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Abstract

In this paper we study the L_1 optimal control problem for linear systems. We will show that by allowing the class of controllers to include nonlinear controllers we can make the closed loop L_1 norm strictly smaller then we could do using only linear controllers.

Keywords: L_1 optimal control, Linear systems, Nonlinear controllers

1 Introduction

The L_1 optimal control problem has been studied extensively in the literature. The L_1 problem was originally formulated in [1, 8]. A solution to the problem was presented in [3] (for discrete time systems) and [2] (for continuous time systems). In these papers it became obvious that, when searching for optimal linear controllers for these problems, we need infinite-dimensional (continuous-time) or very high order (discrete-time) compensators. Especially for discrete time systems there is now a good theory available for the design of linear compensators (see e.g. [5]). However, the approach taken in these papers is a method based on linear programming and the method is therefore essentially constrained to linear compensators.
The objective of the current paper is to study whether we can improve by extending the class of compensators to include nonlinear compensators. For H_∞, it is for instance known that this is not possible: the minimum over all stabilizing linear compensators of the closed loop H_∞ norm is equal to the minimum over all stabilizing (possibly) nonlinear compensators (see [6]).
In [7] it has been shown that, although for the L_1 optimal control problem optimal and near optimal linear state feedback compensators are in general dynamic, there always exists static nonlinear compensators which achieve the same or better performance. Moreover, in [4] it was shown that nonlinear controllers which are differentiable in the origin cannot do better than linear controllers. Via an example it was shown in that paper that nonlinear controller can do better for individual disturbances w. But, for this example, the worst-case L_1 norm could not be improved via nonlinear controllers. The objective of this paper is to show that we can achieve smaller L_1 norm if we allow for nonlinear, continuous controllers. This will be shown by means of a very simple static example. Hence the example applies equally well to

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continuous and discrete time. For ease of exposition we will concentrate on continuous time systems only.
The paper has the following structure. In the next section we will give a problem formulation. Then we will present our example amd we conclude with some final remarks.

2 Problem formulation

We will consider systems of the form:

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew, \\ y = C_1 x + D_{11} u + D_{12} w, \\ z = C_2 x + D_{21} u + D_{22} w. \end{cases} \tag{2.1}$$

We will assume that x, u, w, y and z take values in finite-dimensional vector spaces: $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^l$, $y(t) \in \mathbb{R}^q$ and $z(t) \in \mathbb{R}^p$. A special case are static systems where x is absent ($n = 0$) and we just have:

$$\Sigma : \begin{cases} y = D_{11} u + D_{12} w, \\ z = D_{21} u + D_{22} w. \end{cases} \tag{2.2}$$

For a vector in $\mathbb{R}^n$ we define the L_∞-norm by:

$$\|p\|_\infty = \sup_i |p_i| \tag{2.3}$$

where $p = (p_1, p_2, \ldots, p_n)^T$. We define the function space L_∞ as the class of time-signals f for which the norm:

$$\|f\|_\infty = \sup_{t \in \mathbb{R}^+} \|f(t)\|_\infty$$

is finite. This norm will be referred to as the L_∞ norm. We define the L_∞-induced operator norm of an operator $\mathcal{G}$ mapping w to z by:

$$\|\mathcal{G}\|_1 = \sup_{0 \neq w \in L_\infty} \frac{\|\mathcal{G}w\|_\infty}{\|w\|_\infty}$$

This norm is also referred to as the L_1 norm of $\mathcal{G}$. This is due to the fact that for operators $\mathcal{G}$ described by a linear time invariant system of the form:

$$\Sigma : \begin{cases} \dot{x} = Fx + Gw, \quad x(0) = 0, \\ z = Hx + Jw. \end{cases} \tag{2.4}$$

we find that the L_∞-induced operator norm is equal to the L_1 norm of the impulse response which is defined by:

$$\|H\|_1 = \int_0^\infty \|H(t)\| \, dt,$$

where for a matrix $M = \{M_{ij}\}$ we have:

$$\|M\| = \max_j \sum_i |M_{ij}|.$$

Clearly, this interpretation only holds for linear time-invariant systems and in particular does not hold for the closed loop system we obtain by applying a nonlinear controller to (2.1). Nevertheless, since the L_∞-induced operator norm yields a natural extension of the L_1 norm to nonlinear systems, we will often refer to the L_∞-induced operator norm of a nonlinear system as the L_1 norm.
Note that for a static, time-invariant system like (2.2) it first of all is obviously of no use to consider dynamic compensators. Hence we only consider static, time invariant but possibly non-linear controllers. Then the output at time t of the closed loop system is only affected by the disturbance at time t and the input at time t and we obtain a finite dimensional optimization problem. Find a function K from $\mathbb{R}^q$ to $\mathbb{R}^m$ such that $I - D_{11}K$ is invertible and the closed loop operator G_{cl} from $\mathbb{R}^l$ to $\mathbb{R}^p$ defined by:

$$G_{cl} := D_{22} + D_{21}K(I - D_{11}K)^{-1}D_{12}$$

has minimal L_∞-induced operator norm where the L_∞ norm of the input and output vectors are defined by (2.3).

3 Example

In this section we will study the following very simple example:

$$\Sigma : \begin{cases} y = & w, \\ z = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 & 3 & 0 \\ -1.5 & 1.5 & -3 \\ -3 & 0 & 0 \\ 0 & 0 & -3 \end{pmatrix} w. \end{cases} \tag{3.1}$$

We claim that for this static example the minimal achievable L_1-norm is 3. However, the latter can only be achieved via a nonlinear controller. Via a linear controller the minimal achievable L_1-norm is equal to 3.75.

3.1 Linear controllers

We will study the optimal input u for four specific disturbances:

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \quad w_4 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \tag{3.2}$$

Each of these disturbances has norm 1. Hence in order to achieve an L_1-norm equal to α we have to have that for w_i and the corresponding control input u_i we obtain an output z_i with norm less than or equal to α. For w_1, w_2 and w_3 we find that $\|z_i\|_\infty = 3 + \|u_i\|_\infty$. For z_4 we have $\|z_4\|_\infty = \max\{6 - u_4, u_4 - 3\}$. Since we only allow linear controllers and $w_4 = w_1 - w_2 - w_3$ we must have $u_4 = u_1 - u_2 - u_3$. We find that to make $\|z_4\|_\infty < 3.75$ then either u_1, u_2 or u_3 must be larger then 0.75 and hence z_1, z_2 or z_3 must have norm larger than 3.75. This implies that we can never make the L_1 norm less than 3.75 by linear controllers. Moreover, the linear controller

$$u = \begin{pmatrix} 0.75 & -0.75 & 0.75 \end{pmatrix} w$$

yields the closed loop system:

$$z = \begin{pmatrix} 0.75 & 2.25 & 0.75 \\ -0.75 & 0.75 & -2.25 \\ -2.25 & -0.75 & 0.75 \\ 0.75 & -0.75 & -2.25 \end{pmatrix} w$$

and it clearly achieves a closed-loop L_1 norm of 3.75.

3.2 Non-linear controllers

To find an optimal nonlinear controller for the system (3.1) we again look at the four disturbances from (3.2). Since we have $\|z_i\|_\infty = 3 + \|u_i\|_\infty$ it is obvious that we cannot make the L_∞-induced operator norm less than 3. We will now construct an optimal nonlinear controller which achieves an L_1 norm equal to 3. Consider the eight corners of the cube $\|u\|_\infty = 1$ Besides the four corners given by (3.2) we have:

$$w_5 = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}, \quad w_6 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad w_7 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad w_8 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \tag{3.3}$$

We find that the optimal input for each corner is $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, $u_4 = 3$, $u_5 = 0$, $u_6 = 0$, $u_7 = 0$, $u_8 = -3$. This yields an output with norm 3 for each corner. We next extend our function to each face of the cube. Any point on a given face can be written uniquely as the convex combination of its four corners. Hence for any w with $\|w\|_\infty = 1$ we find

$$w = \lambda_1 w_{i_1} + \lambda_2 w_{i_2} + \lambda_3 w_{i_3} + \lambda_4 w_{i_4}$$

for some $\lambda_i \geq 0$ with $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$. We choose the corresponding input as:

$$u = \lambda_1 u_{i_1} + \lambda_2 u_{i_2} + \lambda_3 u_{i_3} + \lambda_4 u_{i_4}$$

and the resulting output is

$$z = \lambda_1 z_{i_1} + \lambda_2 z_{i_2} + \lambda_3 z_{i_3} + \lambda_4 z_{i_4}$$

Finally we note that

$$\|z\|_\infty \leq \lambda_1 \|z_{i_1}\|_\infty + \lambda_2 \|z_{i_2}\|_\infty + \lambda_3 \|z_{i_3}\|_\infty + \lambda_4 \|z_{i_4}\|_\infty = 3 = 3\|w\|_\infty$$

In this way we have defined a continuous function $\bar{f}$ on the cube $\|w\|_\infty = 1$. We can extend this function to the whole $\mathbb{R}^3$ by:

$$f(w) = \bar{f}\left(\frac{w}{\|w\|_\infty}\right)$$

and the closed loop system resulting from $u = f(w)$ can be easily checked to have L_∞-induced operator norm equal to 3. We have already seen that this feedback is therefore optimal.

4 Conclusion

In previous papers the first step was to use the Youla parameterization to bring the closed loop operator in the form $T_1 + T_2 Q T_3$ where Q is the design variable which should be a stable system and which determines uniquely the corresponding controller. In [4] it has been shown that for $T_2 = I$ nonlinear controllers can not achieve a smaller L_1-norm. This paper gives an example where $T_3 = I$ and we can do better by nonlinear controllers. $T_2 = I$ is connected to estimation problems while $T_3 = I$ is connected to control problems. This shows a clear lack of duality and the best we can hope for is a kind of separation structure for L_1 controllers which will be of the form of a linear estimator interconnected to a nonlinear static state feedback. Nonlinear static state feedbacks were already studied in [7].
This paper basically only tells us that we should study nonlinear compensators if we want to obtain optimal controllers.

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