A stochastic inventory policy with limited transportation capacity

A Stochastic Inventory Policy with Limited Transportation Capacity

S. Dabia, G. P. Kiesmüller, N. Dellaert
Faculty of Technology Management,
Technische Universiteit Eindhoven,
P.O. Box 513,
5600 MB Eindhoven,
The Netherlands

A Stochastic Inventory Policy with Limited Transportation Capacity

Abstract

In this paper we consider a stochastic single-item inventory problem. A retailer keeps a single product on stock to satisfy customers stochastic demand. The retailer is replenished periodically from a supplier with ample stock. For the delivery of the product, trucks with finite capacity are available and a fixed shipping cost is charged whenever a truck is dispatched regardless of its load. Furthermore, linear holding and backorder costs are considered at the end of a review period.
A replenishment policy is proposed to determine order quantities taking into account transportation capacity and aiming at minimizing total average cost. Every period an order quantity is determined based on an order-up-to logic. If the order quantity is smaller than a given threshold then the shipment is delayed. On the other hand, if the order quantity is larger than a second threshold then the initial order size is enlarged and a full truckload is shipped. An order size between these two thresholds results in no adaption of the order quantity and the order is shipped as it is.
We illustrate that this proposed policy is close to the optimal policy and much better than an order-up-to policy without adaptations. Moreover, we show how to compute the cost optimal policy parameters exactly and how to compute them by relying on approximations. In a detailed numerical study, we compare the results obtained by the heuristics with those given by the exact analysis. A very good cost performance of the proposed heuristics can be observed.

Keywords: Stochastic inventory control – Transportation capacity – Full truckloads – Markov chain

1 Introduction

In many practical situations inventory and transportation decisions are significantly correlated. On the one hand, transportation managers aim at low transportation costs. Therefore, few large shipments with highly utilized trucks are required to benefit from economics of scale. In order to obtain highly utilized trucks, more products than required can be shipped, leading to an increased inventory level and higher inventory holding costs. Alternatively, orders may be delayed until a certain truck utilization is reached resulting in more backorders or higher safety stocks. On the other hand, inventory managers aim at low inventory costs which may require many small shipments with probably low truck utilization, and hence high transportation cost. How many trucks are needed for a shipment of an entire order depends on the order quantity as well as on the truck capacity. However, the most common replenishment policies, like the order-up-to policy (see Silver et al. (1998)) do not take into account transportation capacity. Moreover, it is often assumed that the fixed shipment costs are not dependent on the shipmentsize, respectively the number of trucks to be needed. We believe that it is important to include transportation (ordering) capacity in a quantitative model in order to minimize total cost, composed of inventory, backorder and transportation (ordering) costs. But literature on replenishment policies with capacity restrictions is rare.

In case of a periodic inventory systems with unlimited ordering capacity and a fixed ordering cost, Scarf (1960) has proven that the optimal policy is a periodic (s, S)-type policy; at the beginning of each period, when the inventory position (stock on-hand minus backorders plus outstanding orders) drops to or below the level s, enough is ordered to raise the inventory position up to the level S. Federgruen and Zipkin (1986) consider a capacitated inventory system with no ordering costs. They have proven the optimality of the modified base stock policy; if there is enough capacity, order up to S, otherwise, order as much as possible. In the case of capacitated inventory systems with fixed ordering costs, the optimal policy is not that straightforward. Wijngaard (1972) investigates the conditions for which the (s, S)-type policy is optimal. Shaoxiang et al. (1994) consider a single-item, periodic review inventory system with a limited ordering capacity and a fixed ordering cost. They have shown that the optimal policy, for the finite horizon case, has a systematic pattern, which they call the $X-Y$ band structure. The $X-Y$ band structure works as follows. Whenever the inventory position drops below level X, an order up to capacity takes place; when the inventory position exceeds level Y, no action is taken. When the inventory position is between X and Y, the order quantity is different from state to state and no specific structure seems to be optimal. Similar results are obtained for the infinite horizon in Shaoxiang (2004). Gallego et al. (1998) have shown that the $X-Y$ band structure can be characterized with a four regions structure in case of a finite horizon. Nevertheless, when the inventory position falls between X and Y, the optimal decisions can differ from case to case. Chan et al. (2003) focus on the region between X and Y. They provide some properties of this region and develop an efficient algorithm that allows the computation of the optimal ordering quantities. However, none of the above mentioned papers provides easy formulas to compute the values of X and Y.

In this paper, similarly to Shaoxiang (2004), a stochastic single-item, periodic review inventory system is considered, where a truck with fixed and finite capacity is used to ship the orders. A fixed transportation cost is charged as well as linear holding and backorder costs at the end of a period. We present a simpler and similar policy determined by the parameters (S, Q_1, Q_2). The policy is similar in the sense that it has two thresholds Q_1 (the waiting threshold) and Q_2 (the full truckload threshold), and simpler in the sense that the region between Q_1 and Q_2 uses a simple order-up-to policy with order-up-to level S. We illustrate in this paper that in many cases this policy is optimal and if not, it is close to optimal. Furthermore, we show how to compute the cost optimal policy parameters exactly by using a Markov modelling approach. Additional fast and simple heuristics are proposed to compute near optimal policy parameters.

This paper is organized as follows. In section 2, we describe the problem in more detail. In section 3, the proposed policy is explained. Section 4 is devoted to the formulation of the mathematical model enabling the computation of the optimal policy parameters. In section 5, the optimal decisions are compared with the proposed policy. In section 6, two simple and fast heuristics are presented for computing near optimal policy parameters. In section 7, the results obtained from a numerical study are shown. Finally, section 8 concludes this paper with a summary of the main results.

2 Problem description

A single stock location (a retailer) is considered, where a single item is stored to fulfill customers stochastic demand. Time is divided into periods of fixed length (e.g. weeks or days). Demand in period n, D_n, is a discrete random variable and the distribution $P(D_n = k) = p_k$ is supposed to be known. Additionally, demand in subsequent periods are assumed to be independent and identically distributed. Furthermore, we assume that demand can not exceed a truck capacity and that, without loss of generality, only one truck is used to ship the orders (think about the situation where the retailer has contracts that do not allow customers to order more than a certain quantity). Demand which cannot be satisfied is assumed to be backordered.

The retailer is supplied from an external supplier with ample stock which means that there is no delivery delay due to a lack of stock. Furthermore, the retailer is replenished by means of a truck with a finite and fixed capacity V. A truck is assumed to be always available. A fixed shipping cost A is charged whenever the truck is dispatched, regardless of its load. It is assumed, without loss of generality, that deliveries are instantaneous. In other words, the lead time is assumed to be equal to zero, which is often the case in the retail environment where the orders are shipped during the weekend, or during the night, when no demand occurs. Moreover, the analysis can easily be extended to the case of a positive and constant leadtime. The inventory is periodically (e.g. at the beginning of each week) reviewed and the review period is considered to be an exogenous variable and can therefore be assumed to be equal to one. At the end of each period, holding costs are charged per unit of inventory on-hand and penalty costs are charged per unit backordered.

At the retailer the following replenishment policy is used. At the beginning of each review period the inventory position is reviewed and an order may be placed to raise the inventory to a certain level. The first objective is to come up with a "simple" and good replenishment policy which takes into account truck utilization and benefits from economies of scale. The second objective is to compute the policy parameters that minimize the long-run average cost consisting of inventory costs, backorder costs as well as transportation costs.

In the remainder of this paper, the following notation will be used.

V	:	Capacity of the truck
A	:	Fixed cost for dispatching a truck (ordering costs)
D_n	:	Demand during the nth period
X_n	:	The inventory position, before ordering, at the beginning of the nth period
q_n	:	The quantity shipped at the beginning of the nth period
T	:	Time between two successive shipments
$D(i)$	:	Demand during i periods
h	:	Holding cost at the end of a period per item per time unit
p	:	Backorder cost at the end of a period per item per time unit
$E[X]$	:	Expectation of a random variable X
f_X	:	The probability density distribution function of a continuous random variable X
X^+	:	max(0,X)
X^-	:	max(0,-X)
$\lfloor x \rfloor$	:	The whole part of the real number x
$[\![a, b]\!]$	:	The interval of integer numbers between a and b (a and b are also integers).

3 The (S, Q_1, Q_2) replenishment policy

In the following we describe the (S, Q_1, Q_2) policy, which is illustrated in Figure 1. The order-up-to level S determines at the beginning of a review period the initial order size, $O_n = \max\{0, S - X_n\}$. Since this initial order-size can lead to a low truck utilization it is allowed to be adapted. Whenever the initial order-quantity is at or below the waiting threshold Q_1 (see Figure 1 e.g. beginning of periods 2 and 3) then it is reduced to zero, so the order is delayed. In case of an initial order-size larger than the full truckload threshold Q_2 (see Figure 1 e.g. beginning of period 4) a full truck is dispatched. When the initial order size falls between Q_1 and Q_2, it is not adapted and a shipment takes place to raise inventory to the level S (see Figure 1 e.g. beginning of period 6).

The quantity to be shipped at the beginning of a period n (in Figure 1 represented by the circles) is then given on the one hand, if $Q_1 = Q_2$ as

$$q_n = \begin{cases} 0 & , \quad S - X_n \leq Q_1 \\ V & , \quad S - X_n > Q_1 \end{cases} \tag{1}$$

and on the other hand, if $Q_1 \neq Q_2$ as:

$$q_n = \begin{cases} 0 & , \quad S - X_n \leq Q_1 \\ S - X_n & , \quad Q_1 < S - X_n < Q_2 \\ V & , \quad S - X_n \geq Q_2 \end{cases} \tag{2}$$

It is trivial that Q_1 and Q_2 should be in the interval $[\![0, V]\!]$, because it does not make sense to send an empty truck ($Q1 < 0$) or delay the shipment of a full truck ($Q_2 > V$). Furthermore, it does not make sense to have $Q_2 < Q_1$, because otherwise the waiting

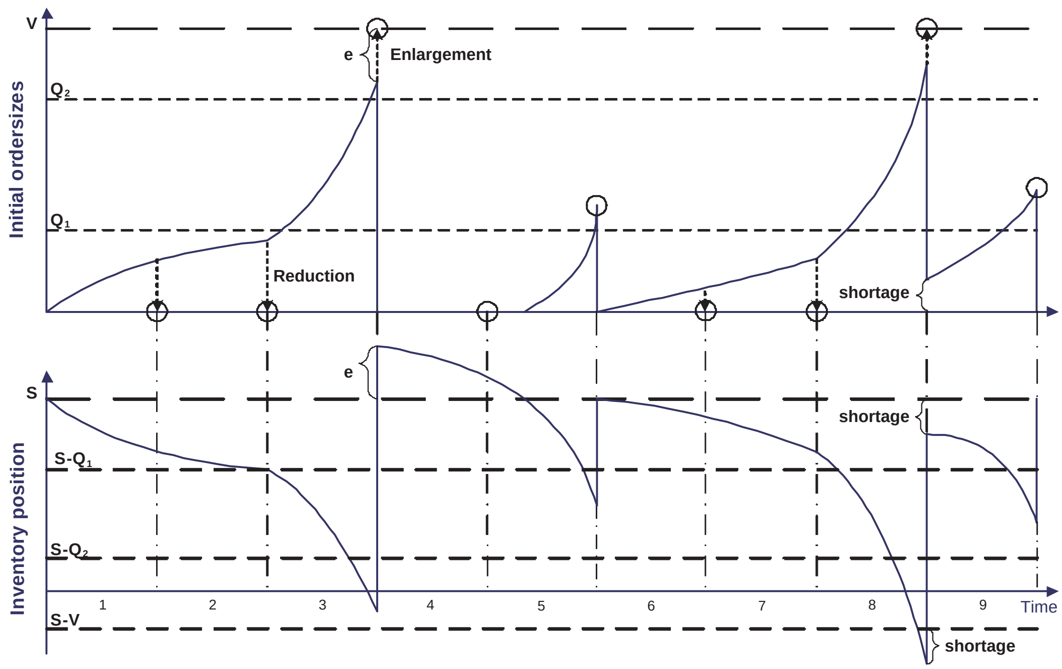

Figure 1: The (S, Q_1, Q_2) policy.

thresholds Q_1 will be superfluous in the sense that it will have no influence on how the policy works.
In the case of multiple-item inventory systems, Van Eijs (1994), Cachon (2001) and Kiesmüller (2006) proposed policies that combine transportation and inventory decisions in a similar way. If we would relate their ideas to the (S, Q_1, Q_2) policy, we observe that Kiesmüller's idea is the special case when $Q_1 = Q_2 = \frac{V}{2}$. Moreover, Van Eijs's idea is the special case with $Q_1 = 0$ where orders can only be enlarged and Cachon's idea is the special case when $Q_2 = V$, which means that an order will never be enlarged. The standard order-up-to policy is obtained when $Q_1 = 0$ and $Q_2 = V$.

4 Model

4.1 Model description

In this section, the mathematical model is formulated to analyze the policy. It is important to emphasize that because of enlargements, the inventory position X_n at the beginning of period n before ordering can exceed the order-up-to level S (see Figure 1 e.g. beginning of period 5). Furthermore, if the inventory position at the beginning of a period n drops below the level $S-V$, a shipment will of course take place, but we will not be able to raise the inventory position to the value S (see Figure 1 e.g. beginning of period 9), because of limited transportation capacity (only one truck is used to ship the orders). Shortages are delivered in the next shipment moment.

The inventory position is modelled as a discrete time Markov chain. The optimal parameters of the (S, Q_1, Q_2) policy, minimizing the total cost consisting of transportation, backorder as well as inventory costs, are exactly computed by an exhaustive search. In fact, for each relevant combination of S, Q_1 and Q_2 the average cost are computed by using this Markov chain approach.

4.2 The exact analysis

Because of demand's independency assumption, the inventory position at the beginning of a period, $\{X_n, n \geq 0\}$ can be modelled using a Discrete Time Markov Chain with state space $SS = [\![S - V - Q_1, S + V - Q_2]\!]$.

From the balance equation, the inventory position is given as:

$$X_{n+1} = X_n + q_n - D_n \tag{3}$$

Hence, by replacing (1) and (2) in (3) we get:

If $Q_1 = Q_2$:

$$X_{n+1} = \begin{cases} X_n + V - D_n & , \quad S - V - Q_1 \leq X_n < S - Q_1 \\ X_n - D_n & , \quad S - Q_1 \leq X_n \leq S + V - Q_2 \end{cases} \tag{4}$$

Otherwise, if $Q_1 \neq Q_2$:

$$X_{n+1} = \begin{cases} X_n + V - D_n & , \quad S - V - Q_1 \leq X_n \leq S - Q_2 \\ S - D_n & , \quad S - Q_2 < X_n < S - Q_1 \\ X_n - D_n & , \quad S - Q_1 \leq X_n \leq S + V - Q_2 \end{cases} \tag{5}$$

The transition probabilities $p_{i,j} = P(X_{n+1} = j | X_n = i)$ can easily be computed as follows. For all $(i, j) \in SS^2$ we have:

If $Q_1 = Q_2$:

$$p_{i,j} = \begin{cases} P(D_n = i + V - j) & , \quad S - V - Q_1 \leq X_n < S - Q_1 \\ P(D_n = i - j) & , \quad S - Q_1 \leq X_n \leq S + V - Q_2 \end{cases} \tag{6}$$

Otherwise, if $Q_1 \neq Q_2$:

$$p_{i,j} = \begin{cases} P(D_n = i + V - j) & , \quad S - V - Q_1 \leq X_n \leq S - Q_2 \\ P(D_n = S - j) & , \quad S - Q_2 < X_n < S - Q_1 \\ P(D_n = i - j) & , \quad S - Q_1 \leq X_n \leq S + V - Q_2 \end{cases} \tag{7}$$

The steady state probabilities are defined as $\pi_i = \lim_{n \to \infty} P(X_n = i)$ and they can be computed by solving the following system of linear equations:

$$\begin{cases} \sum_{i \in SS} \pi_i = 1 & , \quad 0 \leq \pi_i \leq 1 \\ \pi = \pi \times P & , \quad \pi = (\pi_i)_{i \in SS} \end{cases} \tag{8}$$

where $(P = (p_{i,j})_{(i,j) \in SS^2})$ denotes the matrix of the transition probabilities.

4.3 The optimal (S, Q_1, Q_2) policy

So far we have derived everything we need to compute the long-run cost function $C(S, Q_1, Q_2)$, which is defined as:

$$C(S, Q_1, Q_2) = \sum_{i \in SS} \pi_i c(i) \tag{9}$$

where $c(i)$ is the involved cost (transportation, holding and penalty costs) when the system is in the state i. We have:

$$c(i) = A_i + hi^+ + pi^- \tag{10}$$

Moreover, we know that we only ship if the initial order size is larger than Q_1, hence at the end of a period we have:

$$A_i = \begin{cases} A \ , & S - i > Q_1 \\ 0 \ , & otherwise \end{cases} \tag{11}$$

Unfortunately, the cost function is not an explicit function of S, Q_1 and Q_2. As a consequence, it is not straightforward to prove the convexity of the cost function in the policy parameters and find explicit expressions for the optimal values S^*, Q_1^* and Q_2^*. However, using an exhaustive search the optimal values S^*, Q_1^* and Q_2^* can be obtained numerically.

5 The optimal decisions

It is known that the proposed (S, Q_1, Q_2) policy is not always optimal (see Shaoxiang (2004)). In order to determine the relative performance of the proposed policy we compare the (S^*, Q_1^*, Q_2^*) policy with the overall optimal decisions. Therefore, we formulate the single item ordering problem with a capacity constraint and fixed ordering cost as a dynamic programming problem. The inventory position is again used to describe the state of the system and we denote the set of all possible states by I. The number of items to be ordered is non-negative and cannot be larger than the truck capacity. Therefore, the set of possible actions is given as $A_c = [\![0, V]\!]$. If $c_i(a)$ is defined as the one period cost when the system is in state i and action a is taken, then we can compute $c_i(a)$ as follows:

$$c_i(a) = A\delta(a) + h\sum_{k=0}^{i+a}(i + a - k)p_k + p\sum_{k=i+a+1}^{V}(k - i - a)p_k \ , \quad (i, a) \in I \times \ A_c \tag{12}$$

where $\delta(a)$ is defined as follows:

$$\delta(a) := \begin{cases} 0 & \text{if} \quad a = 0 \\ 1 & \text{if} \quad a \neq 0 \end{cases} \tag{13}$$

Finally, let $V_n(i)$ be the expected minimal cost for an n horizon problem, if the beginning inventory level is i. Then we can formulate the following dynamic programming recursion.

$$V_n(i) = \min_{a \in A_c} \left\{ c_i(a) + \sum_{j \in I} p_{ij}(a) V_{n-1}(j) \right\} \tag{14}$$

We will use (14) in order to determine the optimal decisions and the minimal costs numerically by value iteration (see for example Tijms (1994)) and we will compare the results with the optimal (S^*, Q_1^*, Q_2^*) policy (Table 2 and 3) for the parameter values as given in Table 1.

	Case 1	Case 2	Case 3
Demand distribution	Uniform	Linear positive	Two points
A	50	250	250
V	20	20	20
p	100	100	100
h	1	1	10

Table 1: Parameter set for the numerical example in Table 2

Three different demand distributions are considered: uniform distribution, the linear positive distribution (Figure 2), and a two points distribution. The two points demand distribution is chosen such that $p_{16} = 0.95$, $p_{17} = 0.05$ and all other probabilities are zero.

The optimal quantities to be ordered at the beginning of a period q_n^{opt} and the quantities obtained by the optimal (S, Q_1, Q_2) policy are depicted in Table 2 for different states.

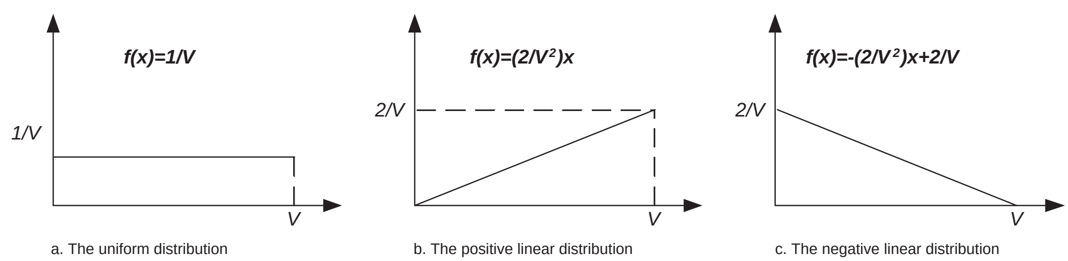

Figure 2: The demand distributions

	Case1		Case2		Case3	
States	q_n^{opt}	q_n	q_n^{opt}	q_n	q_n^{opt}	q_n
.	.	.	.	.	.	.
-3	20	20	20	20	19	20
-2	19	20	20	20	18	20
-1	18	20	19	20	17	17
0	17	20	20	20	16	16
1	20	20	20	20	15	15
2	19	20	20	20	14	14
3	18	20	19	20	13	13
4	20	20	20	20	12	12
5	20	20	20	20	11	11
6	19	20	20	20	10	10
7	18	20	19	20	9	9
8	20	20	20	20	8	8
9	20	20	20	20	7	7
10	19	20	20	20	6	6
11	18	20	19	20	20	0
12	20	20	20	20	20	0
13	20	20	20	20	19	0
14	19	19	20	20	18	0
15	18	18	19	19	0	0
16	0	0	0	0	0	0
.	.	.	.	.	.	.

Table 2: The ordered quantities at the beginning of a period

We observe that, while the (S, Q_1, Q_2) policy has three regions, the optimal decisions have no obvious structure. However, in Table 3 we compare the cost given by the (S^*, Q_1^*, Q_2^*) policy with the cost obtained by the optimal decisions.

As can be observed, differences, if any, are negligible ($< 0.4\%$). Differences are more visible in case of the two points distribution, while for the other distributions, the optimal decisions are identical to the (S^*, Q_1^*, Q_2^*) policy.

6 Heuristics

For large truck capacities (e.g. $V = 50$) the computation time for determining the optimal (S, Q_1, Q_2) policy becomes large because for each combination of S, Q_1 and Q_2 a linear system of equations with dimension $V \times V$ has to be solved (8). Therefore, relatively simple heuristics that give good results in a short time are desirable. In this section, we propose two heuristics, we call the S-Heuristic and the SQ-Heuristic. The S-Heuristic is based on an approximated value of S^* which is determined by a kind of Newsboy equation. Q_1^* and Q_2^* are again obtained by a search procedure, but as a consequence of the pre-determined S^*-value the linear system of equations has to be solved less times

		Uniform		Linear positive		2 pts distribution	
A	h	C_{opt}	$C_{SQ_1Q_2}$	C_{opt}	$C_{SQ_1Q_2}$	C_{opt}	$C_{SQ_1Q_2}$
50	1	43,46	43,46	49,48	49,48	49,18	49,29
	2	60,43	60,43	62,06	62,27	51,90	51,90
	5	91,79	91,79	81,20	81,20	54,75	54,75
250	1	143,46	143,46	186,15	186,15	210,12	210,15
	2	160,43	160,43	200,42	200,42	218,77	218,81
	5	206,25	206,25	239,60	239,62	243,42	244,37

Table 3: Comparison of the (S, Q_1, Q_2) policy and the optimal decisions

which reduces computation times. This heuristic can be used for an arbitrary demand distribution.

In contrast to this, the SQ-Heuristic works different for different demand distributions. In this paper formulas are developed for three different demand distributions: the uniform distribution, the positive linear distribution, and the negative linear distribution (see also Figure 2). By making some additional assumptions, closed form expressions can be derived for the cost functions. The optimal order-up to level S^* is again estimated and a relation between the optimal Q_1 and Q_2 is derived, meaning that only a one dimensional optimization problem has to be be solved.

6.1 The S-Heuristic

In case of the order-up-to policy, the Newsboy's formula can be used to compute a numerical value for S^* and the inventory position at the beginning of a review period, after ordering, is always exactly equal to S. Moreover, it is known that S will cover the future demand occurring during the lead time plus the review period. In the case of the (S, Q_1, Q_2) policy, the inventory position at the beginning of a review period, after ordering, X_n^a, is not necessarily equal to S. As we have already explained, because of the enlargement and the reduction of the initial order size, the value of X_n^a varies in the interval $[\![S - Q_1, S + V - Q_2]\!]$. Furthermore, the time between two successive shipments T is, opposite to the case of the order-up-to level policy, a random variable. T depends on the demand distribution in the sense that T is shorter when demand has been high. Furthermore, T depends on the waiting and the full truckload thresholds. In fact, T increases with the waiting threshold Q_1 (the larger Q_1, the longer we have to wait before a shipment takes place) and decreases with the full truckload threshold Q_2 (the smaller Q_2, the larger is the enlargement, hence, the longer we have to wait because of the excess of inventory).

We want to use Newsboy's equation to estimate S^*. Therefore, the distribution of demand during T, $D(T)$, is needed. The values of T and $D(T)$ are not independent. The following recursive equation can be used as an approximation to the distribution of $D(T)$.

$$\begin{aligned} P(D(T)=m) &= \sum_{i=1}^{\infty} P(D(T)=m|T=i)P(T=i) \\ &= \sum_{i=1}^{\infty}\sum_{k=0}^{m} P(D(i-1)=m-k)P(D(1)=k)P(T=i) \end{aligned} \tag{15}$$

As we can observe, the probability distribution of T is needed. If $X_n^a = S + \theta$ where $\theta \in [\![-Q_1, V - Q_2]\!]$, then $T = i$, if for all $j \in [\![1, i-1]\!]$ holds $0 \leq D_j \leq Q_1 + k - \sum_{k=1}^{j-1} D_k$ and $Q_1 + \theta + 1 - \sum_{j=1}^{i-1} D_j \leq D_i \leq V$. Hence,

$$\begin{aligned} P(T=1) &= \sum_{k=-Q_1}^{V-Q_2} P(T=1|\theta=k)P(\theta=k) \\ &= \sum_{k=-Q_1}^{V-Q_2} P(Q_1+k+1 \leq D_1 \leq V)P(\theta=k) \end{aligned} \tag{16}$$

We introduce the following notation $p_{a_l} = P(D_l = a_l)$ and by using the fact that demands in subsequent periods are independent, we can derive a formula for the distribution of the time between shipments for $i \geq 2$:

$$P(T=i) = \sum_{k=-Q_1}^{V-Q_2} \Big\{ \Big(\prod_{l=1}^{i-1} \sum_{a_l=0}^{B_{l-1}} p_{a_l} \Big) \sum_{a_i=1+B_{i-1}}^{V} p_{a_i}) \Big\} P(\theta=k) \tag{17}$$

where $B_0 = Q_1 + k$ and $B_i = Q_1 + k - \sum_{j=1}^{i} a_j$ for $i \geq 1$.

We further assume that X_n^a is uniformly distributed in $[\![S - Q_1, S + V - Q_2]\!]$, meaning that $P(\theta = k) = \frac{1}{V - Q_2 + Q_1 + 1}$ for all $\theta \in [\![-Q_1, V - Q_2]\!]$. Under this assumption and using (17) the distribution of the time between shipments can easily be computed numerically. Moreover, for the numerical examples considered in this paper large values of T will only occur with a very small probability. Therefore, it was enough to use only five values of T when estimating the distribution of $D(T)$.

The assumption of uniformly distributed X_n^a in $[\![S - Q_1, S + V - Q_2]\!]$ is also used to derive a generalization of the newsboy equation, given as follows:

$$\sum_{k=Q_1}^{V-Q_2} F_{D_T}(S^* + k) \geq \frac{p}{h+p}(V + Q_1 - Q_2 + 1) \tag{18}$$

The optimal S should fulfill the condition (18). Hence, for each Q_1 and Q_2, the optimal order-up-to level $S^*(Q_1, Q_2)$ is estimated using (18). Afterwards the average costs $C(S^*(Q_1, Q_2), Q_1, Q_2)$ are computed using the Markov model (see section 4).

6.2 The SQ-Heuristic

In this section, we develop a heuristic, which we call the SQ-Heuristic, that allows us to explicitly express the estimated cost function as a function of the policy parameters S,

Q_1 and Q_2. Opposite to the exact analysis and the S-Heuristic, the policy parameters will be expressed as a function of the variables involved (e.g. h, p, A, V). As a consequence, the policy parameters are easily estimated.

The demand is now assumed to be continuous and has values in the interval $[0, V]$. We develop the SQ-Heuristic for three different demand distributions (Figure 2) namely the uniform distribution (a), a positive linear probability demand distribution (b) and a negative linear probability demand distribution (c).

- **a. The uniform demand distribution**

The demand is assumed to have a continuous uniform distribution on $[0, V]$. Then we also have $E[D_n] = \frac{V}{2}$. Furthermore, we assume that the inventory position after ordering at the beginning of a review period is uniformly distributed in $[S - Q_1, S + V - Q_2]$. Based on this, the time between two shipments is estimated by:

$$E[T] = 1 + \int_{-Q_1}^{V-Q_2} \frac{Q_1 + x}{E[D_n]}\, dx = \frac{2V + Q_1 - Q_2}{V} \tag{19}$$

The expected inventory on-hand at the end of a period can be calculated by the following double integral:

$$\begin{aligned} E[OH] &= \int_{-Q_1}^{V-Q_2} \int_{0}^{S+\theta} (S + \theta - x) f(x) f(\theta)\, dx\, d\theta \\ &= \frac{(S + V - Q_2)^3 - (S - Q_1)^3}{6V(V + Q_1 - Q_2)} \end{aligned} \tag{20}$$

The expected amount of backorders at the end of a period is:

$$\begin{aligned} E[BO] &= \int_{-Q_1}^{V-Q_2} \int_{S+\theta}^{V} (x - S - \theta) f(x) f(\theta)\, dx\, d\theta \\ &= \frac{(S - Q_2)^3 - (S - V - Q_1)^3}{6V(V + Q_1 - Q_2)} \end{aligned} \tag{21}$$

The estimated long-run cost related to the (S, Q_1, Q_2) policy can be expressed as follows:

$$C_{SQ}(S, Q_1, Q_2) = \frac{A}{E[T]} + hE[OH] + pE[BO] \tag{22}$$

Hence, by replacing (19),(20) and (21) in (22) we get:

$$\begin{aligned} C_{SQ}(S, Q_1, Q_2) &= \frac{AV}{2V + Q_1 - Q_2} + h\frac{(S + V - Q_2)^3 - (S - Q_1)^3}{6V(V + Q_1 - Q_2)} \\ &+ p\frac{(S - Q_2)^3 - (S - V - Q_1)^3}{6V(V + Q_1 - Q_2)} \end{aligned} \tag{23}$$

We have managed to express the estimated long-run cost as an explicit function of the policy parameters S, Q_1 and Q_2. We can prove that this estimated cost is convex in S and that there is an optimal S^*_{SQ}. In fact, we have:

$$\frac{\partial^2 C_{SQ}(S, Q_1, Q_2)}{\partial S^2} = \frac{p+h}{V} > 0 \tag{24}$$

We can search for the optimal value of S by setting the first partial derivative equal to zero. After some basic algebra, we find that the optimal S has to satisfy:

$$S^*_{SQ} = \frac{Q_1 + Q_2}{2} + (\frac{p-h}{p+h})\frac{V}{2} \tag{25}$$

In the remainder of the paper we define the order-up-to region as the region between Q_1 and Q_2, where the order-up-to policy is applied. We characterize this region by the variable $X = Q_2 - Q_1$ such that, $X = 0$ means a full truckload policy, and $X = V$ corresponds to the order-up-to policy.

By replacing (25) in (23), we can express the estimated long-run cost as a function of X. We can also prove that $C_{SQ}(X)$ is convex in X. However, by solving the equation:

$$\frac{dC_{SQ}(X)}{dX} = 0 \tag{26}$$

we find that the optimal X, X^*, such that $0 \leq X \leq V$, is the solution of the following important relation:

$$(2V - X^*)^2 = \frac{12AV^2}{(p+h)(V - X^*)} \tag{27}$$

This is a very important result in the sense that simple and useful conclusions can be drawn out of it. For instance, we can observe that if $A \simeq 0$ (negligible transportation costs), $X^* = V$ which means we always use the order-up-to policy. Moreover, the full truckload policy ($X^* = 0$) is optimal when:

$$\frac{A}{p+h} \geq \frac{V}{3} \tag{28}$$

In Figure 3, we can see how the order-up-to region varies as a function of the cost parameters and the truck capacity. We observe that, for a given V, the smaller is the ratio $\frac{A}{p+h}$, the more X^* tends to V (the order-up-to policy), which is expected.

In Figure 4, we illustrate which policy to use depending on the input parameters. We observe, for instance, that for larger inventory and backorder costs $(p + h)$, it is less attractive to use the full truckload policy. The same thing can be observed when the truck capacity is increased. By a policy with a $Q_1 - Q_2$ band we mean a policy with $X > 0$. However, based on the insights gained in this section, we can develop an algorithm which we expect to allow us to quickly come up with good estimates of the optimal parameters. The algorithm is the following:

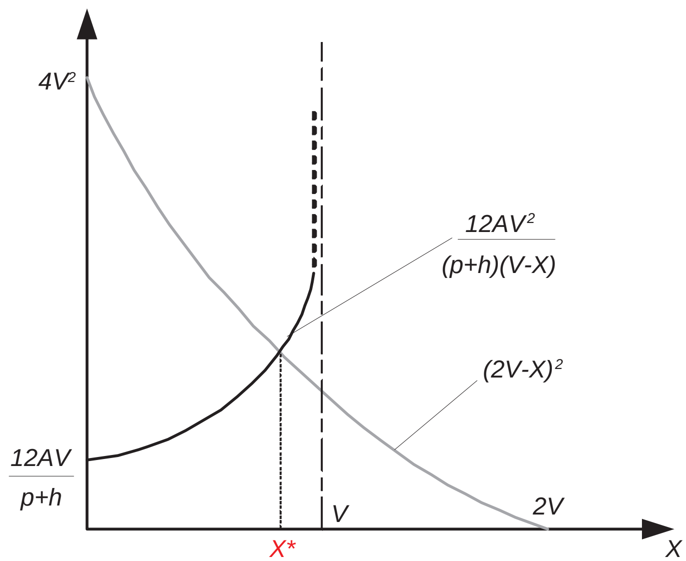

Figure 3: X^* as a function of A, h, p, and V.

Determine X^* **by solving (27)**
for $Q_1 := 0$ **to** V
  $Q_2 := max\{Q_1 + X^*, V\}$
  $S = \frac{Q_1+Q_2}{2} + (\frac{p-h}{p+h})\frac{V}{2}$
  Compute $C(S, Q_1, Q_2)$ **by using the Markov model**
end
Determine $\min\{C(S, Q_1, Q_2)\}$
Give Q_1^* **and** Q_2^*
$S^* = \frac{Q_1^*+Q_2^*}{2} + (\frac{p-h}{p+h})\frac{V}{2}$, $X^* = Q_2^* - Q_1^*$

We should mention that the policy parameters are determined using the formulas developed in this section. Afterwards, these parameters are used to compute the real cost $C(S, Q_1, Q_2)$.

Theoretically, it is possible to develop such formulas for any demand distribution. In the Appendix, we show the formulas for the case of the linear positive and the linear negative distributions. But we should keep in mind that, for many distributions, derivations may become complex. Mathematical softwares can be used to solve the equations, but usually no nice formulas will be obtained.

7 Numerical study

In this section, the results obtained from a numerical study are presented. The exact analysis as well as the heuristics developed in the previous sections have been implemented in a program written in Matlab. The following parameter set has been used for the numerical study:

V=20 ; A={50, 250} ; p=100 ; h={1, 2, 5, 10, 20}.

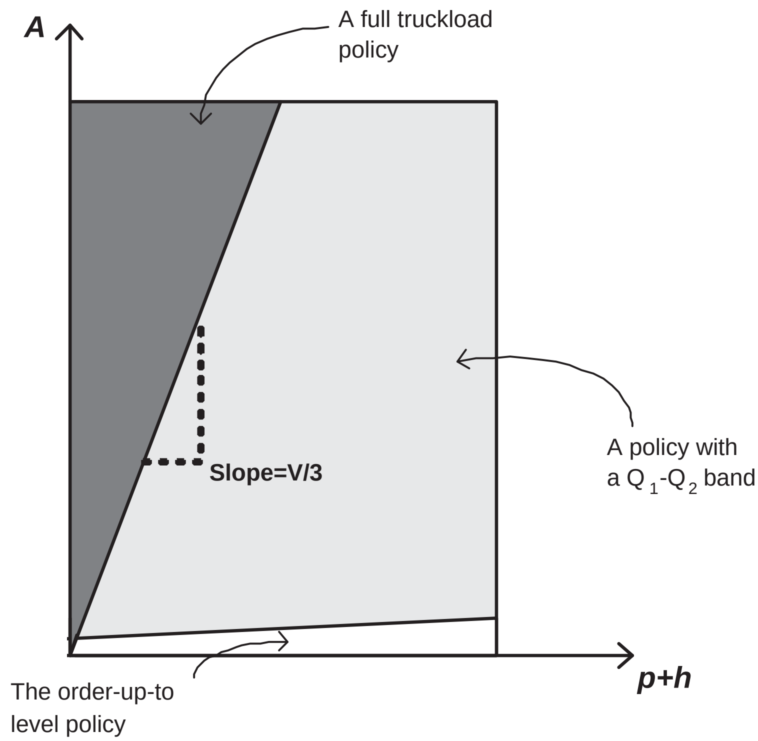

Figure 4: Sensitivity analysis for the optimal policy.

Additionally, the discrete versions of the demand distributions presented in Figure 2 are considered. These three different demand distributions allow us to cover cases with high, middle and low demand (keeping in mind that demand can not exceed V). Furthermore, they allow us to have different demand variabilities. In Tables 4, 5 and 6, the results are presented separately for each demand distribution. The performance of the exact analysis is compared with the heuristics performances and with the performance the order-up-to policy denoted (R, S), with optimal order-up-to level $S^*_{R,S}$ and minimal cost $C_{R,S}$.

From tables 4, 5 and 6, we observe that, in case of the exact analysis, the optimal value of the full truckload threshold Q_2 is always V. This means that it is not profitable to enlarge the initial order size. This is a very important result, in the sense that we can conclude that the optimal parameters are such that our policy always takes the form (S, Q_1, V). Intuitively, this result makes sense. In fact, we have seen that enlargements cause the inventory position after ordering to exceed the value S. Since the exact analysis aims to use the optimal value of S, it does not make sense to take any action that may lead to any deviation from this optimum.

However, we can easily prove that when the optimal policy parameters are such that $Q_1^* = Q_2^* = Q^*$, the optimal policy is also a policy such that $Q_2^* = V$. In fact, we can prove that if (S^*, Q^*, Q^*) is the optimal policy, all the policies in the range (S^*+i, Q^*+i, Q^*+i), $i \in [\![-Q^*, V - Q^*]\!]$ are optimal as well. In fact, if the optimal policy takes the form (S^*, Q^*, Q^*), it is characterized by the set of equations (6) and (11). Now, for each integer $i \in [\![-Q^*, V - Q^*]\!]$, the policy $(S^* + i, Q^* + i, Q^* + i)$ is characterized by the same set of equations, which means that the policy $(S^* + V - Q^*, V, V)$ is also an optimal one.

		Exact analysis				S-Heuristic				SQ-Heuristic			(R,S)	
A	h	Q_1^*	Q_2^*	S^*	$C_{SQ_1Q_2}$	$Q_{1,S}^*$	$Q_{2,S}^*$	S_S^*	C_S	X^*	S_{SQ}^*	C_{SQ}	$S_{R,S}^*$	$C_{R,S}$
50	1	20	20	37	43,46	20	20	38	43,74	1	37	43,46	20	57,62
	2	20	20	36	60,43	20	20	36	60,43	1	37	60,97	20	67,62
	5	4	20	20	91,79	6	13	20	95,25	16	20	92,32	19	97,62
	10	4	20	19	137,38	7	18	19	145,48	16	19	137,74	19	142.85
	20	3	20	17	217,48	6	17	17	224,40	16	17	218,57	17	221.90
250	1	20	20	37	143,46	20	20	38	143,74	0	34	147,92	20	248.09
	2	20	20	36	160,43	20	20	36	160,43	0	34	162,50	20	258.09
	5	20	20	34	206,25	17	20	32	207,29	0	33	206,50	20	288.09
	10	20	20	31	271,43	14	20	27	273,43	1	32	272,00	19	333.33
	20	9	20	19	358,45	8	19	19	358,69	9	19	360,14	17	412.38

Table 4: Results for the uniform distribution

		Exact analysis				S-Heuristic				SQ-Heuristic			(R,S)	
A	h	Q_1^*	Q_2^*	S^*	$C_{SQ_1Q_2}$	$Q_{1,S}^*$	$Q_{2,S}^*$	S_S^*	C_S	X^*	S_{SQ}^*	C_{SQ}	$S_{R,S}^*$	$C_{R,S}$
50	1	20	20	38	49,48	20	20	38	49,48	1	38	49,48	20	56,33
	2	2	20	20	62,27	20	20	37	63,75	19	20	62,47	20	62,67
	5	2	20	20	81,20	6	20	20	85,11	19	20	81,46	20	81,67
	10	3	20	20	112,55	6	20	20	117,82	19	20	113,10	20	113,33
	20	3	20	19	167,47	6	20	19	173,78	19	19	167,87	19	168,10
250	1	20	20	38	186,15	20	20	38	186,15	0	38	186,15	20	256,33
	2	20	20	37	200,42	20	20	37	200,42	1	37	200,42	20	262,67
	5	20	20	35	239,62	18	20	35	240,40	1	37	241,98	20	281,67
	10	20	20	34	296,58	18	20	34	299,67	17	20	310,52	20	313,33
	20	6	20	19	355,87	6	20	19	355,87	17	19	365,21	19	368,10

Table 5: Results for the linear positive distribution

We also observe that in most of the cases, the S-heuristic and the SQ-heuristic give a very good estimation of S^*.

In general, the S-heuristic as well as the SQ-heuristic perform quite good. To get an idea on how much the costs obtained by the heuristics deviate from the cost obtained by the exact analysis, we computed the percentual cost deviation

$$\Delta C_{heuristic} = \frac{C_{heuristic} - C(S,Q_1,Q_2)}{C(S,Q_1,Q_2)} \times 100\% \qquad (29)$$

On the one hand, the maximum value $\Delta C_{S-heuristic}$ can take is 5.9% and the minimum value it can take is 0%. On average the relative deviation is $\Delta C_{S-heuristic}$=1.4%. On the other hand, the maximum value $\Delta C_{SQ-heuristic}$ can take is 5% and the minimum value it can take is 0%, leading to an average of $\Delta C_{SQ-heuristic}$=1.3%. Based on these values we can conclude that the proposed heuristics perform quite good. While the SQ-heuristic is clearly simpler than the S-heuristic, it can not be used for more complicated demand distributions. The S-heuristic is less simple but it is useful whatever the demand

		Exact analysis				S-Heuristic				SQ-Heuristic			(R,S)	
A	h	Q_1^*	Q_2^*	S^*	$C_{SQ_1Q_2}$	$Q_{1,S}^*$	$Q_{2,S}^*$	S_S^*	C_S	X^*	S_{SQ}^*	C_{SQ}	$S_{R,S}^*$	$C_{R,S}$
	1	20	20	33	34,68	17	20	31	34,78	1	34	34,84	18	57,38
	2	14	20	27	50,91	14	19	27	50,91	7	28	52,45	17	68,51
50	5	7	20	18	85,74	7	15	18	86,55	16	18	89,21	16	98,57
	10	5	20	15	129,37	6	18	15	130,45	17	15	134,42	14	140,24
	20	5	20	13	197,71	5	17	13	198,47	18	13	202,25	12	206,57
	1	20	20	33	98,02	20	20	34	98,17	0	32	98,36	18	238,34
	2	20	20	32	114,56	18	20	32	115,59	0	32	114,56	17	249,47
250	5	20	20	29	157,38	15	20	27	162,11	1	30	158,25	16	279,52
	10	16	20	23	216,19	10	20	19	221,54	6	24	221,92	14	321,19
	20	10	20	16	297,22	7	20	14	303,00	13	16	312,47	12	387,52

Table 6: Results for the linear negative distribution

distribution is. Furthermore, the S-heuristic can be made simpler by only using one or two values of T but then $\Delta C_{S-heuristic}$ will increase. Moreover, we observe that the order-up-to policy is clearly outperformed, mainly when transportation costs are high (an increase with regard to the exact analysis reaches 143%).

8 Summary and Conclusions

In this paper, we have proposed a policy that combines replenishment decisions with transportation capacity. Our policy adapts the initial order sizes in the sense that they might be enlarged as well as reduced when beneficial. However, we have seen that when the optimal order-up-to level value of S is used, enlargements are not preferred. We have shown how the optimal policy parameters can be computed. In a numerical study, we illustrate that, while our policy has a quite simple structure, it performs as good as the optimal decisions whose structure is much more complex. Furthermore, two heuristics have been developed to compute near optimal policy parameters. They have proven to perform well, in the sense that good results can be obtained in a very reasonable amount of time. In addition, simple and useful rules could be obtained by applying the heuristics. In fact, we have seen how a simple calculation can help managers to decide on whether always sending full trucks is the best solution. Furthermore, we have shown that our policy clearly outperforms the order-up-to policy, sometimes by even more than 100%.

Based on the results obtained in this paper, a direction to future research could be extending our analysis to the multiple-item multiple-truck situation and consequently also other cost aspects, like line-item costs, have to be taken into account. The transportation capacity will still be a constraint, but then a decision has to be made on how to allocate capacity to the items to be shipped in a smart way.

Appendix

- **b. The positive linear distribution**

In case the demand has a positive linear distribution, as described in Figure 2, $E[D_1] = \frac{2V}{3}$. Hence in the same way as for the uniform distribution, we can have:

$$E[T] = \frac{7V + Q_1 - Q_2}{4V} \tag{30}$$

The expected on-hand and backorders at the end of a period can be similarly expressed as follows:

$$E[OH] = \frac{(S + V - Q_2)^4 - (S - Q_1)^4}{12V^2(V + Q_1 - Q_2)} \tag{31}$$

and,

$$E[BO] = \frac{(S + V - Q_2)^4 - (S - Q_1)^4}{12V^2(V + Q_1 - Q_2)} - S + \frac{Q_1 + Q_2}{2} - \frac{V}{6} \tag{32}$$

The formulas (30), (31) and (32) can be used to express the estimated cost C_{SQ}. Furthermore, we can, in the same way, find S^*_{SQ} such that:

$$S^*_{SQ} = -\frac{V - Q_1 - Q_2}{2} + \sqrt{\frac{p}{p+h}V^2 - \frac{(V + Q_1 - Q_2)^2}{12}} \tag{33}$$

Again we take $X = Q_2 - Q_1$. We can, as in the case of the uniform distribution, express C_{SQ} as a function of X. Finding the X^*, is the same as taking $Q_2 = V$ and finding $Q^*_{1,SQ}$. X^* should be equal to $V - Q^*_{1,SQ}$. $Q^*_{1,SQ}$ is found by solving the following equation:

$$\frac{dC_{SQ}(Q_1)}{dQ_1} = 0 \tag{34}$$

Or, after some algebra, solving:

$$Q^*_{1,SQ}(Q^*_{1,SQ} + \frac{4}{3}V)^2\sqrt{\frac{p}{p+h}V^2 - \frac{Q^{*2}_{1,SQ}}{12}} = \frac{8AV^3}{3(p+h)} \tag{35}$$

- **c. The negative linear distribution**

In case the demand has a positive linear distribution, as described in Figure 2, $ED_n = \frac{V}{3}$. Hence in the same way, we can have:

$$E[T] = \frac{4V + Q_1 - Q_2}{2V} \tag{36}$$

The expected on hand and backorders at the end of a period can be similarly expressed as follows:

$$E[OH] = \frac{(S-Q_1)^4 - (S+V-Q_2)^4}{12V^2(V+Q_1-Q_2)} + \frac{(S+V-Q_2)^3 - (S-Q_1)^3}{3V(V+Q_1-Q_2)} \tag{37}$$

and,

$$E[BO] = E[OH] - S + \frac{Q_1+Q_2}{2} - \frac{V}{6} \tag{38}$$

The formulas (36), (37) and (38) can be used to express the estimated cost C_{SQ}. Furthermore, we can, in the same way, find S^*_{SQ} such that:

$$S^*_{SQ} = \frac{V+Q_1+Q_2}{2} + \sqrt{\frac{h}{p+h}V^2 - \frac{(V+Q_1-Q_2)^2}{12}} \tag{39}$$

Again we take $X = Q_2 - Q_1$. We can, as in the case of the uniform distribution, express C_{SQ} as a function of X. Finding the X^*, is the same as taking $Q_2 = V$ and finding $Q^*_{1,SQ}$. X^* should be equal to $V - Q^*_{1,SQ}$. $Q^*_{1,SQ}$ is found by solving the following equation:

$$Q^*_{1,SQ}(Q^*_{1,SQ} + \frac{2}{3}V)^2\sqrt{\frac{h}{p+h}V^2 - \frac{Q^{*2}_{1,SQ}}{12}} = \frac{4AV^3}{3(p+h)} \tag{40}$$

References

[1] Cachon G (2001) Managing a Retailer's Shelf Space Inventory, and Transportation. Manufacturing and Service Operations Management 3:211-229

[2] Chan GH, Song Y (2003) A dynamic analysis of the single-item periodic stochastic inventory system with order capacity. European Journal of Operational Research 146:529-542

[3] Federgruen A, Zipkin P (1986) An inventory model with limited production capacity and uncertain demands. Mathematics of Operations Research 11:193-215

[4] Gallego G, Scheller-Wolf A (1998) Capacitated inventory probelms with fixed order costs: Some optimal policy structure. European Journal of Operational Research 126:603-613

[5] Kiesmüller GP (2006) A multi-item periodic replenishment policy with full truckloads. Beta working paper series (Rep. No. 174)

[6] Scarf H (1960) The optimality of (s,S) policies in dynamic inventory problems, in: Arrow K, Karlin S, Suppes P, Mathematical Models in the social Sciences, Stanford University Press, Stanford, CA, 1960

[7] Silver EA, Pyke DF, Peterson R (1998) Inventory Management and Production Planning and Scheduling, 3rd edn, John Wiley & Sons

[9] Shaoxiang C, Lambrecht M (1996) X-Y Band and modified (s,S) policy. Operations Research 44:1013-1019

[9] Shaoxiang C (2004) The infinite horizon periodic review problem with setup costs and capacity constraints: a partial characterization of the optimal policy. Operations Research 52: 409-421

[10] Tijms HC (1994) Stochastic models: an algorithmic approach, John Wiley & Sons.

[11] Van Eijs MJG (1994) Multi-item inventory systems with joint ordering and transportation decisions. International Journal of Production Economics 35:285-292

[12] Wijngaard J (1972) An inventory problem with contrained order capacity, TH-Report 72-WSK-63, Eindhoven University of Technology