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# On the application of Rouché's theorem in queueing theory

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## Abstract

For queueing models that can be analyzed as (embedded) Markov chains, many results are presented in terms of the probability generating function (PGF) of the stationary queue length distribution. Queueing models that belong to this category are bulk service queues,  $M/G/1$  and  $G/M/1$ -type queues, and discrete or discrete-time queues. The determination of the PGF typically requires a fixed number of complex-valued zeros on and within the unit circle of some analytic function. Rouché's theorem can be used to prove the existence of these zeros and fulfills as such a prominent role in queueing theory. For most queueing models the analytic function of interest is of the type  $z^s - A(z)$ , where  $A(z)$  is the PGF of a discrete random variable. The standard application of Rouché's theorem requires that  $A(z)$  has a radius of convergence strictly larger than one. However, in some applications this is not true.

In this note we present an elementary proof of the existence of the zeros for  $z^s - A(z)$  that includes functions  $A(z)$  with a radius of convergence of one. The proof is based on applying the classical argument principle to a truncation of the series  $A(z)$ .

**Keywords:** queueing theory, zeros of an analytic function, roots, Rouché's theorem, argument principle, uniform convergence.

**AMS 2000 Subject Classification:** 30C20, 30E20, 40A30, 60K25.

# 1 Introduction

For queueing models that can be analyzed as (embedded) Markov chains, many results are presented in terms of the probability generating function (PGF) of the stationary queue length distribution. Queueing models that belong to this category are bulk service queues,  $M/G/1$  and  $G/M/1$ -type queues, and discrete or discrete-time queues. The determination of the PGF typically requires a fixed number of complex-valued zeros on and within the unit circle of some analytic function.

In 1932, Crommelin [7] was the first to use the technique of deriving a PGF in terms of zeros. Crommelin obtained the PGF of the stationary queue length in the  $M/D/s$  queue that was expressed in terms of the  $s$  zeros on and within the unit circle of the function  $z^s - \exp(\lambda(z-1))$  with  $\lambda < s$ . Since then, Crommelin's technique or a similar generating function technique has been applied to numerous queueing models, see e.g. [3, 5, 8, 14, 15, 16, 17, 19]. Crucial in applying such techniques is to prove the existence of zeros in a certain domain of analyticity of the function of interest. The zeros usually have no explicit representation, due to which one should rely on the specific properties of the analytic function that defines the zeros in an implicit way. Therefore, to prove the existence of the zeros, Rouché's theorem is a natural tool to use (as recognized by Crommelin [7]).

Rouché's theorem is a direct consequence of the *argument principle* and a powerful tool for determining regions of the complex plane in which there may be zeros of a given analytic function. The scope of application of Rouché's theorem goes well beyond the field of queueing theory. While the verification of the conditions needed to apply Rouché's theorem can become rather difficult, in queueing theory this is usually straightforward. For most queueing applications, the region of interest is typically the unit disk  $\{z \in \mathbb{C} : |z| \leq 1\}$ , and the ingredient that makes Rouché's theorem work is oftentimes the stability condition. This is why Rouché's theorem is a popular and standardized tool in queueing theory. However, the standard way in which Rouché's theorem is applied requires the analytic continuation of the function of interest *outside the unit disk*. This can be done for many functions, but definitely not for all.

In the standard setting the number of zeros in the unit disk of the function  $z^s - A(z)$  has to be determined, where  $A(z)$  is the PGF of a discrete random variable  $A$ . In order to apply Rouché's theorem it is then required that  $A(z)$  has a radius of convergence larger than one, which is not true in general. PGF's obey all the rules of power series with non-negative coefficients, and since  $A(1) = 1$  the radius of convergence of any PGF is at least 1. The shoe thus pinches for those PGF's for which the radius of convergence is exactly 1. Examples of PGF's of heavy-tailed distributions with a radius of convergence of 1 are presented in Sec. 4.

For Crommelin [7] this was obviously not an issue, since for the Poisson distribution  $A(z) = \exp(\lambda(z-1))$ , which is an entire function in the complex plane. Another example of suitable distributions are those with finite support, since in that case  $A(z)$  is a polynomial (see e.g. [14]). A problem does occur when  $A(z)$  is assumed to be the PGF of an arbitrary discrete random variable, like in [5, 8, 16, 17, 19]. In these papers, the assumption is made that  $A(z)$  has a radius of convergence larger than 1, which is clearly a restriction.

This restriction of generality has been relieved by Abolnikov & Dukhovny [1] who applied the so-called *generalized principle of the argument* (that was proved by Gakhov et al. [10] in 1973) to prove the existence of the zeros in the unit disk for general  $A(z)$ . Klimenok [13] extended this result to a larger class of functions, again using the generalized principle of the argument. An alternative approach to deal with general  $A(z)$  was presented by Boudreau et al. [4]. Under the condition that all zeros in the unit disk are distinct, they were able to apply the implicit function theorem to prove the existence of the zeros. However, examples can be constructed for which there are multiple zeros, and so this approach does not cover the issue in full generality. The key idea of Boudreau et al. is to study the parameterized function  $z^s - tA(z)$ ,  $0 \leq t < 1$ , and then letting  $t$  tend to one. The same idea, without making the assumption of distinct zeros, has been used by Gail et al. [9] for a larger class of zeros, including  $z^s - A(z)$ .

We present a proof of the existence of the zeros for general  $A(z)$  using the classical argument principle and truncation of  $A(z)$ . We make use of elementary results and techniques. The outcome

of our analysis is that the standard setting based on Rouché's theorem can be extended such that it holds for an arbitrary function  $A(z)$ .

In Sec. 2 we first describe the classical application of Rouché's theorem in queueing theory. In Sec. 3 we give our proof for general  $A(z)$ , and in Sec. 4 we provide some examples of (heavy-tailed) discrete distributions for which the classical approach fails, but to which our result can be applied.

## 2 Classical setting

In the vast majority of queueing problems to which Rouché's theorem is applied, the analytic function of interest is given by  $z^s - A(z)$ , where  $s \in \mathbb{N}$  and  $A(z)$  is the PGF of a nonnegative discrete random variable  $A$ . Denoting  $\mathbb{P}(A = j)$  by  $a_j$ , we have that

$$A(z) = \sum_{j=0}^{\infty} a_j z^j, \quad (1)$$

which is known to be analytic in the open disk  $\{z \in \mathbb{C} : |z| < 1\}$  and continuous up to the unit circle  $\{z \in \mathbb{C} : |z| = 1\}$ . Note that  $A(z)$  is differentiable at  $z = 1$  if and only if  $\sum_{j=1}^{\infty} j a_{j-1} z^{j-1} < \infty$ . If  $A(z)$  is differentiable at  $z = 1$ , it is differentiable at  $z$  for all  $z \in \mathbb{C}$  with  $|z| = 1$ . For continuous-time bulk service queues,  $M/G/1$  and  $G/M/1$ -type queues, the  $A(z)$  is typically of the form  $A(z) = \bar{B}(\lambda(1-z))$ , where  $\bar{B}(s)$  is the Laplace-Stieltjes transform of a continuous random variable and  $\lambda$  is some positive real constant (see e.g. [2, 11, 15]).

Let us first state Rouché's theorem (see e.g. Titchmarsh [18]):

**Theorem 2.1.** (Rouché) *Let the bounded region  $D$  have as its boundary a simple closed contour  $C$ . Let  $f(z)$  and  $g(z)$  be analytic both in  $D$  and on  $C$ . Assume that  $|f(z)| < |g(z)|$  on  $C$ . Then  $f(z) - g(z)$  has in  $D$  the same number of zeros as  $g(z)$ , all zeros counted according to their multiplicity.*

When  $A(z)$  has a radius of convergence larger than one, we can prove the following result concerning the number of zeros on and within the unit circle of  $z^s - A(z)$  by using Rouché's theorem:

**Lemma 2.2.** *Let  $A(z)$  be a PGF that is analytic in  $|z| \leq 1 + \nu$ ,  $\nu > 0$ . Assume that  $A'(1) < s$ ,  $s \in \mathbb{N}$ . Then the function  $z^s - A(z)$  has exactly  $s$  zeros in  $|z| \leq 1$ .*

**Proof** Define the functions  $f(z) := A(z)$ ,  $g(z) := z^s$ . Because  $f(1) = g(1)$  and  $f'(1) = A'(1) < s = g'(1)$ , we have, for sufficiently small  $\epsilon > 0$ ,

$$f(1 + \epsilon) < g(1 + \epsilon). \quad (2)$$

Consider all  $z$  with  $|z| = 1 + \epsilon$ . By the triangle inequality and (2) we have that

$$|f(z)| \leq \sum_{j=0}^{\infty} a_j |z|^j = f(1 + \epsilon) < g(1 + \epsilon) = |g(z)|, \quad (3)$$

and hence  $|f(z)| < |g(z)|$ . Because both  $f(z)$  and  $g(z)$  are analytic for  $|z| \leq 1 + \epsilon$ , Rouché's theorem tells us that  $g(z)$  and  $f(z) - g(z)$  have the same number of zeros in  $|z| \leq 1 + \epsilon$ . Letting  $\epsilon$  tend to zero yields the proof.  $\square$

The application of Lemma 2.2 is limited to the class of functions  $A(z)$  with a radius of convergence larger than 1. In case  $A(z)$  has radius of convergence 1, the results of the next section can be applied.

### 3 New setting

Before we present our main result, we first prove a result on the number and location of zeros of  $z^s - A(z)$  on the unit circle. We define the period  $p$  of a series  $\sum_{-\infty}^{\infty} b_j z^j$  as the largest integer for which  $b_j = 0$  whenever  $j$  is not divisible by  $p$ .

**Lemma 3.1.** *Let  $A(z)$  be a PGF of some nonnegative discrete random variable with  $A(0) > 0$ . Assume  $A(z)$  is differentiable at  $z = 1$  and  $A'(1) < s$ , where  $s$  is a positive integer. If  $z^s - A(z)$  has period  $p$ , then  $z^s - A(z)$  has exactly  $p$  zeros on the unit circle given by the  $p$ -th roots of unity  $\tau_k = \exp(2\pi i k/p)$ ,  $k = 0, 1, \dots, p-1$ . In each of these zeros, the derivative of  $z^s - A(z)$  does not vanish.*

**Proof** Obviously, any zero  $\xi$  of  $z^s - A(z)$  with  $|\xi| = 1$  is simple, since  $|A'(\xi)| \leq A'(|\xi|) = A'(1) < s$  and, thus,  $s\xi^{s-1} - A'(\xi) \neq 0$ . Furthermore, for any  $z$  with  $|z| = 1$ ,  $|A(z)| = A(1)$  iff  $z^k = 1$  whenever  $a_k > 0$ . This easily follows from the fact that  $|a_0 + a_k z^k| < a_0 + a_k$  if  $z^k \neq 1$ . So, for  $z$  with  $|z| = 1$  and  $A(z) - z^s = 0$  it follows that  $z^k = 1$  for all  $k$  with  $a_k > 0$ , and  $z^s = 1$ . This implies that  $z^p = 1$ , which completes the proof.  $\square$

Note that the requirement  $a_0 = A(0) > 0$  involves no essential limitation: If  $a_0$  were zero we would replace the distribution  $\{a_i\}_{i \geq 0}$  by  $\{a'_i\}_{i \geq 0}$  where  $a'_i = a_{i+m}$ ,  $a_m$  being the first non-zero entry of  $\{a_i\}_{i \geq 0}$ , and a corresponding decrease in  $s$  according to  $s' = s - m$ .

We are now in a position to give the main result:

**Theorem 3.2.** *Let  $A(z)$  be a PGF of some nonnegative discrete random variable with  $A(0) > 0$ . Assume  $A(z)$  is differentiable at  $z = 1$  and  $A'(1) < s$ , where  $s$  is a positive integer. Also, let  $z^s - A(z)$  have period  $p$ . Then the function  $z^s - A(z)$  has  $p$  zeros on the unit circle given by  $\tau_k = \exp(2\pi i k/p)$ ,  $k = 0, 1, \dots, p-1$  and exactly  $s - p$  zeros in  $|z| < 1$ .*

**Proof** Lemma 3.1 tells us that  $F(z) = z^s - A(z)$  has  $p$  equidistant zeros on the unit circle, and so it remains to prove that this function has exactly  $s - p$  zeros within the unit circle. Thereto, define, for  $N \in \mathbb{N}$ , the truncated PGF

$$A_N(z) = \sum_{j=0}^{N-1} a_j z^j + \sum_{j=N}^{\infty} a_j z^N, \quad (4)$$

where  $N$  is a multiple of  $p$ . Then  $F_N(z) = z^s - A_N(z)$  has obviously  $s$  zeros in  $z \in D = \{z \in \mathbb{C} : |z| \leq 1\}$ , since  $A_N(z)$  is a polynomial satisfying  $A'_N(1) < s$ , and Lemma 2.2 thus applies. By Lemma 3.1 we know that  $F_N(z)$  has  $p$  simple and equidistant zeros on the unit circle. We further have that

$$|A(z) - A_N(z)| \leq 2 \sum_{j=N}^{\infty} a_j, \quad |z| \leq 1, \quad (5)$$

$$|A'(z) - A'_N(z)| \leq 2 \sum_{j=N}^{\infty} j a_j, \quad |z| \leq 1. \quad (6)$$

Thus,  $A_N(z)$  and  $A'_N(z)$  converge uniformly to  $A(z)$  and  $A'(z)$  on  $z \in D$ , respectively. Moreover, if  $G : D \rightarrow \mathbb{C}$  is continuous, then  $G(A_N(z))$  is uniformly convergent to  $G(A(z))$  on  $z \in D$ .

Let  $z$  on  $C = \{z \in \mathbb{C} : |z| = 1\}$ . If for all  $n \in \mathbb{N}$  there is a  $z_n \in D$  with  $0 < |z - z_n| < \frac{1}{n}$  and  $F(z_n) = 0$ , then  $F(z) = 0$  and

$$F'(z) = \lim_{n \rightarrow \infty} \frac{F(z_n) - F(z)}{z_n - z} = 0. \quad (7)$$

However, this is impossible by Lemma 3.1. Hence, there is an  $\eta > 0$  such that  $F(\xi) \neq 0$  for all  $\xi \in D(z, \eta) := \{\xi \in D : 0 < |\xi - z| < \eta\}$ . Since  $C$  is compact, it can be covered by finitely many  $D(z, \eta)$ 's. Hence, there is a  $0 < r < 1$  such that  $F(z)$  has no zeros in  $r \leq |z| < 1$ .

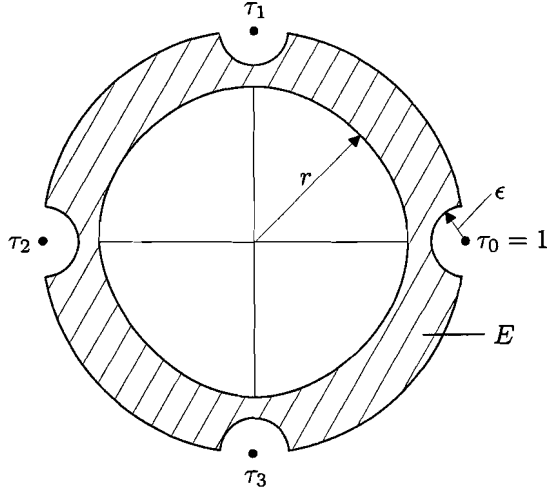


Figure 1: Graphical representation of the compact set  $E$ .

Now we prove that for large  $N$  the function  $F_N(z)$ , as the function  $F(z)$ , has no zeros in  $r \leq |z| < 1$ . Thereto, we show that there is an  $\epsilon > 0$  and  $M \in \mathbb{N}$  such that  $F_N(z) \neq 0$  for all  $N \geq M$  and  $0 < |z - \tau_k| < \epsilon$ ,  $k = 0, 1, \dots, p-1$ . Because  $F'(z)$  is continuous and  $F'_N(z)$  converges uniformly to  $F'(z)$  on  $z \in D$ , there are  $\epsilon > 0$  and  $M \in \mathbb{N}$  such that (for  $k = 0, 1, \dots, p-1$ )

$$|F'_N(z) - F'(\tau_k)| < \delta < |F'(\tau_k)|, \quad 0 < |z - \tau_k| < \epsilon, \quad N \geq M. \quad (8)$$

Furthermore, we have (for  $k = 0, 1, \dots, p-1$ )

$$|F_N(z) - F'(\tau_k)(z - \tau_k)| = \left| \int_{[\tau_k, z]} (F'_N(s) - F'(\tau_k)) ds \right|, \quad (9)$$

where the integration is carried out along the straight line that connects  $\tau_k$  and  $z$ . Hence, for  $0 < |z - \tau_k| < \epsilon$  and  $N \geq M$ , we obtain (for  $k = 0, 1, \dots, p-1$ )

$$\left| \int_{[\tau_k, z]} (F'_N(s) - F'(\tau_k)) ds \right| \leq |z - \tau_k| \max_{s \in [\tau_k, z]} |F'_N(s) - F'(\tau_k)| < |z - \tau_k| \delta. \quad (10)$$

So, it follows that for  $0 < |z - \tau_k| < \epsilon$  and  $N \geq M$  (for  $k = 0, 1, \dots, p-1$ )

$$|F_N(z)| = |F_N(z) - F'(\tau_k)(z - \tau_k) + F'(\tau_k)(z - \tau_k)| \quad (11)$$

$$\geq |F'(\tau_k)| |z - \tau_k| - |F_N(z) - F'(\tau_k)(z - \tau_k)| \quad (12)$$

$$> (|F'(\tau_k)| - \delta) |z - \tau_k| > 0. \quad (13)$$

Since  $F_N(z)$  converges uniformly to  $F(z)$  and  $F(z) \neq 0$  on the compact set (see Fig. 1)

$$E = \{z \in \mathbb{C} : r \leq |z| \leq 1\} \setminus \bigcup_{k=0}^{p-1} D(\tau_k, \epsilon), \quad (14)$$

there exists an  $K \in \mathbb{N}$  such that  $F_N(z) \neq 0$  for all  $N \geq K$  and  $z \in \mathbb{C}$  with  $r \leq |z| < 1$ . Hence, for all  $N \geq K$  the number of zeros of  $F_N(z)$  with  $|z| < r$  is equal to  $s - p$ . This number can be expressed by the argument principle (see e.g. Titchmarsh [18]) as follows

$$s - p = \frac{1}{2\pi i} \oint_{|z|=r} \frac{F'_N(z)}{F_N(z)} dz. \quad (15)$$



The integrand converges uniformly to  $F'(z)/F(z)$ , and thus

$$\frac{1}{2\pi i} \oint_{|z|=r} \frac{F'(z)}{F(z)} dz = \lim_{N \rightarrow \infty} \frac{1}{2\pi i} \oint_{|z|=r} \frac{F'_N(z)}{F_N(z)} dz = s - p. \quad (16)$$

Hence, the number of zeros of  $F(z)$  with  $|z| < r$  is also  $s - p$ . This completes the proof.  $\square$

## 4 Examples

On behalf of Thm. 3.2, the  $A(z)$  with a radius of convergence of 1 do not have to be excluded from the analysis of the zeros of  $z^s - A(z)$ . This further means that these PGF's can be incorporated in the general formulation of the solution to the queueing models of interest. The  $A(z)$  that have radius of convergence 1 are typically those associated with heavy-tailed random variables. Some examples are given below.

- (i) The discrete Pareto distribution (e.g. Johnson et al. [12]), defined by

$$a_j = c \frac{1}{j^{p+1}}, \quad j = 1, 2, \dots, \quad (17)$$

with

$$c = \left( \sum_{j=1}^{\infty} a_j \right)^{-1} = \zeta(p+1)^{-1}, \quad (18)$$

where  $\zeta(\cdot)$  is called the *Riemann zeta function* and  $p > 1$ . For  $k < p$ , the  $k^{\text{th}}$  moment  $\mu_k$  of the discrete Pareto distribution is given by

$$\mu_k = \frac{\zeta(p-k+1)}{\zeta(p+1)}, \quad (19)$$

whereas for  $k \geq p$  the moments are infinite. The discrete Pareto distribution is also known as the Zipf or Riemann zeta distribution

- (ii) The discrete standard lognormal distribution, defined by

$$a_j = ce^{-\frac{(\log j)^2}{2}}, \quad j = 1, 2, \dots, \quad (20)$$

where  $c$  is a normalization constant.

- (iii) The discrete distribution, related to the continuous Weibull distribution, defined by

$$a_j = cp^{-\sqrt{j}}, \quad j = 0, 1, \dots, \quad (21)$$

where  $p > 1$  and  $c$  is a normalization constant.

- (iv) The Haight's zeta distribution (see e.g. Johnson et al. [12]), defined by

$$a_j = \frac{1}{(2j-1)^p} - \frac{1}{(2j+1)^p}, \quad j = 1, 2, \dots, \quad (22)$$

with  $p > 1$ .

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## References

- [1] Abolnikov, L., A. Dukhovny (1987). Necessary and sufficient conditions for the ergodicity of Markov chains with transition  $\Delta_{m,n}$  ( $\Delta'_{m,n}$ )-matrix, *Journal of Applied Mathematics and Simulation* **1** (1): 13-24.
- [2] Adan, I.J.B.F., Y.Q. Zhao (1996). Analyzing  $GI/E_r/1$  queues, *Operations Research Letters* **19**: 183-190.
- [3] Bailey, N.T.J. (1954). On queueing processes with bulk service, *Journal of the Royal Statistical Society* **16**: 80-87.
- [4] Boudreau, P.E., J.S. Griffin, M. Kac (1962). An elementary queueing problem, *American Mathematical Monthly* **69**: 713-724.
- [5] Bruneel, H. (1984). A general model for the behaviour of infinite buffers with periodic service opportunities, *European Journal of Operational Research* **16** (1): 98-106.
- [6] Chaudhry, M.L., C.M. Harris, W.G. Marchal (1990). Robustness of rootfinding in single-server queueing models, *ORSA Journal on Computing* **2**: 273-286.
- [7] Crommelin, C.D. (1932). Delay probability formulae when the holding times are constant, *Post Office Electrical Engineers Journal* **25**: 41-50.
- [8] Darroch, J.N. (1964). On the traffic-light queue, *Ann. Math. Stat* **35**: 380-388.
- [9] Gail, H.R., S.L. Hantler, B.A. Taylor (1996). Spectral analysis of M/G/1 and G/M/1 type Markov chains, *Advances in Applied Probability* **28**, 114-165.
- [10] Gakhov, F.D., E.I. Zverovich, S.G. Samko (1973). Increment of the argument, logarithmic residue and a generalized principle of the argument (in Russian), *Dokl. Akad. Nauk SSSR* **213**: 1233-1236 (translated in *Soviet Math. Dokl.* **14** (6): 1856-1860).
- [11] Harris, C.M., W.G. Marchal, R.W. Tibbs (1992). An algorithm for finding characteristic roots of quasitriangular Markov chains, in *Queueing and Related Models*, U.N. Bhat, Editor, Oxford University Press, New York.
- [12] Johnson, N.L., S. Kotz, A.W. Kemp (1992). *Univariate Discrete Distributions*, 2<sup>nd</sup> edition, John Wiley & Sons, New York.
- [13] Klimenok, V. (2001). On the modification of Rouché's theorem for the queueing theory problems, *Queueing Systems* **38**: 431-434.
- [14] Konheim, A.G. (1975). An elementary solution of the queueing system  $G/G/1$ , *SIAM J. Comput.* **4** (4): 540-545.
- [15] Lee, D.S. (1996). A two-queue model with exhaustive and limited service disciplines, *Stochastic Models* **12** (2): 285-305.
- [16] Powell, W.B., P. Humblet (1986). The bulk service queue with a general control strategy: Theoretical analysis and a new computational procedure, *Operations Research* **34** (2): 267-275.
- [17] Servi, L.D. (1996).  $D/G/1$  queues with vacations, *Operations Research* **34** (4): 619-629.
- [18] Titchmarsh, E.C. (1939). *The Theory of Functions*, 2<sup>nd</sup> edition, Oxford University Press, New York.
- [19] Zhao, Y.Q., L.L. Campbell (1996). Equilibrium probability calculations for a discrete-time bulk queue model. *Queueing Systems* **22**: 189-198.