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Abstract: We consider short-time existence, uniqueness, and regularity for a moving boundary problem describing Stokes flow of a free liquid drop driven by surface tension. The surface tension coefficient is assumed to be a nonincreasing function of the surfactant concentration, and the surfactant is insoluble and moves by convection along the boundary.
The problem is reformulated as a fully nonlinear, nonlocal Cauchy problem for a vector-valued function on a fixed reference manifold. This problem is, in general, degenerate parabolic. Existence and uniqueness results are obtained via energy estimates in Sobolev spaces of sufficiently high order. In the two-dimensional case, the problem is strictly parabolic, and we prove instantaneous smoothing of the free boundary, using maximal regularity results in little Hölder spaces.

1 Introduction and problem formulation

The moving boundary problem of Stokes flow (in its simplest form) of a viscous, incompressible liquid consists of the Stokes equations together with the incompressibility condition. These equations form a linear elliptic system. We will consider surface tension as the only driving mechanism. Mathematically, this is described by an inhomogeneous boundary condition on the normal stress. Up to now, most attention has been given to the case of a constant surface tension coefficient $\gamma > 0$. In this case, γ is just a proportionality factor linking the normal stress to the curvature vector at the boundary (and can be normalized to 1). The two-dimensional version of this problem has been discussed in e.g. [3, 4, 12, 14], and short-time solvability in the general case was proved in [10, 15].

For various applications, it is also of interest to consider a nonconstant surface tension coefficient γ. In this case, the surface gradient of γ also occurs in the boundary condition for the normal stress, giving rise to the so-called Marangoni effect, a well-known cause for which is a temperature dependent γ [4].

The possibly simplest reasonable model with nonconstant surface tension coefficient has been discussed in [11]. There it is assumed that fixed values of γ are assigned to the particles at the boundary, i.e. γ is transported along with these particles. (This situation arises from the temperature-dependent case if heat conduction is negligible.) Recently, there is increasing interest in the case where γ is a function of the concentration of a surfactant, i.e. a surface-active substance which decreases surface tension. For examples of applications we refer to [18], where, in particular, the delivery of surfactant in the human lung for medical purposes is modeled.

We discuss short-time well-posedness and regularity for the problem with a surfactant in a simple topology, namely, for a free drop of finite volume. We assume that the surfactant is insoluble and its diffusion on the surface of the drop is negligible, i.e. it is transported by the tangential component of the liquid velocity (cf. [18]).

The original problem will be transformed to a Cauchy problem for a fully nonlinear pseudodifferential evolution equation on a fixed compact reference manifold. As remarked in [11], the cases of constant and nonconstant surface tension lead to evolution equations of different types: For $\gamma = \text{const}$, the problem can be transformed to a scalar parabolic equation for an unknown representing the local distance of the moving boundary from the reference manifold. In the general case, however, where tangential transport in the boundary has to be considered, one arrives at a vector-valued equation which is degenerate parabolic in the following sense: the principal part of the linearization of the right hand side in the evolution equation has an infinite-dimensional kernel, hence no coercive estimates can be obtained. In the situation discussed here, under some natural assumptions on our data, this kernel is given by the divergence free tangential velocity fields. As there are no nontrivial divergence free vector fields on curves, in the two-dimensional case we obtain a parabolic system for an $\mathbb{R}^2$-valued function.

The structure of this paper is as follows: After the formulation of our moving boundary problem, in Section 2 we transform the problem to a nonlocal evolution equation ((2.20)). In Section 3, this evolution equation is linearized, an L^2-energy estimate for the linearization is derived. This estimate together with a nonlocal chain rule is used in Section 4 to obtain energy estimates for the nonlinear problem in higher order Sobolev norms. The necessary estimates for the lower order terms (Lemma 4.1) are somewhat tedious but essentially straightforward. At the end of Section 4, the existence results are obtained by Galerkin approximations in a fashion oriented at [13]. Sections 2–4 are to some extent parallel to [11], therefore some proofs are just by reference to that paper. The parabolic problem arising in the plane case is addressed in Section 5. Apart from a slightly larger class of admissible initial data, our main interest there is in the proof of the smoothing property one expects for a parabolic equation: For smooth data, the solution will be smooth for all positive times in the existence interval. We prove this, both for C^∞-smoothness and for analyticity, using a general theorem on parabolic smoothing which has been proved in [7]. This theorem is a generalized version of a "parameter trick" due to Angenent [1, 2] which is based on the property of maximal regularity for the linearized problem. Therefore, in Section 5 we change the functional analytic framework and use little Hölder spaces as in these spaces the maximal regularity property can be established in a standard way. Finally, the appendix summarizes some results on solution operators for various standard elliptic boundary value problems on fixed, bounded smooth domains which are used in Section 5 but are otherwise independent of the moving boundary problem.

We start by giving a precise description of our problem (cf. [11]). For given $\Omega(0) \subset \mathbb{R}^{m+1}$ and given $\widetilde{\rho}(\cdot,0) \geq 0$ defined on $\partial\Omega(0)$, one looks for

- a family of bounded domains $\Omega(t) \subset \mathbb{R}^{m+1}$ parametrized by time $t \geq 0$ with C^2-boundaries $\Gamma(t)$ which move with velocity $V_n(t)$ in the direction of the outer normal $n(t)$,
- (nonnegative) functions $\widetilde{\rho}(\cdot,t) \in C^1(\Gamma(t))$, and
- functions $\widetilde{u}(\cdot,t) \in C^2(\overline{\Omega(t)}, \mathbb{R}^{m+1})$, $\widetilde{p}(\cdot,t) \in C^1(\overline{\Omega(t)})$

such that

$$\left.\begin{array}{rcll} -\Delta\widetilde{u}(\cdot,t) + \nabla\widetilde{p}(\cdot,t) &=& 0 & \text{in } \Omega(t), \\ \operatorname{div}\widetilde{u}(\cdot,t) &=& 0 & \text{in } \Omega(t), \\ \big(T(\widetilde{u}(\cdot,t),\widetilde{p}(\cdot,t))n(t)\big)_i &=& \operatorname{div}_{\Gamma(t)}(\widetilde{\gamma}(\cdot,t)\nabla_{\Gamma(t)}x_i(t)) & \text{on } \Gamma(t), \\ \widetilde{\gamma}(\cdot,t) &=& \sigma(\widetilde{\rho}(\cdot,t)), & \\ \int_{\Omega(t)} \widetilde{u}(\cdot,t)\,dx &=& 0, & \\ \int_{\Omega(t)} \operatorname{rot}\widetilde{u}(\cdot,t)\,dx &=& 0, & \\ V_n(t) &=& \widetilde{u}(\cdot,t)\cdot n(t) & \text{on } \Gamma(t). \end{array}\right\} \tag{1.1}$$

Here, $\Omega(t)$ represents the domain occupied by the liquid drop at time $t \geq 0$, $\widetilde{\rho}$ is the surfactant concentration on $\Gamma(t)$, $\widetilde{u}$ and $\widetilde{p}$ represent the velocity and pressure field, T is the stress tensor given by

$$(T(u,p))_{ij} = \partial_i u_j + \partial_j u_i - p\delta_{ij},$$

and $\widetilde{\gamma}(\cdot,t)$ is a scalar function on $\Gamma(t)$ representing the surface tension coefficient. The differential operators Δ, ∇, and div are applied with respect to the spatial coordinates, the operators $\operatorname{div}_{\Gamma(t)}$ and $\nabla_{\Gamma(t)}$ are the divergence and gradient on $\Gamma(t)$ with respect to its Riemannian metric induced from the ambient space, and $x(t) : \Gamma(t) \hookrightarrow \mathbb{R}^{m+1}$ denotes the natural embedding of $\Gamma(t)$ into $\mathbb{R}^{m+1}$.

The surface tension coefficient $\widetilde{\gamma}$ depends on $\widetilde{\rho}$ via $\sigma \in C^\infty([0,\infty))$ satisfying the structural assumptions

$$\sigma \geq 0, \quad \sigma' \leq 0, \tag{1.2}$$

i.e. $\widetilde{\gamma}$ is a nonnegative and nonincreasing function of the surfactant concentration. (The assumption that σ is defined on $[0,\infty)$ is just for the sake of simplicity, its domain of definition may be restricted to a suitable interval.)

The problem is completed by an evolution equation for the surfactant concentration on the moving boundary. As we assume that the surfactant is insoluble and that the surfactant diffusion is negligible, the evolution of $\widetilde{\rho}$ is given by two mechanisms: surfactant transport and local change of the surface area, both induced by the flow at the boundary. To be more precise, let us introduce Lagrangian coordinates by

$$\begin{aligned} \dot{X}(t,\xi) &= \widetilde{u}(X(t,\xi),t), \\ X(0,\xi) &= \xi \end{aligned}$$

for $\xi \in \Gamma(0)$, $t \geq 0$. Write the material derivative of $\widetilde{\rho}$ as

$$D_t\widetilde{\rho}(X(t,\xi),t) := \frac{d}{dt}(\widetilde{\rho}(X(t,\xi),t)),$$

this makes sense as $(1.1)_7$ ensures $X(t,\Gamma(0)) = \Gamma(t)$. Then the mass conservation equation for the surfactant reads (see e.g. [5], Ch. 10)

$$D_t\widetilde{\rho}(\cdot,t) + \widetilde{\rho}(\cdot,t)(\operatorname{div}_{\Gamma(t)}\widetilde{u}_T(\cdot,t) - \kappa(t)\widetilde{u}(\cdot,t)\cdot n(t)) = 0 \text{ on } \Gamma(t), \tag{1.3}$$

$t \geq 0$, where $\widetilde{u}_T$ denotes the component of $\widetilde{u}$ tangential to $\Gamma(t)$ and $\kappa(t)$ is the (m-fold) mean curvature of $\Gamma(t)$ with the sign taken negative if $\Omega(t)$ is convex.

2 Transformation and evolution equation

We fix the following notation and recall some basic estimates; for the proofs see [11].

Let $\Omega \subset \mathbb{R}^{m+1}$ be a bounded smooth domain, $\Gamma := \partial\Omega$ and let Tr_Γ denote the trace operator from function spaces on Ω to the corresponding spaces on Γ. For $\tau \in \mathbb{R}$, we denote by $H^\tau(\Gamma)$ and $H^\tau(\Gamma,\mathbb{R}^{m+1})$ the usual L^2-based Sobolev spaces of order τ with values in $\mathbb{R}$ and $\mathbb{R}^{m+1}$, respectively. The norms of these spaces are denoted by $\|\cdot\|_\tau^\Gamma$. If z is a function defined on Ω, we write $\|z\|_\tau^\Gamma$ instead of $\|\mathrm{Tr}_\Gamma z\|_\tau^\Gamma$. For $\tau \geq 0$, $H^\tau(\Omega)$, $H^\tau(\Omega,\mathbb{R}^{m+1})$, and $\|\cdot\|_\tau^\Omega$ are defined analogously, and for $z \in L^2(\Omega)$ we define

$$\|z\|_{-\tau}^\Omega := \sup_{\|v\|_\tau^\Omega=1} \left| \int_\Omega zv\,dx \right|.$$

We recall the estimates

$$\|\partial_i z\|_\tau^\Omega \leq C\big(\|z\|_{\tau+1}^\Omega + \|z\|_{\tau+\frac{1}{2}}^\Gamma\big) \tag{2.1}$$

for $\tau \leq -1$, $z \in H^1(\Omega)$,

$$\|zv\|_\tau^M \leq C\|z\|_\tau^M\|v\|_s^M \tag{2.2}$$

for $|\tau| \leq s$, $s > \frac{m}{2}$, $z \in H^{\tau^+}(M)$, $v \in H^s(M)$, and

$$\|z_1z_2\dots z_k\|_t^M \leq C\prod_{i=1}^k \|z_i\|_{s_i}^M, \tag{2.3}$$

where M is Ω or Γ, $0 \leq \tau \leq s_i$, $\tau - \frac{\dim M}{2} < \sum_{i=1}^{k}(s_i - \frac{\dim M}{2})$, $z_i \in H^{s_i}(M)$.

Moreover, we introduce a right inverse $\mathcal{E}$ of Tr_Γ by $\mathcal{E}h := w$, $h \in H^{\frac{1}{2}}(\Gamma)$, where w solves

$$\left.\begin{array}{rcll} \Delta w & = & 0 & \text{in } \Omega, \\ w & = & h & \text{on } \Gamma, \end{array}\right\}$$

and recall the estimate

$$\|\mathcal{E}h\|^{\Omega}_{\tau+\frac{1}{2}} + \|\nabla \mathcal{E}h\|^{\Gamma}_{\tau-1} + \left\|\nabla^2 \mathcal{E}h\right\|^{\Gamma}_{\tau-2} \leq C\|h\|^{\Gamma}_{\tau} \tag{2.4}$$

for $h \in H^\tau(\Gamma) \cap H^s(\Gamma)$, $s > 2$.

Let us fix $s_0 > \frac{m}{2}$, $s \geq s_0 + 4$ integer. Let $\mathcal{U}$ be a small open neighborhood of the identity in $H^{s+1}(\Gamma, \mathbb{R}^{m+1})$ which will be shrunken in the sequel whenever necessary without further mentioning. For $\phi \in \mathcal{U}$, set

$$\Phi := \mathcal{E}(\phi - \mathrm{Id}_\Gamma) + \mathrm{Id}_\Omega$$

(with $\mathcal{E}$ acting componentwise on $\mathbb{R}^{m+1}$-valued functions) and note that

$$\Phi \in C^3(\overline{\Omega}, \mathbb{R}^{m+1}) \cap \mathrm{Diff}(\Omega, \Phi(\Omega))$$

due to the Sobolev embedding theorem and the smallness of $\mathcal{U}$.

Let (ξ^i) be a local parametrization of Γ (in Cartesian coordinates.) The mappings Φ and ϕ induce on Ω and Γ the Riemannian metrics $\mathbf{g}$ and $\widetilde{\mathbf{g}}$, respectively, having Cartesian coordinates

$$g_{ij} = \partial_i \Phi^k \partial_j \Phi^k, \quad \widetilde{g}_{\alpha\beta} = \partial_\alpha \xi^i g_{ij} \partial_\beta \xi^j,$$

$i, j = 1, \ldots, m+1$, $\alpha, \beta = 1, \ldots, m$. Furthermore, we set $G = (g_{ij})$, $g = \det G$, $g^{ij} = (G^{-1})_{ij}$, and introduce analogous notation for $\widetilde{\mathbf{g}}$. Moreover, let

$$a^i_k := \partial_k (\Phi^{-1})^i \circ \Phi = ((D\Phi)^{-1})_{ik}.$$

(Clearly, all these quantities depend on ϕ but we are going to suppress this dependence in our notation for the sake of brevity.) In order to transform the moving boundary problem to the fixed domain Ω, we introduce the spaces

$$\begin{array}{rcl} V & := & \left\{(c^{ij}) \,|\, i,j = 1, \ldots, N, c^{ij} \in \mathbb{R}, c^{ij} = -c^{ji}\right\}, \\ X_\tau & := & H^{\tau+\frac{1}{2}}(\Omega, \mathbb{R}^{m+1}) \times H^{\tau-\frac{1}{2}}(\Omega) \times (\mathbb{R}^{m+1} \times V), \\ Y_\tau & := & H^{\tau-\frac{3}{2}}(\Omega, \mathbb{R}^{m+1}) \times H^{\tau-\frac{1}{2}}(\Omega) \times H^{\tau-1}(\Gamma, \mathbb{R}^{m+1}) \times \mathbb{R}^{m+1} \times V \end{array}$$

for $\tau \geq \frac{1}{2}$ and the operator $L : \mathcal{U} \longrightarrow \mathcal{L}(X_s, Y_s)$, using the notation of covariant calculus, by

$$L(\phi)(u, p, \lambda) := \left(\begin{array}{c} -\nabla^k \nabla_k u^i + \nabla^i p + \lambda^k_1 a^i_k \\ \nabla_i u^i \\ (\nabla^i u^j + \nabla^j u^i - g^{ij} p + \lambda^{kl}_2 a^i_k a^j_l) n_j \\ \int_\Omega \sqrt{g} \partial_k \Phi^i u^k \, dx \\ \int_\Omega \sqrt{g} \partial_k \Phi^i \partial_l \Phi^j (\nabla^k u^l - \nabla^l u^k) \, dx \end{array} \right)^T . \tag{2.5}$$

We denote the canonical projection of X_τ onto its i-th component by Π_i and by E_3 the operator in $\mathcal{L}(H^{\tau-1}(\Gamma,\mathbb{R}^{m+1}),Y_\tau)$ mapping H to $E_3H := (0,0,H,0,0)$. For Banach spaces E and F, let $\mathcal{L}_{is}(E,F)$ denote the set of continuous isomorphisms from E to F with the topology inherited from $\mathcal{L}(E,F)$. It is shown in [11], Lemma 2.4., that

$$L \in C^\infty(\mathcal{U}, \mathcal{L}_{is}(X_s, Y_s)). \tag{2.6}$$

Furthermore, fix a nonnegative function $\rho \in H^s(\Gamma)$ and define

$$\gamma := \sigma\left(\rho\frac{\sqrt{\widetilde{g}(\mathrm{Id})}}{\sqrt{\widetilde{g}(\phi)}}\right), \tag{2.7}$$

$$f^k := \frac{1}{\sqrt{\widetilde{g}}}\partial_\alpha(\sqrt{\widetilde{g}}\gamma\widetilde{g}^{\alpha\beta}\partial_\beta\Phi^k), \quad k=1,\dots,m+1, \tag{2.8}$$

$$f := (f^1,\dots,f^{m+1}). \tag{2.9}$$

Using the fact that $H^\tau(\Gamma)$ is a Banach algebra for $\tau > \frac{m}{2}$, one can show as in the proof of Lemma 2.3 in [11] that

$$\left[\phi \mapsto \rho\frac{\sqrt{\widetilde{g}(\mathrm{Id})}}{\sqrt{\widetilde{g}(\phi)}}\right] \in C^\infty(\mathcal{U}, H^s(\Gamma)). \tag{2.10}$$

Applying additionally Theorem II.4.3 in [17] on the smoothness of superposition operators given by smooth functions we also get

$$\gamma \in C^\infty(\mathcal{U}, H^s(\Gamma)), \tag{2.11}$$

$$f \in C^\infty(\mathcal{U}, H^{s-1}(\Gamma,\mathbb{R}^{m+1})). \tag{2.12}$$

To account for the pull-back of vector fields, we introduce the mapping $D\Phi$ given by

$$(D\Phi z)^i = \partial_j\Phi^i z^j$$

and note that

$$\begin{aligned}[\phi \mapsto D\Phi] \in\ & C^\infty(\mathcal{U}, \mathcal{L}_{is}(H^{s+\frac{1}{2}}(\Omega,\mathbb{R}^{m+1}))) \cap C^\infty(\mathcal{U}, \mathcal{L}_{is}(H^{s-\frac{3}{2}}(\Omega,\mathbb{R}^{m+1}))) \\ & \cap C^\infty(\mathcal{U}, \mathcal{L}_{is}(H^{s-1}(\Gamma,\mathbb{R}^{m+1}))).\end{aligned} \tag{2.13}$$

Now we can transform our moving boundary problem to a nonlocal evolution equation on a fixed manifold. For this purpose, we change notation slightly and consider ϕ as a function of time $t \in [0,T]$, valued in $\mathcal{U}$.

Lemma 2.1 *(Transformation) For $\phi \in C^1([0,T],\mathcal{U})$ the following statements are equivalent:*

(i) For the family of domains $\Omega(t) := \Phi(\cdot,t)[\Omega]$, $t \in [0,T]$, there are functions

$$\begin{aligned}\widetilde{u}(\cdot,t) &\in C^2(\overline{\Omega(t)}, \mathbb{R}^{m+1}),\\ \widetilde{p}(\cdot,t) &\in C^1(\overline{\Omega(t)}),\\ \widetilde{\rho}(\cdot,t) &\in C^1(\Gamma(t))\end{aligned}$$

such that

$$\dot{\phi}(\cdot,t) = \widetilde{u}(X(\phi(\cdot,0),t),t)|_\Gamma, \tag{2.14}$$

$$\widetilde{\rho}(\cdot,0) = \phi(0)_* \left(\rho \frac{\sqrt{\widetilde{g}(\mathrm{Id})}}{\sqrt{\widetilde{g}(\phi(0))}}\right), \tag{2.15}$$

and (1.1), (1.3) *hold.*

(ii) There are functions

$$(u,p) \in C^1([0,T], H^{s+\frac{1}{2}}(\Omega,\mathbb{R}^{m+1}) \times H^{s-\frac{1}{2}}(\Omega)) \tag{2.16}$$

such that, with f from (2.8), (2.9),

$$L(\phi)(u,p,0) = E_3(D\Phi)^{-1}f, \tag{2.17}$$

$$\dot{\phi} = (D\Phi)\mathrm{Tr}_\Gamma u =: \mathcal{F}(\phi) =: \mathcal{G}(\phi,\gamma). \tag{2.18}$$

Proof: (i)⇒(ii): Let $\Phi(t) := \mathrm{Id}_\Omega + \mathcal{E}(\phi(t) - \mathrm{Id}_\Gamma)$ and let $\Phi(t)^*$ and $\Phi(t)_*$ denote the pull-back and push-forward operators induced by $\Phi(t)$ (both for scalar functions and for vector fields). Set

$$\begin{aligned}u(\cdot,t) &:= \Phi(t)^*\widetilde{u}(\cdot,t),\\ p(\cdot,t) &:= \Phi(t)^*\widetilde{p}(\cdot,t).\end{aligned}$$

On Γ we introduce the time-dependent functions

$$\begin{aligned}\rho_1(\cdot,t) &:= \rho\frac{\sqrt{\widetilde{g}(\mathrm{Id})}}{\sqrt{\widetilde{g}(\phi(t))}},\\ \rho_2(\cdot,t) &:= \phi(t)^*\widetilde{\rho}(\cdot,t).\end{aligned}$$

Let us denote by $D(\sqrt{\widetilde{g}})$ the Fréchet derivative of the map $\phi \mapsto \sqrt{\widetilde{g}}$. Furthermore, $\Xi(\cdot,t) := X(t,\phi(\cdot,0))$. Then $\Xi(\cdot,t) = \phi(t)$ and

$$\begin{aligned}&\frac{d}{dt}\sqrt{\widetilde{g}(\phi(t))} = D(\sqrt{\widetilde{g}})[\dot{\phi}]\\ =\ & D(\sqrt{\widetilde{g}})[\widetilde{u}_T(\Xi(\cdot,t),t) + \widetilde{u}(\Xi(\cdot,t),t)\cdot n(\Xi(\cdot,t),t)\, n(\Xi(\cdot,t),t)]\\ =\ & [\mathrm{div}_{\Gamma(t)}\widetilde{u}(\Xi(\cdot,t),t) - \kappa(\Xi(\cdot,t),t)\widetilde{u}(\Xi(\cdot,t),t)\cdot n(\Xi(\cdot,t),t)n(\Xi(\cdot,t),t)]\sqrt{\widetilde{g}}\\ =\ & [\mathrm{div}_{\widetilde{\mathbf{g}}}u_T - \kappa_{\mathbf{g}}u^i(n_{\mathbf{g}})_i]\sqrt{\widetilde{g}},\end{aligned}$$

where $\mathrm{div}_{\widetilde{\mathbf{g}}}$ denotes the divergence in the Riemannian manifold $(\Gamma, \widetilde{\mathbf{g}})$, and $\kappa_{\mathbf{g}}$ and $n_{\mathbf{g}}$ are the curvature and the outer unit normal vector of Γ with respect to $\mathbf{g}$. The third equality follows from the transport theorem in a Riemannian manifold (see e.g. [16], Ch. 2.2) and the well-known "first variation of area formula" for variations in normal direction. Hence

$$\dot{\rho}_1 = -\rho \frac{\sqrt{\widetilde{g}(\mathrm{Id})}}{\widetilde{g}(\phi(t))} \frac{d}{dt} \sqrt{\widetilde{g}} = -\rho_1 (\mathrm{div}_{\widetilde{\mathbf{g}}} u_T - \kappa_{\mathbf{g}} u^i (n_{\mathbf{g}})_i). \tag{2.19}$$

If (1.3) is transformed to Γ by $\phi(t)^*$ we get the same equation for ρ_2, and as we have assumed $\rho_1(0) = \rho_2(0)$ we get $\rho_1 = \rho_2$ from the uniqueness of the solution of (2.19) for given initial datum, hence

$$\widetilde{\rho}(\cdot, t) = \phi(t)_* \left(\rho \frac{\sqrt{\widetilde{g}(\mathrm{Id})}}{\sqrt{\widetilde{g}}} \right).$$

Now it is a routine task to check that (1.1) and (2.14) transform (at first formally) to (2.17) and (2.18) (see [11]). The regularity result (2.16) follows then from (2.10) and (2.6).

(ii)⇒(i): We set

$$\begin{aligned} \widetilde{u}(\cdot, t) &:= \Phi(t)_* u(\cdot, t), \\ \widetilde{p}(\cdot, t) &:= \Phi(t)_* p(\cdot, t), \\ \widetilde{\rho}(\cdot, t) &:= \phi(t)_* \left(\rho \frac{\sqrt{\widetilde{g}(\mathrm{Id})}}{\sqrt{\widetilde{g}}} \right). \end{aligned}$$

Then it is straightforward to check all statements in (i). ■

The proof of Lemma 2.1 also provides a translation for any solution to (2.18) to a solution of our original moving boundary problem. Conversely, given (sufficiently smooth) initial data $\Omega(0)$, $\widetilde{\rho}(\cdot, 0)$, one chooses a smooth domain Ω near $\Omega(0)$, an initial function ϕ_0 such that ϕ_0 is near the identity and $\phi_0(\Gamma) = \Gamma(0)$, and $\rho \in H^s(\Gamma)$ such that (2.15) is satisfied. Then, by the above lemma, our problem is reduced to the Cauchy problem

$$\left. \begin{aligned} \dot{\phi} &= \mathcal{F}(\phi), \\ \phi(0) &= \phi_0, \end{aligned} \right\} \tag{2.20}$$

which will be investigated in the sequel.

At first, we conclude from (2.6), (2.11) (2.12) and (2.13) that

$$\begin{aligned} &\mathcal{F} \in C^\infty(\mathcal{U}, H^s(\Gamma, \mathbb{R}^{m+1})), \\ &\mathcal{G} \in C^\infty(\mathcal{U} \times H^s(\Gamma), H^s(\Gamma, \mathbb{R}^{m+1})). \end{aligned} \tag{2.21}$$

3 Linearization

To give the necessary estimates on $\mathcal{F}'(\phi)$, we start with an additional regularity result on the Stokes equations with traction boundary conditions which might be of interest in its own right (Lemma 3.1 below). To ensure uniformity of the estimates with respect to perturbations of the domain, we work with a general metric $\mathbf{g}$ induced by an arbitrary $\phi \in \mathcal{U}$ on the fixed domain Ω.

We will also work with the measure on Γ induced by $\mathbf{g}$, given by $\omega_{\mathbf{g}}\, d\Gamma$ where

$$\omega_{\mathbf{g}} := \frac{\sqrt{\widetilde{g}}}{\sqrt{\widetilde{g}(\mathrm{Id})}}.$$

We denote by $\Delta_{\mathbf{g}} := \nabla^i \nabla_i$ the Laplace-Beltrami operator with respect to $\mathbf{g}$ and introduce the operators $A(\phi)$ and $B(\phi)$ by

$$\begin{aligned} A(\phi) &:= \mathrm{Tr}_\Gamma \Pi_1 L(\phi)^{-1} E_3, \\ B(\phi)\theta &:= \mathrm{Tr}_\Gamma (\Delta_{\mathbf{g}}, \partial_n)^{-1}(0, \theta - \overline{\theta}) \end{aligned}$$

with

$$\overline{\theta} := \frac{\int_\Gamma \omega_{\mathbf{g}} \theta \, d\Gamma}{\int_\Gamma \omega_{\mathbf{g}} \, d\Gamma}$$

and $\partial_n := n^i_{\mathbf{g}} \mathrm{Tr}_\Gamma \partial_i$. Moreover, we will write $P^{\mathbf{g}}$ for the $\mathbf{g}$-orthogonal projection of $\mathbb{R}^{m+1}$–valued vector fields onto $T\Gamma$, given by

$$(P^{\mathbf{g}} v)^j = \partial_\alpha \phi^i \widetilde{g}^{\alpha\beta} \partial_\beta \phi^j v_i,$$

and $\nabla^{\mathbf{g}}$ for the gradient in $(\Omega, \mathbf{g})$. Note that then $\nabla^{\widetilde{\mathbf{g}}} := P^{\mathbf{g}} \nabla^{\mathbf{g}} \mathcal{E}$ is the gradient in $(\Gamma, \widetilde{\mathbf{g}})$ if tangent vectors on Γ are identified with their images under the embedding in the ambient space.

We recall that the operator A maps the traction boundary data of solutions to the homogeneous Stokes equations to the corresponding Dirichlet data. The operator B is the solution operator for the Neumann problem of the Laplace equation, for details we refer to [11]. In particular, it follows from [11], Lemma 3.1. that

$$\|B\theta\|_0^\Gamma \le C \|\theta\|_{-1}^\Gamma \tag{3.1}$$

with C independent of θ and $\phi \in \mathcal{U}$.

The following lemma can be informally stated as follows: the subspace of potential tangential vector fields is invariant under the principal part of A, and its restriction to this space is conjugate to $\frac{1}{2}B$ under the gradient map $\nabla^{\widetilde{\mathbf{g}}}$. Note that standard regularity results would only provide an estimate with $C\|\theta\|_0^\Gamma$ on the right.

Lemma 3.1 *(The operator A on gradients) There is a $C > 0$ such that for all $\phi \in \mathcal{U}$ and all $\theta \in L^2(\Gamma)$*

$$\|(A\nabla^{\widetilde{\mathbf{g}}} - \tfrac{1}{2}\nabla^{\widetilde{\mathbf{g}}} B)\theta\|_0^\Gamma \le C\|\theta\|_{-1}^\Gamma.$$

Proof: Suppressing the argument ϕ again, we set

$$\begin{aligned}(u,p,\lambda) &:= L^{-1}E_3\nabla^{\tilde{\mathbf{g}}}\theta,\\ \psi &:= \tfrac{1}{2}(\Delta_{\mathbf{g}},\partial_n)^{-1}(0,\theta-\overline{\theta}).\end{aligned}$$

Applying Lemma 3.5.(ii) in [11] and writing n instead of $n_{\mathbf{g}}$, we get

$$\begin{aligned}\|(A\nabla^{\tilde{\mathbf{g}}}-\tfrac{1}{2}\nabla^{\tilde{\mathbf{g}}}B)\theta\|_0^\Gamma &\leq \left\|n\,n_iA^i\nabla^{\tilde{\mathbf{g}}}\theta\right\|_0^\Gamma+\left\|P^{\mathbf{g}}(A\nabla^{\tilde{\mathbf{g}}}-\tfrac{1}{2}\nabla^{\tilde{\mathbf{g}}}B)\theta\right\|_0^\Gamma\\ &\leq C\|\theta\|_{-1}^\Gamma+\|P^{\mathbf{g}}(u-\nabla^{\mathbf{g}}\psi)\|_0^\Gamma.\end{aligned}$$

It remains to estimate the last term on the right. For this purpose, we extend n as a C^2-function to the interior of Ω, such that $\|n\|_{C^2(\overline{\Omega})}$ is bounded uniformly with respect to $\phi\in\mathcal{U}$, and calculate

$$\begin{aligned}\Delta_{\mathbf{g}}(\nabla^i\psi) &= \nabla^k\nabla_k\nabla^i\psi=R^{ij}\nabla_j\psi,\\ \operatorname{div}_{\mathbf{g}}(\nabla^{\mathbf{g}}\psi) &= \Delta_{\mathbf{g}}\psi=0,\end{aligned}$$

in Ω, where R^{ij} are the coordinates of the Ricci tensor of $\mathbf{g}$, and

$$\begin{aligned}(\nabla^i\nabla^j\psi+\nabla^j\nabla^i\psi)n_j &= 2\nabla^i\nabla^j\psi n_j=2(\nabla^i(\nabla^j\psi n_j)-\nabla^i\psi\nabla^j n_j)\\ &= 2(\nabla^{\mathbf{g}})^i(\nabla^j\psi n_j)-2\nabla^i\psi\nabla^j n_j\\ &= 2(\nabla^{\tilde{\mathbf{g}}})^i(\nabla^j\psi n_j)+2\nabla^k(\nabla^j\psi n_j)n_kn^i-2\nabla^i\psi\nabla^j n_j\\ &= (\nabla^{\tilde{\mathbf{g}}})^i\theta+2\nabla^k(\nabla^j\psi n_j)n_kn^i-2\nabla^i\psi\nabla^j n_j\end{aligned}$$

on Γ. Hence

$$L(u-\nabla^{\mathbf{g}}\psi,p,\lambda)=\begin{pmatrix}-R^{ij}\nabla_j\psi\\ 0\\ -2\nabla^k(\nabla^j\psi n_j)n_kn^i+2\nabla^i\psi\nabla^j n_j\\ \int_\Omega\sqrt{g}\partial_k\Phi^i\nabla^k\psi\,dx\\ 0\end{pmatrix}^T,$$

and applying first [11], Lemmas 3.2 with $t=1$ and 3.5.(i) with $t=0$ and then [11], Lemma 3.1. with $t=0$ we get

$$\begin{aligned}\|P^{\mathbf{g}}(u-\nabla^{\mathbf{g}}\psi)\|_0^\Gamma &\leq C\Big(\left\|R^{ij}\nabla_j\psi\right\|_{-\frac{3}{2}}^\Omega+\left\|\nabla^k(\nabla^j\psi n_j)n_k\right\|_{-2}^\Gamma+\left\|\nabla^{\mathbf{g}}\psi\nabla^j n_j\right\|_{-1}^\Gamma\\ &\qquad+\left|\int_\Omega\sqrt{g}\partial_k\Phi^i\nabla^k\psi\,dx\right|\Big)\\ &\leq C\left(\|\psi\|_{-\frac{1}{2}}^\Omega+\|\psi\|_0^\Gamma\right)\leq C\|\theta\|_{-1}^\Gamma.\end{aligned}$$

■

We start the linearization by linearizing the right side of (2.17) with respect to an (vector-valued) perturbation h. For this purpose, we introduce the notation

$$\widehat{\rho} := \rho \frac{\sqrt{\widetilde{g}(\mathrm{Id})}}{\widetilde{g}} = \frac{\rho}{\omega_{\mathbf{g}}}, \quad k := (D\Phi)^{-1} h, \quad \widetilde{\nu} := D\Phi n_{\mathbf{g}} = D\Phi n.$$

Lemma 3.2 *(Linearization of f) We have*

$$(D\Phi)^{-1} f'(\phi)[h] = \gamma \Delta_{\widetilde{\mathbf{g}}}(h^i \widetilde{\nu}^i) n - \nabla^{\widetilde{\mathbf{g}}}(\sigma'(\widehat{\rho})\widehat{\rho}\, \mathrm{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k) + R_1(\phi) h,$$

where $R_1(\phi)$ is a first order differential operator whose coefficients are smooth functions of ϕ and its derivatives up to order 3.

Proof: As in the proof of Lemma 3.4 in [11] we have

$$f = \widetilde{g}^{\alpha\beta} \partial_\alpha \gamma \partial_\beta \phi + \gamma \widetilde{g}^{\alpha\beta} \partial_{\alpha\beta} \phi^j \widetilde{\nu}^j \widetilde{\nu},$$

and, using the same results on $D(\sqrt{\widetilde{g}})$ as in the proof of Lemma 2.1,

$$\begin{aligned} f'[h] &= \widetilde{g}^{\alpha\beta} \partial_\alpha \left(-\sigma'(\widehat{\rho}) \frac{\widehat{\rho}}{\sqrt{\widetilde{g}}} D(\sqrt{\widetilde{g}})[h] \right) \partial_\beta \phi + \gamma \widetilde{g}^{\alpha\beta} \partial_{\alpha\beta} h^j \widetilde{\nu}^j \widetilde{\nu} + R_2(\phi) h \\ &= -\widetilde{g}^{\alpha\beta} \partial_\alpha \left(\sigma'(\widehat{\rho}) \widehat{\rho} (\mathrm{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k - \kappa_{\mathbf{g}} k^i n_i) \right) \partial_\beta \phi + \gamma \widetilde{g}^{\alpha\beta} \partial_{\alpha\beta} h^j \widetilde{\nu}^j \widetilde{\nu} + R_2(\phi) h \\ &= -D\Phi \nabla^{\widetilde{\mathbf{g}}} \left(\sigma'(\widehat{\rho}) \widehat{\rho} \mathrm{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k \right) + \gamma \widetilde{g}^{\alpha\beta} \partial_{\alpha\beta} h^j \widetilde{\nu}^j D\Phi n + R_2(\phi) h, \end{aligned}$$

where $R_2(\phi)h$ denotes varying first-order differential operators whose coefficients are smooth functions of ϕ and its derivatives up to order 3. Thus

$$(D\Phi)^{-1} f'[h] = -\nabla^{\widetilde{\mathbf{g}}} \left(\sigma'(\widehat{\rho}) \widehat{\rho} \mathrm{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k \right) + \gamma \widetilde{g}^{\alpha\beta} \partial_{\alpha\beta} h^j \widetilde{\nu}^j n + R_2(\phi) h,$$

and the proof proceeds further as the proof of Lemma 3.4 in [11]. ∎

Now the key estimate for the linearized evolution operator can be proved:

Lemma 3.3 *(L^2-energy estimate for $\mathcal{F}'(\phi)$) There is a $C > 0$ such that for all $\phi \in \mathcal{U}$ and all $h \in H^{s+1}(\Gamma, \mathbb{R}^{m+1})$*

$$(\mathcal{F}'(\phi)[h], h)_{H^0(\Gamma, \mathbb{R}^{m+1})} \leq C \|h\|_0^{\Gamma\ 2}.$$

Proof: We introduce the notation

$$\begin{aligned} \widetilde{A}(\phi) &:= D\Phi A(\phi) (D\Phi)^{-1}, \\ \mathcal{A}(\phi) &:= \partial_n (\Delta_{\mathbf{g}}, \mathrm{Tr}_\Gamma)^{-1}, \end{aligned}$$

and recall from the proof of Lemma 3.6 in [11] that for any nonnegative $\chi \in C^2(\Gamma)$ we have the estimate

$$\int_\Gamma \omega_{\mathbf{g}} \chi \psi \mathcal{A} \psi \, d\Gamma \geq -C \|\psi\|_0^{\Gamma\ 2} \tag{3.2}$$

for all $\psi \in H^0(\Gamma)$, where C depends only on $\|\chi\|_{C^2(\Gamma)}$. We have

$$\mathcal{F}(\phi) = \widetilde{A}(\phi) f(\phi)$$

and hence, using Lemma 3.2,

$$\begin{aligned} \mathcal{F}'(\phi)[h] &= \mathcal{F}'_{(1)}(\phi)[h] + \mathcal{F}'_{(2)}(\phi)[h], \\ \mathcal{F}'_{(1)}(\phi)[h] &:= \widetilde{A}'(\phi)[h] f(\phi) + \widetilde{A}(\phi) D\Phi(\gamma \Delta_{\mathbf{g}} k^i n_i n + R_1[h]), \\ \mathcal{F}'_{(2)}(\phi)[h] &:= -D\Phi A(\phi) \nabla^{\widetilde{\mathbf{g}}} \sigma'(\widehat{\rho})\widehat{\rho} \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k. \end{aligned}$$

It can be shown in complete analogy to the proof of Lemma 3.6 in [11] that

$$(\mathcal{F}'_{(1)}(\phi)[h], h)_{H^0(\Gamma,\mathbb{R}^{m+1})} \leq C \|h\|_0^{\Gamma\,2},$$

thus it remains to show a parallel estimate for $\mathcal{F}'_{(2)}$. Setting $\chi := -\frac{1}{2}\sigma'(\widehat{\rho})\widehat{\rho}$ and noting that this is nonnegative due to (1.2), one gets

$$\begin{aligned} (\mathcal{F}'_{(2)}(\phi)[h], h)_{H^0(\Gamma,\mathbb{R}^{m+1})} &= 2\int_\Gamma \omega_{\mathbf{g}} A^i (\nabla^{\widetilde{\mathbf{g}}}(\chi \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k)) k_i \, d\Gamma \\ &= \int_\Gamma \omega_{\mathbf{g}} \mathbf{g}(\nabla^{\widetilde{\mathbf{g}}} B(\chi \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k), k) \, d\Gamma + R_3 \\ &= \int_\Gamma \omega_{\mathbf{g}} \mathbf{g}(\nabla^{\widetilde{\mathbf{g}}} B(\chi \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k), P^{\mathbf{g}} k) \, d\Gamma + R_3 \\ &= -\int_\Gamma \omega_{\mathbf{g}} B(\chi \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k) \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k \, d\Gamma + R_3 \end{aligned}$$

with

$$R_3 := \int_\Gamma \omega_{\mathbf{g}} \mathbf{g}(2A\nabla^{\widetilde{\mathbf{g}}} - \nabla^{\widetilde{\mathbf{g}}} B)(\chi \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k), k) \, d\Gamma,$$

and thus, due to Lemma 3.1,

$$|R_3| \leq C \left\| (2A\nabla^{\widetilde{\mathbf{g}}} - \nabla^{\widetilde{\mathbf{g}}} B) \chi \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k \right\|_0^\Gamma \|k\|_0^\Gamma \leq C \left\| \chi \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k \right\|_{-1}^\Gamma \|k\|_0^\Gamma \leq C \|h\|_0^{\Gamma\,2}.$$

It follows from Green's formula that B is symmetric with respect to the inner product

$$(u, v) \mapsto \int_\Gamma \omega_{\mathbf{g}} \mathbf{g}(u, v) \, d\Gamma.$$

Hence, using $\mathcal{A}B = \mathrm{Id}$, (3.2) and (3.1), we obtain

$$\begin{aligned} (\mathcal{F}'_{(2)}(\phi)[h], h)_{H^0(\Gamma,\mathbb{R}^{m+1})} &= -\int_\Gamma \omega_{\mathbf{g}} \chi B(\operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k) \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k \, d\Gamma + R_3 \\ &= \int_\Gamma \omega_{\mathbf{g}} \chi B(\operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k) \, \mathcal{A} B \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k \, d\Gamma + R_3 \\ &\leq C \left\| B \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k \right\|_0^{\Gamma\,2} \leq C \left\| \operatorname{div}_{\widetilde{\mathbf{g}}} P^{\mathbf{g}} k \right\|_{-1}^{\Gamma\,2} \leq C \|h\|_0^{\Gamma\,2}. \end{aligned}$$

■

4 Main result in the general case

On the basis of Lemma 3.3 we can give corresponding energy estimates in higher Sobolev norms for the nonlinear operator. This will be done essentially parallel to [11], and not all the details given there will be repeated here. However, we will describe the main arguments and point out the new aspects.

Let $D_1, \dots, D_{m+1}$ be $m+1$ smooth vector fields on Γ, identified with first order differential operators, such that

$$\operatorname{span}\{D_1, \dots, D_{m+1}\} = T_x\Gamma \quad \forall x \in \Gamma.$$

We recall the definition of the operator $\mathcal{G}$ from (2.18) and the following facts from [11]:

- For all $n \in \mathbb{N}$, the scalar product $(\,\cdot\,,\cdot\,)_n$ defined by

$$(u,v)_n := \sum_{|\alpha|\le n} (D^\alpha u, D^\alpha v)_{H^0(\Gamma)},$$

  $D^\alpha := D_1^{\alpha_1} \dots D_{m+1}^{\alpha_{m+1}}, \quad |\alpha| := \alpha_1 + \dots + \alpha_{m+1}$, generates a norm on $H^n(\Gamma)$ which is equivalent to the usual one.

- For sufficiently smooth $\phi \in \mathcal{U}$ and γ, we have

$$\begin{aligned} &D^\alpha \mathcal{G}(\phi,\gamma) = \mathcal{G}'(\phi,\gamma)\,[(D^\alpha\phi, D^\alpha\gamma)] \\ &+\sum_{k=2}^{|\alpha|} \sum_{\alpha_1+\dots+\alpha_k=\alpha} C_{\alpha_1\dots\alpha_k} \mathcal{G}^{(k)}(\phi,\gamma)\,[(D^{\alpha_1}\phi, D^{\alpha_1}\gamma), \dots, (D^{\alpha_k}\phi, D^{\alpha_k}\gamma)]\,. \end{aligned} \tag{4.1}$$

- As $\mathcal{G}$ is linear in the second argument,

$$\begin{aligned} &\mathcal{G}^{(k)}(\phi,\gamma)\,[(D^{\alpha_1}\phi, D^{\alpha_1}\gamma), \dots, (D^{\alpha_k}\phi, D^{\alpha_k}\gamma)] = \mathcal{F}^{(k)}(\phi)[D^{\alpha_1}\phi, \dots, D^{\alpha_k}\phi] \\ &+\sum_{j=1}^{k} \left(\partial_\phi^{k-1}\mathcal{G}\right)(\phi, D^{\alpha_j}\gamma)[D^{\alpha_1}\phi, \dots, D^{\alpha_{j-1}}\phi, D^{\alpha_{j+1}}\phi, \dots, D^{\alpha_k}\phi], \end{aligned} \tag{4.2}$$

  where ∂_ϕ^l denotes the l-th Fréchet derivative with respect to the first argument.

Now our main effort is to give estimates for the higher Fréchet derivatives of $\mathcal{G}$. Essentially, we will show that the terms can be estimated like terms of order zero.

For any multiindex $\alpha \in \mathbb{N}^{m+1}$ we define the "nonlinear commutator"

$$R_\alpha := D^\alpha(\gamma(\phi)) - \gamma'(\phi)[D^\alpha\phi]. \tag{4.3}$$

Recall that we have fixed $s_0 > \frac{m}{2}$, $s \ge s_0 + 4$ integer.

Lemma 4.1 *(Lower order terms) Assume $k, r \in \mathbb{N}$, $r \geq s$, $\phi \in \mathcal{U} \cap H^{r+1}(\Gamma)$, $\rho \in H^r(\Gamma)$, $\alpha_0, \ldots, \alpha_k \in \mathbb{N}^{m+1}$, $\alpha_1, \ldots, \alpha_k > 0$, $\alpha := \alpha_0 + \ldots + \alpha_k$, $1 \leq |\alpha| \leq r$. Then there is a $C = C_{\rho,r,\mathcal{U},\Gamma}$ such that*

(i)

$$\left\| \partial_\phi^k \mathcal{G}(\phi, D^{\alpha_0}\gamma)[D^{\alpha_1}\phi, \ldots, D^{\alpha_k}\phi] \right\|_0^\Gamma \leq C(\|\phi\|_r^\Gamma + 1) \tag{4.4}$$

for $\alpha \neq \alpha_0$ and $\alpha \neq \alpha_1$,

(ii)

$$\|\mathcal{G}(\phi, R_\alpha)\|_0^\Gamma \leq C(\|\phi\|_r^\Gamma + 1). \tag{4.5}$$

Proof: 1. Define the operators $S(\phi)$, $T(\phi)$, $\widetilde{L}(\phi)$ by

$$\begin{aligned} S(\phi)(u, p, \lambda) &:= (D\Phi u, p, \lambda), \\ T(\phi)(F, K, H, M_1, M_2) &:= (D\Phi F, K, D\Phi H, M_1, M_2), \\ \widetilde{L}(\phi) &:= T(\phi)L(\phi)S(\phi)^{-1}. \end{aligned}$$

We recall from [11] that $\widetilde{L} \in C^\infty(\mathcal{U}, \mathcal{L}_{is}(X_s, Y_s))$ and note that $\mathcal{G}(\phi, \gamma) = u$, where

$$\widetilde{L}(u, p, 0) = E_3 f(\phi, \gamma). \tag{4.6}$$

This equation defines u and p implicitly as functions of ϕ and γ, and we introduce the notation

$$(u, p)^{(j)}(\phi, \gamma)[h_1, \ldots, h_j] := (\partial_\phi^j u(\phi, \gamma)[h_1, \ldots, h_j], \partial_\phi^j p(\phi, \gamma)[h_1, \ldots, h_j]).$$

Moreover, for $t \in [0, s_0 + 2]$, $(u, p) \in H^{s+\frac{1}{2}}(\Omega, \mathbb{R}^{m+1}) \times H^{s-\frac{1}{2}}(\Omega)$, we introduce the aggregated norm

$$|||(u, p)|||_t := \|u\|_t^\Gamma + \|u\|_{t+\frac{1}{2}}^\Omega + \|\nabla u\|_{t-1}^\Gamma + \|p\|_{t-1}^\Gamma + \|p\|_{t-\frac{1}{2}}^\Omega.$$

We are going to prove the estimate

$$|||(u, p)^{(k)}(\phi, D^{\alpha_0}\gamma)[D^{\alpha_1}\phi, \ldots, D^{\alpha_k}\phi]|||_t \leq \begin{cases} C(\|\phi\|_{t+|\alpha|}^\Gamma + 1), & \alpha \neq \alpha_0 \wedge \alpha \neq \alpha_1, \\ C(\|\phi\|_{t+|\alpha|+1}^\Gamma + 1), & \alpha = \alpha_0 \vee \alpha = \alpha_1 \end{cases} \tag{4.7}$$

for all $t \in [0, s_0 + 2]$ and all multiindices $\alpha, \alpha_0, \ldots, \alpha_k \in \mathbb{N}^{m+1}$ such that

$$\alpha = \alpha_0 + \ldots + \alpha_k, \quad \alpha_1, \ldots, \alpha_k > 0, \quad t + |\alpha| \leq r. \tag{4.8}$$

(Note that we also allow $\alpha = \alpha_0 = 0$ here.) The estimate (4.7) with $t = 0$ implies (4.4).

2. Fix $t \in [0, s_0+2]$, $\alpha, \alpha_0, \dots, \alpha_k \in \mathbb{N}^{m+1}$ such that (4.8) holds. As a first step, we will show

$$\left\|\partial_\phi^k f(\phi, D^{\alpha_0}\gamma)[D^{\alpha_1}\phi, \dots, D^{\alpha_k}\phi]\right\|_{t-1}^\Gamma \le \begin{cases} C(\|\phi\|_{t+|\alpha|}^\Gamma + 1), & \alpha \neq \alpha_0 \wedge \alpha \neq \alpha_1, \\ C(\|\phi\|_{t+|\alpha|+1}^\Gamma + 1), & \alpha = \alpha_0 \vee \alpha = \alpha_1, \end{cases} \tag{4.9}$$

and

$$\|f(\phi, R_\alpha)\|_{-1}^\Gamma \le C(\|\phi\|_r^\Gamma + 1) \qquad (\alpha \neq 0). \tag{4.10}$$

From (2.7) we find that $D^{\alpha_0}\gamma$ is a finite sum of terms of the form

$$a(\nabla\Phi, \rho)\prod_{i=1}^{l} D^{\alpha_{0i}}\rho \prod_{i=1}^{\iota} \partial^{\nu_0^i}\mathcal{E}D^{\alpha_0^i}\phi$$

with $|\nu_0^1| = 1$, $\sum \alpha_{0i} + \sum \alpha_0^i = \alpha_0$, and a is smooth. The nonlinear commutator R_α also has this structure, and we have $\alpha_0^1 \neq \alpha_0$ for all summands occurring there. Consequently, both $\partial_\phi^k f(\phi, D^{\alpha_0}\gamma)[D^{\alpha_1}\phi, \dots, D^{\alpha_k}\phi]$ and $f(\phi, R_\alpha)$ are finite sums of terms

$$\mathcal{T} := a(\nabla\Phi, \nabla^2\Phi, \rho)\prod_{i=1}^{l} \partial^{\nu_{0i}}D^{\alpha_{0i}}\rho \prod_{i=1}^{\iota} \partial^{\nu_0^i}\mathcal{E}D^{\alpha_0^i}\phi \prod_{i=1}^{k} \partial^{\nu_i}\mathcal{E}D^{\alpha_i}\phi$$

with $0 \le |\nu_{0i}| \le 1$, $1 \le |\nu_0^i| \le 2$, $1 \le |\nu_i| \le 2$, $\sum|\nu_{0i}| + \sum|\nu_0^i| + \sum|\nu_i| = l + \iota + k + 1$. To estimate these terms, we distinguish three cases:
Case 1: $\alpha = 0$: Then

$$\|\mathcal{T}\|_{t-1}^\Gamma = \|f(\phi, \gamma)\|_{t-1}^\Gamma \le C(\|\phi\|_{t+1}^\Gamma + 1).$$

Case 2: $\alpha = \alpha_0^1 > 0$ or $\alpha = \alpha_1$. In this case, $\mathcal{T}$ contains just one factor with a derivative of ϕ of order at most $|\alpha| + 2$, hence

$$\|\mathcal{T}\|_{t-1}^\Gamma \le C\|\phi\|_{t+|\alpha|+1}^\Gamma.$$

Case 3: Otherwise, set

$$\begin{aligned} \beta_i &:= |\alpha_{0i}| + |\nu_{0i}|, & 1 \le i \le l, \\ \beta_{l+i} &:= |\alpha_0^i| + |\nu_0^i|, & 1 \le i \le \iota, \\ \beta_{l+\iota+i} &:= |\alpha_i| + |\nu_i|, & 1 \le i \le k. \end{aligned}$$

Then $\beta_i \le |\alpha| + 1$ for $i = 1, \dots, l + \iota + k$. Define $I := \{j \,|\, \beta_j > 3\}$, $m := \#I$.
Case 3.1.: $m \le 1$: Then $I \subset \{i\}$ for some i, and, using (2.3) we get

$$\|\mathcal{T}\|_{t-1}^\Gamma \le C\max\{\|\rho\|_{t+\beta_i-1}^\Gamma, \|\phi\|_{t+\beta_i-1}^\Gamma\} \le C(\|\phi\|_{t+|\alpha|}^\Gamma + 1).$$

Case 3.2: $m \ge 2$: We proceed as in step 1.2. of the proof of Lemma 4.1 in [11] and set

$$b := \sum_{j\in I}\beta_j, \quad \lambda_j := \frac{(\beta_j - 3)^+}{b - 3m}, \quad \tau := (t-1)^+, \quad s_j := (1-\lambda_j)(s_0+1) + \lambda_j\tau.$$

Then $0 \le \tau \le s_0 + 1$, $\tau \le s_j$, $\sum_{j=1}^k s_j = \tau + (k-1)(s_0+1) > \tau + (k-1)\frac{m}{2}$, and by (2.3)

$$\|\mathcal{T}\|_{t-1}^{\Gamma} \le \|\mathcal{T}\|_{\tau}^{\Gamma} \le C \prod_{j\in I, j\le l} \|\rho\|_{s_j+\beta_j}^{\Gamma} \prod_{j\in I, j>l} \|\phi\|_{s_j+\beta_j}^{\Gamma}.$$

Using now that

$$s_j + \beta_j \le (1-\lambda_j)(s_0+4) + \lambda_j(\tau + |\alpha|) \le r$$

we get, using the corresponding interpolation inequalities for the Sobolev scale $H^t(\Gamma)$,

$$\|\mathcal{T}\|_{t-1}^{\Gamma} \le C \prod_{j\in I, j>l} \|\phi\|_{s_j+\beta_j}^{\Gamma} \le C {\|\phi\|_{s_0+4}^{\Gamma}}^{1-\lambda} {\|\phi\|_{t+|\alpha|}^{\Gamma}}^{\lambda} \le C(\|\phi\|_{t+|\alpha|}^{\Gamma} + 1),$$

where $\lambda := \sum_{j>l} \lambda_j \in [0,1]$.

Thus, $\|\mathcal{T}\|_{t-1}^{\Gamma}$ is estimated in all possible cases. Assume now $\alpha \ne 0$ and consider only the terms from $f(\phi, R_\alpha)$. This excludes cases 1 and 2, and (4.10) follows. Assume now $\alpha \ne \alpha_0$ and $\alpha \ne \alpha_1$. This also excludes cases 1 and 2, and (4.9) follows.

3. The estimate (4.10) and Lemma 3.2 in [11] imply

$$\begin{aligned} \|\mathcal{G}(\phi, R_\alpha)\|_0^{\Gamma} &= \left\|\Pi_1 \widetilde{L}(\phi)^{-1} E_3 f(\phi, R_\alpha)\right\|_0^{\Gamma} = \left\|\Pi_1 S(\phi) L^{-1}(\phi) T(\phi) E_3 f(\phi, R_\alpha)\right\|_0^{\Gamma} \\ &\le C\|f(\phi, R_\alpha)\|_{-1}^{\Gamma} \le C(\|\phi\|_r^{\Gamma} + 1), \end{aligned}$$

thus (4.5) is proved.

4. Finally, we prove (4.7) by induction over k. For $k = 0$, one immediately gets from Lemma 3.2 in [11] and (4.9)

$$|||(u,p)(\phi, D^\alpha \gamma)|||_t \le C\|f(\phi, D^\alpha\gamma)\|_{t-1}^{\Gamma} \le C(\|\phi\|_{t+|\alpha|+1}^{\Gamma} + 1).$$

Assume now (4.7) for the derivatives up to order $k-1$. Taking the k-th Fréchet derivative of (4.6) with respect to ϕ and applying it to $(D^{\alpha_1}\phi, \dots, D^{\alpha_k}\phi)$ yields

$$\begin{aligned} &\widetilde{L}(\phi)(u,p)^{(k)}(\phi, D^{\alpha_0}\gamma)[D^{\alpha_1}\phi, \dots, D^{\alpha_k}\phi] = E_3 f^{(k)}(\phi, D^{\alpha_0}\gamma)[D^{\alpha_1}\phi \dots, D^{\alpha_k}\phi] \\ &- \sum_{l=1}^{k} \sum_{\pi\in \mathsf{S}_k} \widetilde{L}^{(l)}(\phi, D^{\alpha_0}\gamma)[D^{\alpha_{\pi(1)}}\phi, \dots, D^{\alpha_{\pi(l)}}\phi] \\ &\qquad (u,p)^{(k-l)}(\phi, D^{\alpha_0}\gamma)[D^{\alpha_{\pi(l+1)}}\phi, \dots, D^{\alpha_{\pi(k)}}\phi]. \end{aligned}$$

Now all the terms on the right can be estimated by a similar technique as in step 2, and (4.7) for $(u,p)^{(k)}$ follows from [11], Lemma 3.2. For the details we refer to the analogous arguments in the proof of Lemma 4.1 in [11]. ■

Considering now the nonlinear commutator

$$[D^\alpha, \mathcal{F}] := D^\alpha \mathcal{F}(\phi) - \mathcal{F}'(\phi)[D^\alpha \phi]$$

and taking into account (4.1), (4.2), and (4.3) we find

$$\begin{aligned}
[D^\alpha, \mathcal{F}] &= D^\alpha \mathcal{G}(\phi,\gamma) - \partial_\phi \mathcal{G}(\phi,\gamma)[D^\alpha \phi] - \partial_\gamma \mathcal{G}(\phi,\gamma)[\gamma'[D^\alpha \phi]] \\
&= \mathcal{G}(\phi, R_\alpha) + \sum_{k=2}^{|\alpha|} \sum_{\alpha_1+\ldots+\alpha_k=\alpha} C_{\alpha_1\ldots\alpha_k} \partial_\phi^{(k)} \mathcal{G}(\phi,\gamma)[D^{\alpha_1}\phi, \ldots, D^{\alpha_k}\phi] \\
&\quad + \sum_{k=2}^{|\alpha|} \sum_{\alpha_1+\ldots+\alpha_k=\alpha} C_{\alpha_1\ldots\alpha_k} \sum_{j=1}^{k} \partial_\phi^{(k-1)} \mathcal{G}(\phi,\gamma)[D^{\alpha_1}\phi, \ldots, \\
&\qquad\qquad D^{\alpha_{j-1}}\phi, D^{\alpha_{j+1}}\phi, \ldots, D^{\alpha_k}\phi].
\end{aligned}$$

To all the terms on the right hand side we can apply Lemma 4.1 and obtain

$$\|[D^\alpha, \mathcal{F}]\|_0^\Gamma \leq C(\|\phi\|_r^\Gamma + 1) \tag{4.11}$$

for $\alpha \leq r$.

As in [11], this implies the following a priori estimate:

Lemma 4.2 *(H^r - a priori estimate for $\mathcal{F}$) Let $r \geq s+1$ be integer. Then*

$$(\mathcal{F}(\phi), \phi)_r \leq C_r \left(1 + \|\phi\|_r^{\Gamma\,2}\right), \qquad \phi \in \mathcal{U} \cap H^{r+1}(\Gamma, \mathbb{R}^{m+1}).$$

Proof: It is sufficient to show the estimate for smooth $\phi \in \mathcal{U}$. For such ϕ, we have from Lemma 3.3, the definition of $(\cdot,\cdot)_r$, and (4.11)

$$\begin{aligned}
& (\mathcal{F}(\phi), \phi)_r \\
= \; & \sum_{|\alpha|\leq r} (D^\alpha \mathcal{F}(\phi), D^\alpha \phi)_0 = (\mathcal{F}(\phi), \phi)_0 + \sum_{1\leq|\alpha|\leq r} (D^\alpha \mathcal{F}(\phi), D^\alpha \phi)_0 \\
= \; & (\mathcal{F}(\phi), \phi)_0 + \sum_{1\leq|\alpha|\leq r} (\mathcal{F}'(\phi)[D^\alpha \phi], D^\alpha \phi)_0 + \sum_{1\leq|\alpha|\leq r} ([D^\alpha, \mathcal{F}](\phi), D^\alpha \phi)_0 \\
\leq \; & C_r \left(1 + \|\phi\|_r^{\Gamma\,2}\right).
\end{aligned}$$

■

For the formulation of our main result, we introduce the following notation: Let $r \geq r_0 := s+1$ be integer and set $\mathcal{V} := \mathcal{U} - \mathrm{Id}$, where we assume that $\mathcal{V}$ is a ball of radius $\delta > 0$ around 0 in $H^{r_0}(\Gamma, \mathbb{R}^{m+1})$. As usual, we will denote the open ball in X around 0 with radius K by $B_0(K, X)$.

Setting $\psi := \phi - \mathrm{Id}$, instead of (2.20) we consider the equivalent problem

$$\left.\begin{aligned} \dot{\psi} &= \mathcal{F}(\psi + \mathrm{Id}), \\ \psi(0) &= \psi_0 := \phi_0 - \mathrm{Id}. \end{aligned}\right\} \tag{4.12}$$

Theorem 4.3 *(Existence, uniqueness, and regularity of solutions to* (4.12)*)* *(i)* *For any* $\psi_0 \in \mathcal{V} \cap H^r(\Gamma, \mathbb{R}^{m+1})$ *with* $\|\psi_0\|^\Gamma_{r_0} \le K < \delta$ *there is a* $T = T(K,r) > 0$ *such that* (4.12) *has a unique solution*

$$\psi = \Psi(\cdot, \psi_0) \in C\left([0,T], \mathcal{V} \cap H^r(\Gamma, \mathbb{R}^{m+1})\right) \cap C^1\left([0,T], H^{r-1}(\Gamma, \mathbb{R}^{m+1})\right).$$

(ii) *For any* $r \ge r_0$, $K \in (0, \delta)$, *and* $t \in [0, T(K,r)]$, *the mappings* $\Psi(t, \cdot)$ *are continuous from* $B_0\left(K, H^{r_0}(\Gamma, \mathbb{R}^{m+1})\right) \cap H^r(\Gamma, \mathbb{R}^{m+1})$ *to* $H^r(\Gamma, \mathbb{R}^{m+1})$, *uniformly with respect to* t.

(iii) *Suppose* $\psi \in C([0,T], \mathcal{V})$ *is a solution to* (4.12) *with* $\psi_0 \in H^r(\Gamma, \mathbb{R}^{m+1})$. *Then* $\psi \in C\left([0,T], H^r(\Gamma, \mathbb{R}^{m+1})\right)$.

The proof is literally the same as for Theorem 4.3 in [11]. ■

We repeat the remark from [11] that this result implies, in particular, the fact that solutions starting from smooth initial data are smooth in space and time.

5 The plane case

In the case $m = 1$, Γ is a smooth curve which can be parametrized by its arclength s. If smooth functions and vector fields on Γ are identified via $\psi \widehat{=} \psi\partial_s$, both div_Γ and ∇_Γ reduce to the arclength derivative ∂_s, and the second order differential operator $\nabla_\Gamma \mathrm{div}_\Gamma$, which is degenerate elliptic for $m > 1$, reduces to the strongly elliptic operator $\partial_s^2 = \Delta_\Gamma$.

This observation enables us to prove, under natural assumptions, that the plane version of our problem (1.1) yields a parabolic surface motion having the smoothing property, i.e. the moving boundary becomes smooth in space and time immediately after the initial time. For this purpose we use the approach described in [7] which is also applicable to the proof of analyticity. In particular, we will use (continuous) maximal regularity, therefore it is convenient to change our analytic framework to the so-called little Hölder spaces $h^\theta(\Gamma)$, $h^\theta(\Omega)$ which for $\theta \in \mathbb{R}_+ \setminus \mathbb{N}$ are defined as closures of $C^\infty(\Gamma)$ and $C^\infty(\overline{\Omega}) := BUC^\infty(\Omega)$ in the usual C^θ-Hölder norms which we will denote by $\|\cdot\|_\theta$ and $\|\cdot\|_{\theta,\Omega}$, respectively.

In order to discuss the C^∞-case and the real-analytic case simultaneously, we fix $K \in \{\infty, \omega\}$ and demand that Γ is a C^K-manifold. On our data, we impose the smoothness assumptions

$$\rho \in C^K(\Gamma), \quad \sigma \in C^K[0,\infty), \tag{5.1}$$

and the nondegeneracy assumptions

$$\sigma > 0, \quad \sigma' < 0, \quad \rho > 0, \tag{5.2}$$

which are sharpenings of our earlier demands.

We start with a brief description of the abstract results which will be used; for further details, we refer to [7], Sections 2 and 3. Let E_0 and E_1 be two Banach spaces with continuous and dense embedding $E_1 \hookrightarrow E_0$. Let $D \subset E_1$ be open, and suppose

$$P \in C^1(D, E_0). \tag{5.3}$$

We fix $u_0 \in D$ and consider the abstract Cauchy problem

$$\left.\begin{array}{rcl} \partial_t u + P(u) &=& 0, \\ u(0) &=& u_0. \end{array}\right\} \tag{5.4}$$

The parabolicity assumption on (5.4) is

$$P'(v) \in \mathcal{H}(E_1, E_0), \qquad v \in D \tag{5.5}$$

i.e. $-P'(v)$ (considered as an in general unbounded operator on E_0 with domain E_1) generates a strongly continuous analytic semigroup on E_0.

For fixed $T > 0$ we introduce the Banach spaces

$$\begin{array}{rcl} \mathbb{E}_0 &:=& C([0,T], E_0), \\ \mathbb{E}_1 &:=& C([0,T], E_1) \cap C^1([0,T], E_0), \end{array}$$

and the trace operator at $t = 0$, $\mathrm{Tr}_{t=0} \in \mathcal{L}(\mathbb{E}_0, E_0)$, given by $w \mapsto w(0)$. We assume that the linearization of (5.4) has the so-called maximal regularity property:

$$(\partial_t + P'(v), \mathrm{Tr}_{t=0}) \in \mathcal{L}_{is}(\mathbb{E}_1, \mathbb{E}_0 \times E_1), \qquad v \in D. \tag{5.6}$$

(Note that the validity of this condition does not depend on T.) Then, the following holds:

Theorem 5.1 *(*Da Prato-Grisvard [6]*) Assume* (5.3), (5.5), (5.6). *There is a* $t^+ = t^+(u_0) > 0$ *such that* (5.4) *has a unique maximal solution in*

$$C([0, t^+), D) \cap C^1([0, t^+), E_0).$$

To show the smoothing property, we make the following further assumptions:

$$E_1 \hookrightarrow (C^1(\Gamma))^n, \quad E_0 \hookrightarrow (C(\Gamma))^n \tag{A_1}$$

for some $n \in \mathbb{N}$. Next we fix a suitable $N \in \mathbb{N}$ and a mapping

$$S \in C^K(\mathbb{R}^N \times \mathbb{R} \times \Gamma, \Gamma)$$

having the following properties:

$$\begin{array}{l} S(\mu, \cdot, \cdot) \text{ is a flow on } \Gamma \text{ for all } \mu \in \mathbb{R}^N, \\ S(\mu, t, \cdot) \in \mathrm{Diff}^K(\Gamma), \quad (\mu, t) \in \mathbb{R}^N \times \mathbb{R}, \\ \left\{ \dfrac{\partial}{\partial t} S(\mu, t, p)|_{t=0} \,;\, \mu \in \mathbb{R}^N \right\} = T_p\Gamma, \quad p \in \Gamma, \\ \left[\mu \mapsto \dfrac{\partial}{\partial t} S(\mu, t, \cdot)|_{t=0} \right] \in \mathrm{Hom}(\mathbb{R}^N, \mathcal{V}^K(\Gamma)), \end{array}$$

where $\mathcal{V}^K(\Gamma)$ denotes the set of C^K-vector fields on Γ. For $K = \omega$, the existence of such an S is shown in [7], Lemma 3.1, for $K = \infty$ the proof is simpler and proceeds along the same lines. (For our application, we will give a simple explicit construction of S with $N = 1$ below. Here, however, we prefer to describe the general result independently from this.)

As $S(\mu, \cdot, \cdot)$ is a flow, the family of mappings $W_\mu(t) \in \mathrm{Isom(E_j)}$, $j = 0, 1$ (parametrized by t) given by

$$W_\mu(t)v := S(\mu, t, \cdot)^* v := v \circ S(\mu, t, \cdot), \qquad v \in E_j,\ j = 0,\ 1,$$

form a one-parameter group with respect to composition. We demand its strong continuity:

$$[t \mapsto W_\mu(t)] \text{ is a strongly continuous group on } E_j,\ j = 0,\ 1. \tag{A_2}$$

Let V_μ denote the infinitesimal generator of $t \mapsto W_\mu(t)$, considered as a group of operators on E_0. We assume that

$$E_1 \hookrightarrow \mathrm{dom}(A_\mu) \quad \text{for any} \quad \mu \in \mathbb{R}^N. \tag{A_3}$$

and

$$[(\mu, w) \mapsto A_\mu w] \in \mathcal{L}^2(\mathbb{R}^N \times E_1, E_0). \tag{A_4}$$

In order to formulate the crucial condition on P we set for $\mu \in \mathbb{R}^N$

$$\begin{aligned} S_\mu &:= S(\mu, 1, \cdot) \in \mathrm{Diff^K}(\Gamma), \\ S_\mu^* v &:= v \circ S_\mu, \quad v \in E_j,\ j = 0, 1, \\ S_*^\mu &:= (S_\mu^*)^{-1}. \end{aligned}$$

Recall that S_0 is the identity. Hence, by continuity, there is an $r_0 > 0$ and and open neighborhood $D_0 \subset D$ of u_0 such that $W_\mu(1)[D_0] \subset D$ for all $\mu \in \mathbb{B}_{\mathbb{R}^N}(0, r_0)$. Hence we can define the mapping

$$Q : \mathbb{B}_{\mathbb{R}^N}(0, r_0) \times D_0 \longrightarrow E_0$$

by

$$Q(\mu, v) := S_\mu^* P S_*^\mu v.$$

We shall assume

$$Q \in C^K(\mathbb{B}_{\mathbb{R}^N}(0, r_0) \times D_0, E_0). \tag{A_5}$$

Note that (A_5) implies $Q(0, \cdot) = P|_{D_0} \in C^K(D_0, E_0)$. More generally, ($\mathrm{A}_5$) can be seen as a compatibility condition between P and "spatial shifts" given by S_μ. If P is a (linear) differential operator, (A_5) implies a smoothness demand on the coefficients of P. (For a simple but illuminating example, see Remark 3.7 b) in [9].)

To formulate now the result on the smoothing property which we will apply, let u be as in Theorem 5.1 and define $\widehat{u} \in C([0, t^+) \times \Gamma, \mathbb{R}^n)$ by

$$\widehat{u}(t, p) := u(t)(p).$$

The following result holds:

Theorem 5.2 *(Smoothing property,* [7]*, Theorem* 3.9*) Under the assumptions of Theorem* 5.1 *and* (A_1)–(A_5)*, we have*

$$\widehat{u}|_{(0,t^+)\times\Gamma} \in C^K((0,t^+)\times\Gamma, \mathbb{R}^n).$$

(Note that in [7] only the case $n = 1$ is considered. The result in the vector-valued case follows from a straightforward modification of the proof of Theorem 3.9 of that paper.)

We are going to apply Theorems 5.1 and 5.2 to (2.20), where we will set $n = 2$, $\theta \in (0, 1)$,

$$E_0 := (h^{2+\theta}(\Gamma))^2, \quad E_1 := (h^{3+\theta}(\Gamma))^2, \quad P = -\mathcal{F}, \quad u_0 := \phi_0. \tag{5.7}$$

The little Hölder spaces are stable under continuous interpolation. It follows from this fact and Théorème 3.1 in [6] that in our application, the validity of (5.5) for all $\theta \in (0, 1)$ implies (5.6) for all $\theta \in (0, 1)$, see Remark 2.2.j) in [7]. Moreover, it is straightforward to check that conditions (A_1)–(A_4) hold. Hence, it remains to check (A_5) and (5.5).

To simplify the technicalities, suppose Γ is parametrized by arclength s. Then $\widetilde{g}(\mathrm{Id}) \equiv 1$. Moreover, let $\mathbf{t}$ denote the (positively oriented) unit tangent vector field on Γ, let $N = 1$ and let $S(\mu, \cdot, \cdot)$ be the flux generated by $\mu\mathbf{t}$, thus S_μ is a translation along Γ by arclength μ. This implies, in particular, that ∂_s and S_μ^* commute.

We recall the definition of $\mathcal{G}$ from (2.18) and begin the proof of (A_5) by showing smoothness of $\mathcal{G}$ in little Hölder spaces. (Although we use different function spaces we keep the same notation for corresponding functions and operators.)

Lemma 5.3 *(Smoothness of* $\mathcal{G}$ *in little Hölder spaces) There is an open neighborhood* D *of the identity in* $(h^{3+\theta}(\Gamma))^2$ *such that*

$$\mathcal{G} \in C^K(D \times h^{2+\theta}(\Gamma), (h^{2+\theta}(\Gamma))^2).$$

Here and in the sequel, we shrink D when necessary without explicit mentioning.

Proof: The proof proceeds along the same lines as the proof of (2.21). First we note that the spaces $h^\theta(\Gamma)$, $h^\theta(\Omega)$, $\theta \in \mathbb{R}_+\backslash\mathbb{N}$, are Banach algebras with respect to pointwise multiplication. This fact is the basis of the following considerations.

We introduce the Banach spaces

$$\begin{aligned} V &:= \{(c^{ij}) \,|\, i,j = 1,\ldots,m+1,\ c^{ij} \in \mathbb{R},\ c^{ij} = -c^{ji}\}, \\ \mathcal{X}^\theta &:= (h^{2+\theta}(\Omega))^{m+1} \times h^{1+\theta}(\Omega) \times (\mathbb{R}^{m+1} \times V), \\ \mathcal{Y}^\theta &:= (h^{\theta}(\Omega))^{m+1} \times h^{1+\theta}(\Omega) \times (h^{1+\theta}(\Gamma))^{m+1} \times \mathbb{R}^{m+1} \times V, \end{aligned}$$

and for $\phi \in D$ we will consider the operator $L(\phi)$ given by (2.5) as an operator on $\mathcal{X}^\theta$ now. Using the analyticity of the inversion of regular matrices and square root of positive functions, one straightforwardly checks

$$\begin{aligned} g_{ij}, \sqrt{g}, \frac{1}{\sqrt{g}} &\in C^K(D, h^{2+\theta}(\Omega)), \\ \widetilde{g}, \sqrt{\widetilde{g}}, \frac{1}{\sqrt{\widetilde{g}}} &\in C^K(D, h^{2+\theta}(\Gamma)), \\ L &\in C^K(D, \mathcal{L}(\mathcal{X}^\theta, \mathcal{Y}^\theta)). \end{aligned}$$

By Lemma A.1 in [7] we have

$$L(\mathrm{Id}) \in \mathcal{L}_{is}\left(\mathcal{X}^\theta, \mathcal{Y}^\theta\right) \tag{5.8}$$

and consequently, as $\mathcal{L}_{is}(\mathcal{X}^\theta, \mathcal{Y}^\theta)$ is open in $\mathcal{L}(\mathcal{X}^\theta, \mathcal{Y}^\theta)$,

$$L \in C^K(D, \mathcal{L}_{is}(\mathcal{X}^\theta, \mathcal{Y}^\theta)).$$

Moreover, we have

$$[L \mapsto L^{-1}] \in C^K(\mathcal{L}_{is}(\mathcal{X}^\theta, \mathcal{Y}^\theta)), \mathcal{L}_{is}(\mathcal{Y}^\theta, \mathcal{X}^\theta))$$

and $D\Phi \in C^K(D, \mathcal{L}_{is}(h^\tau(\Omega))^2)$ for $\tau \leq 2+\theta$, $\tau \notin \mathbb{N}$. Considering

$$f := \frac{1}{\sqrt{\widetilde{g}}}\partial_s(\gamma \frac{1}{\sqrt{\widetilde{g}}}\partial_s \phi^k) e_k$$

as a function of ϕ and γ we get

$$[(\phi, \gamma) \mapsto f] \in C^K(D \times h^{2+\theta}(\Gamma), (h^{1+\theta}(\Gamma))^2).$$

Now the assertion of the lemma follows from

$$\mathcal{G}(\phi, \gamma) = D\Phi \mathrm{Tr}_\Gamma L(\phi)^{-1} E_3 (D\Phi)^{-1} f(\phi, \gamma).$$

■

We recall that in our application we have

$$\gamma = \gamma(\phi) = \sigma\left(\frac{\rho}{\sqrt{\widetilde{g}}}\right)$$

and prove a result similar to (A_5) for the mapping $[\phi \mapsto \gamma]$.

Lemma 5.4 *(Compatibility of γ and S_μ) Assume* (5.1). *Then there is a $r_0 > 0$ such that*

$$[(\mu, \zeta) \mapsto S_\mu^* \gamma(S_*^\mu \zeta)] \in C^k((-r_0, r_0) \times D, h^{2+\theta}(\Gamma)).$$

Proof: Set $\phi := S^\mu_* \zeta$ and note that

$$\begin{aligned} S^*_\mu \gamma(S^\mu_* \zeta) &= \gamma(\phi) \circ S_\mu = \sigma\left(\frac{\rho \circ S_\mu}{\sqrt{\tilde{g} \circ S_\mu}}\right) = \sigma\left(\frac{\rho \circ S_\mu}{\sqrt{(\partial_s(\phi + \mathrm{Id})^i \partial_s(\phi + \mathrm{Id})^i) \circ S_\mu}}\right) \\ &= \sigma\left(\frac{\rho \circ S_\mu}{\sqrt{\partial_s(\zeta + S_\mu)^i \partial_s(\zeta + S_\mu)^i}}\right). \end{aligned}$$

It follows from (5.1) and Lemma 4.4 in [7] together with the analyticity of the square root operation that

$$\left[(\mu, \zeta) \mapsto \frac{\rho \circ S_\mu}{\sqrt{\partial_s(\zeta + S_\mu)^i \partial_s(\zeta + S_\mu)^i}}\right] \in C^K((-r_0, r_0) \times D, h^{2+\theta}(\Gamma)).$$

The lemma follows from this by the fact that the superposition operator induced by σ is C^K in Hölder spaces (and thus, by an approximation argument, also in little Hölder spaces), see e.g. [17], Theorems II.4.4 or II.5.2, respectively. ■

Now it is straightforward to check the validity of (A_5). Writing $\phi := S^\mu_* \zeta$ again and using the equivariance of $\mathcal{G}$ with respect to arbitrary smooth diffeomorphisms of Γ (cf. [11], Eq. (4.1)) we get

$$Q(\mu, \zeta) = \mathcal{G}(\phi, \gamma(\phi)) \circ S_\mu = \mathcal{G}(\zeta, \gamma(\phi) \circ S_\mu) = \mathcal{G}(\zeta, S^*_\mu \gamma(S^\mu_* \zeta)),$$

and (A_5) follows from Lemmas 5.3 and 5.4.

Our next aim is to show (5.5) in the case of our application. As

$$\mathcal{H}((h^{3+\theta}(\Gamma))^2, (h^{2+\theta}(\Gamma))^2) \text{ is open in } \mathcal{L}((h^{3+\theta}(\Gamma))^2, (h^{2+\theta}(\Gamma))^2),$$

it is sufficient for this purpose to show

$$-\mathcal{F}'(\mathrm{Id}) \in \mathcal{H}((h^{3+\theta}(\Gamma))^2, (h^{2+\theta}(\Gamma))^2). \tag{5.9}$$

Parallel to the calculations in Section 3, we obtain

$$\mathcal{F}'(\mathrm{Id})[h] = A_0(\sigma(\rho)\partial_s^2(h \cdot n)n - \partial_s \sigma'(\rho)\rho \partial_s(h \cdot \mathbf{t})\mathbf{t}) + R_4 h \tag{5.10}$$

with

$$\begin{aligned} A_0 &:= A(\mathrm{Id}) = \mathrm{Tr}_\Gamma \Pi_1 L(\mathrm{Id})^{-1} E_3, \\ R_4 &\in \mathcal{L}((h^{3+\theta}(\Gamma))^2, (h^{3+\theta}(\Gamma))^2). \end{aligned}$$

We introduce the operators A_0^t, $A_0^n \in \mathcal{L}(h^{3+\theta}(\Gamma), h^{2+\theta}(\Gamma))$ by

$$\begin{aligned} A_0^t \psi &:= \mathbf{t} \cdot A_0(\psi \mathbf{t}), \\ A_0^n \psi &:= n \cdot A_0(\psi n) \end{aligned}$$

and write $\mathbf{Q} \in C^\infty(\Gamma, \mathbb{R}^{2\times 2})$ for the orthogonal matrix mapping the unit vectors to $\mathbf{t}$ and n, respectively. Then we find from (5.10)

$$\mathcal{F}'(\mathrm{Id})[h] = \mathbf{Q} \begin{bmatrix} -A_0^t \partial_s(\sigma'(\rho)\rho\partial_s) & 0 \\ 0 & A_0^n(\sigma(\rho)\partial_s^2) \end{bmatrix} \mathbf{Q}^T h + R_5 h \tag{5.11}$$

with

$$R_5 h := R_4 h - n \cdot A_0(\partial_s(\sigma'(\rho)\rho\partial_s(h\cdot\mathbf{t})\mathbf{t})n + \mathbf{t}\cdot A_0(\sigma(\rho)\partial_s^2(h\cdot n)n]\mathbf{t}.$$

From Lemma A.1 of the present paper and Lemma A.2 in [7] we conclude that

$$R_5 \in \mathcal{L}((h^{3+\theta}(\Gamma))^2, (h^{3+\theta}(\Gamma))^2). \tag{5.12}$$

The crucial step in the proof of (5.9) is the following generation result for the scalar operators on the diagonal in the first term of (5.11).

Lemma 5.5 *(Scalar generators) Assume* (5.2). *Then we have*

$$\begin{array}{cc} (i) & [\psi \mapsto A_0^t\partial_s(\sigma'(\rho)\rho\partial_s\psi)] \in \mathcal{H}(h^{3+\theta}(\Gamma), h^{2+\theta}(\Gamma)), \\ (ii) & [\psi \mapsto -A_0^n(\sigma(\rho)\partial_s^2\psi)] \in \mathcal{H}(h^{3+\theta}(\Gamma), h^{2+\theta}(\Gamma)). \end{array}$$

Proof: (i): We write $\alpha := -\sigma'(\rho)\rho$ and note that α is a positive smooth function. Moreover, we introduce the operators B_0 and $\mathcal{A}_0$ by

$$\begin{aligned} B_0\psi &:= B(\mathrm{Id})\psi = \mathrm{Tr}_\Gamma(\Delta, \partial_n)^{-1}(0, \psi - \overline{\psi}), \quad \overline{\psi} := \frac{\int_\Gamma \psi\, d\Gamma}{\int_\Gamma d\Gamma}, \\ \mathcal{A}_0 &:= \mathcal{A}(\mathrm{Id}) = \partial_n(\Delta, \mathrm{Tr}_\Gamma)^{-1}(0, \cdot). \end{aligned}$$

Identifying tangential vector fields and scalar functions on Γ, we get

$$\begin{aligned} A_0^t\partial_s(\alpha\partial_s\psi) &= \tfrac{1}{2}\partial_s B_0(\alpha\partial_s\psi) + (t\cdot A_0\partial_s(\alpha\partial_s\psi) - \tfrac{1}{2}\partial_s B_0(\alpha\partial_s\psi)) \\ &= -\tfrac{\alpha}{2}\mathcal{A}_0\psi + \tfrac{1}{2}(\alpha\mathcal{A}_0\psi + \partial_s B_0(\alpha\partial_s\psi)) \\ &\quad +(\mathbf{t}\cdot A_0\partial_s(\alpha\partial_s\psi) - \tfrac{1}{2}\partial_s B_0(\alpha\partial_s\psi)). \end{aligned}$$

Consequently, by Lemmas A.2 and A.4,

$$\|\tfrac{\alpha}{2}\mathcal{A}_0\psi + A_0^t\partial_s(\alpha\partial_s\psi)\|_{3+\theta} \le C\|\psi\|_{3+\theta}, \qquad \psi \in h^{3+\theta}(\Gamma).$$

It can be shown as in [7], Appendix B, that for any positive $\beta \in C^\infty(\Gamma)$ we have

$$\beta\mathcal{A}_0 \in \mathcal{H}(h^{3+\theta}(\Gamma), h^{2+\theta}(\Gamma)). \tag{5.13}$$

(The result is shown there only for $\beta = \mathrm{const}$, the proof is by "freezing of coefficients" and therefore immediately generalizable to the case of variable β.)

By continuous interpolation, for any given $\theta \in (0,1)$, $\theta' \in (0,\theta)$, and any $\varepsilon > 0$ there is a C such that

$$\begin{aligned} & \|\tfrac{\alpha}{2}\mathcal{A}_0\psi - A_0^t\partial_s(\sigma'(\rho)\rho\partial_s\psi)\|_{2+\theta} \\ \leq \; & \|\tfrac{\alpha}{2}\mathcal{A}_0\psi - A_0^t\partial_s(\sigma'(\rho)\rho\partial_s\psi)\|_{3+\theta'} \leq \|\psi\|_{3+\theta'} \leq \varepsilon\|\psi\|_{3+\theta} + C\|\psi\|_{2+\theta}. \end{aligned}$$

By a well-known perturbation result for analytic generators, this estimate together with (5.13) implies (i).

(ii): Writing $\sigma := \sigma(\rho)$ for brevity, we have

$$\begin{aligned} A_0^n(\sigma\partial_s^2\psi) &= A_0^n\partial_s^2(\sigma\psi) + A_0^n[\partial_s^2,\sigma]\psi \\ &= -\tfrac{1}{2}\mathcal{A}_0(\sigma\psi) + (\tfrac{1}{2}\mathcal{A}_0 + A_0^n\partial_s^2)(\sigma\psi) + A_0^n[\partial_s^2,\sigma]\psi \\ &= -\tfrac{\sigma}{2}\mathcal{A}_0\psi + [\tfrac{\sigma}{2},\mathcal{A}_0] + (\tfrac{1}{2}\mathcal{A}_0 + A_0^n\partial_s^2)(\sigma\psi) + A_0^n[\partial_s^2,\sigma]\psi. \end{aligned}$$

Now we obtain from Lemma A.3 of the present paper and Lemma 5.4 in [7] that

$$\|\tfrac{\sigma}{2}\mathcal{A}_0\psi + A_0^n(\sigma\partial_s^2\psi)\|_{3+\theta} \leq C\|\psi\|_{3+\theta}, \qquad \psi \in h^{3+\theta}(\Gamma).$$

From this and (5.13) we conclude the assertion by the same argument as in (i). ■

To finish the proof of (5.9), we note that Lemma 5.5 obviously implies

$$\begin{bmatrix} -A_0^t\partial_s(\sigma'(\rho)\rho\partial_s) & 0 \\ 0 & A_0^n(\sigma(\rho)\partial_s^2) \end{bmatrix} \in \mathcal{H}((h^{3+\theta}(\Gamma))^2, (h^{2+\theta}(\Gamma))^2)$$

and, as $\mathbf{Q}$ is an isomorphism both on $(h^{2+\theta}(\Gamma))^2$ and $(h^{3+\theta}(\Gamma))^2$, also

$$\mathbf{Q}\begin{bmatrix} -A_0^t\partial_s(\sigma'(\rho)\rho\partial_s) & 0 \\ 0 & A_0^n(\sigma(\rho)\partial_s^2) \end{bmatrix}\mathbf{Q}^T \in \mathcal{H}((h^{3+\theta}(\Gamma))^2, (h^{2+\theta}(\Gamma))^2).$$

Now (5.9) and, consequently, (5.5) for our application follows from (5.11) by a perturbation argument parallel to the ones in the proof of Lemma 5.5.

We have shown that all assumptions of Theorems 5.1 and 5.2 are satisfied in our situation given by (5.7). These theorems yield the following final result:

Theorem 5.6 *(Well-posedness and smoothing for* (2.20) *in the plane case)*
Assume $K \in \{\infty,\omega\}$ and (5.1), (5.2). *There is an open neighborhood D of the identity in $(h^{3+\theta}(\Gamma))^2$ such that for any ϕ_0, there are a $t^+ = t^+(\phi_0) > 0$ and a unique maximal solution*

$$\phi \in C([0,t^+), D) \cap C^1([0,t^+), (h^{2+\theta}(\Gamma))^2)$$

of (2.20). *Moreover,*

$$[(t,p) \mapsto \phi(t)(p)]\big|_{(0,t^+)\times\Gamma} \in C^K((0,t^+)\times\Gamma, \mathbb{R}^2).$$

Conclusion

Let us compare the approaches and the resulting evolution equations for the cases $\gamma = \text{const.}$, as discussed in [10], with the case $\gamma = \sigma(\rho)$ (and purely convective surfactant transport), as discussed here. (For simplicity, we assume (5.2).)

	$\gamma = \text{const.}$	$\gamma = \sigma(\rho)$
Description of moving boundary	normal perturbation	Lagrangian coordinates
Evolution equation valued in	$\mathbb{R}$	$\mathbb{R}^{m+1}$
Type	parabolic in any space dimension	parabolic if $m = 1$ degenerate parabolic if $m > 1$

It would clearly be interesting to investigate the effect of surfactants on the long time behavior of the flow, at least in the neighborhood of an equilibrium. However, one easily sees that our model (1.1), (1.3) does not posess equlibria if γ is not constant on the moving surface. This property results from neglecting the surfactant diffusion. An analysis including this effect would result in a coupled system of evolution equations where (2.18) is augmented by a diffusion equation for ρ whose coefficients depend on ϕ. It seems reasonable to conjecture that in this case the equilibria are given by balls with constant surfactant density, and that these are stable, at least in the strictly parabolic case.

A Appendix: Some auxiliary estimates for the Stokes and Laplace equations in Hölder spaces

In this appendix, we collect the results on linear, nonlocal operators arising from solutions of the Stokes and Laplace equations on a fixed, smooth domain that are needed in Section 5. All these results can also be proved using the calculus of pseudodifferential operators on (smooth, compact) manifolds, however, to carry out the necessary details would be more technical than direct proofs. Although we will apply the results in the case $m = 1$ only, we prefer to give the results in arbitrary dimension where this is possible without additional difficulties.

We recall the following definitions: For $\theta \in \mathbb{R}_+ \backslash \mathbb{N}$, we set

$$\begin{aligned} V &:= \{(c^{ij}) \,|\, i,j = 1,\ldots,m+1,\ c^{ij} \in \mathbb{R},\ c^{ij} = -c^{ji}\}, \\ \mathcal{X}^\theta &:= (h^{2+\theta}(\Omega))^{m+1} \times h^{1+\theta}(\Omega) \times (\mathbb{R}^{m+1} \times V), \end{aligned}$$

$$\mathcal{Y}^\theta := (h^\theta(\Omega))^{m+1} \times h^{1+\theta}(\Omega) \times (h^{1+\theta}(\Gamma))^{m+1} \times \mathbb{R}^{m+1} \times V.$$

and we have $L_0 := L(\mathrm{Id}) \in \mathcal{L}_{is}\left(\mathcal{X}^\theta, \mathcal{Y}^\theta\right)$ given by

$$L_0(u,p,\lambda) := \begin{pmatrix} -\Delta u + \nabla p + \lambda_1 \\ \mathrm{div} u \\ (\partial_i u_j + \partial_j u_i - p\delta_{ij} + \lambda_2^{ij})n_j \\ \int_\Omega u\,dx \\ \int_\Omega (\partial_i u_j - \partial_j u_i)\,dx \end{pmatrix}^T$$

(cf. (5.8)). Moreover, we have defined A_0, B_0 and $\mathcal{A}_0$ by

$$\begin{aligned} A_0 &:= A(\mathrm{Id}) = \mathrm{Tr}_\Gamma \Pi_1 L_0^{-1} E_3, \\ B_0 h &:= B(\mathrm{Id})h = \mathrm{Tr}_\Gamma(\Delta, \partial_n)^{-1}(0, h - \overline{h}), \\ \mathcal{A}_0 &:= \mathcal{A}(\mathrm{Id}) = \partial_n(\Delta, \mathrm{Tr}_\Gamma)^{-1}(0, \cdot), \end{aligned}$$

and P denotes the orthogonal projection of $\Gamma \times \mathbb{R}^{m+1}$ onto $T\Gamma$.

Our first result concerns the Neumann problem for Stokes flow and is complementary to Lemma A.2 in [7]. It is completely parallel to Lemma 3.5 (i) in [11], but here we work in Hölder spaces, and can restrict ourselves to the unperturbed domain.

Lemma A.1 *(Diagonal structure of A_0) We have $f \mapsto PA_0(fn) \in \mathcal{L}\left(h^{1+\theta}(\Gamma), (h^{3+\theta}(\Gamma))^{m+1}\right)$.*

Proof: Assume $f \in C^\infty(\Gamma)$ and let (u, p, λ) be given by

$$L_0(u,p,\lambda) = (0, 0, fn, 0, 0).$$

As A_0 maps smooth functions to smooth functions, it is sufficient to show

$$\|Pu\|_{3+\theta} \le C\|f\|_{1+\theta}$$

with C independent of f.

Define $\psi \in C^\infty(\overline{\Omega})$ by

$$\left.\begin{aligned} \Delta\psi &= 0 && \text{in } \Omega, \\ \partial_n \psi &= \tfrac{1}{2}(f - \overline{f}) && \text{on } \Gamma, \\ \textstyle\int_\Omega \psi &= 0. && \end{aligned}\right\} \tag{A.1}$$

Let $d \in C^\infty(\overline{\Omega})$ be such that $d = 0$ and $\nabla d = n$ at Γ. (Such a d can be constructed by using the signed distance function near Γ and cutting it off away from Γ.) We extend n into the interior of Ω by ∇d. Define now

$$\begin{aligned} v &:= \psi n - d\nabla\psi, \\ q &:= -2\nabla\psi \cdot n. \end{aligned}$$

Then it is straightforward to calculate (with summation over indices occurring twice)

$$\begin{aligned} -\Delta v_i + \partial_i q &= -2\partial_j\psi\partial_j n_i - \psi\Delta n_i + \Delta d\partial_i\psi - 2\partial_k\psi\partial_i n_k, \\ \mathrm{div} v &= \psi\,\mathrm{div} n, \end{aligned}$$

and on Γ

$$(\partial_i v_j + \partial_j v_i)n_j - q n_i = \psi n_j(\partial_i n_j + \partial_j n_i) + (f - \overline{f})n_i.$$

Moreover,

$$\int_\Omega (\partial_i v_j - \partial_j v_i)\,dx = \int_\Gamma (n_i v_j - n_j v_i)\,d\Gamma = 0.$$

Hence

$$L_0(u-v, p-q, \lambda) = \begin{pmatrix} 2\partial_j\psi\partial_j n_i + \psi\Delta n_i - \Delta d\partial_i\psi + 2\partial_k\psi\partial_i n_k \\ -\psi\,\mathrm{div} n \\ -\psi n_j(\partial_i n_j + \partial_j n_i) + \overline{f} n_i \\ -\int_\Omega(\psi n - d\nabla\psi)\,dx \\ 0 \end{pmatrix}^T.$$

Note that $Pv = 0$ on Γ, hence by (5.8) and well-known regularity results on the Neumann problem for the Laplacian

$$\begin{aligned} \|Pu\|_{3+\theta} &= \|P(u-v)\|_{3+\theta} \le C\|u-v\|_{3+\theta} \le C\|(u-v, p-q, \lambda)\|_{\mathcal{X}^{1+\theta}} \\ &\le C\|L_0(u-v, p-q, \lambda)\|_{\mathcal{Y}^{1+\theta}} \le C(\|\psi\|_{2+\theta,\Omega} + \|f\|_0) \le C\|f\|_{1+\theta}. \end{aligned}$$

■

The following lemma, as well as its proof, is parallel to Lemma 3.1. We will write ∇_Γ for the surface gradient in Γ and recall that $\nabla_\Gamma \mathrm{Tr}_\Gamma = P\mathrm{Tr}_\Gamma\nabla$.

Lemma A.2 *(The operator A_0 on gradients II) We have* $A_0\nabla_\Gamma - \frac{1}{2}\nabla_\Gamma B_0 \in \mathcal{L}\left(h^{2+\theta}(\Gamma), (h^{3+\theta}(\Gamma))^{m+1}\right)$.

Proof: Fix $f \in C^\infty(\Gamma)$, define ψ as in (A.1), $v := \nabla\psi$, and (u, p, λ) by

$$L_0(u, p, \lambda) = (0, 0, \nabla_\Gamma f, 0, 0).$$

Then (cf. the proof of Lemma 3.1)

$$L_0(u-v, p, \lambda) = \begin{pmatrix} 0 \\ 0 \\ -\partial_n^2\psi n + 2\partial_j\psi\nabla n_j \\ \int_\Omega \nabla\psi\,dx \\ 0 \end{pmatrix}^T.$$

Hence, by (5.8), Lemma A.1, and Lemma A.2 in [7]

$$\begin{aligned} &\|(A_0\nabla_\Gamma - \frac{1}{2}\nabla_\Gamma B_0)f\|_{3+\theta} \le \|(n\cdot A_0\nabla_\Gamma f)n\|_{3+\theta} + \|P(u-v)\|_{3+\theta} \\ \le\;& C(\|\partial_n^2\psi\|_{1+\theta} + \|\nabla\psi\|_{2+\theta,\Omega}) \le C\|\psi\|_{3+\theta,\Omega} \le C\|f\|_{2+\theta}. \end{aligned}$$

This implies the result. ■

Our next lemma concerns a simple commutator property of the Dirichlet-Neumann operator $\mathcal{A}_0$ (cf. e.g. [8] Lemma 6.5.)

Lemma A.3 *(Commutator estimate for $\mathcal{A}_0$) For any $\alpha \in C^\infty(\Gamma)$, we have $\alpha\mathcal{A}_0 - \mathcal{A}_0\alpha \in \mathcal{L}\left(h^{2+\theta}(\Gamma)\right)$.*

Proof: Fix $f \in C^\infty(\Gamma)$ and let $u, v \in C^\infty(\overline{\Omega})$ be the harmonic extensions of f and αf into Ω, respectively. We extend both α and n smoothly into Ω and define $w := \alpha u - v$. Then w vanishes at Γ and, by straightforward calculations and standard estimates,

$$\|\partial_n w\|_{2+\theta,\Omega} \leq C\|w\|_{3+\theta,\Omega} \leq C\|\Delta w\|_{1+\theta,\Omega} \leq C\|\nabla u\|_{1+\theta,\Omega} \leq C\|u\|_{2+\theta,\Omega},$$

and

$$\begin{aligned} \|(\alpha\mathcal{A}_0 - \mathcal{A}_0\alpha)f\|_{2+\theta} &\leq \|\alpha\partial_n u - \partial_n v\|_{2+\theta,\Omega} = \|\partial_n w + [\alpha, \partial_n]u\|_{2+\theta,\Omega} \\ &\leq C\|u\|_{2+\theta,\Omega} \leq C\|f\|_{2+\theta}. \end{aligned}$$

This implies the lemma. ■

In our last lemma, we restrict ourselves to the case $m = 1$ and denote by ∂_s the tangential derivative along Γ with respect to arclength.

Lemma A.4 *(Relating B_0 and $\mathcal{A}_0$) Assume $m = 1$, $\alpha \in C^\infty(\Gamma)$. Then we have $\partial_s B_0 \alpha \partial_s + \alpha\mathcal{A}_0 \in \mathcal{L}\left(h^{2+\theta}(\Gamma)\right)$.*

Proof: For arbitrary $f \in C^\infty(\Gamma)$, let $u \in C^\infty(\overline{\Omega})$ be given by

$$\left.\begin{array}{rcll} \Delta u &=& 0 & \text{in } \Omega, \\ \partial_n u &=& \alpha\partial_s f - \overline{\alpha\partial_s f} & \text{on } \Gamma, \\ \int_\Gamma u\, d\Gamma &=& 0, & \end{array}\right\}$$

and let v be the harmonic extension of f into Ω. Extend α, n, and the positively oriented unit tangent vector field from Γ to smooth functions on $\overline{\Omega}$ such that we can consider now ∂_s and ∂_n as first-order differential operators on $\overline{\Omega}$. Define $w := \partial_s u + \alpha\partial_n v$. Then

$$\|\Delta w\|_{\theta,\Omega} \leq C(\|u\|_{2+\theta,\Omega} + \|v\|_{2+\theta,\Omega})$$

and on Γ

$$\begin{aligned} \partial_n w &= \partial_s(\alpha\partial_s f) + \partial_n(\alpha\partial_n v) + [\partial_n, \partial_s]u \\ &= \alpha(\partial_s^2 f + \partial_n^2 v) + [\partial_s, \alpha]\partial_s f + [\partial_n, \alpha]\partial_n v + [\partial_n, \partial_s]u \\ &= \alpha(\partial_s^2 + \partial_n^2)v + [\partial_s, \alpha]\partial_s f + [\partial_n, \alpha]\partial_n v + [\partial_n, \partial_s]u \\ &= -\alpha\kappa\partial_n v + [\partial_s, \alpha]\partial_s f + [\partial_n, \alpha]\partial_n v + [\partial_n, \partial_s]u, \end{aligned}$$

where we have used the identity

$$\mathrm{Tr}_\Gamma \Delta = (\partial_s^2 + \partial_n^2 + \kappa\partial_n)\mathrm{Tr}_\Gamma$$

with κ denoting the curvature of Γ. Consequently, using the standard Schauder estimate for the Laplacian with Neumann boundary conditions, we get

$$\begin{aligned}\|(\partial_s B_0 \alpha \partial_s + \alpha \mathcal{A}_0) f\|_{2+\theta} &\leq \|w\|_{2+\theta,\Omega} \leq C(\|\Delta w\|_{\theta,\Omega} + \|\partial_n w\|_{1+\theta} + \|w\|_{0,\Omega}) \\ &\leq C(\|f\|_{2+\theta} + \|u\|_{2+\theta,\Omega} + \|v\|_{2+\theta,\Omega}) \leq C\|f\|_{2+\theta}.\end{aligned}$$

This implies the lemma. ■

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