Duality and exact correlations for a model of heat conduction

JOURNAL OF MATHEMATICAL PHYSICS **48**, 033301 (2007)

Duality and exact correlations for a model of heat conduction

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(Received 21 December 2006; accepted 16 January 2007; published online 23 March 2007)

We study a model of heat conduction with stochastic diffusion of energy. We obtain a dual particle process which describes the evolution of all the correlation functions. An exact expression for the covariance of the energy exhibits long-range correlations in the presence of a current. We discuss the formal connection of this model with the simple symmetric exclusion process. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2711373]

I. INTRODUCTION

Simple systems of particles on a lattice have received considerable attention in the last years, as they are a testing ground for exploring the properties of far from equilibrium states, for which at present no general theory is available.

Amongst the most studied models are the so-called exclusion processes in which particles diffuse amongst empty sites on a lattice. The problem has been attacked with techniques of statistical mechanics and probability, and many analytic results are available,[15,17,20] in particular, for the simple symmetric exclusion process (SEP).

One is also interested in the transport of continuous quantities, especially energy. Kipnis, Marchioro, and Presutti[11] (KMP) have introduced a model of energy transport in which the energy of neighboring sites is randomly redistributed. This model has also been thoroughly studied, although analytic solutions, in particular, for the steady-state correlation functions, are harder to obtain than in the SEP. Both the KMP and the SEP have an interesting (and rather exceptional) feature: the evolution of the K-point correlation functions can be exactly mapped onto a diffusion equation for K particles. This so-called "duality property"[15] is a powerful tool, which in the study of the SEP yields the complete ergodic theory (see Ref. 15).

Despite the resemblance between energy and particle transport, studying both models is not a redundant exercise, as the SEP and the KMP models also show intriguing differences,[4] mainly in their large-deviation functions. Perhaps the most striking is the fact that in the implementation of an "additivity principle" of the large deviation functions of two subsystems, the density at the interface has to be maximized in one case, and minimized on the other, in order to obtain the correct result.[6]

In this paper we shall study a family of models of energy transport which, we shall argue, are the natural counterpart of the SEP models. In each site i of the lattice there is a free particle with

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momentum x_i. Between any two neighboring sites $(i,i+1)$ and for any small time interval there is a random exchange of momentum that leaves $\{x_i^2+x_{i+1}^2\}$ invariant. The same transport terms were already considered in models of wave propagation in random media.[21,1] They later appeared as the high-energy limit of a chain with deterministic dynamics[9] and were also used as a stochastic perturbation (mimicking nonlinearities) of an oscillator chain.[3,8]

In the present paper we construct for our energy diffusion model a dual process that expresses the evolution of the K-point correlation functions of the kinetic energies in terms of a process of K interacting random walkers. We also give a closed expression for the stationary covariance $\langle x_i^2;x_j^2\rangle$ which confirms the presence of long-range correlations in the nonequilibrium stationary state [as already found before in the SEP Ref. 19].

The energy diffusion model is clearly very close from the physical point of view to the KMP model. But at the same time, as we have mentioned, it can be viewed as the continuation of the SEP family in the following sense: on the one hand, the SEP can naturally be generalized[18] to processes in which up to n particles are allowed per site (the usual SEP has $n=1$), and hopping rates are proportional to the number of particles at the departure and the number of "holes" at the arrival site. On the other hand, the energy diffusion model can be generalized to having in each site i of the lattice m free particles with momenta $x_{i,\alpha}$, $\alpha=1,\ldots,m$, and the random exchange between $x_{i,\alpha}$ and $x_{i+1,\beta}$ leaves $\Sigma_\gamma\{x_{i,\gamma}^2+x_{i+1,\gamma}^2\}$ invariant. We shall see below that the energy transport model with m particles per site formally corresponds to the continuation of the SEP family to *negative* occupation number: $m=-n/2$. In other words, the energy diffusion model as defined above is, formally, *the SEP with* $-1/2$ *particles per site*.

The rest of our paper is organized as follows. In Sec. II we define our energy diffusion model. In Sec. III we construct the dual process of interacting random walkers and derive the basic corollaries of duality, namely, existence and uniqueness of the stationary measure and expression of the stationary correlation functions in terms of absorption probabilities. In Secs. IV and V we derive the exact stationary temperature profile and energy-energy correlation function. In Sec. VI we show that our model satisfies local equilibrium. In Sec. VII we show a formal connection between our model and the SEP family, showing that the energy diffusion model can be viewed as a bosonic version of the SEP. In Sec. VIII we discuss possible asymmetric extensions of our model.

II. THE MODEL

The model is defined as a stochastic process on N-dimensional vectors $(x_1,\ldots,x_N)\in\mathbb{R}^N$ which have to be interpreted as momenta associated with lattice sites $\{1,\ldots,N\}$. Additionally, lattice sites 1 and N are in contact with a heat reservoir at temperature T_L, respectively, T_R.

The process is defined by its generator L (acting on the core of $\mathcal{C}^\infty$ functions f with compact support),

$$L=L_1+L_N+\sum_{i=1}^{N-1}L_{i,i+1}, \tag{2.1}$$

with

$$L_1 f=T_L\frac{\partial^2 f}{\partial x_1^2}-x_1\frac{\partial f}{\partial x_1}, \tag{2.2}$$

$$L_N f=T_R\frac{\partial^2 f}{\partial x_N^2}-x_N\frac{\partial f}{\partial x_N}, \tag{2.3}$$

$$L_{i,i+1}f = \left(x_i \frac{\partial}{\partial x_{i+1}} - x_{i+1}\frac{\partial}{\partial x_i}\right)^2 (f). \tag{2.4}$$

This corresponds in the language of Fokker-Planck equation (or master equation) to the following evolution equation for the time-dependent probability density $p(x,t)$:

$$\frac{\partial p(x,t)}{\partial t} = L^* p(x,t), \tag{2.5}$$

where L^* is the adjoint [in $L^2(\mathrm{d}x)$] of L, i.e., more explicitly,

$$L^* = L_1^* + L_N^* + \sum_{i=1}^{N-1} L_{i,i+1},$$

$$L_1^* f = T_L \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial}{\partial x_1}(x_1 f),$$

$$L_N^* f = T_R \frac{\partial^2 f}{\partial x_N^2} + \frac{\partial}{\partial x_N}(x_N f).$$

Let us first motivate the choice of this generator (2.1). The L_1 and L_N part is the generator of the usual Ornstein-Uhlenbeck process which represents thermaliting noise corresponding to heat baths at temperatures T_L, respectively, T_R.

To explain the other part, consider the operator

$$\mathcal{A} = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial y}\right)^2. \tag{2.6}$$

In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, this operator simply reads

$$\mathcal{A} = \frac{\partial^2}{\partial \theta^2},$$

which means that in the process $(x(t), y(t))$ corresponding to, Eq. (2.6), $r(t) = r(0)$ and $\theta(t)$ performs a Brownian motion on the interval $[0, 2\pi]$. More precisely, from the Itô formula, it follows that the generator (2.6) corresponds to the stochastic differential equations (in Itô sense)

$$\mathrm{d}x(t) = -x(t)\mathrm{d}t + \sqrt{2}y(t)\mathrm{d}B(t),$$

$$\mathrm{d}y(t) = -y(t)\mathrm{d}t - \sqrt{2}x(t)\mathrm{d}B(t), \tag{2.7}$$

where $B(t)$ is the standard Brownian motion. In other words $r^2(t) = x^2(t) + y^2(t) = r^2(0)$ with probability 1 and the angular variable $\theta(t) = \arctan(y(t)/x(t))$ is a martingale.

The bulk part $\Sigma_i L_{i,i+1}$ of the process $(x_1(t), \ldots, x_N(t))$ corresponds then to the stochastic differential equations

$$\mathrm{d}x_1(t) = -x_1(t)\mathrm{d}t + \sqrt{2}x_2(t)\mathrm{d}B_{1,2}(t),$$

$$\mathrm{d}x_i(t) = -2x_i(t)\mathrm{d}t + \sqrt{2}x_{i+1}(t)\mathrm{d}B_{i,i+1}(t) - \sqrt{2}x_{i-1}(t)\mathrm{d}B_{i-1,i}(t), \quad i \in [2, N-1],$$

$$dx_N(t) = -x_N(t)dt - \sqrt{2}x_{N-1}(t)dB_{N-1,N}(t), \tag{2.8}$$

where $B_{i,i+1}(t)(i=1,\ldots,N-1)$ are independent Brownian motions and $B_{0,1}(t)=B_{N,N+1}(t)=0$. In this process, the total kinetic energy $\Sigma_i x_i^2(t)$ is conserved, i.e., $\Sigma_i x_i^2(t)=\Sigma_i x_i^2(0)$ with probability 1, as can be seen easily from Itô's formula.

The full process, i.e., the process with generator (2.1) (including the boundary terms L_1, L_N), corresponds to the system of stochastic differential equations

$$dx_1(t) = -2x_1(t)dt + \sqrt{2}x_2(t)dB_{1,2}(t) + \sqrt{2T_L}dW(t),$$

$$dx_i(t) = -2x_i(t)dt + \sqrt{2}x_{i+1}(t)dB_{i,i+1}(t) - \sqrt{2}x_{i-1}(t)dB_{i-1,i}(t), \quad i \in [2,N-1],$$

$$dx_N(t) = -2x_N(t)dt - \sqrt{2}x_{N-1}(t)dB_{N-1,N}(t) + \sqrt{2T_R}dW'(t), \tag{2.9}$$

where $W(t)$, $W'(t)$ are two independent Brownian motions, independent of all the other $B_{i,i+1}(t)$.

III. DUALITY

The main tool which considerably simplifies the analysis of this model is duality. First note that the equations for the evolution of correlation functions of degree n for the x process are *closed*, i.e., the time derivative of the expectation of a polynomial of degree n in the variables $x_1,\ldots,x_n$ does not involve expectations of polynomials of higher order.[9]

We now show that the evolution in time of well-chosen polynomials reduces to a Markovian evolution of their *indices*. If we interpret the indices of the polynomials as discrete (indistinguishable) particle configurations (i.e., specifying for each site $i \in \{0,\ldots,N+1\}$ the number of particles), then the evolution of the indices turns out to become a jump process that conserves the total number of particles. This jump process is called the dual process. A similar situation arises in the case of the SEP (which is self-dual),[15] in the case of an infinite system of independent random walkers (where the Poisson polynomials have the self-dual property),[10] and in the KMP model.[11]

We index our polynomials by a vector $\xi=(\xi_0,\ldots,\xi_{N+1})$, $\xi_i \in \mathbb{N}$ and introduce the notation $\mathbf{x}=(x_0,x_1,\ldots,x_N,x_{N+1})$, where the first and last components are *fixed* by $x_0=\sqrt{T_L}$ and $x_{N+1}=\sqrt{T_R}$. The polynomial $D(\xi,\mathbf{x})$ is then defined by

$$D(\xi,\mathbf{x}) = T_L^{\xi_0}T_R^{\xi_{N+1}}\prod_{i=1}^{N}\frac{x^{2\xi_i}}{(2\xi_i-1)!!}. \tag{3.1}$$

Only even powers of x_i need to be considered, since other stationary correlations vanish due to the invariance of the generator under the transformation $x_i \to -x_i$. We interpret $\xi \in \Omega=\mathbb{N}^{N+2}$ as prescribing the number of particles in each lattice site $i \in \{0,\ldots,N+1\}$.

In order to introduce the generator of the dual process, we define, for $\xi \in \Omega$, $i,j \in \{0,\ldots,N+1\}$, the configuration $\xi^{i,j}$ to be the configuration obtained from ξ by removing one particle at i and adding one particle at j. On the dual variables $\xi \in \Omega$, we then define the following generator:

$$\begin{aligned}\mathcal{L}\varphi(\xi) :=\ & 2\xi_1[\varphi(\xi^{1,0})-\varphi(\xi)] + 2\xi_1(2\xi_2+1)[\varphi(\xi^{1,2})-\varphi(\xi)] \\ & + \sum_{i=2}^{N-1}(2\xi_i(2\xi_{i-1}+1)[\varphi(\xi^{i,i-1})-\varphi(\xi)] + 2\xi_i(2\xi_{i+1}+1)[\varphi(\xi^{i,i+1})-\varphi(\xi)]) \\ & + 2\xi_N(2\xi_{N-1}+1)[\varphi(\xi^{N,N-1})-\varphi(\xi)] + 2\xi_N[\varphi(\xi^{N,N+1})-\varphi(\xi)],\end{aligned} \tag{3.2}$$

where $\varphi:\Omega\to\mathbb{R}$ is an arbitrary function of the finite particle configurations.

In other words, we can describe the process generated by $\mathcal{L}$ as follows: a configuration $\xi=(\xi_0,\ldots,\xi_{N+1})$ represents K particles (or walkers) on $\{0,1,\ldots,N+1\}$, with $K=\Sigma_{i=0}^{N+1}\xi_i$. The walkers can only jump to neighboring sites and are stuck when arriving to sites 0 or $N+1$. The rate at which there is a jump of a walker depends on how many walkers there are at neighboring sites. If

we have ξ_i walkers at site i, ξ_{i-1} walkers at site $i-1$, and ξ_{i+1} walkers at site $i+1$ (for $i=2,\ldots N-1$) then each of the walkers at site i jumps to site $i-1$ at rate $2(2\xi_{i-1}+1)$ and to site $i+1$ at rate $2(2\xi_{i+1}+1)$. At the boundaries, each of the ξ_1 walkers at site 1 is absorbed at site 0 at rate 2 and it jumps to site 2 at rate $2(2\xi_2+1)$; each of the ξ_N walkers at site N is absorbed at site $N+1$ at rate 2 and it jumps to site $N-1$ at rate $2(2\xi_{N-1}+1)$. Note that this process conserves the number of particles, i.e.,

$$|\xi(t)| = \sum_{i=0}^{N+1} \xi_i(t) = \sum_{i=0}^{N+1} \xi_i(0) = |\xi(0)|.$$

For a single particle, i.e., $\xi=\delta_i$, the dual process is then $\xi(t)=\delta_{X(t)}$, where $X(t)$ is a continuous-time simple symmetric nearest neighbor random walk jumping at rate 2 and absorbed upon hitting 0 or $N+1$. For two particles, i.e., $\xi=\delta_i+\delta_j$, the particles perform independent symmetric nearest neighbor random walks at rate 2, *except* when they are sitting at *neighboring sites*. In that case, i.e., if the two walkers are at neighboring places, one of them jumps to the place of the other one at rate 6 (and other jumps are still at rate 2).

Remark 1: *This attractive interaction between the dual walkers has to be compared with the repulsive (hard-core) interaction between the walkers in the SEP.*

In order to formulate our duality result and its consequences, we denote by $\hat{\mathbb{E}}_\xi$ expectation in the dual process [i.e., the process with generator (3.2)] starting from ξ. We can now formulate the duality result.

Theorem 1: *Let $\xi(t)$ denote the process with generator (3.2) and $\mathbf{x}(t)$ the process $(x_0(t),x_1(t),x_2(t),\ldots,x_N(t),x_{N+1}(t))$, where $(x_1(t),\ldots,x_N(t))$ is the process with generator (2.1) and where $x_0(t)=\sqrt{T_L}$, $x_{N+1}(t)=\sqrt{T_R}$. Then we have*

$$\mathbb{E}_{\mathbf{x}}[D(\xi,\mathbf{x}(t))] = \hat{\mathbb{E}}_\xi[D(\xi(t),\mathbf{x})]. \tag{3.3}$$

Proof: Start from Eq. (2.1). For $i=1,\ldots,N-1$ we have

$$\begin{aligned} L_{i,i+1}D(\xi,x) = {} & T_L^{\xi_0}T_R^{\xi_{N+1}}\left(\prod_{k\notin\{i,i+1\}}\frac{x_k^{2\xi_k}}{(2\xi_k-1)!!}\right) \\ & \times\left(2\xi_{i+1}(2\xi_{i+1}-1)\frac{x_i^{2\xi_i+2}}{(2\xi_i-1)!!}\frac{x_{i+1}^{2\xi_{i+1}-2}}{(2\xi_{i+1}-1)!!} - 2\xi_i(2\xi_{i+1}+1)\frac{x_i^{2\xi_i}}{(2\xi_i-1)!!}\frac{x_{i+1}^{2\xi_{i+1}}}{(2\xi_{i+1}-1)!!}\right. \\ & \left. - 2\xi_{i+1}(2\xi_i+1)\frac{x_i^{2\xi_i}}{(2\xi_i-1)!!}\frac{x_{i+1}^{2\xi_{i+1}}}{(2\xi_{i+1}-1)!!} + 2\xi_i(2\xi_i-1)\frac{x_i^{2\xi_i-2}}{(2\xi_i-1)!!}\frac{x_{i+1}^{2\xi_{i+1}+2}}{(2\xi_{i+1}-1)!!}\right), \end{aligned}$$

which implies

$$L_{i,i+1}D(\xi,x) = (2\xi_{i+1}(2\xi_i+1)[D(\xi^{i+1,i},x)-D(\xi,x)] + 2\xi_i(2\xi_{i+1}+1)[D(\xi^{i,i+1},x)-D(\xi,x)]).$$

Furthermore

$$\begin{aligned} L_1D(\xi,x) &= T_R^{\xi_{N+1}}\left(\prod_{k\notin\{1\}}\frac{x_k^{2\xi_k}}{(2\xi_k-1)!!}\right)\times\left(T_L^{\xi_0+1}2\xi_1(2\xi_1-1)\frac{x_1^{2\xi_1-2}}{(2\xi_1-1)!!} - T_L^{\xi_0}2\xi_1\frac{x_1^{2\xi_1}}{(2\xi_1-1)!!}\right) \\ &= 2\xi_1[D(\xi^{1,0},x)-D(\xi,x)] \end{aligned}$$

and

$$\begin{aligned} L_ND(\xi,x) &= T_L^{\xi_0}\left(\prod_{k\notin\{N\}}\frac{x_k^{2\xi_k}}{(2\xi_k-1)!!}\right)\times\left(T_R^{\xi_{N+1}+1}2\xi_N(2\xi_N-1)\frac{x_N^{2\xi_N-2}}{(2\xi_N-1)!!} - T_R^{\xi_{N+1}}2\xi_N\frac{x_N^{2\xi_N}}{(2\xi_N-1)!!}\right) \\ &= 2\xi_N[D(\xi^{N,N+1},x)-D(\xi,x)]. \end{aligned}$$

Therefore, we obtain

$$LD(\xi,\mathbf{x}) = \mathcal{L}D(\xi,\mathbf{x}), \tag{3.4}$$

where in the lhs of Eq. (3.4) the operator L is working on the $\mathbf{x}$ variable, and in the rhs the operator $\mathcal{L}$ is working on the ξ variable.

This relation then lifts to the semigroups and the processes via a standard argument, see, e.g., the proof of Theorem 1.1 page 363 in Ref. 15. [Notice that informally by Eq. (3.4) $\mathbb{E}_x D(\xi,\mathbf{x}(t)) = e^{tL}D(\xi,\cdot)(\mathbf{x}) = e^{t\mathcal{L}}D(\cdot,\mathbf{x})(\xi) = \hat{\mathbb{E}}_\xi D(\xi(t),\mathbf{x})$]. □

Notice that in the dual process all particles are eventually absorbed at one of the boundaries 0, $N+1$. Hence, the limiting configuration of the dual process starting from $\xi \in \Omega$ will always be of the form $k\delta_0 + l\delta_{N+1}$, where $k+l=\Sigma_i \xi_i$. We say that a function $f:\Omega \to \mathbb{R}$ is harmonic for the process generated by $\mathcal{L}$ if $f(\xi) = \hat{\mathbb{E}}_\xi[f(\xi(t))]$. Since the configurations of the type $k\delta_0 + l\delta_{N+1}$ are absorbing, the corresponding absorption probabilities

$$c_{kl}(\xi) = \hat{\mathbb{P}}_\xi(\xi(\infty) = k\delta_0 + l\delta_{N+1}) \tag{3.5}$$

are harmonic.

The following propositions give some consequences of duality.

Proposition 1: *If $T_L = T_R = T$, then the unique stationary measure of the process $x(t) = (x_1(t),\ldots,x_N(t))$ is the Gaussian product measure with probability density function*

$$\rho_T(x) = \frac{1}{(2\pi T)^{N/2}} \exp\left(-\sum_{i=1}^{N} \frac{x_i^2}{2T}\right). \tag{3.6}$$

From every initial condition $(x_1,\ldots,x_N)$ the process converges in distribution to ρ_T.

Proof: We first show stationarity of ρ_T. If $T_L = T_R = T$, for the Gaussian measure (3.6) we have

$$\int D(\xi,\mathbf{x})\rho_T(x)\mathrm{d}x = T^{|\xi|},$$

where $|\xi| = \Sigma_{i=1}^N \xi_i$. Therefore, using Eq. (3.3)

$$\int \mathbb{E}_{\mathbf{x}}[D(\xi,\mathbf{x}(t))]\rho_T(x)\mathrm{d}x = \hat{\mathbb{E}}_\xi \int D(\xi(t),\mathbf{x})\rho_T(x)\mathrm{d}x = \hat{\mathbb{E}}_\xi T^{|\xi(t)|} = T^{|\xi|} = \int D(\xi,\mathbf{x})\rho_T(x)\mathrm{d}x, \tag{3.7}$$

where in the third equality we used that in the dual process the number of particles is conserved. Next, to prove the convergence, remember that in the dual process eventually all particles are absorbed at 0 or at $N+1$. Therefore, if $T_L = T_R = T$,

$$\lim_{t\to\infty} \mathbb{E}_{\mathbf{x}}(D(\xi,\mathbf{x}(t))) = \sum_{k,l:k+l=|\xi|} \hat{\mathbb{P}}_\xi(\xi(\infty) = k\delta_0 + l\delta_{N+1}) T^k T^l = T^{|\xi|} = \int D(\xi,\mathbf{x})\rho_T(x)\mathrm{d}x. \tag{3.8}$$

□

Proposition 2: *For all $T_L \neq T_R$, there is a unique stationary measure μ_{T_L,T_R} which has finite moments of every order, and for this measure one has*

$$\mu_{T_L,T_R}(D(\xi,\mathbf{x})) = \sum_{k,l:k+l=|\xi|} T_L^k T_R^l c_{kl}(\xi), \tag{3.9}$$

where $c_{kl}(\xi)$ are the absorption probabilities defined in Eq. (3.5).

Proof: If μ_{T_L,T_R} is invariant and has finite moment of all order, then, by Eq. (3.3) we have

$$\int D(\xi,\mathbf{x})\mu_{T_L,T_R}(\mathrm{d}x) = \lim_{t\to\infty}\int \mathbb{E}_x(D(\xi,\mathbf{x}(t)))\mu_{T_L,T_R}(\mathrm{d}x) = \int (\lim_{t\to\infty}\hat{\mathbb{E}}_\xi(D(\xi(t),\mathbf{x})))\mu_{T_L,T_R}(\mathrm{d}x)$$
$$= \sum_{k,l:k+l=|\xi|} T_L^k T_R^l \hat{\mathbb{P}}_\xi(\xi(\infty) = k\delta_0 + l\delta_{N+1}), \tag{3.10}$$

where in the last equation we used that eventually all particle are absorbed at one of the absorbing states 0 or $N+1$. Conversely, if Eq. (3.9) holds, then, since the absorption probabilities are harmonic for the dual process, by Eq. (3.3) we have

$$\int D(\xi,\mathbf{x})\mu_{T_L,T_R}(\mathrm{d}x) = \int \hat{\mathbb{E}}_\xi[D(\xi(t),\mathbf{x})]\mu_{T_L,T_R}(\mathrm{d}x) = \int \mathbb{E}_\mathbf{x}[D(\xi,\mathbf{x}(t))]\mu_{T_L,T_R}(\mathrm{d}x),$$

which shows invariance of μ_{T_L,T_R}. □

IV. TEMPERATURE PROFILE

The following proposition shows the convergence to and linearity of the limiting temperature profile (linearity of the stationary temperature profile has been shown before in Ref. 9).

Proposition 3:

a) *Starting from $x \in \mathbb{R}^N$, the local temperature at site i satisfies*

$$\mathbb{E}_x(x_i^2(t)) = \hat{\mathbb{E}}_i(x_{X(t)}^2) = \sum_j p_t(i,j)x_j^2, \tag{4.1}$$

where $X(t)$ is continuous-time simple symmetric random walk jumping at rate 2, and absorbed at 0, $N+1$, and where $p_t(i,j)$ denotes the transition probability of this random walk to go from i to j in time t.

b) *The stationary temperature profile is given by*

$$T_i := \mu_{T_L,T_R}(x_i^2) = T_L\hat{\mathbb{P}}_i(X(\infty)=0) + T_R\hat{\mathbb{P}}_i(X(\infty)=N+1) = T_L\left(1-\frac{i}{N+1}\right) + T_R\left(\frac{i}{N+1}\right). \tag{4.2}$$

c) *For the stationary local temperature autocorrelation function we have*

$$\mathbb{E}_{\mu_{T_L,T_R}}(x_i^2(0)x_j^2(t)) - T_iT_j = \sum_{k=1}^N \hat{\mathbb{P}}_j(X_t=k,\tau>t)\mu_{T_L,T_R}(x_i^2x_k^2) - T_iT_j\hat{\mathbb{P}}_j(\tau>t), \tag{4.3}$$

where τ is the absorption time of the random walk $\{X_t: t\geqslant 0\}$. As a consequence,

$$\lim_{t\to\infty}\frac{\mathbb{E}_{\mu_{T_L,T_R}}(x_i^2(0)x_j^2(t)) - T_iT_j}{\hat{\mathbb{P}}_j(\tau>t)} = -T_iT_j + \frac{1}{N}\sum_{k=1}^N \mu_{T_L,T_R}(x_i^2x_k^2). \tag{4.4}$$

Proof:

a) This follows from Eq. (3.3) and the fact that the dual process starting from a single particle at i is given by the random walk $X(t)$.

b) This follows by taking the limit $t\to\infty$ in Eq. (4.1) and the well-known formula for the absorption probabilities of the random walk $X(t)$.

c) The first statement [Eq. (4.3)] follows from Eqs. (4.1) and (4.2). The second statement [Eq. (4.4)] follows from the fact that the quasistationary distribution of the random walk $X(t)$ is the uniform measure on $\{1,\ldots,N\}$, i.e., for any function $f:\{1,\ldots,N\}\to\mathbb{R}$:

$$\lim_{t\to\infty} \hat{\mathbb{E}}_i(f(X_t)|\tau > t) = \frac{1}{N}\sum_{j=1}^{N} f(j).$$

□

V. STATIONARY TWO POINT CORRELATION FUNCTION

We now study the two point function $\mu_{T_L,T_R}(x_i^2 x_j^2)$. In the dual process we then have two particles starting at sites i and j of which we denote the positions at time t by (X_t, Y_t). From Eq. (3.2) we see that these two particles behave as two independent random walkers (with rate 2) as long as they are not at neighboring sites. When they are at neighbor sites then on the next step they prefer to be on the same site compared to being further separated.

From Eq. (3.9) we infer

$$\mu_{T_L,T_R}(x_i^2 x_j^2) = (1+2\delta_{i,j})(T_L^2 \hat{\mathbb{P}}_{ij}(\{X_\infty, Y_\infty\} = \{0\}) + T_R^2 \hat{\mathbb{P}}_{ij}(\{X_\infty, Y_\infty\} = \{N+1\})$$
$$+ T_L T_R \hat{\mathbb{P}}_{ij}(\{X_\infty, Y_\infty\} = \{0, N+1\})). \tag{5.1}$$

Unfortunately, now the absorption probabilities cannot be obtained by elementary probabilistic considerations since the two random walkers of the dual process are interacting (attracting each other when they are at nearest neighbor sites). We therefore proceed by directly solving the linear equations for $y_{i,j} = \mu_{T_L,T_R}(x_i^2 x_j^2)$. We restrict ourself to the case $1 \leqslant i \leqslant j \leqslant N$ since $\mu_{T_L,T_R}(x_i^2 x_j^2)$ is obviously symmetric. We further denote $y_i = T_i = \mu_{T_L,T_R}(x_i^2)$.

The $y_{i,j}$ then satisfy the following system of linear equations:

$$3y_{i-1,i} - 2y_{i,i} + 3y_{i,i+1} = 0, \quad j = i,$$

$$y_{i-1,i+1} + y_{i+1,i+1} + y_{i,i} + y_{i,i+2} - 8y_{i,i+1} = 0, \quad j = i+1,$$

$$y_{i-1,j} + y_{i+1,j} + y_{i,j-1} + y_{i,j+1} - 4y_{i,j} = 0, \quad j \geqslant i+2, \tag{5.2}$$

with boundary conditions given by

$$y_{i,N+1} = y_i y_{N+1} = y_i T_R, \quad i = 0, \ldots, N+1,$$

$$y_{0,j} = y_0 y_j = T_L y_j, \quad j = 0, \ldots, N+1. \tag{5.3}$$

Plugging in the ansatz

$$y_{i,j} = A + Bi + Cj + Dij,$$

we find that these equations have the following solution: if $j > i$ then

$$y_{i,j} = T_L^2 + i\frac{T_R - T_L}{N+1}\left(T_L + 2\frac{T_R - T_L}{N+3}\right) + j\frac{T_L(T_R - T_L)}{N+1} + ij\frac{(T_R - T_L)^2}{(N+1)(N+3)}, \tag{5.4}$$

and for the diagonal $i = j$ we have

$$y_{i,i} = 3\left(T_L^2 - \frac{(T_R - T_L)^2}{(N+1)(N+3)}\right) + 6i\frac{T_R - T_L}{N+1}\left(T_L + \frac{T_R - T_L}{N+3}\right) + 3i^2\frac{(T_R - T_L)^2}{(N+1)(N+3)}. \tag{5.5}$$

The solution can also be written as a quadratic form in T_L and T_R and then the coefficients of the quadratic form add up to 1 outside of diagonal and to 3 on the diagonal [as can be seen directly from Eq. (5.1)]. Explicitly

$$y_{i,j} = \left(1 - \frac{i}{N+3}\right)\left(1 - \frac{j}{N+1}\right)T_L^2 + \frac{i(2+j)}{(N+1)(N+3)}T_R^2 + \left[1 - \left(1 - \frac{i}{N+3}\right)\left(1 - \frac{j}{N+1}\right) - \frac{i(2+j)}{(N+1)(N+3)}\right]T_L T_R \tag{5.6}$$

and

$$y_{i,i} = 3\left[\left(1 - \frac{i}{N+3}\right)\left(1 - \frac{i}{N+1}\right) - \frac{1}{(N+1)(N+3)}\right]T_L^2 + 3\left[\frac{i(2+i)}{(N+1)(N+3)} - \frac{1}{(N+1)(N+3)}\right]T_R^2 + 3\left[1 - \left(1 - \frac{i}{N+3}\right)\left(1 - \frac{i}{N+1}\right) - \frac{i(2+i)-2}{(N+1)(N+3)}\right]T_L T_R. \tag{5.7}$$

Comparing this with Eq. (5.1) we can *read of* the expressions for the absorption probabilities. For example, for $i<j$

$$\hat{\mathbb{P}}_{ij}(X_\infty = 0, Y_\infty = 0) = \left(1 - \frac{i}{N+3}\right)\left(1 - \frac{j}{N+1}\right). \tag{5.8}$$

Remark that this probability is larger than in the case of independent random walkers where the expression would be $(1-(i/N+1))(1-(j/N+1))$. This means that due to the attractive interaction of the walkers, they have a (small) preference to be absorbed at the same site. This effect is, however, negligible (of order $1/N$) as $N\to\infty$.

Finally, we compute the covariance

$$c_{i,j} = \mu_{T_L,T_R}(x_i^2 x_j^2) - \mu_{T_L,T_R}(x_i^2)\mu_{T_L,T_R}(x_j^2)$$

and we obtain

$$c_{i,j} = \frac{2i(N+1-j)}{(N+3)(N+1)^2}(T_R - T_L)^2, \tag{5.9}$$

$$c_{i,i} = \frac{(2N^2+8N+3)T_L^2 + 6T_L T_R - 3T_R^2}{(N+1)(N+3)} + i\frac{2(T_R - T_L)((3+2N)T_L + 3T_R)}{(N+1)(N+3)} + i^2\frac{2N(T_R - T_L)^2}{(N+1)^2(N+3)}. \tag{5.10}$$

Remark 2: *Notice that the covariance $c_{i,j}$ manifests the presence of long-range correlations which are believed to be typical for nonequilibrium steady states.*[19] *The fact that the covariance is positive is due to the attractive character of the interaction between the dual walkers. This has to be contrasted with the case of the SEP model where the covariance is negative and the interaction between walkers is repulsive.*

We notice that in $c_{i,j}$ appears Green's function of the simple (continuous time) random walk X_t,

$$G(i,j) = \frac{2i(N+1-j)}{(N+1)}, \tag{5.11}$$

i.e., the expected total time spent at i starting at j before being absorbed at 0 or $N+1$. Moreover, for $0<\alpha_1<\alpha_2<1$,

$$\lim_{N\to\infty} N c_{N\alpha_1, N\alpha_2} = 2(T_R - T_L)^2\alpha_1(1-\alpha_2). \tag{5.12}$$

From Eq. (5.12), in the spirit of Ref. 14, one expects that in the limit $N\to\infty$ the random distributions

$$Y_\alpha^N := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (x_i^2 - T_\alpha)\delta(\alpha - i/N)$$

converge jointly to a Gaussian random field with covariance

$$C(\alpha_1, \alpha_2) = 2(T_L - T_R)^2 \alpha_1 (1 - \alpha_2) + 2T^2(\alpha_1)\delta(\alpha_1 - \alpha_2),$$

where

$$T(\alpha) = \lim_{N\to\infty} T_{\lfloor \alpha N \rfloor} = (1-\alpha)T_L + \alpha T_R, \tag{5.13}$$

with $\lfloor \alpha N \rfloor$ denoting the integer part of αN.

VI. LOCAL EQUILIBRIUM

From the computation of the stationary two point correlation, we infer, in particular, that for $\alpha \in (0,1)$,

$$\lim_{N\to\infty} \mu_{T_L,T_R}(x_{\lfloor \alpha N \rfloor}^4) = 3T^2(\alpha) = \frac{1}{\sqrt{2\pi T(\alpha)}} \int x^4 e^{-x^2/2T(\alpha)} dx,$$

while for each $i \in \mathbb{N}, i > 0$,

$$\lim_{N\to\infty} \mu_{T_L,T_R}(x_{\lfloor \alpha N \rfloor}^2 x_{\lfloor \alpha N \rfloor + i}^2) = T^2(\alpha) = \left(\frac{1}{\sqrt{2\pi T(\alpha)}} \int x^2 e^{-x^2/2T(\alpha)} dx \right)^2,$$

where T_α is defined in Eq. (5.13).

This suggests that "around each macroscopic point α," which we associate with the micropoint $\lfloor \alpha N \rfloor$, there is a Gaussian "local equilibrium" distribution with variance $T(\alpha)$. More precisely we give the following definition cf. Ref. 11.

Definition 1: *Let T_L and T_R be* fixed *and let $\alpha \in (0,1)$. We say that local equilibrium holds if for all $n \in \mathbb{N}$, for all $k_1, \ldots, k_n \in \mathbb{N}$, and for all $i_1, \ldots, i_n \in \{1, \ldots, N\}$* fixed, *we have*

$$\lim_{N\to\infty} \int D\left(\sum_{l=1}^{n} k_l \delta_{\lfloor \alpha N \rfloor + i_l}, \mathbf{x} \right) \mu_{T_L,T_R}(dx) = \prod_{l=1}^{n} \int \frac{x^{2k_l}}{(2k_l - 1)!!} \rho_{T(\alpha)}(dx) = [T(\alpha)]^{\sum_{l=1}^n k_l}, \tag{6.1}$$

where $\rho_{T(\alpha)}$ denotes the measure of a centered Gaussian variable with variance $T(\alpha)$.

This is equivalent with the requirement that

$$\lim_{N\to\infty} \tau_{\lfloor \alpha N \rfloor}(\mu_{T_L,T_R}) = \mathcal{G}(T(\alpha)), \tag{6.2}$$

where $\tau_{\lfloor \alpha N \rfloor}$ denotes spatial shift, $\mathcal{G}(\sigma)$ denotes the product measure on $\mathbb{R}^{\mathbb{N}}$ with marginals that are normally distributed with mean zero and variance σ^2, and where the limit is in the sense that expectations of polynomials of type $D(\xi, \mathbf{x})$ converge to the corresponding expectations in the Gaussian measure.

The following lemma shows that factorization of the absorption probabilities is sufficient for local equilibrium.

Lemma 1: *Let T_L and T_R be* fixed *and let $\alpha \in (0,1)$. Suppose that for all $n \in \mathbb{N}$, for all $k_1, \ldots, k_n \in \mathbb{N}$, for all $i_1, \ldots, i_n \in \{1, \ldots, N\}$* fixed, *and for all $K, L \in \mathbb{N}$ with $K+L = \Sigma_{l=1}^n k_l$,*

$$\lim_{N\to\infty} \hat{\mathbb{P}}_{\Sigma_{l=1}^n k_l \delta_{\lfloor \alpha N \rfloor + i_l}}(\xi(\infty) = K\delta_0 + L\delta_{N+1}), \tag{6.3}$$

$$= \lim_{N\to\infty} \hat{\mathbb{P}}_{\lfloor \alpha N \rfloor}(X(\infty) = 0)^K \hat{\mathbb{P}}_{\lfloor \alpha N \rfloor}(X(\infty) = N+1)^L, \tag{6.4}$$

then local equilibrium holds.

Proof: Combination of Eq. (3.9) with Eq. (6.4) gives

$$\begin{aligned}
&\lim_{N\to\infty} \int D\left(\sum_{l=1}^{n} k_l \delta_{\lfloor \alpha N\rfloor + i_l, \mathbf{x}}\right) \mu_{T_L,T_R}(\mathrm{d}x) \\
&= \sum_{K,M:K+M=k_1+\cdots+k_n} \lim_{N\to\infty} \hat{\mathbb{P}}_{\lfloor \alpha N\rfloor}(X(\infty)=0)^M \hat{\mathbb{P}}_{\lfloor \alpha N\rfloor}(X(\infty)=N+1)^K T_L^M T_R^K \\
&= \lim_{N\to\infty} (\hat{\mathbb{P}}_{\lfloor \alpha N\rfloor}(X(\infty)=0)T_L + \hat{\mathbb{P}}_{\lfloor \alpha N\rfloor}(X(\infty)=N+1)T_R)^{k_1+\cdots+k_n} \\
&= \lim_{N\to\infty} T_{\lfloor \alpha N\rfloor}^{k_1+\cdots+k_n} = T(\alpha)^{k_1+\cdots+k_n}. \qquad (6.5)
\end{aligned}$$

□

In order to see heuristically why the factorization property for the absorption probabilities holds, it suffices to see that if we start n dual random walkers $X_1(t),\ldots,X_n(t)$ from initial positions $\lfloor \alpha N\rfloor + i_l$, $l=1,\ldots,n$ then we can couple them with n independent random walkers $X_1'(t),\ldots,X_n'(t)$ started at the same initial positions such that for all $\epsilon>0$ and for all $l\in\{1,\ldots,n\}$,

$$X_l(t) - X_l'(t) \leqslant \epsilon\sqrt{t}, \qquad (6.6)$$

with probability close to 1 for t large enough. Indeed, if dual walker l is absorbed at 0, then this happens at a time of the order N^2, and then, for large N, Eq. (6.6) tells that the corresponding independent walker is at distance less than ϵN from 0 at that time. Therefore, the probability that this independent walker is absorbed at $N+1$ is at most $\epsilon N/(N+1)\leqslant\epsilon$. So the probability (in the coupling) that dual walker X_l and corresponding independent walker X_l' are absorbed at different points is less than ϵ. Therefore, if we have the coupling with property (6.6), we have for all $\theta_1,\ldots,\theta_n\in\{0,N+1\}$,

$$\lim_{N\to\infty} (\hat{\mathbb{P}}(X_1(\infty)=\theta_1,\ldots,X_n(\infty)=\theta_l) - \mathbb{P}'(X_1'(\infty)=\theta_1,\ldots,X_n'(\infty)=\theta_l)) = 0,$$

where $\hat{\mathbb{P}}$ refers to the probability measure on path space for dual walkers starting at positions $\lfloor \alpha N\rfloor + i_1,\ldots,\lfloor \alpha N\rfloor + i_n$, and $\mathbb{P}'$ refers to the probability for independent walkers starting at positions $\lfloor \alpha N\rfloor + i_1,\ldots,\lfloor \alpha N\rfloor + i_n$.

To see that such a coupling exists, we observe that in the dual process there is only interaction of the walkers when they are at neighboring positions. In that case they jump with higher rate (than independent walkers) to the same position. The coupling then consists in letting the independent walkers and the dual walkers perform the same jumps and having extra jumps for the dual walkers (which are not performed by the independent walkers) when they are at neighboring positions. The total time that dual walkers are at neighboring positions in the time interval $[0,t]$ is less than $t^{(1/2)+\delta}$ with probability close to 1. So the difference between the position of the independent walker and the dual walker is a sum of the order of $t^{(1/2)+\delta}$ independent increments taking values ±1, which is bounded by $(t^{(1/2)+\delta})^{(1/2)+\delta}\leqslant t^{(1/2)-\delta'}$ with high probability.

A similar idea of coupling has been implemented in the context of the simple SEP, see Ref. 5. For the full proof of the factorization property along these lines we, however, refer to future work.

VII. THE ENERGY-DIFFUSION MODEL AND THE SYMMETRIC EXCLUSION PROCESS FAMILY: BOSONS VERSUS FERMIONS

In this section we shall see the formal relation between the SEP family and the energy-diffusion model. This can help as a guide to see the similarities in methods for treating both cases.

We shall first consider a generalization of the energy-diffusion model to the case in which there are m momenta per site, and kinetic energy is exchanged between any two momenta in neighboring sites. The generator is again Eq. (2.1), with now

$$L_1 = \sum_{\alpha=1}^{m} \left[T_L \frac{\partial^2}{\partial x_{1,\alpha}^2} - x_{1,\alpha} \frac{\partial}{\partial x_{1,\alpha}} \right], \tag{7.1}$$

$$L_N = \sum_{\alpha=1}^{m} \left[T_R \frac{\partial^2}{\partial x_{N,\alpha}^2} - x_{N,\alpha} \frac{\partial}{\partial x_{N,\alpha}} \right], \tag{7.2}$$

$$L_{i,i+1} = \frac{1}{m} \sum_{\alpha,\beta=1}^{m} \left(x_{i,\alpha} \frac{\partial}{\partial x_{i+1,\beta}} - x_{i+1,\beta} \frac{\partial}{\partial x_{i,\alpha}} \right)^2. \tag{7.3}$$

The factor $1/m$ multiplying $L_{i,i+1}$ is rather arbitrary, it sets the time scale. We wish to show the connection of this family of models (labeled by m) with the (partial) exclusion process, with maximal occupancy n, and in which the jumping rate is proportional both to the number of particles on the departure configuration and to the number of holes (n minus the number of particles) in the arrival configuration. The evolution operator of this process can be written as the Hamiltonian H of the spin $j=n/2$ ferromagnet[18]

$$H = -L_{\mathrm{SEP}}^{*}, \tag{7.4}$$

with

$$L_{\mathrm{SEP}}^{*} = \frac{1}{j} \sum_i (J_i^+ J_{i+1}^- + J_i^- J_{i+1}^+ + 2J_i^o J_{i+1}^o - 2j^2) + \alpha(J_1^- - J_1^o - j) + \gamma(J_1^+ + J_1^o - j)$$
$$+ \delta(J_L^- - J_L^o - j) + \beta(J_L^+ + J_L^o - j). \tag{7.5}$$

The factor $1/j$ is analogous to the factor $1/m$ in Eq. (7.3). The operators J_i^+, J_i^-, J_i^o act on the Hilbert space corresponding to $0 \leqslant r \leqslant n$ particles per site $\otimes_i |r\rangle_i$ as follows:

$$J_i^+ |r\rangle_i = (2j - r)|r\rangle_i,$$

$$J_i^- |r\rangle_i = r|r\rangle_i,$$

$$J_i^o |r\rangle_i = (r - j)|r\rangle_i. \tag{7.6}$$

They satisfy the commutation relations of the SU(2) algebra:

$$[J_i^o, J_i^{\pm}] = \pm J_i^{\pm}, \quad [J_i^-, J_i^+] = -2J_i^o, \tag{7.7}$$

and they can be transformed to the conventional SU(2) matrices by a similarity transformation.

Representations are labeled by the squared angular momentum operator

$$J^2 |jM\rangle = j(j+1)|jM\rangle, \tag{7.8}$$

with $j=n/2$, so that the ordinary SEP [with (0, 1) occupation] corresponds to a representation of spin 1/2.

Going back to the generalized model defined above, we can rewrite the generator of the energy diffusion process as the Hamiltonian $H=-L^*$ with

$$L^* = \frac{4}{m} \sum_i \left(K_i^+ K_{i+1}^- + K_i^- K_{i+1}^+ - 2K_i^o K_{i+1}^o + \frac{m^2}{8} \right) + 2\left(T_1 K_1^+ - K_1^o - \frac{m}{4} \right) + 2\left(T_L K_L^+ - K_L^o - \frac{m}{4} \right), \tag{7.9}$$

where we have defined, in each site,

$$K_i^+ = \frac{1}{2}\sum_\alpha x_{i,\alpha}^2,$$

$$K_i^- = \frac{1}{2}\sum_\alpha \frac{\partial^2}{\partial x_{i,\alpha}^2},$$

$$K_i^o = \frac{1}{4}\sum_\alpha \left\{ \frac{\partial}{\partial x_{i,\alpha}} x_{i,\alpha} + x_{i,\alpha}\frac{\partial}{\partial x_{i,\alpha}} \right\}. \tag{7.10}$$

These satisfy the SU(1,1) relations

$$[K_i^o, K_i^\pm] = \pm K_i^\pm, \quad [K_i^-, K_i^+] = 2K_i^o \tag{7.11}$$

[note the sign difference with respect to SU(2)]. Representations are labeled in a manner analogous to SU(2):

$$K_i^2|kM\rangle = [K_i^o]^2 - \frac{1}{2}[K_i^+K_i^- + K_i^-K_i^+]|kM\rangle = k(k-1)|kM\rangle,$$

$$K_i^o|kM\rangle = m|kM\rangle. \tag{7.12}$$

To identify which is the representation (i.e., the value of k), we check the value of K_i^2 as applied to the constant (which is the zero eigenvalue eigenvector of L^*):

$$K_i^-|1\rangle = 0, \quad K_i^o|1\rangle = k|1\rangle = \frac{m}{4}|1\rangle, \quad K_i^2|1\rangle = k(k-1)|1\rangle = \frac{m}{4}\left(\frac{m}{4} - 1\right)|1\rangle. \tag{7.13}$$

Hence $k=m/4$, and, in particular, $k=1/4$ for the process with one velocity per site.

Consider the coherent-state representation of vectors and operators (cf. Refs. 16, 12, and 13):

$$\psi(z_i) = \langle z_i|\psi\rangle \quad \text{with } |z_i\rangle = e^{z_i^* K_i^+}|0\rangle,$$

$$\psi(z_i) = \langle z_i|\psi\rangle \quad \text{with } |z_i\rangle = e^{z_i^* J_i^+}|0\rangle \tag{7.14}$$

(where $|0\rangle$ is the state annihilated by J_i^- or K_i^-). The group operators act on such states as

$$\langle z_i|K_i^+|\psi\rangle = \left[z_i^2\frac{\partial}{\partial z_i} + 2kz_i \right]\langle z_i|\psi\rangle,$$

$$\langle z_i|K_i^-|\psi\rangle = \frac{\partial}{\partial z_i}\langle z_i|\psi\rangle,$$

$$\langle z_i|K_i^o|\psi\rangle = \left[z_i\frac{\partial}{\partial z_i} + k \right]\langle z_i|\psi\rangle, \tag{7.15}$$

and

$$\langle z_i|J_i^+|\psi\rangle = \left[-z_i^2\frac{\partial}{\partial z_i} + 2jz_i \right]\langle z_i|\psi\rangle,$$

$$\langle z_i|J_i^-|\psi\rangle = \frac{\partial}{\partial z_i}\langle z_i|\psi\rangle,$$

$$\langle z_i|J_i^o|\psi\rangle = \left[z_i\frac{\partial}{\partial z_i} - j\right]\langle z_i|\psi\rangle. \tag{7.16}$$

Now, writing the generator of the energy diffusion model and the SEP models in these representations, it is easy to check that models with $k=m/4$ of one class formally map into models with $-j=-n/2$ of the other class, albeit with somewhat different boundary terms.

Another way to see the relation between these two families of models is to note that we can rewrite the generator L of the energy transport model *with even m* in terms of $m/2$ bosons $a_{i\alpha}$ and $m/2$ bosons $b_{i\alpha}$ in each site

$$a_{i\alpha}a_{j\beta}^\dagger - a_{j\beta}^\dagger a_{i\alpha} = \delta_{ij}\delta_{\alpha\beta}, \quad b_{i\alpha}b_{j\beta}^\dagger - b_{j\beta}^\dagger b_{i\alpha} = \delta_{ij}\delta_{\alpha\beta} \tag{7.17}$$

(all other commutators vanish) and writing

$$K_i^+ = \sum_\alpha a_{i\alpha}^\dagger b_{i\alpha}^\dagger, \quad K_i^- = \sum_\alpha b_{i\alpha}a_{i\alpha},$$

$$K_i^o = \frac{1}{2}\sum_\alpha (a_{i\alpha}^\dagger a_{i\alpha} + b_{i\alpha}^\dagger b_{i\alpha}) + r. \tag{7.18}$$

We can similarly write the generalized SEP with allowed occupancy n with n *fermions* $a_{i\alpha}$ and n fermions $b_{i\alpha}$ in each site

$$a_{i\alpha}a_{j\beta}^\dagger + a_{j\beta}^\dagger a_{i\alpha} = \delta_{ij}\delta_{\alpha\beta}, \quad b_{i\alpha}b_{j\beta}^\dagger + b_{j\beta}^\dagger b_{i\alpha} = \delta_{ij}\delta_{\alpha\beta} \tag{7.19}$$

(all other anti-commutators vanish) and writing

$$J_i^+ = \sum_\alpha a_{i\alpha}^\dagger b_{i\alpha}^\dagger, \quad J_i^- = \sum_\alpha b_{i\alpha}a_{i\alpha},$$

$$J_i^o = \frac{1}{2}\sum_\alpha (a_{i\alpha}^\dagger a_{i\alpha} + b_{i\alpha}^\dagger b_{i\alpha}) - n. \tag{7.20}$$

Hence, SEP and energy diffusion models are essentially fermionic and bosonic counterparts of the same models. This is perhaps not surprising, in view of the fact that Poisson processes (as the SEP) are related to Fermions, just as Gaussian processes are related to bosons.[7,2]

VIII. ASYMMETRIC GENERALIZATIONS

In this last section we briefly indicate how to introduce asymmetry in our model without breaking the bulk energy conservation law. This is in the spirit of introducing a bias to move in a preferred direction in the simple SEP.

Consider as the first example two neighboring sites with momenta x,y. Going back to Eq. (2.6), we may add to the diffusion term a bias:

$$\mathcal{A}^{\text{drift}} = \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)^2 + E(x,y)\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \tag{8.1}$$

and its obvious generalization to a chain then becomes the asymmetric analog of the bulk part of Eq. (2.1),

$$\sum_i L_{i,i+1}^{\text{drift}} = \sum_i \left(x_i\frac{\partial}{\partial x_{i+1}} - x_{i+1}\frac{\partial}{\partial x_i}\right)^2 + E(x_i,x_{i+1})\left(x_i\frac{\partial}{\partial x_{i+1}} - x_{i+1}\frac{\partial}{\partial x_i}\right). \tag{8.2}$$

A possible choice for the drift function $E(x,y)$ is $E(x,y)=Exy$. This form of the drift has the advantage of being a product of two K operators in Eq. (7.10), which is also the case if one goes from the symmetric to the asymmetric exclusion process (in the quantum spin chain language).

ACKNOWLEDGMENTS

The authors thank B. Derrida and E. Presutti for helpful discussions. We wish to thank H. Spohn for a remark about the asymmetric generalizations. One of the authors (C.G.) acknowledges the kind hospitality of Ecole Superieure de Physique e Chimie Industrielles and the support of European Science Foundation (ESF) and Gruppo Nazionale di Fisica Matematica (GNFM).

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