# Iterative solution methods of the Maxwell equations using staggered grid spatial discretization

**EINDHOVEN UNIVERSITY OF TECHNOLOGY**
Department of Mathematics and Computing Science

RANA 04-18
August 2004

# Iterative solution methods of the Maxwell equations using staggered grid spatial discretization

by

R. Horváth, I. Faragó, W.H.A. Schilders

Reports on Applied and Numerical Analysis
Department of Mathematics and Computer Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

# Iterative Solution Methods of the Maxwell Equations Using Staggered Grid Spatial Discretization

Róbert Horváth* István Faragó† W.H.A. Schilders‡

**Abstract**

In this paper we solve the Maxwell equations with finite difference methods. We employ the staggered spatial discretization in order to obtain a system of first order linear ordinary differential equations with a sparse coefficient matrix. This system is solved by different iterative methods. Namely, we apply the explicit Euler method, the implicit Euler method combining with the gradient iteration, the Namiki-Zheng-Chen-Zhang (NZCZ) alternating direction implicit method, the Kole-Figge-de Raedt method and a Krylov-space method. The considered methods are compared with the classical Yee-method from the point of view of computational speed, stability and accuracy. Our result is that the NZCZ and the Krylov space methods can be more efficient than the Yee-method.

**Keywords.** FDTD, Maxwell-equations, Krylov-space methods, Arnoldi orthogonalization

**AMS subject classifications.** 35L05, 65M06, 65M12, 78M20

## 1 Introduction

The 3D Maxwell equations, which describe the behavior of time-dependent electromagnetic fields, in the absence of free charges and currents, can be written in the form

$$-\nabla \times \mathbf{H} + \varepsilon \partial_t \mathbf{E} = \mathbf{0}, \tag{1}$$

*University of West-Hungary, Hungary, Institute of Mathematics and Statistics, H-9400 Sopron, Erzsébet u. 9., E-mail: rhorvath@ktk.nyme.hu

†Eötvös Loránd University, Hungary, Department of Applied Analysis, H-1117 Budapest, Pázmány P. sétány 1/C., E-mail: faragois@cs.elte.hu.

‡Philips Research Laboratories Eindhoven, the Netherlands, 5656AA Eindhoven, Prof. Holstlaan 4., E-mail: wil.schilders@philips.com

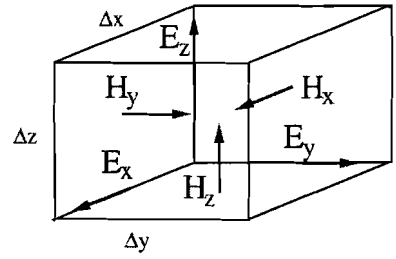


Figure 1: *Standard Yee cell.*

$$\nabla \times \mathbf{E} + \mu \partial_t \mathbf{H} = \mathbf{0}, \tag{2}$$

$$\nabla(\varepsilon \mathbf{E}) = 0, \tag{3}$$

$$\nabla(\mu \mathbf{H}) = 0, \tag{4}$$

where

$$\mathbf{E} = (E_x(t,x,y,z), E_y(t,x,y,z), E_z(t,x,y,z)) \tag{5}$$

is the electric field strength,

$$\mathbf{H} = (H_x(t,x,y,z), H_y(t,x,y,z), H_z(t,x,y,z)) \tag{6}$$

is the magnetic field strength, $\varepsilon$ is the electric permittivity and $\mu$ is the magnetic permeability (see [9] and [13] for more details). The two material parameters can depend on the spatial coordinates. It is well-known that the divergence equations (3) and (4) follow from the curl equations (1) and (2) if we suppose that the fields in question were divergence-free at the initial point of time. This means that we must solve only the curl equations applying divergence-free initial conditions for $\mathbf{E}$ and $\mathbf{H}$.

A lot of numerical Maxwell solution methods are known. Some of them are the finite difference method, the variational method, the method of moments, the finite element method, the transmission line matrix method, the Monte Carlo method and the method of lines ([9, 11, 13]). In this paper we investigate those finite difference methods that employ the staggered spatial discretization in definition of a system of linear first order ordinary differential equations.

In order to obtain the above mentioned semi-discretized system we define a rectangular mesh with the step-sizes $\Delta x, \Delta y$ and $\Delta z$ for the electric field and another staggered (by $\Delta x/2, \Delta y/2$ and $\Delta z/2$) grid for the magnetic field in the computational domain. The building blocks of these meshes are the so-called Yee-cells (see Figure 1). The staggered grid structure was firstly successfully applied by Yee ([16]) in 1966.

Let us consider the

$$\partial_t(\sqrt{\varepsilon}\mathbf{E}) = \frac{1}{\sqrt{\varepsilon}}\nabla \times \frac{(\sqrt{\mu}\mathbf{H})}{\sqrt{\mu}}, \tag{7}$$

$$\partial_t(\sqrt{\mu}\mathbf{H}) = -\frac{1}{\sqrt{\mu}}\nabla \times \frac{(\sqrt{\varepsilon}\mathbf{E})}{\sqrt{\varepsilon}}, \tag{8}$$

rearranged form of the curl equations. Discretizing these equations at the points shown in Figure 1, we arrive at the system of ordinary differential equations

$$\frac{\mathrm{d}\Psi(t)}{\mathrm{d}t} = \mathbf{A}\Psi(t),\ t > 0, \Psi(0) \text{ is given} \tag{9}$$

(see [1] for details). $\Psi(t) \in \mathbb{R}^{6N}$, $N$ is the number of the Yee-cells in the computational domain, gives the approximations of the components $\sqrt{\varepsilon}E_x, ..., \sqrt{\mu}H_z$ at the discretization points at the time instant $t$. Every row of $\mathbf{A} \in \mathbb{R}^{6N\times 6N}$ consists at most four nonzero elements in the forms $\pm 1/(\sqrt{\varepsilon_{.,.,.}\mu_{.,.,.}}\Delta_.)$, that is $\mathbf{A}$ is a sparse matrix. Moreover, $\mathbf{A}$ is a skew-symmetric matrix ($\mathbf{A}^\top = -\mathbf{A}$). Because of the skew-symmetry of $\mathbf{A}$ the eigenvalues can be written in the form $\pm \mathtt{i}\lambda_k$, where $k = 1, \ldots, 3N$, $\lambda_k > 0$ and $\mathtt{i} = \sqrt{-1}$. Applying the Gerschgoren theorem we obtain the upper bound $|\lambda_k| \leq 4c/h$ ($k = 1, \ldots, 3N$), where $h = \min\{\Delta x, \Delta y, \Delta z\}$ and $c$ is the maximal speed of light in the computational domain ($c = \max\{1/\sqrt{\varepsilon\mu}\}$).

We know from the theory of ordinary differential equations that the solution of (9) can be written in the form

$$\Psi(t) = \exp(t\mathbf{A})\Psi(0), \tag{10}$$

where $\exp(t\mathbf{A})$ denotes the matrix exponential and it is well-defined with the Taylor-series of the exponential function. This matrix exponential cannot be computed directly because $\mathbf{A}$ is, in real life problems, a very large matrix. According to the form (10), the numerical methods for the Maxwell equations are based on some approximation of the matrix exponential $\exp(t\mathbf{A})$. With the choice of a time-step $\Delta t > 0$

$$\Psi(t + \Delta t) = \exp(\Delta t\mathbf{A})\Psi(t) \tag{11}$$

follows from (10). Using this equality the one-step iteration

$$\Psi^{n+1} = U_n(\Delta t\mathbf{A})\Psi^n,\ \Psi^0 \text{ is given} \tag{12}$$

can be defined, where $U_n(\Delta t\mathbf{A})$ is the approximation of the exponential $\exp(\Delta t\mathbf{A})$ (this approximation may depend on $n$) and $\Psi^n$ is the approximation of the function $\Psi$ at the time-level $n\Delta t$.

In this paper we investigate several time-integration schemes for (9). These methods will differ only in the definition of the exponential approximation $U_n(\Delta t\mathbf{A})$. At the end of this paper we will compare the schemes from the point of view of computational speed, stability and accuracy.

# 2 Time integration schemes for the semi-discretized Maxwell equations

In this section we list some possible time-integration methods for (9). In order to compare the methods, we calculate the number of operations per one iteration step, moreover, we discuss the question of stability of the methods. We call the numerical solution method stable, when the relation $\| \Psi^n \|_2 \leq K \cdot \| \Psi^0 \|_2$ is valid for some fixed constant $K$ and for all natural number $n$. Let us introduce the notation $q = c\Delta t/h$.

## 2.1 Explicit Euler method

The most evident method, the explicit Euler method, is investigated first. The method approximates the exponential $\exp(\Delta t\mathbf{A})$ by the first two terms of the series of the exponential function. That is, $U_n(\Delta t\mathbf{A}) = \mathbf{I} + \Delta t\mathbf{A}$. The matrix $\mathbf{I}$ denotes the identity matrix. This iteration method is very fast (the number of operation per time-step is $36N$), but the method is not stable, so it is not usable in practice.

**Theorem 2.1** *The numerical solution of the Maxwell equations using staggered spatial discretization and explicit Euler time-integration is unstable.*

*Proof.* It is sufficient to show that the modulus of the eigenvalues of $\mathbf{A}$ are greater than one. We have

$$|\Lambda_{\pm k}| = |1 \pm \mathrm{i}\Delta t\lambda_k| = 1 + \Delta t^2\lambda_k^2 > 1.\blacksquare \tag{13}$$

## 2.2 Implicit Euler method

The second solution method is the implicit Euler method, where we employ the approximation $U_n(\Delta t\mathbf{A}) = (I - \Delta t\mathbf{A})^{-1}$.

**Theorem 2.2** *The numerical solution of the Maxwell equations using staggered spatial discretization and implicit Euler time-integration is unconditionally stable.*

*Proof.* The unconditional stability can be shown with the inequality

$$\| \Psi^{n+1} \|_2^2 = \| (\mathbf{I} - \Delta t\mathbf{A})^{-1}\Psi^n \|_2^2 \leq \| (\mathbf{I} - \Delta t\mathbf{A})^{-1} \|_2^2 \cdot \| \Psi^n \|_2^2 = \tag{14}$$

$$= \frac{1}{1 + \Delta t^2 \cdot \min_{k=1,\ldots,3N}\{\lambda_k^2\}} \| \Psi^n \|_2^2 \leq \| \Psi^n \|_2^2 .\blacksquare \tag{15}$$

Naturally, in practice, we do not compute the inverse of $\mathbf{I} - \Delta t\mathbf{A}$, but we solve a system of linear algebraic equations in the form

$$(\mathbf{I} - \Delta t\mathbf{A})\Psi^{n+1} = \Psi^n \tag{16}$$

in every time-step. Because the coefficient matrix of the system is a sparse one, we prefer the iterative solution method. Let us choose the simple gradient iteration

$$\Psi^{n+1}_{(l+1)} = [(1-\omega)\mathbf{I} + \omega\Delta t\mathbf{A}]\,\Psi^{n+1}_{(l)} + \omega\Psi^n, \quad l = 0, 1, \ldots, \quad \Psi^{n+1}_{(0)} = \Psi^n, \tag{17}$$

where $\omega$ is a suitably chosen positive constant. The method is convergent if and only if the spectral radius of the iteration matrix is less than one. So, we obtain the necessary and sufficient condition of the convergence

$$0 < \omega < \frac{2}{1 + \Delta t^2 \max_{k=1,\ldots,3N}\{\lambda_k^2\}}. \tag{18}$$

The smaller the spectral radius of the iteration matrix, the faster the convergence. Analyzing the second order (in $\omega$) form of the spectral radius we can find that to achieve the fastest convergence the parameter $\omega$ must be chosen according to the equality

$$\omega = \frac{1}{1 + \Delta t^2 \max_{k=1,\ldots,3N}\{\lambda_k^2\}}. \tag{19}$$

Inserting this parameter into the expression of the spectral radius we have

$$\varrho([(1-\omega)\mathbf{I} + \omega\Delta t\mathbf{A}]) = \sqrt{\omega^2(1+\Delta t^2)\max_{k=1,\ldots,3N}\{\lambda_k^2\} + 1 - 2\omega} = \tag{20}$$

$$= 1 - \frac{1}{1 + \Delta t^2 \max_{k=1,\ldots,3N}\{\lambda_k^2\}} \leq 1 - \frac{1}{1+16q^2}. \tag{21}$$

Although the implicit Euler method is unconditionally stable, which would make possible the choice of arbitrarily large time-steps, increasing $\Delta t$ the iteration method will be slower because of the relatively large spectral radius. When we would like to solve the system of linear equations decreasing the error of the initial approximation with a factor of $10^6$, then we have to perform about 75 iterations (choosing $q = 1/\sqrt{3}$, which will be the maximal value for $q$ in the Yee-method). So we would have $75 \cdot 7 \cdot 6N = 3150N$ operations per time-step. This number of operations increases dramatically increasing the time-step (for $q = 2$ we have $18816N$ operations per time-step).

## 2.3 The Yee-method

Yee derived the first efficient finite difference solution method for the Maxwell equations in 1966 (see [16]). This method uses a so-called leap-frog time integration scheme, for which the electric field at $t = 0$ and the magnetic field at $t = \Delta t/2$ must be given. Now we show that the Yee-method is also based on the matrix exponential approximation. Let us define two matrices, $\mathbf{A}_{1Y}$ and $\mathbf{A}_{2Y}$, as follows. The matrix $\mathbf{A}_{1Y}$ is composed from the matrix $\mathbf{A}$ changing the rows belonging to the electric field variables to zero rows. $\mathbf{A}_{2Y}$ can be derived, in similar manner, zeroing the rows belonging to the magnetic field variables. It can be easily shown that $\mathbf{A}_{1Y}$ and $\mathbf{A}_{2Y}$ do not commute and that the relations $\mathbf{A} = \mathbf{A}_{1Y} + \mathbf{A}_{2Y}$, $\exp(\Delta t\mathbf{A}_{1Y}) = \mathbf{I} + \Delta t\mathbf{A}_{1Y}$ and $\exp(\Delta t\mathbf{A}_{2Y}) = \mathbf{I} + \Delta t\mathbf{A}_{2Y}$ are fulfilled. In the case of the Yee-method we can apply the exponential approximation $U_n(\Delta t\mathbf{A}) = (\mathbf{I} + \Delta t\mathbf{A}_{1Y})(\mathbf{I} + \Delta t\mathbf{A}_{2Y})$, which comes from the relations

$$\exp(\Delta t\mathbf{A}) = \exp(\Delta t(\mathbf{A}_{1Y} + \mathbf{A}_{2Y})) \approx \exp(\Delta t\mathbf{A}_{1Y})\exp(\Delta t\mathbf{A}_{2Y}) = (\mathbf{I} + \Delta t\mathbf{A}_{1Y})(\mathbf{I} + \Delta t\mathbf{A}_{2Y}). \quad (22)$$

It can be proven applying Von Neumann analysis, that the Yee-method can be kept to be stable choosing the time-step sufficiently small.

**Theorem 2.3** *(e.g. [13]) The numerical solution of the Maxwell equations using staggered spatial discretization and leap-frog time integration is stable if and only if the condition*

$$\Delta t < \frac{1}{c\sqrt{(1/\Delta x)^2 + (1/\Delta y)^2 + (1/\Delta z)^2}} \quad (23)$$

*is fulfilled.*

From the theorem we obtain the upper bound $q < 1/\sqrt{3}$. The number of operations is $36N$ in one time-step, that is the same like in the explicit Euler method. Thus this method is very fast, but because of the strict stability condition it proceeds relatively slowly.

## 2.4 The Namiki-Zheng-Chen-Zhang method

A lot of effort has been invested during the last decade to bridge the stability problem of the Yee-method. The main goal was to construct methods, where $\Delta t$ can be chosen based on accuracy considerations instead of stability reason ([3, 4]). The first papers that described unconditionally stable methods were written by Namiki and by the triple Zheng, Chen and Zhang ([8, 15]).

The Namiki-Zhang-Chen-Zhang (NZCZ) method is based on the explicit and implicit Euler method. Let us define the matrices $\mathbf{A}_{1N}$ and $\mathbf{A}_{2N}$ such a way that $\mathbf{A}_{1N}$ comes from the discretization of the first items in the curl operator, and $\mathbf{A}_{2N}$ comes from the second ones. Then we can

define the iteration process

$$\frac{\Psi^{n+1/2}-\Psi^{n}}{\Delta t/2} = \mathbf{A}_{1N}\Psi^{n+1/2}+\mathbf{A}_{2N}\Psi^{n}, \tag{24}$$

$$\frac{\Psi^{n+1}-\Psi^{n+1/2}}{\Delta t/2} = \mathbf{A}_{1N}\Psi^{n+1/2}+\mathbf{A}_{2N}\Psi^{n+1}, \tag{25}$$

which gives the exponential approximation

$$U_n(\Delta t\mathbf{A}) = (\mathbf{I}-(\Delta t/2)\mathbf{A}_{2N})^{-1}\cdot(\mathbf{I}+(\Delta t/2)\mathbf{A}_{1N})\cdot(\mathbf{I}-(\Delta t/2)\mathbf{A}_{1N})^{-1}\cdot(\mathbf{I}+(\Delta t/2)\mathbf{A}_{2N}). \tag{26}$$

The unconditional stability of the method was previously demonstrated on test problems or its proof was given that used computer algebraic tools. Applying the fact $\mathbf{A}_{1N}+\mathbf{A}_{2N}=\mathbf{A}$ and the skew-symmetry of the matrices $\mathbf{A}_{1N}$ and $\mathbf{A}_{2N}$, a pure mathematical proof of the stability was given in [5].

**Theorem 2.4** *(see [5]) Let $h=\min\{\Delta x,\Delta y,\Delta z\}$ and let $q=c\Delta t/h$ be an arbitrary fixed number. The numerical solution of the Maxwell equations is unconditionally stable using staggered spatial discretization and using the Namiki-Zheng-Chen-Zhang time integration method.*

Let us notice that the constant $q$ must be chosen according to the inequality $q<1/\sqrt{3}$ (here $h=\Delta x=\Delta y=\Delta z$) in 3D problems in the case of the classical Yee-method to guarantee the stability of the method. According to the previous theorem in the NZCZ-method the parameter $q$ can be set arbitrarily, which shows the unconditional stability of the method.

In every time-step we have to apply the explicit and implicit Euler method twice. The implicit method is used with a symmetric tridiagonal matrix, so the solution can be obtained by the so-called Thomas algorithm. The number of operation is $144N$ in one time-step. The NZCZ-method is four times slower than the Yee-method for a fixed time-step $\Delta t$. Because the NZCZ-method is unconditionally stable, we can choose time-steps beyond the stability bound of the Yee-method. Thus in the long run the NZCZ-method can be faster than the Yee-method.

## 2.5 The Kole-Figge-de Raedt method

Kole, Figge and de Raedt noticed (see [6]) that it is possible to split the matrix $\mathbf{A}$ into the sum of skew-symmetric matrices, for which the matrix exponential can be computed exactly (KFR-method). When we have such splitting in the form $\mathbf{A}=\mathbf{A}_1+\ldots+\mathbf{A}_p$ ($p\in\mathbb{N}$), then we have $U_n(\Delta t\mathbf{A})$ as a product of exactly computed exponentials $\exp(\xi_{i_l}\Delta t\mathbf{A}_i)$, where $\xi_{i_l}$ is some suitable constant ($i\in\{1,\ldots,p\}$). The computation of the matrix exponentials is based on the equality

$$\exp\left(\begin{bmatrix}0 & \alpha\\ -\alpha & 0\end{bmatrix}\right)=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}, \tag{27}$$

where $\alpha$ is an arbitrary constant. Since the matrices $\mathbf{A}_1, \ldots, \mathbf{A}_p$ are skew-symmetrical and only the products of the exponents of these matrices are used in the approximation, the iteration matrix will be orthogonal. That is its 2-norm is exactly one. Thus the KFR-method is also unconditionally stable.

**Theorem 2.5** *(see [6]) The numerical solution of the Maxwell equations using staggered spatial discretization and using products of exactly calculated matrix exponentials of skew-symmetric matrices in the time integration is unconditionally stable.*

With the splitting $\mathbf{A} = \mathbf{A}_1 + \ldots + \mathbf{A}_p$ we can define, for instance, the approximation

$$U_n^{(1)}(\Delta t\mathbf{A}) = \exp(\Delta t\mathbf{A}_1)\exp(\Delta t\mathbf{A}_2)\cdot\ldots\cdot\exp(\Delta t\mathbf{A}_p), \tag{28}$$

which is called KFR1-method, because this method has order one. A second order approximation can be achieved with

$$U_n^{(2)}(\Delta t\mathbf{A}) = \left(U_n^{(1)}(-\frac{\Delta t}{2}\mathbf{A})\right)^{\top}\cdot\left(U_n^{(1)}(\frac{\Delta t}{2}\mathbf{A})\right) \tag{29}$$

(KFR2-method), while we get a fourth order method with

$$U_n^{(4)}(\Delta t\mathbf{A}) = U_n^{(2)}(\theta\Delta t\mathbf{A})\cdot U_n^{(2)}(\theta\Delta t\mathbf{A})\cdot U_n^{(2)}((1-4\theta)\Delta t\mathbf{A})\cdot U_n^{(2)}(\theta\Delta t\mathbf{A})\cdot U_n^{(2)}(\theta\Delta t\mathbf{A}), \tag{30}$$

$\theta = 1/(4-\sqrt[3]{4})$ (KFR4-method). For more details regarding the splitting methods consult [7] and [14].

The number of operations per time-step is $108N$ for the KFR1-, $216N$ for the KFR2-, and $1080N$ for the KFR4-method.

## 2.6 The Krylov-space method

In the previous methods we approximated the matrix exponential $\exp(\Delta t\mathbf{A})$ and used this approximation to generate a matrix iteration. Changing the philosophy of the matrix exponential approximation we can proceed as follows. We do not approximate the matrix exponential itself but the product of the matrix exponential and the previous state vector ([1, 2, 10, 12]).

Let us suppose that an initial vector $\Psi^0$ and a fixed natural number $m$ are given. We are interested in finding the best approximation to $\exp(\Delta t\mathbf{A})\Psi^0$ from the Krylov-space

$$\mathcal{K}(\Delta t\mathbf{A}, \Psi^0, m) = \mathrm{span}\{\Psi^0, \Delta t\mathbf{A}\Psi^0, \ldots, (\Delta t\mathbf{A})^{m-1}\Psi^0\}. \tag{31}$$

(span{} denotes the set of all possible linear combinations of the vectors). Because $\mathbf{A}$ is skew-symmetric, it is possible to find an orthonormal matrix $\mathbf{V}_m$ and a skew-symmetric tridiagonal matrix $\mathbf{T}_m$ such that the relation

$$\mathbf{V}_m^\top \mathbf{A} \mathbf{V}_m = \mathbf{T}_m \tag{32}$$

is satisfied. The matrices $\mathbf{V}_m$ and $\mathbf{T}_m$ can be calculated applying the modified Arnoldi-method. With the help of these matrices the best approximation for $\exp(\Delta t\mathbf{A})\Psi^0$ can be written in the form

$$\Psi^1 = \beta \mathbf{V}_m \exp(\Delta t \mathbf{T}_m)\mathbf{e}_1, \tag{33}$$

where $\beta = \| \Psi^0 \|_2$ and $\mathbf{e}_1$ is the first unit vector. This method is also unconditionally stable.

**Theorem 2.6** *([5]) The numerical solution of the Maxwell equations using staggered spatial discretization and using the Krylov-method with a modified Arnoldi orthogonalization in the time integration is unconditionally stable.*

The main advantage of the method is that choosing $m$ relatively small ($m << 6N$) we need to compute the matrix exponential only for the small matrix $\Delta t\mathbf{T}_m$ and we get the next approximation as a linear combination of the $m$ columns of $\mathbf{V}_m$. The number of operations per time-steps is $72mN$ (if we neglect the number of operations in the computation of $\exp(\Delta t\mathbf{T}_m)$).

**Remark 2.7** *Let us suppose that* $\dim(\mathcal{K}(\Delta t\mathbf{A}, \Psi^0, m)) = m_0 < m$. *Then in the above relations we must use* $m_0$ *instead of* $m$. *Moreover, in this case* $\mathbf{A}^{m_0}\Psi^0 \in \text{span}\{\Psi^0, \Delta t\mathbf{A}\Psi^0, \ldots, (\Delta t\mathbf{A})^{m_0-1}\Psi^0\}$ *and* $\Psi^1$ *gives the exact value of* $\exp(\Delta t\mathbf{A})\Psi^0$, *which means that* $\exp(\Delta t\mathbf{A})\Psi^0$ *can be computed exactly for arbitrary time-steps. This shows that the Krylov-method in special cases* ($m_0 << 6N$) *can be a very efficient one.*

**Remark 2.8** *Considering Theorem 4 in [2] we can give an estimation for the error of this method in the form*

$$\| \exp(\Delta t\mathbf{A})\Psi^0 - \beta \mathbf{V}_m \exp(\Delta t\mathbf{T}_m)\mathbf{e}_1 \|_2 \leq 12 e^{-(2q)^2/m} \left(\frac{2eq}{m}\right)^m, \quad m \geq 4q \ (q = c\Delta t/h). \tag{34}$$

*With this relation we are able to choose* $m$ *or* $\Delta t$ *to guarantee a certain accuracy level of the computations.*

# 3 Comparison of the methods

In this section we compare the previously listed numerical schemes for the Maxwell equations. We are not going to present numerical tests here. Instead of this we summarize our experience in the

| | | |
|---|---|---|
| Euler explicit | 100% | unstable |
| Euler implicit | min. 8750% | stable |
| YEE | 100% | stable iff $q < 1/\sqrt{3}$ |
| NZCZ | 400% | stable |
| KFR1 | 300% | stable |
| KFR2 | 600% | stable |
| KFR4 | 3000% | stable |
| Krylov | $200m$% | stable |

Table 1: *Number of computations and stability.*

use of the methods (see papers [1, 5, 6, 8, 10, 13]). We consider the Yee-method as a standard solution method, so we compare the methods with the Yee-method. Table 1 shows the number of operations per one time-step (100% =Yee-method) and indicates the stability properties for the methods.

The explicit Euler method is unstable so it cannot be used in practice. The implicit Euler method is also unpractical because of the expenses of the computations.

The NZCZ-method is stable, so the time-step can be chosen arbitrarily. Of course, the increasing time-step decreases the accuracy of the method. We have to find the appropriate balance between the accuracy and the computational speed. We have found that with acceptable accuracy the NZCZ-method can be faster with a factor 10 than the Yee-method.

The KFR-method seems to be a very efficient one, because, like in the Yee-method, it computes the matrix exponentials exactly. Instead if this, the method appears to be very inaccurate. In order to lift the accuracy of the method we have to apply the fourth order version of it, which makes it slower than the Yee-method in the long run. However, because the 2-norm of the iteration matrix is exactly one, the method behaves nicely in spectrum computations.

The number of operations of the Krylov-method is about $2m$ times greater than in the Yee-method (the number of the iterations $m$ can be estimated from Remark 2.8). As we noticed in Remark 2.7 if $m_0$ is sufficiently small, then we can obtain the exact solution of the semi-discrtetized system for all $\Delta t$ step-sizes with an acceptable computational time.

# References

[1] I. FARAGÓ, R. HORVÁTH, W.H.A. SCHILDERS, *Investigation of Numerical Time Integrations of the Maxwell Equations Using the Staggered Grid Spatial Discretization*, RANA

report 02-15, University of Technology Eindhoven, Department of Mathematics and Computer Science.

[2] M. HOCHBRUCK, C. LUBICH, *On Krylov-Subspace Approximations to the Matrix Exponential Operator*, SIAM J. Numer. Anal., Vol. 34, No. 5, 1911-1925, 1997.

[3] R. HOLLAND, *Implicit Three-Dimensional Finite Differencing of Maxwell's Equations*, IEEE Trans. Nuclear Science, Vol. NS-31, 1322-1326, 1984.

[4] R. HOLLAND, K.S. CHO, *Alternating-Direction Implicit Differencing of Maxwell's Equations: 3D Results*, Computer Science Corp., Albuquerque, NM, technical report to Harry Diamond Labs., Adelphi, MD, Contract DAAL02-85-C-0200, June 1, 1986.

[5] R. HORVÁTH, *Uniform Treatment of the Numerical Time-Integration of the Maxwell Equations* (to appear in Proceedings Scientific Computing in Electrical Engineering).

[6] J. S. KOLE, M. T. FIGGE, H. DE RAEDT, *Unconditionally Stable Algorithms to Solve the Time-Dependent Maxwell Equations*, Phys. Rev. E 64, 066705, 2001.

[7] G.I. MARCHUK, *Methods of Splitting*, Nauka, Moscow, 1988.

[8] T. NAMIKI, *A New FDTD Algorithm Based on Alternating-Direction Implicit Method*, IEEE Transactions on Microwave Theory and Techniques, Vol. 47, No. 10, 2003-2007, Oct. 1999.

[9] P. NEITTAANMÄKI, M. KRIZEK, *Mathematical and Numerical Modelling in Electrical Engineering: Theory and Applications*, Kluwert, 1996.

[10] R. F. REMIS, P. M. VAN DEN BERG, *A Modified Lanczos Algorithm for the Computation of Transient Electromagnetic Wavefields*, IEEE Transactions on Microwave Theory and Techniques, vol. 45, no. 12 (1997), pp. 2139-2149.

[11] M. N. O. SADIKU, *Numerical Techniques in Electromagnetics*, Second edition, CRC Press, 2000.

[12] R. B. SIDJE, *EXPOKIT: Software Package for Computing Matrix Exponentials*, ACM - Transactions on Mathematical Software, 1998.

[13] A. TAFLOVE, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, 2 ed., Artech House, Boston, MA, 2000.

[14] V. S. VARADARAJAN, *Lie Groups, Lie Algebras and Their Representations*, Prentice-Hall Inc., Englewood Cliffs, New Jersey 1974.

[15] F. ZHENG, Z. CHEN, J. ZHANG, *Toward the Development of a Three-Dimensional Unconditionally Stable Finite-Difference Time-Domain Method*, IEEE Trans. Microwave Theory and Techniques, Vol. 48, No. 9, 1550-1558, September 2000.

[16] K. S. YEE, *Numerical Solution of Initial Boundary Value Problems Involving Maxwell Equations in Isotropic Media*, IEEE Transactions on Antennas and Propagation, vol. 14, no. 3, 302-307, March 1966.