{c_{j,j} (e^{-1} \sqrt{2} j)^{2j} j^{\nu-\beta-1}\} = \frac{1}{2\pi} \Gamma(\frac{1}{2} \nu) \Gamma(\frac{1}{2} \nu - \beta).$$

*Proof.*

(i) Take  $n = j - 1$  in the recurrence relation, then  $(2j + \nu - 2) (j + \frac{1}{2} \nu - \beta - 1) c_{j,j} = c_{j-1, j-1}$ . Noting that  $c_{0,0} = 1$  the result follows.

(ii) Put  $d_{j,n} = \frac{c_{j,n}}{c_{j,j}}$ ,  $0 \leq j \leq n < \infty$ . Then from the recurrence relation we obtain,

$$\begin{aligned} d_{j+1, n+1} &= \frac{(2j + \nu) (j + \frac{1}{2} \nu - \beta)}{(2n + \nu) (n + \frac{1}{2} \nu - \beta)} d_{j,n} + \\ &+ \frac{2n^2 + 2n(2\alpha + 1) + \nu(2\alpha + \beta + 2 - \frac{1}{2} \nu) - (\alpha + \beta + 1)^2}{(2n + \nu) (n + \frac{1}{2} \nu - \beta)} d_{j+1, n} + \\ &+ \frac{2n(2\alpha + \beta + 2 - \nu - n)}{(2n + \nu) (n + \frac{1}{2} \nu - \beta)} d_{j+1, n-1} + \frac{2n(n-1)}{(2n + \nu) (n + \frac{1}{2} \nu - \beta)} d_{j+1, n-2}. \end{aligned}$$

Since  $\frac{1}{2} \nu, \frac{1}{2} \nu - \beta \neq 0, -1, -2, \dots$ , there exists  $\epsilon > 0$  such that  $|n + \frac{1}{2} \nu| > \epsilon$  and  $|n + \frac{1}{2} \nu - \beta| > \epsilon$  for all  $n \in \mathbb{N}_0$ . Applying the triangle inequality we estimate, for instance,

$$\left| \frac{2j + \nu}{2n + \nu} \right| \leq 1 + \frac{n}{|n + \frac{1}{2} \nu|} \leq 2 + \frac{|\nu|}{2\epsilon}, \quad 0 \leq j \leq n.$$

So it easily follows that there exists  $\xi_{\alpha, \beta, \nu} > 1$  such that

$$\begin{aligned} |d_{j+1, n+1}| &\leq \frac{1}{4} \xi_{\alpha, \beta, \nu} (|d_{j,n}| + |d_{j+1, n}| + |d_{j+1, n-1}| + |d_{j+1, n-2}|), \\ & \quad , -1 \leq j \leq n. \end{aligned}$$

Now apply induction.

(iii) Follows from (i) and Stirling's formula. □

We gather the numbers  $c_{j,n}$  in an upper triangular matrix  $C = [c_{j,n}]_{j,n=0}^{\infty}$ . The next theorem gives some results on the inverse  $C^{-1}$  of  $C$  which is also an upper triangular matrix. The proof does not differ much from the preceding proofs.

**Theorem 2.3.**

(i) The elements  $a_{k,j}$ ,  $0 \leq k \leq j < \infty$  of  $C^{-1}$  satisfy

$$D_{\alpha,\beta}^j (\cosh^{-\nu} t) = \sum_{k=0}^j a_{k,j} \cosh^{-\nu} t (2 \tanh^2 t - 1)^k, \quad t \in \mathbb{R}.$$

(ii) The numbers  $a_{k,j}$ ,  $0 \leq k \leq j$ , satisfy the recurrence relation

$$\begin{aligned} a_{k,j+1} &= (2k + \nu - 2) \left(k + \frac{1}{2} \nu - \beta - 1\right) a_{k-1,j} + \\ &- (2k^2 + 2k(2\alpha + 1) + \nu(2\alpha + \beta + 2 - \frac{1}{2} \nu) - (\alpha + \beta + 1)^2) a_{k,j} + \\ &+ 2(k+1) (2\alpha + \beta + 1 - \nu - k) a_{k+1,j} + 2(k+2) (k+1) a_{k+2,j}, \quad 0 \leq k \leq j+1 \end{aligned}$$

with boundary conditions

$$a_{0,0} = 1, \quad a_{k,j} = 0 \text{ if } k < 0 \text{ or } k > j.$$

(iii) There exists a positive real number  $\eta_{\alpha,\beta,\nu}$  such that

$$|c_{j,j} a_{k,j}| \leq \eta_{\alpha,\beta,\nu}^j, \quad 0 \leq k \leq j < \infty.$$

*Proof.*

(i) Follows from Lemma 2.1 (i).

(ii) Applying the differential operator  $D_{\alpha,\beta}$  we get

$$D_{\alpha,\beta}^{j+1} (\cosh^{-\nu} t) = \sum_{k=0}^j a_{k,j} D_{\alpha,\beta} \{ \cosh^{-\nu} t (2 \tanh^2 t - 1)^k \}.$$

Evaluating the right hand side of this equality yields the asserted recurrence relation.

(iii) Put  $b_{k,j} = c_{j,j} a_{k,j}$ ,  $0 \leq k \leq j < \infty$ . Noting that  $(2j + \nu) (j + \frac{1}{2} \nu - \beta) c_{j+1,j+1} = c_{j,j}$  it follows from the recurrence relation for the  $a_{k,j}$  that

$$\begin{aligned} b_{k,j+1} &= \frac{(2k + \nu - 2) \left(k + \frac{1}{2} \nu - \beta - 1\right)}{(2j + \nu) \left(j + \frac{1}{2} \nu - \beta\right)} b_{k-1,j} + \\ &- \frac{(2k^2 + 2k(2\alpha + 1) + \nu(2\alpha + \beta + 2 - \frac{1}{2} \nu) - (\alpha + \beta + 1)^2)}{(2j + \nu) \left(j + \frac{1}{2} \nu - \beta\right)} b_{k,j} + \end{aligned}$$

$$+ \frac{2(k+1)(2\alpha+\beta+1-\nu-k)}{(2j+\nu)(j+\frac{1}{2}\nu-\beta)} b_{k+1,j} + \frac{2(k+2)(k+1)}{(2j+\nu)(j+\frac{1}{2}\nu-\beta)} b_{k+2,j}.$$

Preceding as in the proof of Theorem 2.2 (ii) yields the result. □

### 3. The growth behaviour of the Fourier-Jacobi transform of a class of analytic functions.

We start with some auxiliary results. From [E, 2.3 (9)] we extract the following asymptotic formula for the hypergeometric function  $z \mapsto {}_2F_1(a, b; c; z)$  for large values of  $|z|$ . Unless  $a-b$  is an integer, there exist  $\lambda_1, \lambda_2$  such that

$${}_2F_1(a, b; c; z) = \lambda_1 z^{-a} + \lambda_2 z^{-b} + O(z^{-a-1}) + O(z^{-b-1}), \quad |z| \rightarrow \infty.$$

If  $a-b$  is an integer,  $z^{-a}$  or  $z^{-b}$  has to be multiplied by a factor  $\log z$ . Using these asymptotic formulas noting that  $\text{Re}(\nu) > \alpha + \beta + 1$  and that the Jacobi function  $\phi_\lambda^{(\alpha, \beta)}$  is the solution of the eigenvalue problem (3) it follows by partial integration that

$$\int_0^\infty D_{\alpha, \beta}(\cosh^{-\nu} t) \phi_\lambda^{(\alpha, \beta)}(t) \Delta_{\alpha, \beta}(t) dt = -\lambda^2 \int_0^\infty \cosh^{-\nu} t \phi_\lambda^{(\alpha, \beta)}(t) \Delta_{\alpha, \beta}(t) dt.$$

Substituting  $x = \sinh^2 t$  in the latter integral, and using the integral formula [P, 2.21.1 (16)] we obtain the following explicit formula for the Fourier-Jacobi transform of  $t \mapsto \cosh^{-\nu} t$ ,

$$\begin{aligned} & \int_0^\infty \cosh^{-\nu} t \phi_\lambda^{(\alpha, \beta)}(t) \Delta_{\alpha, \beta}(t) dt = \\ & = \frac{2^{2\alpha+2\beta+1} \Gamma(\alpha+1)}{\Gamma(\frac{1}{2}\nu) \Gamma(\frac{1}{2}\nu-\beta)} \Gamma(\frac{1}{2}(\nu-\alpha-\beta-1+i\lambda)) \Gamma(\frac{1}{2}(\nu-\alpha-\beta-1-i\lambda)). \end{aligned}$$

Let  $\phi(z) = \sum_{n=0}^\infty a_n z^n$  be an entire analytic function and let  $f(t) = \phi(1-2 \tanh^2 t) \cosh^{-\nu} t$ . Consider the following formal computation

$$\begin{aligned} \hat{f}^{(\alpha, \beta)}(\lambda) &= \int_0^\infty \phi(1-2 \tanh^2 t) \cosh^{-\nu} t \phi_\lambda^{(\alpha, \beta)}(t) \Delta_{\alpha, \beta}(t) dt = \\ &= \int_0^\infty \left[ \sum_{n=0}^\infty a_n (1-2 \tanh^2 t)^n \right] \cosh^{-\nu} t \phi_\lambda^{(\alpha, \beta)}(t) \Delta_{\alpha, \beta}(t) dt = \\ &= \sum_{n=0}^\infty a_n \int_0^\infty (1-2 \tanh^2 t)^n \cosh^{-\nu} t \phi_\lambda^{(\alpha, \beta)}(t) \Delta_{\alpha, \beta}(t) dt = \\ &= \sum_{n=0}^\infty (-1)^n a_n \sum_{j=0}^n c_{j,n} \int_0^\infty D_{\alpha, \beta}^j(\cosh^{-\nu} t) \phi_\lambda^{(\alpha, \beta)}(t) \Delta_{\alpha, \beta}(t) dt = \end{aligned}$$



$$\begin{aligned}
 &= \sum_{n=0}^{\infty} \sum_{j=0}^n (-1)^n a_n c_{j,n} (-\lambda^2)^j \int_0^{\infty} \cosh^{-\nu} t \phi_{\lambda}^{(\alpha,\beta)}(t) \Delta_{\alpha,\beta}(t) dt = \\
 &= \frac{2^{2\alpha+2\beta+1} \Gamma(\alpha+1)}{\Gamma(\frac{1}{2}\nu) \Gamma(\frac{1}{2}\nu-\beta)} \Gamma(\frac{1}{2}(\nu-\alpha-\beta-1+i\lambda)) \Gamma(\frac{1}{2}(\nu-\alpha-\beta-1-i\lambda)) \psi(\lambda)
 \end{aligned}$$

with

$$\psi(\lambda) = \sum_{j=0}^{\infty} b_j \lambda^{2j} \quad \text{and} \quad b_j = \sum_{n=j}^{\infty} (-1)^{n+j} c_{j,n} a_n.$$

In order to justify this formal calculation we proceed as follows. Introduce the vectors

$$\underline{a} = \text{column } (a_0, a_1, a_2, \dots) \quad \text{and} \quad \underline{b} = \text{column } (b_0, b_1, b_2, \dots)$$

and the infinite diagonal matrix

$$\bar{I} = \text{diag } (1, -1, 1, -1, \dots, (-1)^n, \dots).$$

Now the relation between the, supposed, Taylor coefficients of the functions  $\phi$  and  $\psi$  can be written as

$$\underline{b} = \bar{I} C \bar{I} \underline{a} \quad \text{and} \quad \underline{a} = \bar{I} C^{-1} \bar{I} \underline{b}.$$

We introduce the following terminology. An entire analytic function  $g(z)$  is called sub-exponential if

$$\forall_{\varepsilon > 0} : \sup_{z \in \mathcal{L}} |g(z)| \exp(-\varepsilon |z|) < \infty.$$

The proof of the following characterization is elementary.

### Characterization 3.1.

(i) Consider the Taylor series  $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ . The function  $\phi$  is entire analytic if and only if

$$\forall_{t > 0} : (a_n e^{nt})_{n=0}^{\infty} \in l_2.$$

(ii) Consider the Taylor series  $\psi(z) = \sum_{n=0}^{\infty} b_n z^{2n}$ . The function  $\psi$  is entire and sub-exponential if

$$\text{and only if } \forall_{t > 0} : (b_n n^{2n} e^{nt})_{n=0}^{\infty} \in l_2. \quad \square$$

In the next theorem we derive some fundamental estimates for the matrices  $C$  and  $C^{-1}$ .

### Theorem 3.2.

For each  $t > 0$  there exists  $\tau > t$  such that the infinite upper triangular matrices

$$\Theta(t, \tau) := \text{diag } (n^{2n} e^{nt}) \bar{I} C \bar{I} \text{diag } (e^{-n\tau})$$

$$\Xi(t, \tau) := \text{diag } (e^{nt}) \bar{I} C^{-1} \bar{I} \text{diag } (e^{-n\tau} n^{-2n})$$

are bounded as  $l_2$ -operators.

*Proof.*

For  $0 \leq j \leq n$  we have

$$|\Theta_{j,n}(t, \tau)| = j^{2j} e^{jt} |c_{j,j}| \left| \frac{c_{j,n}}{c_{j,j}} \right| e^{-n\tau}$$

$$|\Xi_{k,j}(t, \tau)| = e^{kt} |c_{j,j} a_{k,j}| \frac{1}{|c_{j,j}|} j^{-2j} e^{-j\tau}.$$

Taking  $\tau$  sufficiently large the results follow with the aid of Theorems 2.2 and 2.3 and the estimate  $\|K\| \leq \sum_{k=-\infty}^{\infty} \sup_{n-j=k} |K_{j,n}|$  for the  $l_2$ -operator norm  $\|K\|$  of an infinite matrix  $K$ . □

Finally, our main result.

**Theorem 3.3.**

The mapping  $F_{\alpha, \beta, \delta, \mu}$  which maps the space of polynomials bijectively on the space of even polynomials can be extended to a bijective continuous linear mapping between the space of entire functions and the space of even entire functions of sub-exponential growth.

*Proof.*

Let  $t > 0$ . Consider

$$\text{diag} (n^{2n} e^{nt}) \underline{b} = \text{diag} (n^{2n} e^{nt}) \bar{I} C \bar{I} \underline{a} =$$

$$= \{ \text{diag} (n^{2n} e^{nt}) \bar{I} C \bar{I} \text{diag} (e^{-n\tau}) \} \text{diag} (e^{n\tau}) \underline{a}.$$

According to Theorem 3.2 the operator between  $\{ \}$  is bounded in  $l_2$  for  $\tau$  sufficiently large. Furthermore,  $\text{diag} (e^{n\tau}) \underline{a} \in l_2$  for all  $\tau > 0$  (see Characterization 3.1(i)). So  $\text{diag} (n^{2n} e^{nt}) \underline{b} \in l_2$ . From Characterization 3.1(ii) we conclude that  $\psi$  is an entire analytic function of sub-exponential growth.

The inverse  $F_{\alpha, \beta, \delta, \mu}^{-1}$ , which corresponds to the equality  $\underline{a} = \bar{I} C^{-1} \bar{I} \underline{b}$  can be dealt with in a similar way.

Thus all formal calculations at the beginning of this section become justified. □

**Corollary 3.4.**

The Fourier-Jacobi transform of index  $(\alpha, \beta)$  establishes a bijection between the functions

$$\phi(1 - 2 \tanh^2 t) \cosh^{-\nu} t, \phi \text{ entire}$$

and the functions

$$\Gamma\left(\frac{1}{2}(\nu - \alpha - \beta - 1 + iz)\right) \Gamma\left(\frac{1}{2}(\nu - \alpha - \beta - 1 - iz)\right) \psi(z)$$

with  $\psi$  entire, even,  $\psi$  of sub-exponential growth. □

**Corollary 3.5.**

Comparison with the general formula in Section 1 shows

$$\begin{aligned}
 & W_N\left(\frac{1}{4}x^2; \frac{1}{2}(\delta+i\mu+1), \frac{1}{2}(\delta-i\mu+1), \frac{1}{2}(\alpha+\beta+1), \frac{1}{2}(\alpha-\beta+1)\right) = \\
 & = \frac{(-1)^N \Gamma\left(\frac{1}{2}(\alpha+\beta+\delta+i\mu+2)+N\right) \Gamma\left(\frac{1}{2}(\alpha-\beta+\delta+i\mu+2)+N\right) N!}{\Gamma\left(\frac{1}{2}(\alpha+\beta+\delta+i\mu+2)\right) \Gamma\left(\frac{1}{2}(\alpha-\beta+\delta+i\mu+2)\right)} \\
 & \cdot \sum_{j=0}^N \left[ \sum_{n=j}^N (-1)^{n+j} c_{j,n} \alpha_n \right] x^{2j}
 \end{aligned}$$

with  $\alpha_n$  such that  $P_N^{(\alpha,\delta)}(z) = \sum_{k=0}^N \alpha_k z^k$ .

□

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