Multi-item production control for production to order

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MULTI-ITEM PRODUCTION CONTROL FOR PRODUCTION TO ORDER

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Abstract: this paper deals with the problem of production control in situations in which several types of products are produced on one machine and in which only the ordered goods can be produced. The demand is stochastic and depends on the average delivery-time. We will describe two decomposition methods: a method based on queuing theory and a method for discrete demand and discrete service-times. Both methods will be compared with a cyclic production strategy.

1. Introduction

We consider a situation in which several types of products are produced on one machine. If the production is changed from one type to another, a set-up is needed. For some reasons, such as a large assortment of products which is subject to regular changes, a highly uncertain demand, or unique products, no safety stocks can be kept and we have to produce according to customers specifications. Due to this production to order, delivery-times have to be set for each order, since no orders can be delivered from stock. Some clients may not be content with the promised delivery-dates, therefore the demand is influenced by the delivery-times. In this paper we assume that the demand is a linear function of the average delivery-time.

In this situation it is obvious that the production control is very important. The delivery-times have to be short and the number of set-ups should be limited. Two different decomposition approaches are presented. In the first approach a queuing model is used with exponentially distributed service- and arrival-times.Such models have been presented for instance by Cohen and Boxma (1983) and Watson (1984), who studied cyclic service strategies, and Yadin (1970), who studied a queuing system with two queues and alternating priorities. There are also well-known models for queuing situations with a different service time distribution for the first client. In the second approach Markov-chains are used, assuming constant service-times. Both approaches share the assumption that the mutual influence of different types of products can be reflected in two probabilities: the probability that we start the production of a type if there is a certain number of orders for this type and the probability that we do not yet start a new set-up if the production of a type is finished. The results of the approaches are compared with a cyclic production strategy.

2. Exponential model

We assume that N types of products can be produced on one machine, each with a service rate μ_i and a potential arrival rate ε_i , $(i=1,..,N)$. The set-up time is exponentially distributed with mean s^{-1}. We also assume that all potential clients can be served, that is:

$$\sum_{i=1}^{N}\frac{\varepsilon_i}{\mu_i}<1 \tag{1}$$

However, some of the clients may not be content with their delivery-times and they may not order new demand for a while. This behaviour will be considered in a very simple form:

$$\lambda_i=\varepsilon_i\,(1-a_iS_i) \qquad i=1,2..,N \qquad (2)$$

which implies that the arrival rate for products of type i, denoted by λ_i, decreases according to a linear function of the average delivery-time of the type in a stationary situation, denoted by S_i. Here a_i is a constant expressing how strongly the demand will be influenced by the delivery-times.

Now the problem is to schedule the demand in such a way that as many clients as possible are content with their delivery-times. Assuming that the profit for one product of type i is r_i and that every set-up costs one unit, the object is to maximise the total profit :

$$P=\sum_{i=1}^{N}\lambda_i r_i-\sum_{i=1}^{N}u_i \tag{3}$$

Hereby we assume that the average set-up rate for type i is given by u_i, while the other production costs depend linearly on the demand.

3. Scheduling model

A natural element of the scheduling is the clustering of orders from the same type. Every time a certain type is produced, we will produce all demand for that type, in order to avoid set-ups. Another element that is quite obvious is that we will schedule the most important or most urgent type first. The most important type is that type for which the number of orders, or the number of orders weighted with the profit and the arrival rate, is the highest among all types. If the importance of a type is measured by its profit, we have an instrument for controlling the delivery-times. Nevertheless an additional instrument can be useful. Therefore we will only start the production of a type, say type i, if the number of orders for that type equals at least a minimum level, m_i. In this way we can both limit the amount of set-ups and favour the most profitable types.

Resuming the elements described above, the scheduling takes the following form: each time when the production of a type is finished, we determine the most important type among those types for which the demand is at least the minimum level m_i. If no such type can be found, we will wait for further demand, which may possibly lead to a continued production of the type that was produced last. Otherwise we produce all demand of the most important type, including the demand arriving during the production.

Although this scheduling model may perhaps not lead to the optimal solution, it seems reasonable to assume that optimal scheduling model will show much resemblance with this model. In this paper we will limit ourselves to the situation in which the importance of a type is solely measured by the number of orders. Then the remaining problem is to determine the optimal value of $M=\{m_1,..,m_N\}$. Determining M by means of analysis will be impossible in complex situations and simulation studies may take very much time. Therefore we will describe a decomposition approach, which may give a lot of information without too much effort.

4. Decomposition model

In the decomposition model, we will consider each type separately, using the following approximations of the scheduling model:

1 - if for type i the minimum amount of orders, m_i, is reached, it will take an exponentially distributed time, with average $(b_i\mu_i)^{-1}$, before the production of the type starts. This time includes a set-up and the waiting for other types that will be produced before type i;

2 - if the production of type i is finished, the probability that for no other type the demand is sufficient, is c_i;

3 - if the production of type i is finished and the demand for the other types is insufficient, it will take an exponentially distributed time, with average $(d_i\mu_i)^{-1}$, before the demand of one of the other types reaches its minimum.

Using these approximations, we can model the demand for each type separately as a continuous-time Markov chain. Let us consider one type, with a resulting demand rate λ, service rate μ and a production minimum m. In the Markov chain two elements are playing a role: the number of orders for the type and the state of the machine. The machine can be set for the production of the type or not set for the production. The states will be denoted by k or k^*, where k denotes the number of orders for the type and * indicates that the machine is ready to produce orders for the type. The steady-state probabilities for the states will be denoted by p_k or p_k^* respectively. We now have to solve the following set of equations:

$$\lambda p_0 = \mu(dp_0^* + (1-c)p_1^*) \tag{4}$$

$$\lambda p_k = \lambda p_{k-1} \qquad k=1,2,..,m-1 \tag{5}$$

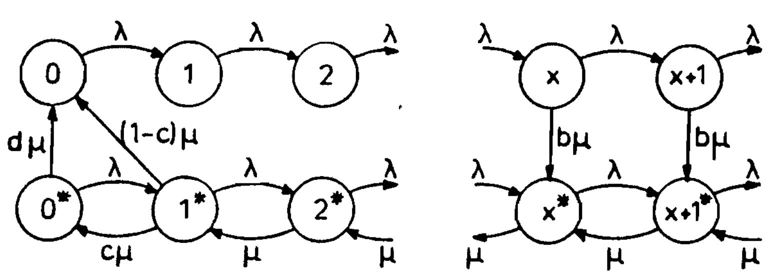
$$(\lambda + b\mu)p_k = \lambda p_{k-1} \qquad k=m,m+1,... \tag{6}$$

$$(\lambda + d\mu)p_0^* = c\mu p_1^* \tag{7}$$

$$(\lambda+\mu)p_k^* = \lambda p_{k-1}^* + \mu p_{k+1}^* \qquad k=1,2,..,m-1 \tag{8}$$

$$(\lambda+\mu)p_k^* = \lambda p_{k-1}^* + \mu p_{k+1}^* + b\mu p_k \qquad k=m,m+1,... \tag{9}$$

The states and the traffic intensities for this set are given in the following figure:



Solving this system yields the following solution for p_0:

$$p_0 = \frac{b(1-\rho)(d+\rho(1-c))}{(mb+\rho)(d+\rho(1-c))+bc\rho} \tag{10}$$

and for the average number of orders in the queue:

$$L=\frac{p_0}{1-\rho}\left[\frac{m(m+2\rho-1)}{2}+\frac{m\rho^2}{1-\rho}+\frac{m\rho}{b}+\frac{\rho^2(1-\rho+b)}{b(1-\rho)^2}+\frac{\rho^2}{b^2}+\right. \tag{11}$$

$$\left.+\frac{\rho^2(c-\rho-d)}{(1-\rho)^2(d+\rho(1-c))}\right]$$

where $\rho=\frac{\lambda}{\mu}$. Since we have Poisson-arrivals the average delivery-time is given by:

$$S=\frac{L}{\lambda} \tag{12}$$

and the set-up rate

$$u=\lambda p_0 \tag{13}$$

Of course the choice of b, c and d, the parameters that incorporate the relationship between the different types, is very important for the accuracy of the model. The best results were obtained for the following choice:

$$b_i=\left[\mu_i(\sum_{j\neq i}w_j+s^{-1})\right]^{-1} \tag{14}$$

where w_j is a measure for the waiting time due to the production of orders for type j, as far as they will delay the production of type i:

$$w_j=\frac{(L_j-0.5m_j(m_j-1)p_{0j})}{\mu_j-\lambda_j} \tag{15}$$

Due to the assumption of independence, the obvious choice for c_i is given by:

$$c_i=\prod_{j\neq i}m_jp_{0j} \tag{16}$$

For the choice of d_i, we consider the transition rates for all types:

$$d_i=\frac{\sum_{j\neq i}(\frac{\lambda_j}{m_j})}{\mu_i} \tag{17}$$

The choice of these functions for b, c and d, was based on simulation studies in which we tried several forms for b,c and d and in which the forms described by (14)-(17) yield the most accurate estimates for the delivery-times. This choice had nothing to do with maximising the profit. The values of λ_i, b_i, c_i and d_i are determined by means of iteration for some set of $\{m_1,..,m_N\}$. In order to maximise the profit, we try other sets of $\{m_1,..,m_N\}$, starting with increasing the m for the products with the smallest profit per unit or decreasing the m for the products with the largest profit per unit.

5. Numerical results and comparison with fixed cycle : 1

In a fixed cycle we have a fixed time T_i available for the production of type i including one set-up. Sometimes this time may not be used entirely, but at other times the time will not be enough to produce all orders, leading to orders that have to wait until the next cycle. By means of iteration, we can determine the optimal values for T_i, $i=1,..,N$ and the corresponding values for the average amount of orders and the profit.

Now we will compare the results of the scheduling method with the decomposition method and with the fixed cycle. For different choices of N, ε, μ, a and r, we will determine the set of values M_{opt} with the highest profit and we will also determine the profit of the set of values M_d, proposed by the decomposition approach. This will be done by means of simulation. We will compare the profit with the maximum profit for the fixed cycle strategy. The following examples will be studied:

Example 1: $N=2$, $s=20/3$ and identical types with $\varepsilon=3$, $\mu=20$, $r=2$ and $a=0.3$.

Example 2: As *Example* 1, but now $N=4$.

Example 3: $N=3$, $s=17$, $\varepsilon_1=5$, $\varepsilon_2=4$, $\varepsilon_3=3$, $\mu_1=17$, $\mu_2=85/6$, $\mu_3=34/3$, $r_i=17/\mu_i$, $a_i=0.1$, $i=1,2,3$.

Example 4: As *Example* 3, but now $r_i=34/\mu_i$, $i=1,2,3$.

For these examples we will first give the proposed M_d, by the decomposition approach, then the optimal M_{opt} for the scheduling method , followed by the profit calculated by the decomposition approach and then the values of the profit for the M_d, M_{opt} and for the fixed cycle.

Example	M_d	M_{opt}	decomp.	M_d	M_{opt}	cycle
1	{4,4}	{4,4}	7.31	8.14	8.14	7.85
2	{4,4,4,4}	{4,4,4,4}	12.82	13.15	13.15	11.22
3	{6,5,4}	{7,5,4}	11.74	11.85	11.88	11.19
4	{4,3,3}	{4,3,3}	24.83	24.92	24.92	23.28

Table 1: profit for 4 examples in the exponential case.

In these examples we can see that the decomposition approach succeeds in finding good values for M; in only one example there is a small difference between M_d and M_{opt}. Therefore the profit is also the same in three of the four situations. The profit estimated by the decomposition approach tends to be more accurate if the traffic intensity increases. The use of a fixed cycle always leads to a lower profit. This difference seems to increase with the number of types.

6. Discrete model

In the discrete model we also assume that N types of products can be produced on one machine. Now the service time is one time unit for all types and the potential arrival average is ε_i. The set-up time is an integer constant s. As in the exponential model we assume that

$$\sum_{i=1}^{N} \varepsilon_i < 1 \tag{18}$$

and

$$\lambda_i = \varepsilon_i(1-a_i S_i) \qquad i=1,2,..,N \qquad (19)$$

An important difference with the exponential model is the fact that we assume a fixed length for the production period: every c time units we can change the production from one type to another. If not all products of a type are manufactured at the end of such a period, overtime will be done to finish them. However this will only be done for demand that arrived before the production period started. Of course this overtime involves extra costs, therefore the formula for the profit is slightly different from (3):

$$P = \sum_{i=1}^{N} \lambda_i r_i - \sum_{i=1}^{N} u_i - ot \sum_{i=1}^{N} o_i, \qquad (20)$$

where o_i is the average amount of overtime per time unit for type i and ot the costs involved with the overtime.

The scheduling model we will use for this problem is more or less the same as the scheduling model for the exponential model: at the beginning of each period we determine the most important type among those types for which the demand is at least the minimum level m_i. If no such type can be found, we do not produce during that period. Otherwise, we will produce all demand of the most important type, except the demand that arrives during the production period.

The remaining problem is to determine the optimal value of $M=\{m_1,..,m_N\}$ and the length of the production period, c. To solve this, we will use a decomposition approach, that will be described in the next section.

7. Analysis

To analyse this model we make use of Markov-chains and we assume the following:
- the demand per period per type is integer valued, finite, independent from other periods or types and stationary stochastic. The demand for the product type is the state in the Markov chain.
-the probability that we produce this type is the probability that the total demand for other types is smaller (or equal if the index of that type is higher) than the total demand for this type.
- the assumption that the total demand is independent from the total demand for other types. Simulation showed that this assumption generally does not lead to large errors.

We use the following notation:
- b_{ij} is the probability that in an arbitrary period the new demand for type i in that period equals j, such that $\sum_{j=1}^{\infty} j b_{ij} = \lambda_i$.
- m_i is a positive integer, indicating the minimum demand needed to start production of type i.
- $p(i,j)$ is the average time between two production periods during which the demand for type i equals j.
- $q(i,j)$ is the probability that we do not produce type i if the demand for this type equals j.

For a start we set:

$$q(i,j)=1 \qquad \text{for } j=0,1,..,m_{i-1} \qquad (21)$$

$$q(i,j)=0 \qquad \text{for } j=m_i,m_{i+1},.. \qquad (22)$$

Then

$$p(i,0)=\frac{b_{i0}}{1-b_{i0}} \tag{23}$$

$$p(i,j)=(1+p(i,0))\,b_{ij}+\sum_{k=1}^{j}p(i,k)\,q(i,k)\,b_{i,j-k} \tag{24}$$

Let $T(i)=\sum_{j\geq 0}p(i,j)$ be the average time between two production periods. Then we can determine the new values $q(i,j)$ for $j \geq m_i$ by:

$$q(i,j)=1-\prod_{k<>i}\frac{\sum_{l=0}^{g(k,i,j)}p(k,l)}{T(k)}, \tag{25}$$

where

$$g(k,i,j)=max(j-1,m_k-1) \qquad \textit{if } k<i \tag{26}$$

$$g(k,i,j)=max(j,m_k-1) \qquad \textit{if } k>i \tag{27}$$

Using the new q-values we determine again $p(i,j)$ for $j \geq m_i$ by (22) and then again the new q-values. This procedure is repeated until the changes in the p- and q-values are negligible. We can limit the state space and thereby the computational efforts by assuming that there is some $X_{\max}$ for which: $q(i,j)=0$ for all i and for all $j \geq X_{\max}$.

From these steady-state probabilities we can determine the probabilities that the delivery-time of an order of a certain type equals k using: the probability that at the end of the arrival period of an order of a certain type, the demand for this type equals j and the probability that it takes k periods before we produce if the demand equals j. From these delivery-times, and also from the average amount of orders for a certain type, we can determine the average delivery-time S_i for every type i . Then we repeat the procedure for the new λ-values, based on the average delivery-times, until the changes in the λ-values are negligible. Then we can determine the amount of overtime:

$$o_i=\frac{\sum_{j\geq c}(j-c+1)p(i,j)(1-q(i,j))}{T(i)} \tag{28}$$

and the set-up rate

$$u_i=\frac{s}{T(i)} \tag{29}$$

and find the average profit by using equation (20).

8. Numerical results and comparison with fixed cycle : 2

To make a good comparison between the fixed cycle and the discrete model, we assume that in the fixed cycle model we also have periods with length c in which only one type of product will be produced. Now we determine a long cycle in which the number of times that a certain type will be produced is proportional to the potential demand for that type. If at the end of a period not all demand for a type is produced, overtime will be done to finish that part of the demand that arrived before the beginning of the production period.

For both models, we assume that the demand has a Poisson-like distribution, which is truncated at $2c\,\varepsilon$ and then corrected for the proper mean. We will compare the results of the discrete model with the results of the fixed cycle for different choices of N and ε. In *Table* 2 we will give the optimal set of values M, the corresponding profit and the profit using the fixed cycle for the following examples:
Example 1: $N=2$, $\varepsilon_1=0.5$, $\varepsilon_2=0.3$.
Example 2: $N=3$, $\varepsilon_1=0.4$, $\varepsilon_2=0.25$, $\varepsilon_3=0.15$.
Example 3: $N=3$, $\varepsilon_1=0.35$, $\varepsilon_2=0.25$, $\varepsilon_3=0.2$.
Example 4: $N=4$, $\varepsilon_1=0.1$, $\varepsilon_2=0.2$, $\varepsilon_3=0.25$, $\varepsilon_4=0.25$.
In all examples we have capacity $c=10$, parameter for discontent clients $a=0.01$, revenues per order $r=1$ and overtime costs $ot=2$.

Example	M_{opt}	discrete model	fixed cycle
1	{3,2}	.518	.459
2	{3,2,1}	.507	.424
3	{3,3,2}	.512	.423
4	{1,2,3,3}	.496	.360

Table 2: profit for 4 examples in the discrete case.

The examples showed us that with rather small efforts in both the discrete situation as well as in the exponential situation good results are obtained. In the discrete case we can also notice that the difference between the scheduling model and the fixed cycle increases as the number of types increases. The models described above can of course be refined in order to adapt them more to a real-life situation or to increase the profit. This has been done in Dellaert (1988). However, especially when we compare these models with a fixed cycle strategy, they seem to be a good starting point for further research.

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