New upper bounds for the football pool problem for 6, 7, and 8 matches

Reprinted from JOURNAL OF COMBINATORIAL THEORY, Series A
All Rights Reserved by Academic Press, New York and London

Vol. 52, No. 2, November 1989
Printed in Belgium

Note

New Upper Bounds for the Football Pool Problem for 6, 7, and 8 Matches

P. J. M. VAN LAARHOVEN

Philips Research Laboratories,
P. O. Box 80.000, 5600 JA Eindhoven, The Netherlands

E. H. L. AARTS AND J. H. VAN LINT

Philips Research Laboratories,
P. O. Box 80.000, 5600 JA Eindhoven, The Netherlands and
Eindhoven University of Technology,
Department of Mathematics and Computer Science,
P. O. Box 513, 5600 MB Eindhoven, the Netherlands

AND

L. T. WILLE

University of California,
Department of Materials Science and Mineral Engineering,
Berkeley, California 94720

Communicated by the Managing Editors

Received March 2, 1988

Consider the set V_3^n of all n-tuples $\mathbf{x} = (x_1, ..., x_n)$ with $x_i \in \{0, 1, 2\}$. We are interested in σ_n, the minimal size of a subset W of V_3^n, such that for any element $\mathbf{x} \in V_3^n$ there exists at least one element $\mathbf{y} \in W$ at a Hamming distance $d_H(\mathbf{x}, \mathbf{y}) \leqslant 1$. σ_n can also be considered as the minimal number of forecasts in a football pool of n matches, such that at least one forecast has at least $n-1$ correct results. In this note we present new upper bounds on σ_6, σ_7, and σ_8: 73, 186, and 486, respectively. The bounds have been obtained by an approximation algorithm based on simulated annealing. A closer analysis of the result for the 8-matches problem has led to a simple way to construct a large number of subsets W of V_3^8, each consisting of 486 8-tuples and each having the aforementioned property.

0097-3165/89 $3.00

1. Introduction

Let V_k^n denote the set of all n-tuples $\mathbf{x} = (x_1, ..., x_n)$ with $x_i \in \mathbb{Z}_k = \{0, ..., k-1\}$. The *Hamming distance* $d_H(\mathbf{x}, \mathbf{y})$ between two n-tuples $\mathbf{x}, \mathbf{y} \in V_k^n$ is defined as the number of positions in which the two n-tuples differ. For all $\mathbf{x} \in V_k^n$, the *rook domain* of $\mathbf{x}$, $B(\mathbf{x})$, is defined as

$$B(\mathbf{x}) = \{\mathbf{y} \in V_k^n \mid d_H(\mathbf{x}, \mathbf{y}) \leqslant 1\}. \tag{1}$$

A subset W of V_k^n is called a *covering by rook domains* if

$$V_k^n = \bigcup_{\mathbf{x} \in W} B(\mathbf{x}). \tag{2}$$

We are interested in upper bounds on the number of n-tuples in a minimal covering of V_k^n, denoted by $\sigma(n, k)$. In this note we only consider the case $k = 3$, which is known as the *football pool problem* (Kamps and Van Lint, [4]), and write σ_n instead of $\sigma(n, 3)$.

In [8], the problem is considered as an optimization problem and a *simulated-annealing*-based algorithm [5] is used to find approximate solutions to it. More specifically, for a fixed number p, the algorithm generates subsets W of V_3^n consisting of p n-tuples, while trying to minimize the amount of *uncovered* n-tuples, i.e., the size of the set $V_k^n \setminus \bigcup_{\mathbf{x} \in W} B(\mathbf{x})$. If a subset W is found for which $V_k^n \setminus \bigcup_{\mathbf{x} \in W} B(\mathbf{x})$ is empty, i.e., if a covering by rook domains consisting of p n-tuples is found, then clearly p is an upper bound on σ_n. In that case, p is decreased by 1 and the algorithm is executed again. Execution of the algorithm is terminated when p is such that the algorithm is not able to find a subset W for which $V_k^n \setminus \bigcup_{\mathbf{x} \in W} B(\mathbf{x})$ is empty. Clearly then, $p + 1$ is an upper bound on σ_n.

The optimization technique employed, simulated annealing, is a randomization version of the well-known *iterative improvement* approach to combinatorial optimization problems. A detailed description of the application of simulated annealing to the football pool problem can be found in (Wille, [8]); for a review of the theory and other applications of simulated annealing the reader is referred to Van Laarhoven and Aarts, [6].

2. The 6- and 7-Matches Problems

By using the same algorithm as in [8], but with slower "cooling" (1000 iterations per temperature step, $\chi_T = 0.995$), we were able to find a covering of V_3^6 by 73 rook domains—a slight improvement on the upper bound of 74 reported in [8]. The positions of the 73 6-tuples are shown in Fig. 1.

A large number of calculations with various cooling rates was performed in an attempt to find a covering with 72 rooks. However, no such structure could be found, in spite of a good deal of computational effort. The best results, obtained on several occasions, leave two 6-tuples uncovered. Such results were found 10 out of 10 times with 100,000 iterations per temperature step, $\Delta T = 0.01$ and linear cooling scheme. Thus, at least from a computational point of view, there is strong evidence that $\sigma_6 = 73$.

An even slower cooling rate (10,000 iterations per temperature step, $\Delta T = 0.01$, linear cooling scheme) was necessary to find a covering of V_3^7 by 186 rook domains. This result was found by executing the algorithm 50 times; in two cases the final covering consisted of 186 rook domains. Both coverings are displayed in Fig. 2. The bound of 186 is a significantly sharper upper bound on σ_7 than the best result published so far, which is 2.16 [2], though in 1958 a group in Finland constructed an apparently unpublished covering proving that $\sigma_7 \leqslant 189$ [7].

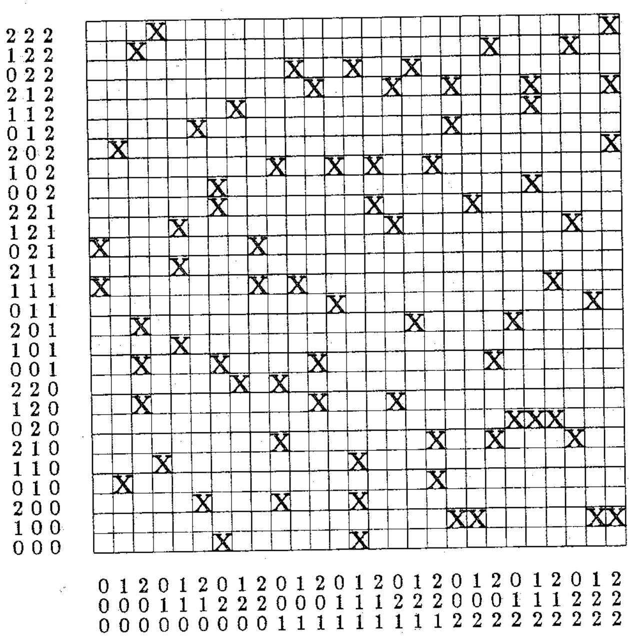

FIG. 1. A covering of V_3^6 by 73 6-tuples. The positions of the 6-tuples are indicated by crosses.

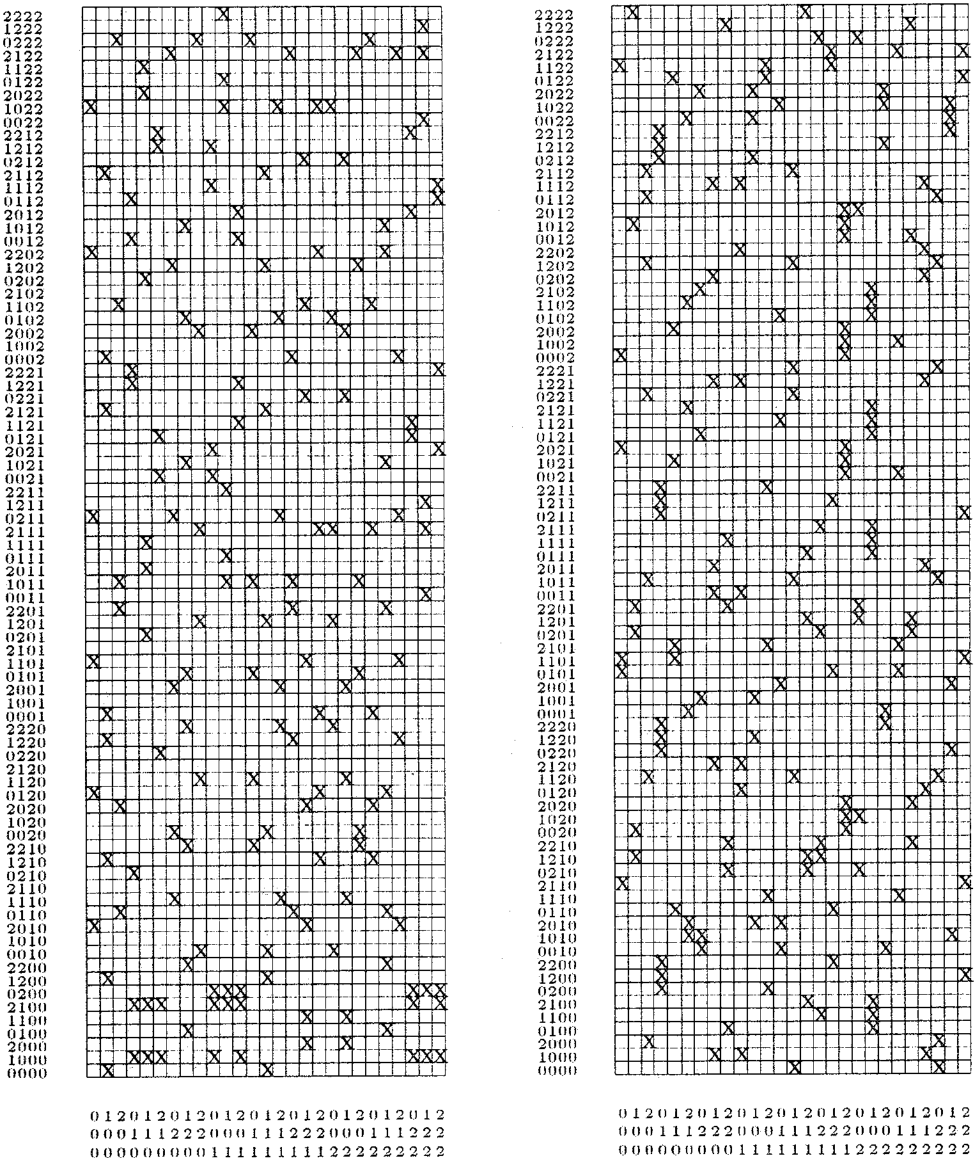

FIG. 2. Two coverings of V_3^7 by 186 7-tuples. The positions of the 7-tuples are indicated by crosses.

3. The 8-Matches Problem

Since the computation time taken by our algorithm quickly increases with growing problem size (each of the 50 runs for the 7-matches problem, for instance, took approximately 2 h of CPU time on a VAX 11/785 computer), we adopt a different (and computationally more efficient) approach to find coverings of larger problems than the ones considered so far. Our approach is based on a result due to Blokhuis and Lam [2]. In order to state this result, we need the following definition. Let A be an $r \times n$ matrix with entries from $\mathbb{Z}_k$. A subset S of V_k^r is called a *covering using* A if

$$V_k^r = \{\mathbf{s} + \alpha \cdot \mathbf{a}_i \mid \mathbf{s} \in S, \alpha \in \mathbb{Z}_k, 1 \leqslant i \leqslant n\}, \tag{3}$$

where $\mathbf{a}_i$ denotes the ith column vector of the matrix A. Note that if $r = n$ and A is the identity matrix, then S is a covering by rook domains.

If $A = (I; M)$, where I is the $r \times r$ identity matrix and M an $r \times (n-r)$ matrix with entries from $\mathbb{Z}_k$, then Blokhuis and Lam prove that $W = \{\mathbf{w} \in V_k^n \mid A\mathbf{w} \in S\}$ is a covering of V_k^n by rook domains if S is a covering of V_k^r using A. Furthermore, they prove that $|W| = |S| \cdot k^{n-r}$. The restriction on the form of the matrix A is not used in the proof that W is a covering by rook domains, so that this result also holds for an arbitrary matrix A. In that case, however, the number of n-tuples in W is not necessarily given by $|S| \cdot k^{n-r}$.

We use the result of Blokhuis and Lam to approximate σ_n. Consequently, we put $k = 3$ and try to find an $r \times n$ matrix A and a subset S of V_3^r such that S is a covering of V_3^r using A. We formulate the latter problem again as an optimization problem; i.e., for a given value p and a choice of r we try to find a subset S of V_3^r consisting of p r-tuples and an $r \times n$ matrix A such that the size of the set

$$V_k^r \backslash \{\mathbf{s} + \alpha \cdot \mathbf{a_i} \mid \mathbf{s} \in S, \alpha \in \mathbb{Z}_k, 1 \leqslant i \leqslant n\} \tag{4}$$

is minimal. Again, we use simulated annealing to solve this problem—a move in this case is either the replacement of a randomly chosen tuple in S by a randomly chosen tuple not in S or the replacement of a randomly chosen column of A by a column not yet in A. If the algorithm finds a subset S and a matrix A for which (3) is satisfied, p is decreased by 1 and the algorithm is executed again. Execution of the algorithm is terminated when p is such that the algorithm does not find a subset S and a matrix A for which the set in (4) is empty.

If $S_{\min}$ is the smallest covering of V_3^r found throughout execution of the algorithm (using a matrix $A_{\min}$), then W is constructed through $W = \{\mathbf{w} \in V_k^n \mid A_{\min}\mathbf{w} \in S_{\min}\}$. Clearly then, $|W|$ is an upper bound on σ_n.

By applying the *cooling schedule* described in Aarts and Van Laarhoven,

[1], using 225 iterations per temperature step and a distance parameter $\delta = 0.01$, we were able to find a covering

$$S = \{(2, 2, 2, 2), (2, 1, 2, 1), (2, 0, 1, 1), (0, 2, 1, 1), (2, 0, 1, 2), (1, 1, 2, 2)\} \tag{5}$$

of V_3^4 using the 4×8 matrix

$$A = \begin{pmatrix} 2 & 0 & 2 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 & 0 & 2 & 2 \\ 2 & 0 & 2 & 0 & 1 & 2 & 2 & 1 \\ 0 & 2 & 2 & 2 & 1 & 0 & 1 & 1 \end{pmatrix}. \tag{6}$$

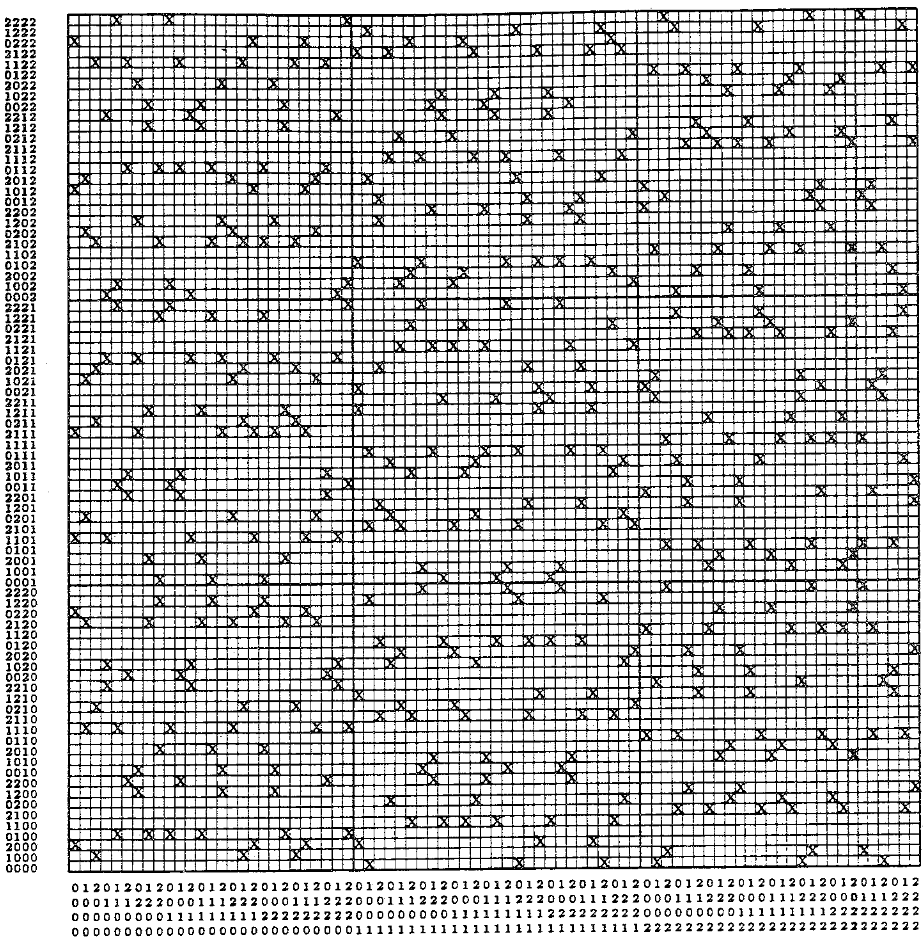

FIG. 3. A covering of V_3^8 by 486 8-tuples. The positions of the 8-tuples are indicated by crosses.

This result was found in 705 s of CPU time on a VAX 11/785 computer; in this case $r=4$ and $p=6$.

Constructing W through $W=\{\mathbf{w}\in V_k^n \mid A\mathbf{w}\in S\}$, with S and A given by (5) and (6), respectively, yields a covering of V_3^8 by 486 rook domains (since the matrix A has maximum rank, the size of W can also be found by applying the result of Blokhuis and Lam: $|W|=|S|\cdot k^{n-r}=6\cdot 3^4=486$). This is again a significantly sharper upper bound than the best result published so far, which is 567 (Fernandes and Rechtschaffen, [3]), although the Finnish group mentioned before also constructed a covering with 546 8-tuples [7]. The positions of the 8-tuples in W are shown in Fig. 3.

4. Discussion

At first glance the result presented in Section 3 does not seem to have any structure. However, a closer analysis of this result led us to a simple way to construct a large number of coverings of V_3^8, each consisting of 486 8-tuples. In order to describe this construction, we need the following definitions.

For $\mathbf{x}\in V_3^3$, we define the set $D(\mathbf{x})$ as

$$D(\mathbf{x}) \stackrel{\text{def}}{=} \{\mathbf{y}\in V_3^3 \mid \mathbf{y}=\mathbf{x}+\alpha\cdot\mathbf{a},\ \alpha\in\mathbb{Z}_3,\ \mathbf{a}\in V_2^3\}. \tag{7}$$

Thus, an element of $D(\mathbf{x})$ is obtained by adding to $\mathbf{x}$ a multiple of a vector for which the non-zero coordinates are all 1.

Two tuples $\mathbf{x}$ and $\mathbf{y}$ in V_3^3 are said to form a *pair* if they are the same in one coordinate position and if $y_i-x_i=1$ for the other two values of i. Finally, two pairs $(\mathbf{x}_1,\mathbf{y}_1)$ and $(\mathbf{x}_2,\mathbf{y}_2)$ are *related* if $\mathbf{x}_2$ and $\mathbf{y}_2$ are not in the union of $D(\mathbf{x}_1)$ and $D(\mathbf{y}_1)$ (or, equivalently, if $\mathbf{x}_1$ and $\mathbf{y}_1$ are not in the union of $D(\mathbf{x}_2)$ and $D(\mathbf{y}_2)$).

It is now easy to check that for a pair $(\mathbf{x}_0,\mathbf{y}_0)$, the set $V_3^3\backslash(D(\mathbf{x}_0)\cup D(\mathbf{y}_0))$ is the union of two related pairs $(\mathbf{x}_1,\mathbf{y}_1)$ and $(\mathbf{x}_2,\mathbf{y}_2)$: without loss of generality we may put $\mathbf{x}_0=(0,0,0)$ and $\mathbf{y}_0=(0,1,1)$, and then find that $V_3^3\backslash(D(\mathbf{x}_0)\cup D(\mathbf{y}_0))$ is the union of the pairs $(\mathbf{x}_1,\mathbf{y}_1)=((1,2,0),(2,2,1))$ and $(\mathbf{x}_2,\mathbf{y}_2)=((1,0,2),(2,1,2))$.

As a consequence of this property, we can write

$$V_3^3=D(\mathbf{x}_0)\cup D(\mathbf{y}_0)\cup\{\mathbf{x}_1,\mathbf{y}_1\}\cup\{\mathbf{x}_2,\mathbf{y}_2\}, \tag{8}$$

$$V_3^3=\{\mathbf{x}_0,\mathbf{y}_0\}\cup D(\mathbf{x}_1)\cup D(\mathbf{y}_1)\cup\{\mathbf{x}_2,\mathbf{y}_2\}, \tag{9}$$

$$V_3^3=\{\mathbf{x}_0,\mathbf{y}_0\}\cup\{\mathbf{x}_1,\mathbf{y}_1\}\cup D(\mathbf{x}_2)\cup D(\mathbf{y}_2). \tag{10}$$

The following theorem is an immediate consequence of the aforementioned property of a pair and the choice of S and A.

THEOREM 1. *If* $S \subset V_3^4$ *is given by*

$$S = \{(\mathbf{x}_i, i), (\mathbf{y}_i, i) \mid i = 0, 1, 2\}, \tag{11}$$

where $(\mathbf{x}_0, \mathbf{y}_0)$ *is an arbitrary pair in* V_3^3 *and the pairs* $(\mathbf{x}_1, \mathbf{y}_1)$ *and* $(\mathbf{x}_2, \mathbf{y}_2)$ *are given by* $V_3^3 \backslash (D(\mathbf{x}_0) \cup D(\mathbf{y}_0))$, *and*

$$A = (I_4; M), \tag{12}$$

where

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{13}$$

then S *covers* V_3^4 *using* A.

Proof. We divide the set V_3^4 into three disjoint subsets, U_0, U_1, and U_2, respectively, where

$$U_i = \{(\mathbf{x}, i) \mid \mathbf{x} \in V_3^3\}, \qquad i = 0, 1, 2. \tag{14}$$

Consider an arbitrary element $(\mathbf{x}, 0)$ of U_0. According to (8), either $\mathbf{x} = \mathbf{x}_0 + \alpha \cdot \mathbf{a} \vee \mathbf{x} = \mathbf{y}_0 + \alpha \cdot \mathbf{a}$ for some $\alpha \in \mathbb{Z}_3$ and $\mathbf{a} \in V_3^2 \backslash (0, 0, 0)$ or $\mathbf{x} \in \{\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_2, \mathbf{y}_2\}$. In the first case, we can write

$$(\mathbf{x}, 0) = (\mathbf{x}_0, 0) + \alpha \cdot (\mathbf{a}, 0) \vee (\mathbf{x}, 0) = (\mathbf{y}_0, 0) + \alpha \cdot (\mathbf{a}, 0), \tag{15}$$

where $(\mathbf{a}, 0)$ is a column of A, since $\mathbf{a} \in V_3^2 \backslash (0, 0, 0)$. In the second case, we use the fourth column of A to write

$$(\mathbf{x}, 0) = (\mathbf{x}_i, i) + (3 - i) \cdot (0, 0, 0, 1) \vee (\mathbf{x}, 0) = (\mathbf{y}_i, i) + (3 - i) \cdot (0, 0, 0, 1), \tag{16}$$

for some $i \in \{1, 2\}$. The same line of reasoning can be applied to the sets U_1 and U_2.

An immediate consequence of the results of Blokhuis and Lam and Theorem 1 is that for an arbitrary pair $(\mathbf{x}_0, \mathbf{y}_0)$ in V_3^3, the set $W = \{\mathbf{w} \in V_k^n \mid A\mathbf{w} \in S\}$, with S and A given by (11) and (12), respectively, yields a covering of V_3^8 by 486 rook domains.

One might be tempted to try to show by a straightforward generalization that $\sigma_6 \leqslant 8 \cdot 3^2$. The idea would be to find 8 3-tuples $\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{x}_1$, $\mathbf{y}_1$, $\mathbf{z}_1$, $\mathbf{x}_2$, $\mathbf{y}_2$, and $\mathbf{z}_2$, such that

$$V_3^3 = D'(\mathbf{x}_0) \cup D'(\mathbf{y}_0) \cup T_0, \tag{17}$$

$$V_3^3 = D'(\mathbf{x}_1) \cup D'(\mathbf{y}_1) \cup D'(\mathbf{z}_1) \cup T_1, \tag{18}$$

$$V_3^3 = D'(\mathbf{x}_2) \cup D'(\mathbf{y}_2) \cup D'(\mathbf{z}_2) \cup T_2, \tag{19}$$

where $T_0 \subset \{\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2\}$, $T_1 \subset \{\mathbf{x}_0, \mathbf{y}_0, \mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2\}$, and $T_2 \subset \{\mathbf{x}_0, \mathbf{y}_0, \mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1\}$, and

$$D'(\mathbf{x}) \stackrel{\text{def}}{=} \{\mathbf{y} \in V_3^3 \mid \mathbf{y} = \mathbf{x} + \alpha \cdot \mathbf{a}_i, \alpha \in \mathbb{Z}_3, i \in \mathbb{Z}_6\}, \tag{20}$$

the vectors $\mathbf{a}_0, ..., \mathbf{a}_5$ being six vectors to be chosen from V_3^3 (including $(0, 0, 0)$). However, as the reader can verify, it is not possible to write V_3^3 in the form of (17).

References

1. E. H. L. Aarts and P. J. M. van Laarhoven, Statistical cooling: A general approach to combinatorial optimization problems, *Philips J. Res.* **40** (1985), 193–226.
2. A. Blokhuis and C. W. H. Lam, More coverings by rook domains, *J. Combin. Theory Ser. A* **36** (1984), 240–244.
3. H. Fernandes and E. Rechtschaffen, The football pool problem for 7 and 8 matches, *J. Combin. Theory Ser. A* **35**, (1983), 109–114.
4. H. J. L. Kamps and J. H. van Lint, The football problem for 5 matches, *J. Combin. Theory* **3** (1967), 325–325.
5. S. Kirkpatrick, C. D. Gelatt, Jr., and M. P. Vecchi, Optimization by simulated annealing, *Science* **220** (1983), 671–680.
6. P. J. M. van Laarhoven and E. H. L. Aarts, "Simulated Annealing: Theory and Applications," Reidel, Dordrecht, 1987.
7. A. Lobstein, private communication, (1987).
8. L. T. Wille, The football pool problem for 6 matches, A new upper bound obtained by simulated annealing, *J. Combin. Theory Ser. A* **45** (1987), 171–177.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium