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ON A NONLINEAR ELLIPTIC-PARABOLIC PARTIAL DIFFERENTIAL EQUATION SYSTEM IN A TWO-DIMENSIONAL GROUNDWATER FLOW PROBLEM*

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Abstract. In this paper a nonlinear elliptic-parabolic system which arises in a two-dimensional groundwater flow problem is studied. Abstract results on evolution equations are employed to obtain existence and uniqueness results. Regularity and stability properties of the solution are also considered.

Key words. elliptic-parabolic system, analytic semigroups, semilinear and quasilinear evolution equations

AMS(MOS) subject classifications. 35K50, 47D05

1. Introduction. Let $\Omega \in \mathbf{R}^2$ be a bounded domain with smooth boundary. In this paper we study the following nonlinear elliptic-parabolic system:

$$(E)\begin{cases} -\Delta v = \partial_1 u & \text{in } \Omega \times (0,\infty), \\ v = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

and

$$(P)\begin{cases} \partial_t u + \operatorname{div}\vec{F} = 0 & \text{in } \Omega \times (0,\infty), \\ \vec{F}\cdot\vec{\nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0(\cdot) & \text{in } \Omega. \end{cases}$$

Here we have

$$\vec{F} = \vec{q}\,u - D\cdot \operatorname{grad} u,$$
$$\vec{q} = \operatorname{curl} v,$$
$$D = (D_{ij}),$$

where $D_{ij}(q_1, q_2)$ are uniformly Lipschitz continuous functions on $\mathbf{R}^2$.

This system arises in the description of the movement of a fluid of variable density (u) through a porous medium under the influence of gravity and hydrodynamic dispersion. In §2 we set up the model and we discuss the physical background.

In a slightly different form, Problem (E), (P) was studied by Su [16] using classical partial differential equation (PDE) methods. In this paper we present an approach in the spirit of abstract evolution equations in Banach spaces. This turns out to be quite efficient because of the particular form of the problem.

We consider two cases of the model separately. In the first (approximate) case we take $D_{ij} = \delta_{ij}$ (δ_{ij} is the Kronecker symbol). Then the system can be considered as a semilinear evolution equation. Clearly, there are many results on abstract semilinear evolution equations, and these results can be well applied to partial differential equations of parabolic type; see, e.g., Friedman [7], Henry [9], Pazy [12], or von Wahl [19]. Here we choose one theorem from von Wahl [20], which fits precisely to the abstract formulation of Problem (E), (P) with constant (D). By this theorem we obtain the global existence of the solution in $L^p(\Omega)$. This is done in §3. There we also study the

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regularity and asymptotic properties of the solution. We show that the solution is in fact a classical solution of $(E), (P)$, and u converges to the mean value in sup-norm as $t \to \infty$. A first draft of §3 was made by de Roo [13].

In §4, we study the full problem, i.e., D is nonconstant and velocity dependent. Then the abstract formulation leads to a quasilinear evolution equation. The abstract results on such equations are not as complete as the results on semilinear equations. Moreover the application to partial differential equations is much harder. In this paper we use the framework of quasilinear evolution equations due to Amann [2], see also Sobolevskii [15]. As a result, we obtain local existence of weak solutions in $W^{1,p}(\Omega)$. As for this moment, we are not able to obtain global existence. Because the coefficients D_{ij} are not differentiable at the origin, see (2.13), we can not expect to have classical solutions.

2. The physical background. Let $\Omega = (-L, L) \times (0, H)$, with $L, H > 0$, denote a rectangular region in the x_1, x_2 plane which is occupied by a homogeneous and isotropic porous medium. This medium is characterized by a permeability $\kappa \in (0, \infty)$ and a porosity $\phi \in (0, 1)$. It is saturated by an incompressible fluid. The fluid is characterized by a constant viscosity $\mu \in (0, \infty)$ and a variable density ρ (or a specific weight $\gamma = \rho g$, where g is the accelaration of gravity). Here the coordinate system is chosen such that the gravity is pointing in the negative x_2-direction. A typical example of this situation arises in the flow of fresh and salt groundwater in a two-dimensional vertical aquifer. In this application it is natural to assume that γ satisfies

$$0 < \gamma_f \leq \gamma(x_1, x_2, t) \leq \gamma_s \quad \forall (x_1, x_2, t) \in \Omega \times (0, \infty). \tag{2.1}$$

Here γ_f and γ_s are constants, denoting the specific weight of the fresh and the salt groundwater, respectively.

The basic equations for flow in a porous medium are the continuity equation

$$\operatorname{div} \vec{q} = 0 \quad \text{in } \Omega \times (0, \infty) \tag{2.2}$$

and the momentum balance equation (Darcy's law), see, e.g., Bear [5],

$$\frac{\mu}{\kappa} \vec{q} + \operatorname{grad} p + \gamma \vec{e}_2 = 0 \quad \text{in } \Omega \times (0, \infty). \tag{2.3}$$

Here we denote by the vector $\vec{q}$ the specific discharge of the fluid and by the scalar p the fluid pressure. Finally, $\vec{e}_2$ denotes the unit vector in the positive x_2-direction (i.e., pointing upwards).

In this paper we are interested in describing the distribution of the specific weight γ in the domain Ω under the action of gravity and hydrodynamic dispersion, without any other influence from outside. Therefore, we impose on the boundary $\partial\Omega$ the no-flow condition

$$\vec{q} \cdot \vec{\nu} = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{2.4}$$

where $\vec{\nu}$ is the outward normal unit vector on $\partial\Omega$.

For a given specific weight distribution γ, (2.2)–(2.4) determine the discharge field $\vec{q}$. To obtain a single equation for this relation we can use either the pressure or, because of (2.2), the stream function. Here we use a formulation in terms of the stream function. It satisfies

$$\vec{q} = (q_1, q_2) = \operatorname{curl} \psi := (-\partial_2 \psi,\ \partial_1 \psi), \tag{2.5}$$

where ∂_i denotes the partial derivative with respect to the variable x_i for $i = 1, 2$. Note that the operator curl in (2.5) acts on a scalar function. Therefore this definition differs from the usual one. It is introduced here only for convenience.

Substituting (2.5) into Darcy's law (2.3) and taking the curl in the usual sense (i.e., curl $\vec{q} = \partial_2 q_1 - \partial_1 q_2$) gives

$$-\Delta\psi = \frac{\kappa}{\mu}\partial_1\gamma \quad \text{in } \Omega \times (0,\infty). \tag{2.6}$$

Combining (2.4) and (2.5) implies that ψ is constant on the boundary $\partial\Omega$. Without loss of generality, we take the boundary condition

$$\psi = 0 \quad \text{on } \partial\Omega \times (0,\infty). \tag{2.7}$$

The boundary value problem (2.6), (2.7) gives the stream function and thus the specific discharge, in terms of the specific weight γ. Conversely, the mass balance equation for the fluid gives the density ρ (and thus the specific weight) in terms of the fluid field $\vec{q}$. According to Bear [5], we have

$$\phi\partial_t\rho + \operatorname{div} \vec{F} = 0 \quad \text{in } \Omega \times (0,\infty), \tag{2.8}$$

where the flux $\vec{F}$ is given by

$$\vec{F} = \vec{q}\,\rho - D \cdot \operatorname{grad} \rho. \tag{2.9}$$

In (2.9), $D = (D_{ij})_{2\times 2}$ is the hydrodynamic dispersion matrix with $D_{ij} : \mathbf{R}^2 \to \mathbf{R}$ given by

$$D_{ij}(q_1, q_2) = \begin{cases} (\alpha_T \mid \vec{q} \mid + \tau\, \phi D_{\mathrm{mol}})\delta_{ij} + (\alpha_L - \alpha_T)\dfrac{q_i q_j}{\mid \vec{q} \mid} & \text{if } (q_1, q_2) \neq 0, \\ \tau\phi D_{\mathrm{mol}}\delta_{ij} & \text{if } (q_1, q_2) = 0. \end{cases} \tag{2.10}$$

Here $\alpha_L, \alpha_T, D_{\mathrm{mol}}$ and τ are positive constants: α_L is the longitudinal and α_T is the the transversal dispersion length ($\alpha_T < \alpha_L$), D_{mol} is the molecular diffusion coefficient and the constant τ describes the tortuosity of the porous medium. Further, $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^2$ and δ_{ij} the Kronecker symbol.

In order to determine ρ (or γ) from (2.8) we have to specify boundary and initial conditions. We consider the no-flux condition

$$\vec{F} \cdot \vec{\nu} = 0 \quad \text{on } \partial\Omega \times (0,\infty), \tag{2.11}$$

and initially

$$\rho(\cdot, 0) = \rho_0(\cdot) \quad \text{on } \Omega. \tag{2.12}$$

Next we rescale the equations into a dimensionless form.
Setting

$$\begin{aligned} x_1 &:= x_1/H, \\ x_2 &:= x_2/H, \\ t &:= t\frac{\kappa}{\mu}(\gamma_s - \gamma_f)/(H\phi), \\ u &:= (\gamma - \gamma_f)/(\gamma_s - \gamma_f), \\ v &:= \psi/(\frac{\kappa}{\mu}(\gamma_s - \gamma_f)H), \\ \Omega &:= (-L/H, L/H) \times (0,1), \end{aligned}$$

we find for u, v the elliptic-parabolic system

$$(E)\begin{cases} -\Delta v = \partial_1 u & \text{in } \Omega \times (0,\infty), \\ v = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

$$(P)\begin{cases} \partial_t u + \operatorname{div}\vec{F} = 0 & \text{in } \Omega \times (0,\infty), \\ \vec{F}\cdot\vec{\nu} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot,0) = u_0(\cdot) & \text{on } \Omega. \end{cases}$$

Here we have

$$\vec{F} = \vec{q}\,u - D\cdot \operatorname{grad} u,$$
$$\vec{q} = \operatorname{curl} v,$$
$$D = (D_{ij})$$

with

$$D_{ij}(q_1,q_2) = \begin{cases} (a\mid\vec{q}\mid + m)\delta_{ij} + (b-a)\dfrac{q_iq_j}{\mid\vec{q}\mid} & \text{if } (q_1,q_2)\neq 0, \\ m\delta_{ij} & \text{if } (q_1,q_2) = 0, \end{cases} \tag{2.13}$$

where $a = \alpha_T/H$, $b = \alpha_L/H$ and $m = \phi D_{\text{mol}}\tau/[\frac{\kappa}{\mu}(\gamma_s - \gamma_f)H]$.

The dispersion matrix D satisfies the following.

PROPOSITION 2.1. *Let $D = (D_{ij})$ be given by* (2.13). *Then*

(i) *D is uniformly positive definite on $\mathbf{R}^2$, i.e., there exists $\mu > 0$ such that*

$$\sum_{i,j=1}^{2} D_{ij}(q_1,q_2)\xi^i\xi^j \geq \mu|\xi|^2 \quad \forall\xi = (\xi^1,\xi^2), (q_1,q_2)\in\mathbf{R}^2;$$

(ii) *D_{ij} is uniformly Lipschitz continuous.*

Proof. The proof of (i) is immediate. To prove (ii) we have to show that the functions $f_{ij} : \mathbf{R}^2 \to \mathbf{R}$, defined by

$$f_{ij}(x) = \begin{cases} \dfrac{x_ix_j}{|x|} & \text{if } x \neq (0,0), \\ 0 & \text{if } x = (0,0), \end{cases}$$

are uniformly Lipschitz continuous. A straightforward computation shows that there exists a constant $L > 0$ such that

$$|\nabla f_{ij}(x)| \leq L \quad \forall x \in \mathbf{R}^2\backslash\{0\}$$

and

$$|f_{ij}(x) - f_{ij}(0)| \leq |x - 0| \quad \forall x \in \mathbf{R}^2.$$

This implies the Lipschitz continuity for f_{ij} and thus for D_{ij}. □

The purpose of this paper is to study the elliptic-parabolic system $(E), (P)$. We do this in two steps. First, in §3 we consider the case, where

$$a = b = 0 \quad \text{and} \quad m = 1.$$

This situation describes the mixing of fresh and salt groundwater with dominant molecular diffusion. It implies $D_{ij} = \delta_{ij}$ which means that the problem is of semilinear type. In §4, we consider the full problem, where

$$0 < a < b < \infty \quad \text{and} \quad m > 0.$$

In this case the mixing is due to mechanical dispersion and molecular diffusion. It implies that D is velocity dependent which means that the problem is of quasilinear type.

3. The semilinear case.

3.1. The abstract setting. In this section we consider the case where the dispersion matrix D is independent of the velocity $\vec{q}$. This can be achieved by setting $a = b = 0$ in (2.13). For simplicity, we also set $m = 1$, which implies that $D_{ij} = \delta_{ij}$. Noting that $\vec{q} \cdot \vec{\nu} = 0$ on $\partial\Omega$, we arrive at the problem

$$(E)\begin{cases} -\Delta v = \partial_1 u & \text{in } \Omega \times (0,\infty), \\ v = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}$$

$$(P')\begin{cases} \partial_t u - \Delta u + \operatorname{grad} u \cdot \operatorname{curl} v = 0 & \text{in } \Omega \times (0,\infty), \\ \dfrac{\partial u}{\partial \vec{\nu}} = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(\cdot, 0) = u_0(\cdot) & \text{in } \Omega. \end{cases}$$

Throughout this section we suppose that Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary $\partial\Omega$.

In order to formulate problem (E), (P') into an abstract form, we need to introduce some operators and Banach spaces.

Throughout this paper all vector spaces are over $\mathbf{R}$. If we use complex quantities (for example, in connection with spectral theory), it is always understood that we work with the natural complexifications (of spaces and operators). Thus by $\rho(A)$, the resolvent set of a linear operator with domain $D(A)$ and range $R(A)$, we mean always the resolvent set of its complexifications.

Let $p \in (2, \infty)$. By inverting (E) we obtain the operator (see the appendix)

$$E_p : D(E_p) = W^{1,p}(\Omega) \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

given by

$$E_p v = (-\Delta)^{-1} \partial_1 v.$$

Then we define

$$M_p(u) = (\partial_1 E_p u)\partial_2 u - (\partial_1 u)\partial_2 E_p u - u$$

for $u \in W^{1,p}(\Omega)$. Furthermore, we define operator A_p by

$$D(A_p) = \left\{ u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \vec{\nu}} = 0 \right\},$$

$$A_p : D(A_p) \to L^p(\Omega)$$

with

$$A_p u = -\Delta u + u.$$

Observe that in the definition of A_p, due to the imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, the boundary condition $\partial u/\partial \vec{\nu} = 0$ is satisfied in the classical sense. By using the operators introduced above, Problem (E), (P') can be formulated as

$$(CP)\begin{cases} u' + A_p u + M_p(u) = 0 & \text{for } t \in (0,\infty), \\ u(0) = u_0. \end{cases}$$

Here u' denotes the derivative of u with respect to t.

It is known that $-A_p$ generates an analytic semigroup on $L^p(\Omega)$. We shall show that M_p is a locally Lipschitz perturbation (in an appropriate sense) of A_p. Then we can apply abstract results for proving existence of solutions of (CP).

We recall the following results.

Let $\Sigma_\omega := \{\lambda \in \mathbf{C} : \mathrm{Re}\lambda \geq \omega\}$ for $\omega \in \mathbf{R}$. Furthermore, let X be a Banach space with norm $\|\cdot\|$, and let A be a given linear operator satisfying

(A1) A is densely defined and closed;
(A2) $\Sigma_0 \subset \rho(-A)$, where $\rho(-A)$ is the resolvent set of $-A$;
(A3) There exists a constant $M > 0$, such that

$$\|(\lambda + A)^{-1}\| \leq \frac{M}{1+|\lambda|} \quad \forall \lambda \in \Sigma_0.$$

The fractional powers A^α of A are well defined for $0 < \alpha \leq 1$, and A^α is a closed linear operator whose domain $D(A^\alpha) \supset D(A)$. In this section we denote by X_α the Banach space obtained by endowing $D(A^\alpha)$ with the graph norm of A^α. Since $0 \in \rho(A)$, A^α is invertible and the norm of X_α is equivalent to $\|u\|_\alpha := \|A^\alpha u\|$ for $u \in D(A^\alpha)$. Also, for $0 < \beta < \alpha \leq 1$, $X_\alpha \hookrightarrow X_\beta$ with continuous imbedding.

Concerning the solvability of semilinear evolution equations of the form

$$u' + Au + M(u) = 0 \tag{3.1}$$

with initial value $u(0) = \varphi$, under the assumptions (A1)–(A3), we recall the following result (see von Wahl [20]).

THEOREM 3.1. *Let $0 \leq \beta < \alpha < 1$, and let $M : X_\alpha \to X$ satisfy $M(0) = 0$ and*

$$\|M(u) - M(v)\| \leq g(\|u\|_\beta + \|v\|_\beta)[\|u-v\|_\alpha + \|u-v\|_\beta(\|u\|_\alpha + \|v\|_\alpha + 1)]$$

for some continuous function g: $\mathbf{R}^+ \to \mathbf{R}^+$ and for all u, $v \in X_\alpha$. For $\varphi \in X_\beta$, there exists a $T = T(\varphi) \in (0,\infty]$ such that

(i) *there is one and only one mapping u: $[0,T) \to X$ fulfilling*

$$u \in C([0,T), X_\beta) \cap C((0,T), X_\alpha),$$

and

$$\sup_{0<t\leq T'} \|t^{\alpha-\beta} A^\alpha u(t)\| < \infty$$

for all $0 < T' < T$;

(ii)

$$u(t) = e^{-tA}\varphi - \int_0^t e^{-(t-s)A} M(u(s))ds$$

for $t \in (0,T)$;

(iii)

$$u(0) = \varphi;$$

(iv) *if* $T < \infty$, *then*

$$\lim_{t \uparrow T} \|u(t)\|_\beta = \infty.$$

Moreover, on $(0,T)$, u *fulfills* (3.1) *in the sense that* $u \in C^1((0,T),X)$, $u(t) \in D(A)$ *for* $t \in (0,T)$ *and* $Au(\cdot) \in C((0,T),X)$.

About the solution obtained in Theorem 3.1 we also have the following (see Henry [9]).

PROPOSITION 3.2. *Under the assumptions of Theorem* 3.1, *the solution* u *satisfies*

$$u'(t) \in X_\gamma$$

for $t \in (0,T)$ *and for any* $\gamma \in (0,1)$.

3.2. The existence results. It follows from Agmon [1] that A_p satisfies (A1)–(A3). Moreover, we have the imbedding properties (see Henry [9]):

PROPOSITION 3.3. (i) $D(A_p^\alpha) \hookrightarrow W^{1,p}(\Omega)$ *for* $\alpha \in (\frac{1}{2},1)$;

(ii) $D(A_p^\alpha) \hookrightarrow W^{1,\infty}(\Omega)$ *for* $\alpha \in (\frac{1}{2}+\frac{1}{p},1)$.

We use Theorem 3.1 to obtain the existence for (CP). In this application we take $X = L^p(\Omega)$ with norm $\|\cdot\|_p$, X_α ($\alpha \in (0,1)$) the Banach space induced by the operator A_p and $\beta = 0$ with $\|\cdot\|_\beta = \|\cdot\|_p$.

PROPOSITION 3.4. *Let* $\alpha \in (\frac{1}{2}+\frac{1}{p},1)$. *Then there exists a constant* $C \geq 1$ *such that*

$$\|M_p(u) - M_p(v)\|_p \leq C[\|u-v\|_\alpha \|u\|_p + \|u-v\|_p(\|v\|_\alpha + 1)]$$

for all $u, v \in D(A_p^\alpha)$.

Proof. By the definition of M_p we have

$$\begin{aligned} \|M_p(u) - M_p(v)\|_p \leq \|u-v\|_p &+ \|\text{grad}\,(u-v)\cdot \text{curl}\, E_p u\|_p \\ &+ \|\text{grad}\, v \cdot \text{curl}\, E_p(u-v)\|_p. \end{aligned} \tag{3.2}$$

From the Appendix and Proposition 3.3, we obtain:

$$\|\text{curl}\, E_p u\|_p \leq C\|u\|_p \tag{3.3}$$

and

$$\|E_p u\|_{1,\infty} \leq C\|u\|_\alpha \tag{3.4}$$

for all $u \in D(A_p^\alpha)$ and for some constant $C \geq 1$. Combining (3.2), (3.3), and (3.4), the desired estimate follows. □

Combining Theorem 3.1 and Proposition 3.4, we obtain that, for every $u_0 \in L^p(\Omega)$, there exists a solution u of (CP) on some interval $[0,T)$.

According to Theorem 3.1 (iv), the global existence of the solution follows if we can show that

$$\overline{\lim_{t\uparrow T}}\|u(t)\|_p < \infty.$$

By the imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, we can define $u(x,t) = u(t)(x)$ pointwise on $\Omega \times (0,T)$. Further, we have the following.

PROPOSITION 3.5. *Let $u_0 \in L^p(\Omega)$ and u be the corresponding solution of (CP) on $[0,T)$ in the sense of Theorem 3.1. Let $J \in C^2(\mathbf{R}, \mathbf{R}^+)$ be a convex function; then we have*

$$\int_\Omega J(u(x,t))dx \le \int_\Omega J(u(x,s))dx$$

for any $0 < s \le t < T$.

Proof. Note that $J(u)$ is well defined due to the imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$.

Multiplying the differential equation in (P') by $J'(u)$ and integrating the result over Ω gives

$$\frac{d}{dt}\int_\Omega J(u)dx = \int_\Omega J'(u)\Delta u\, dx + \int_\Omega J'(u)\text{grad}\, u \cdot \text{curl}\, v\, dx$$

for $0 < t < T$. Using Green's formula we know that

$$\int_\Omega J'(u)\Delta u\, dx = -\int_\Omega J''(u)[(\partial_1 u)^2 + (\partial_2 u)^2]dx \le 0$$

and

$$\int_\Omega J'(u)\text{grad}\, u \cdot \text{curl}\, v\, dx = \int_{\partial\Omega} J(u)\frac{\partial v}{\partial \vec{\tau}}ds = 0,$$

where $\vec{\tau}$ is the tangential unit vector along $\partial\Omega$. Therefore,

$$\frac{d}{dt}\int_\Omega J(u)dx \le 0,$$

which implies the required inequality. □

COROLLARY 3.6. *Let $u_0 \in L^p(\Omega)$ with $p \in (2,\infty]$ and u be the solution of (CP) on $[0,T)$ in the sense of Theorem 3.1. For any $q \in [2,p]$ we have*

$$\|u(t)\|_q \le \|u_0\|_q \tag{3.5}$$

for $t \in [0,T)$.

Proof. This estimate follows directly from Proposition 3.5 by taking $J(s) = |s|^q$ and from the fact that $u \in C([0,T), X)$ for $p < \infty$. We obtain the estimate (3.5) for $p = q = \infty$ by using a limit argument. □

Using Theorem 3.1, Proposition 3.4, and Corollary 3.6, we obtain the following existence result for (CP).

THEOREM 3.7. *Let $\alpha \in (\frac{1}{2}+\frac{1}{p}, 1)$ and $u_0 \in L^p(\Omega)$. Then the initial value problem (CP) has a unique global solution $u(\cdot)$, i.e.,*

$$u \in C([0,\infty), X) \cap C((0,\infty), X_\alpha),$$

$$\sup_{0<t\le 1} \|t^\alpha A_p^\alpha u(t)\| < \infty,$$

$$u(t) = e^{-tA_p}u_0 - \int_0^t e^{-(t-s)A_p} M(u(s))ds$$

for $t \in (0,\infty)$, *and*

$$u(0) = u_0.$$

Moreover, u *fulfills the equation* $u' + A_p u + M_p(u) = 0$ *on* $(0,\infty)$ *in the sense that* $u \in C^1((0,\infty), X), u(t) \in D(A_p)$ *for* $t \in (0,\infty)$ *and* $A_p u \in C((0,\infty), X)$.

3.3. Regularity and asymptotic properties. In the preceding section we obtained the solution of the abstract problem (CP). Here we consider the original system (E), (P'). Let u be the solution of (CP). Then we have

$$u(t) \in W^{2,p}(\Omega), v(t) = E_p u(t) \in W^{2,p}(\Omega) \quad \forall t \in (0,\infty).$$

By the imbedding $W^{2,p}(\Omega) \hookrightarrow C^1(\overline{\Omega})$, we can define $u(x,t) = u(t)(x)$ and $v(x,t) = E_p u(t)(x)$ for $(x,t) \in \overline{\Omega} \times (0,\infty)$. The pair (u,v) satisfies the following

THEOREM 3.8. *Let* $\theta \in (0, 1 - \frac{2}{p})$, $\partial\Omega \in C^{2+\theta}$ *and suppose* $u_0 \in L^p(\Omega)$. *Let* u,v *be defined as above. Then* (u,v) *is the unique classical solution of the system* (E), (P'), *which satisfies*

(i) $u(\cdot,t) \in C^{2+\theta}(\overline{\Omega})$, $\partial_t u(\cdot,t) \in C^\theta(\overline{\Omega})$, $\forall t \in (0,\infty)$;
(ii) $u(x,\cdot) \in C^{1+\frac{\theta}{2}}(0,\infty)$ $\forall x \in \overline{\Omega}$;
(iii) $v(\cdot,t) \in C^{2+\theta}(\overline{\Omega})$, $\forall t \in (0,\infty)$.

Proof. (i) By the imbedding $W^{2,p}(\Omega) \hookrightarrow C^{1+\theta}(\overline{\Omega})$, we have

$$u(\cdot,t), v(\cdot,t) \in C^{1+\theta}(\overline{\Omega}) \quad \forall t \in (0,\infty).$$

Using Propositions 3.2 and 3.3, we also have

$$\partial_t u(\cdot,t) \in C^\theta(\overline{\Omega}) \quad \forall t \in (0,\infty).$$

Let $t_0 \in (0,\infty)$ be fixed. The regularity for u and v implies that

$$F(\cdot) = -\text{grad}\, u(\cdot,t_0) \cdot \text{curl}\, v(\cdot,t_0) + u(\cdot,t_0) - \partial_t u(\cdot,t_0)$$

satisfies

$$F(\cdot) \in C^\theta(\overline{\Omega}).$$

Next, consider the problem

$$\begin{cases} -\Delta w + w = F & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \vec{\nu}} = 0 & \text{on } \partial\Omega. \end{cases}$$

By Gilbarg and Trudinger [8] this problem has a unique solution $w \in C^{2+\theta}(\overline{\Omega})$. A standard argument gives $w(\cdot) = u(\cdot,t_0)$, hence $u(\cdot,t_0) \in C^{2+\theta}(\overline{\Omega})$.

(ii) This is a direct result of (i) and Ladyzenskaja et al. [10, Thm. 5.3].

(iii) The regularity for v is a direct result of the Dirichlet problem (E). □

Remark. If the boundary $\partial\Omega$ is smooth, then the solution is smooth in $\overline{\Omega}\times(0,\infty)$. This follows from Theorem 3.8 together with a bootstrapping argument.

Let (u,v) be the solution of (E), (P'), a straightforward computation shows

$$\overline{u} := \frac{1}{|\Omega|}\int_\Omega u(x,t)dx = \frac{1}{|\Omega|}\int_\Omega u_0(x)dx$$

for all $t\in(0,\infty)$. Here $|\Omega|$ denotes the measure of Ω.

LEMMA 3.9. *We have*

$$\lim_{t\to\infty}\|u(\cdot,t)-\overline{u}\|_2 = 0.$$

Proof. Taking $J(s)=s^2$ in the proof of Propositon 3.5, we obtain

$$\frac{d}{dt}\|u(\cdot,t)-\overline{u}\|_2^2 \le -(\|\partial_1 u\|_2^2+\|\partial_2 u\|_2^2).$$

We estimate the right-hand side by Poincaré's inequality. This gives

$$\|u(\cdot,t)-\overline{u}\|_2^2 \le K(\|\partial_1 u\|_2^2+\|\partial_2 u\|_2^2)$$

for some constant $K>0$. Therefore,

$$\frac{d}{dt}\|u(\cdot,t)-\overline{u}\|_2^2 \le -\frac{1}{K}\|u(\cdot,t)-\overline{u}\|_2^2,$$

which can be integrated to yield

$$\|u(\cdot,t)-\overline{u}\|_2^2 \le e^{-t/K}\|u_0(\cdot)-\overline{u}\|_2^2, \tag{3.6}$$

for all $t\ge 0$. □

We now consider the asymptotic behavior of the solution in the sup-norm.

THEOREM 3.10. *Let $u_0\in L^p(\Omega)$ for any $p\in(2,\infty]$. Then*

$$\lim_{t\to\infty}\|u(\cdot,t)-\overline{u}\|_\infty = 0.$$

Proof. We put

$$\omega = \{U\in C(\overline{\Omega}) : \exists\{t_m\}, \text{ s.t. } \lim_{m\to\infty} t_m=\infty \text{ and } \lim_{m\to\infty}\|u(\cdot,t_m)-U(\cdot)\|_\infty = 0\}$$

and

$$F=\{u(\cdot,t) : t\in(0,\infty)\}.$$

From Corollary 3.6 and Theorem 3.8, it follows that F is a uniformly bounded and equicontinuous subset in $C(\overline{\Omega})$. Therefore, ω is nonempty. Next we show that ω contains only one single point. Let $U\in\omega$. Then there exists a sequence $\{t_m\}$ with

$$\lim_{m\to\infty} t_m = \infty$$

and

$$\lim_{m\to\infty}\|u(\cdot,t_m)-U(\cdot)\|_\infty = 0.$$

This implies

$$u(x,t_m) \to U(x)$$

as $m \to \infty$, uniformly in $x \in \overline{\Omega}$.

On the other hand, we obtain from Lemma 3.9 that

$$u(x,t_m) \to \overline{u}$$

as $m \to \infty$, for almost all $x \in \Omega$. Thus

$$U(x) = \overline{u} \quad \forall x \in \overline{\Omega},$$

which completes the proof. □

4. The quasilinear case.

4.1. The abstract setting. In this section we study Problem (E), (P). As in §3, we treat this system as an abstract evolution equation in a suitably chosen Banach space. In this part we collect some results on quasilinear evolution equations.

Let $\overline{E} = (E_0, E_1)$ be a pair of Banach spaces with E_1 continuously and densely imbedded in E_0. We denote by $\mathcal{H}(\overline{E})$ the set of all $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as a linear operator on E_0, is the infinitesimal generator of a strongly continuous analytic semigroup on E_0. For $\theta \in (0,1)$, let E_θ be the complex interpolation space $[\overline{E}]_\theta$, and $\|\cdot\|_\theta$ be the norm on E_θ. (The notation here is different from the previous section.)

Let $T > 0$ be fixed. We assume (Q) $\beta \in (0,1), V \subset E_\beta$ is open and $A \in C^{1-}(V, \mathcal{H}(\bar{E}))$, i.e., A is locally Lipschitz continuous.

Under these assumptions, we consider the following quasilinear Cauchy problem

$$(QCP)_{(u_0)} : \begin{cases} \dot{u}(t) + A(u(t))u(t) = 0, \ 0 < t \le T, \\ u(0) = u_0, \end{cases}$$

where $u_0 \in V$.

Let $\tau \in (0, T]$; u is called a solution of $(QCP)_{(u_0)}$ on $[0,\tau]$ if the following conditions are satisfied:

(i) $u \in C([0,\tau], V) \cap C((0,\tau], E_1) \cap C^1((0,\tau], E_0)$,
(ii) $\dot{u}(t) + A(u(t))u(t) = 0, \forall t \in (0,\tau]$,
(iii) $u(0) = u_0$.

A solution u is *maximal* if there does not exist a solution of $(QCP)_{(u_0)}$ which is a proper extension of u. In this case the interval of existence is called the *maximal interval of existence.*

The following fundamental theorem can be found in Amann [2] (see also Sobolevskii [15]).

THEOREM 4.1. *Suppose that* $0 < \beta < \alpha < 1$, *and* $u_0 \in V_\alpha := E_\alpha \cap V$. *Furthermore, suppose that the assumption* (Q) *holds. Then there exists* $\tau > 0$, *such that* $(QCP)_{u_0}$ *has a unique solution* $u(\cdot)$ *on* $[0,\tau]$, *satisfying* $u \in C([0,\tau], V_\alpha)$. *Moreover, the maximal interval of existence is open in* $[0,T]$. □

4.2. Local existence. Again we put the system into an abstract form.

Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with smooth boundary $\partial\Omega$. For $p \in (1,\infty)$ and $r \in (-\infty,\infty)$, we denote by $H_p^r(\Omega)$ the so-called Lebesgue spaces (see Triebel [17] or Bergh and Löfström [6]). In this section the norm on $H_p^r(\Omega)$ is denoted by $\|\cdot\|_{r,p}$.

It should be observed that $H_p^m(\Omega) = W^{m,p}(\Omega)$ for integer m. Moreover, we have the interpolation property

$$[H_{p_0}^{s_0}(\Omega), H_{p_1}^{s_1}(\Omega)]_\theta = H_p^s(\Omega) \tag{4.1}$$

for $s_0, s_1 \in \mathbf{R}$, $p_0, p_1 \in (1,\infty)$ with $\frac{1}{p} = (1-\theta)/p_0 + \theta/p_1$ and $s = (1-\theta)s_0 + \theta s_1$.

Let $a_{jk} := D_{jk} \circ Q$ and $a_j = -Q_j$ for $j,k = 1,2$ (see Appendix). Then problem $(E),(P)$ can be formulated as

$$(QCP)\begin{cases} \partial_t u - \partial_j(a_{jk}(u)\partial_k u + a_j(u)u) = 0 & \text{in } \Omega \times (0,T], \\ \nu^j a_{jk}(u)\partial_k u + a_j(u)\nu^j u = 0 & \text{on } \partial\Omega \times (0,T], \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases}$$

Here $T > 0$ and $\vec{\nu} = (\nu^1, \nu^2)$. Note that in this section the summation convention is used and the indices run from 1 to 2.

We use Theorem 4.1 to obtain the existence result for Problem (QCP). In this application we take

$$E_0 = (H_{p'}^1(\Omega))'$$

and

$$E_1 = H_p^1(\Omega),$$

where $p \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$. It should be observed that

$$E_\theta = [E_0, E_1]_\theta \hookrightarrow L^p(\Omega) \tag{4.2}$$

for $\theta \in [\frac{1}{2}, 1]$; see Amann [4, Thm. 3.3].

Let $\mathcal{M}(\Omega) \subset C(\overline{\Omega})^4 \times C(\overline{\Omega})^2$ be the subset whose elements $m(\cdot) = (b_{jk}(\cdot), b_j(\cdot))$ are chosen such that $(b_{jk}(\cdot))_{2\times 2}$ is uniformly positive definite on $\overline{\Omega}$. Assume we set

$$\langle f, g\rangle = \int_\Omega f(x)g(x)dx$$

for $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$. With this notation we define

$$a(m)(v,u) = \langle \partial_j v, b_{jk}\partial_k u + b_j u\rangle$$

for $v \in H_{p'}^1(\Omega)$, $u \in H_p^1(\Omega)$, and $m \in \mathcal{M}(\Omega)$.

Furthermore, given $m \in \mathcal{M}(\Omega)$, we define the operator

$$A(m) : E_1 \to E_0$$

such that

$$\langle A(m)u, v\rangle = a(m)(v,u) \quad \forall v \in H_{p'}^1(\Omega).$$

Then we have the following generation theorem; see Amann [3] or Lunardi and Vespri [11].

THEOREM 4.2.

$$[m \to A(m)] \in C^{1-}(\mathcal{M}(\Omega), \mathcal{H}(\overline{E})).$$

For $p \in (2, \infty)$ and $r > \frac{2}{p}$, we have

$$H_p^r(\Omega) \hookrightarrow C(\overline{\Omega}), \tag{4.3}$$

Therefore, the coefficients $a_{jk}(u)$, $a_j(u)$ are defined pointwise on $\overline{\Omega}$ for each $u \in H_p^r(\Omega)$. Consequently,

$$m(u)(\cdot) := (a_{jk}(u)(\cdot), a_j(u)(\cdot))$$

is well defined on $\overline{\Omega}$. For m we also have the following.

LEMMA 4.3. *Let $p \in (2, \infty)$ and $1 \geq r > \frac{2}{p}$. Then $[u \to m(u)] : H_p^r(\Omega) \to \mathcal{M}(\Omega)$ is uniformly Lipschitz continuous.*

Proof. From the appendix we have

$$Q_i \in \mathcal{L}(H_p^r(\Omega)). \tag{4.4}$$

We combine this with imbedding (4.3) and Proposition 2.1 to obtain

$$m(u) \in \mathcal{M}(\Omega).$$

On the other hand, by Proposition 2.1, (4.3), and (4.4), there exists a constant $C > 0$ such that

$$\|a_{jk}(u) - a_{jk}(v)\|_{C(\overline{\Omega})} \leq C\|u - v\|_{r,p}$$

and

$$\|a_j(u) - a_j(v)\|_{C(\overline{\Omega})} \leq C\|u - v\|_{r,p}$$

for any $u, v \in H_p^r(\Omega)$ and for $j, k = 1, 2$. This completes the proof. $\square$

Let us put $A(u) := A(m(u)(\cdot))$ We are now in a position to prove the main existence result.

THEOREM 4.4. *Let $p \in (2, \infty)$ and $\frac{1}{2} + \frac{1}{p} < \beta < \alpha < 1$. For every $u_0 \in E_\alpha$, there exists a $\tau > 0$ such that*

$$\begin{cases} \dot{u}(t) + A(u(t))u(t) = 0, & 0 < t \leq \tau, \\ u(0) = u_0, \end{cases}$$

has a unique solution $u(\cdot)$ on $[0, \tau]$, i.e.,

(i) $u \in C([0, \tau], E_\alpha) \cap C((0, \tau], E_1) \cap C^1((0, \tau], E_0)$,
(ii) $\dot{u}(t) + A(u(t))u(t) = 0, \forall t \in (0, \tau]$,
(iii) $u(0) = u_0$.

Proof. For $\beta = \frac{1}{2} + \frac{r}{2} \in (0, 1)$ we have

$$E_\beta \hookrightarrow [E_{\frac{1}{2}}, E_1]_r,$$

by the reiteration theorem (see Triebel [17] or Bergh and Löfström [6]). Using (4.2), we have

$$E_\beta \hookrightarrow [L^p(\Omega), H_p^1(\Omega)]_r = H_p^r(\Omega)$$

with $r \in (0, 1)$. Finally, if $1 > \beta > \frac{1}{2} + \frac{1}{p}$, then $1 > r > \frac{2}{p}$ and $H_p^r(\Omega) \hookrightarrow C(\overline{\Omega})$. From Lemma 4.3 we know $[u \to m(u)]$ is uniformly Lipschitz continuous from E_β to $\mathcal{M}(\Omega)$.

On the other hand, it follows from Theorem 4.2 that

$$[m \to A(m)] \in C^{1-}(\mathcal{M}(\Omega), \mathcal{H}(\overline{E})).$$

Hence

$$[u \to A(u)] \in C^{1-}(E_\beta, \mathcal{H}(\overline{E})).$$

The conclusion then follows directly from Theorem 4.1. □

4.3. Some properties of the weak solution. Up to now we have obtained a local solution for Problem (QCP) in $H^1_p(\Omega)$-sense. We now come back to the original system.

Let $\tau > 0$, $u_0 \in E_\alpha$ for some $\alpha \in (\frac{1}{2} + \frac{1}{p}, 1)$ and we suppose $u \in C^1((0,\tau], E_0) \cap C((0,\tau], E_1)$ is the weak solution mentioned in Theorem 4.4. By the appendix we know that $v(t) = K \circ \partial_1 u(t) \in H^2_p(\Omega)$. Using the imbedding $H^1_p(\Omega) \hookrightarrow C(\overline{\Omega})$, we can define

$$u(x,t) := u(t)(x)$$

and

$$v(x,t) := v(t)(x)$$

pointwise on $\overline{\Omega} \times (0,\tau]$. Obviously, we have

$$\partial_t u(x,t) = \dot{u}(t) \in L^p(\Omega).$$

From Theorem 4.4 we know that problem (P) is satisfied in the following sense:

$$\frac{d}{dt}\int_\Omega u(x,t) f(x)dx - \int_\Omega \vec{F}(x,t) \text{grad } f(x) dx = 0 \tag{4.5}$$

for all $f \in H^1_{p'}(\Omega)$ and $t \in (0,\tau]$. Moreover, $u(x,0) = u_0$.

As in the semilinear case we can prove the following.

THEOREM 4.5. *Let (u,v) be the weak solution of $(E),(P)$ as constructed above. Then*

$$\|u(\cdot,t)\|_p \le \|u_0\|_p$$

for all $t \in (0,\tau]$.

Proof. Using the facts

$$u(\cdot,t) \in C(\overline{\Omega})$$

and

$$L^p(\Omega) \hookrightarrow L^{p'}(\Omega),$$

we obtain immediately that $f := p|u|^{p-1}\text{sgn } u \in H^1_{p'}(\Omega)$.

Substitution into (4.5) gives

$$\frac{d}{dt}\|u(\cdot,t)\|_p^p = \int_\Omega (u \text{ curl } v - D \cdot \text{grad } u) \cdot p(p-1)|u|^{p-2} \text{grad } u \, dx \tag{4.6}$$

Since the matrix D is positive definite,

$$-\int_\Omega (D\cdot \operatorname{grad} u)\cdot p(p-1)|u|^{p-2}\operatorname{grad} u\, dx \le 0.$$

On the other hand, by using the Green's formula we have

$$\int_\Omega (u \operatorname{curl} v)\cdot p(p-1)|u|^{p-2}\operatorname{grad} u\, dx = 0.$$

Therefore, the conclusion follows directly from (4.6). □

Final remark. In this paper we assumed Ω to be a bounded domain of $\mathbf{R}^2$ with smooth boundary. On the other hand, the domain in the motivating problem is a rectangle. For such a domain, the same existence results will hold. This is a consequence of the fact that the generation theorems for the operators A_p in §3 and A in §4.2, as well as the proposition in Appendix also hold for such a domain (Vespri [18]).

Appendix. Here we state some results on the Laplace operator with Dirichlet boundary condition, which are related to problem (E).

Let γ denote the trace operator. It is known that the operator $-\Delta$ with Dirichlet boundary condition zero is invertible in $L^p(\Omega)$. We denote this inverse operator by

$$K := (-\Delta \mid \gamma)^{-1}.$$

Further, we introduce operator

$$Q = (Q_1, Q_2) = \operatorname{curl} K\, \partial_1.$$

Let $H_p^r(\Omega)$ be the Lebesgue spaces, with indices $-\infty < r < +\infty$, $1 < p < \infty$.

The operator Q satisfies the following.

PROPOSITION. *Let $r \in [0,1]$ and $1 < p < \infty$. Then*

$$Q_i \in \mathcal{L}(H_p^r(\Omega))$$

for $i = 1, 2$.

Proof. Let $f \in L^p(\Omega)$. We define

$$Fv = \int_\Omega f\partial_1 v\, dx$$

for $v \in W_0^{1,p'}(\Omega)$. Clearly, $F \in (W_0^{1,p'}(\Omega))'$. By the representation theorem in Simader [14, p. 91], we know that there exists $u \in W_0^{1,p}(\Omega)$ such that

$$Fv = \int_\Omega \operatorname{grad} u \cdot \operatorname{grad} v\, dx$$

for $v \in W_0^{1,p'}(\Omega)$. Moreover, there exists a constant C independent of u and f such that

$$\|u\|_{1,p} \le C\|f\|_p.$$

Therefore,

$$Q_i \in \mathcal{L}(L^p(\Omega)).$$

On the other hand, it is well known that

$$Q_i \in \mathcal{L}(H^1_p(\Omega)).$$

By the interpolation property,

$$[H^{s_0}_p(\Omega), H^{s_1}_p(\Omega)]_\theta = H^s_p(\Omega)$$

for $\theta \in [0,1]$ and s_0, s_1, $s \in \mathbf{R}$ with $s = (1-\theta)s_0 + \theta s_1$; the conclusion follows.

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