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# Nonlinear Diffusion on the 2D Euclidean Motion Group 

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#### Abstract

Linear and nonlinear diffusion equations are usually considered on an image, which is in fact a function on the translation group. In this paper we study diffusion on orientation scores, i.e. on functions on the Euclidean motion group SE(2). An orientation score is obtained from an image by a linear invertible transformation. The goal is to enhance elongated structures by applying nonlinear left-invariant diffusion on the orientation score of the image. For this purpose we describe how we can use Gaussian derivatives to obtain regularized left-invariant derivatives that obey the non-commutative structure of the Lie algebra of $\mathrm{SE}(2)$. The Hessian constructed with these derivatives is used to estimate local curvature and orientation strength and the diffusion is made nonlinearly dependent on these measures. We propose an explicit finite difference scheme to apply the nonlinear diffusion on orientation scores. The experiments show that preservation of crossing structures is the main advantage compared to approaches such as coherence enhancing diffusion.


## 1 Introduction

A scale space of a scalar-valued image is obtained by solving an evolution equation on the additive group $\left(\mathbb{R}^{n},+\right)$, i.e. the translation group. The most widely used evolution equation is the diffusion equation, which in the linear case leads to the Gaussian scale space [1] [2. In the nonlinear case with an isotropic diffusion tensor it leads to a nonlinear scale space of Perona and Malik type [3]. An anisotropic diffusion tensor leads to edge- or coherence-enhancing diffusion 4 .

Recently, processing of tensor images gains attention, for instance in Diffusion Tensor Imaging (DTI). A related type of data are orientation scores [5] [6], where orientation is made an explicit dimension. Orientation scores arise naturally in high angular resolution diffusion imaging, but can also be created out of an image by applying a wavelet transform [6]. Both tensor images and orientation scores have in common that they contain richer information on local orientation. They

[^0]can both be considered as functions on the Euclidean motion group $S E(2)=$ $\mathbb{R}^{2} \rtimes \mathbb{T}$, i.e. the group of all 2 D rotations and translations. This richer structure is often overlooked, e.g. if one applies component-wise nonlinear diffusion on tensor images [7]. When processing tensor images or orientation scores, it is actually more natural to define the evolution equation on the Euclidean motion group, leading to scale spaces on the Euclidean motion group.

In this paper we will introduce the analogue of nonlinear diffusion on the Euclidean motion group, with the goal to enhance oriented structures or patterns in two-dimensional images. We will start with introducing orientation scores and (nonlinear) diffusion in orientation scores in more detail. We will propose nonlinear conductivity functions to enable a coherence enhancing diffusion operation in orientation scores, which can handle crossings and adapts to the curvature of line structures. An explicit numerical finite difference scheme will be presented that has good rotational invariance. Finally we will show examples of coherence enhancing diffusion in orientation scores on applications with crossing and curved line structures.

This paper focusses on nonlinear diffusion on $\mathrm{SE}(2)$ and how to operationalize this. Scale spaces on Lie groups in general are treated in [8].

## 2 Orientation Scores

An orientation score is a function $U \in \mathbb{L}_{2}(S E(2))$. Such a function has one additional dimension compared to the original image, which explicitly encodes information on local orientations in the image. An example is shown in Figure 1 a-b. The domain of the orientation score can be parameterized by the group elements $g=(\mathbf{x}, \theta)$ where $\mathbf{x}=(x, y) \in \mathbb{R}^{2}$ are the two spatial variables that label the domain of the image $f$, and $\theta \bmod 2 \pi$ is the orientation angle that captures the orientation of structures in image $f$. The group product and group inverse of elements in $S E(2)$ are given by

$$
\begin{equation*}
g g^{\prime}=(\mathbf{x}, \theta)\left(\mathbf{x}^{\prime}, \theta^{\prime}\right)=\left(\mathbf{x}+\mathbf{R}_{\theta} \mathbf{x}^{\prime}, \theta+\theta^{\prime} \bmod 2 \pi\right), \quad g^{-1}=\left(-\mathbf{R}_{\theta}^{-1} \mathbf{x},-\theta\right) \tag{1}
\end{equation*}
$$

We will use both short notation $g$ and explicit notation $(\mathbf{x}, \theta)$ for group elements.
An orientation score $U_{f}: \mathbb{R}^{2} \rtimes \mathbb{T} \rightarrow \mathbb{C}$ of an image ${ }^{1} f \in \mathbb{L}_{2}\left(\mathbb{R}^{2}\right)$ is obtained by convolving the image with an anisotropic convolution kernel $K \in \mathbb{L}_{2}\left(\mathbb{R}^{2}\right)$,

$$
\begin{equation*}
U_{f}(\mathbf{x}, \theta)=\left(K^{\theta} * f\right)(\mathbf{x})=\int_{\mathbb{R}^{2}} K\left(\mathbf{R}_{\theta}^{-1}\left(\mathbf{x}-\mathbf{x}^{\prime}\right)\right) f\left(\mathbf{x}^{\prime}\right) d \mathbf{x}^{\prime} \tag{2}
\end{equation*}
$$

where $K(\mathbf{x})$ is the kernel with orientation $\theta=0$, and $\mathbf{R}_{\theta}$ is the rotation matrix $\mathbf{R}_{\theta}=\left(\begin{array}{cc}\cos \theta & -\sin \theta \\ \sin \theta & \cos \theta\end{array}\right)$. For some choices of $K$ there exists a stable inverse transformation [6], which is obtained by either convolving $U(\cdot, \theta)$ with the mirrored conjugate kernel of $K$ followed by integration over $\theta$, or simply by $f=\int_{0}^{2 \pi} U(\mathbf{x}, \theta) d \theta$.

[^1]

Fig. 1. (a) Example of an image with concentric circles. (b) The structure of the corresponding orientation score. The circles become spirals and all spirals are situated in the same helicoid-shaped plane. Note that the $\theta$-dimension is periodic. (c) Real part of the orientation score $U$ displayed for 4 different orientations. (d) The absolute vale $|U|$ yields a phase-invariant response displayed for 4 orientations. (e) Real part of the kernel with $\theta=0$ and parameter values $k=2, q=8, t=400, s=10, n_{\theta}=64$. (f) Imaginary part. (g) Fourier tranform. (h) Fourier transform of the net operation, i.e. orientation score transformation followed by the inverse orientation score transformation.

For our purpose, invertibility is required to be able to obtain an enhanced image after applying non-linear diffusion in the orientation score of an image.

In practice the $\theta$-dimension is sampled with steps $\frac{2 \pi}{n_{\theta}}$ where $n_{\theta}$ the number of samples. To emphasize discretization we will use the notation $U[\mathbf{x}, l]=U\left(\mathbf{x}, l \cdot s_{\theta}\right)$ with $\mathbf{x} \in\left[0,1, \ldots, N_{x}-1\right] \times\left[0,1, \ldots, N_{y}-1\right], l \in\left[0,1, \ldots n_{\theta}-1\right]$, and $s_{\theta}=\frac{2 \pi}{n_{\theta}}$.

Note that if the operation in performed in the orientation score is linear, the net operation is just a linear filter operation on the original image. Therefore it is very natural to consider nonlinear evolution equations on orientation scores.

### 2.1 An Invertible Orientation Score Transformation

To transform images to orientation scores using (2) for the purpose of nonlinear diffusion we need a kernel $K$ with the following properties

1. A finite number of orientations.
2. Reconstruction by summing all orientations.
3. Directional kernel, i.e. the kernel should be a convex cone in the Fourier domain 9 .
4. Localization in the spatial domain.
5. Quadrature property [10. This is especially useful since the absolute value $|U|$ of the resulting complex-valued orientation score will render a phase invariant signal responding to both edges and ridges.

Based on these properties we propose the following kernel

$$
\begin{equation*}
K(\mathbf{x})=\frac{1}{N} \mathcal{F}^{-1}\left[\boldsymbol{\omega} \mapsto B^{k}\left(\frac{(\varphi \bmod 2 \pi)-\pi / 2}{s_{\theta}}\right) f(\rho)\right](\mathbf{x}) G_{s}(\mathbf{x}) \tag{3}
\end{equation*}
$$

where $N$ is the normalization constant, $\boldsymbol{\omega}=(\rho \cos \varphi, \rho \sin \varphi), B^{k}$ denotes the $k$ th order B-spline given by

$$
B^{k}(x)=\left(B^{k-1} * B^{0}\right)(x), \quad B^{0}(x)= \begin{cases}1 & \text { if }-1 / 2<x<+1 / 2  \tag{4}\\ 0 & \text { otherwise }\end{cases}
$$

Function $f(\rho)$ specifies the radial function in the Fourier domain, chosen as the Gaussian divided by its Taylor series up to order $q$ to ensure a slower decay, i.e.

$$
\begin{equation*}
f(\rho)=G_{t}(\rho)\left(\sum_{i=0}^{q}\left(\left.\frac{d}{d \rho^{\prime}} G_{t}\left(\rho^{\prime}\right)\right|_{\rho^{\prime}=0}\right) \frac{\rho^{i}}{i!}\right)^{-1}, \quad G_{t}(\rho)=\frac{1}{2 \sqrt{\pi t}} e^{-\frac{\rho^{2}}{4 t}} \tag{5}
\end{equation*}
$$

Function $G_{s}$ in (3) is a Gaussian kernel with scale $s$, which ensures spatial locality. Figure 1 shows an example of this orientation score transformation.

## 3 Diffusion on the Euclidean Motion Group

### 3.1 Left-Invariant Derivatives

We want to construct the diffusion equation from left-invariant differential operators on orientation scores. An operator $\Upsilon: \mathbb{L}_{2}(S E(2)) \rightarrow \mathbb{L}_{2}(S E(2))$ is leftinvariant if $\mathcal{L}_{g} \Upsilon U=\Upsilon \mathcal{L}_{g} U$, for all $g \in G$ and for all $U \in \mathbb{L}_{2}(S E(2))$, where $\mathcal{L}_{g}: \mathbb{L}_{2}(S E(2)) \rightarrow \mathbb{L}_{2}(S E(2))$ is given by $\left(\mathcal{L}_{g} U\right)(h)=U\left(g^{-1} h\right)$. This property is important because a left-invariant operator in an orientation score implies that the net operation on the corresponding image is rotation invariant. The differential operators $\partial_{x}$ and $\partial_{y}$ (we will consistently use the shorthand notation $\partial_{x}$ for derivative operators corresponding to partial derivative $\partial / \partial x$ ) on the orientation score are not left-invariant and are therefore unsuitable. However, the differential operators $\left\{\partial_{\xi}, \partial_{\eta}, \partial_{\theta}\right\}$, where

$$
\begin{equation*}
\partial_{\xi}(g)=\cos \theta \partial_{x}+\sin \theta \partial_{y}, \quad \partial_{\eta}(g)=-\sin \theta \partial_{x}+\cos \theta \partial_{y}, \quad \partial_{\theta}(g)=\partial_{\theta} \tag{6}
\end{equation*}
$$

with $g=(\mathbf{x}, \theta)$, are all left-invariant, see [11] for a derivation. Consequently, all combinations of the operators $\left\{\partial_{\xi}, \partial_{\eta}, \partial_{\theta}\right\}$ are also left-invariant. The tangent space at $g$ is spanned by $\left\{\partial_{\xi}, \partial_{\eta}, \partial_{\theta}\right\}$. To distinguish between the derivative operator at $g$ and the basis of the tangent space at $g$ we will use the following notation for the latter

$$
\begin{equation*}
\left\{\mathbf{e}_{\xi}(g), \mathbf{e}_{\eta}(g), \mathbf{e}_{\theta}(g)\right\}=\left\{\cos \theta \mathbf{e}_{x}+\sin \theta \mathbf{e}_{y},-\sin \theta \mathbf{e}_{x}+\cos \theta \mathbf{e}_{y}, \mathbf{e}_{\theta}\right\} \tag{7}
\end{equation*}
$$

For notational simplicity the dependency on $g$ is omitted further on.
It is very important to note that not all the derivatives $\left\{\partial_{\xi}, \partial_{\eta}, \partial_{\theta}\right\}$ commute. The nonzero commutators (definition $[A, B]=A B-B A$ ) are given by

$$
\begin{equation*}
\left[\partial_{\theta}, \partial_{\xi}\right]=\partial_{\eta}, \quad\left[\partial_{\theta}, \partial_{\eta}\right]=-\partial_{\xi} \tag{8}
\end{equation*}
$$



Fig. 2. Illustrations of Green's functions for different parameter values, obtained using an explicit iterative numerical scheme (Section 7) with end time $t=70$. (a) Shows the effect of a nonzero $D_{11}^{\prime}$ in the spatial plane, i.e. all orientations are summed. Parameters $D_{11}^{\prime}=0.003, D_{22}=1, D_{33}=0$ and $\kappa=0$. (b) Isosurface of (a) in the orientation score. (c) Shows the effect of nonzero $\kappa$. The superimposed circle shows the curvature. Parameters $D_{11}^{\prime}=0, D_{22}=1, D_{33}=0$ and $\kappa=0.06$. (d) Isosurface of (c) in the orientation score, showing the typical spiral shape of the Green's function.

### 3.2 Diffusion Equation

The general diffusion equation for orientation scores using left-invariant derivative operators is

$$
\partial_{t} u=\left(\begin{array}{lll}
\partial_{\theta} & \partial_{\xi} & \partial_{\eta}
\end{array}\right)\left(\begin{array}{lll}
D_{11} & D_{12} & D_{13}  \tag{9}\\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{33}
\end{array}\right)\left(\begin{array}{c}
\partial_{\theta} \\
\partial_{\xi} \\
\partial_{\eta}
\end{array}\right) u=\mathcal{A} u
$$

with $u(\mathbf{x}, \theta ; 0)=U(\mathbf{x}, \theta)$, and $u(\mathbf{x}, 0 ; t)=u(\mathbf{x}, 2 \pi ; t)$. This equation constitutes a scale space on $\operatorname{SE}(2)$ [8]. The solution can be written as $u(\cdot, \cdot ; t)=e^{t} \mathcal{A} U$.

In practice, it makes no sense to consider the full diffusion tensor. If we want the diffusion to be optimal for straight lines with any orientation, we only have to consider the diagonal elements. In that case $D_{22}$ determines the diffusion along the line structure, $D_{33}$ determines the diffusion orthogonal to the line structure, and $D_{11}$ accounts for diffusion between different orientations. For curved lines, diffusion with a diagonal diffusion tensor is not optimal. We can obtain a diffusion process with a curvature $\kappa$ by replacing $\partial_{\xi}$ in the diagonal diffusion equation by $\partial_{\xi}+\kappa \partial_{\theta}$ (i.e. the generator of a curved line), yielding

$$
\partial_{t} u=\left(\begin{array}{lll}
\partial_{\theta} & \partial_{\xi} & \partial_{\eta}
\end{array}\right)\left(\begin{array}{ccc}
D_{11}^{\prime}+D_{22} \kappa^{2} & D_{22} \kappa & 0  \tag{10}\\
D_{22} \kappa & D_{22} & 0 \\
0 & 0 & D_{33}
\end{array}\right)\left(\begin{array}{l}
\partial_{\theta} \\
\partial_{\xi} \\
\partial_{\eta}
\end{array}\right) u
$$

When $\kappa$ is nonzero, the resulting kernels will be curved in the image plane. Figure 2 shows examples of Green's functions of linear evolutions of this type.

## 4 Using Gaussian Derivatives in Orientation Scores

Regularized derivatives on the orientation score are operationalized by $\mathcal{D} e^{t \mathcal{A}} u$ where $\mathcal{D}$ is a derivative of any order constructed from $\left\{\partial_{\xi}, \partial_{\eta}, \partial_{\theta}\right\}$. The order of
the regularization operator and differential operators matters in this case, i.e. the diffusion should come first.

In this paper we restrict ourselves to $D_{22}=D_{33}$ and $\kappa=0$, leading to

$$
\begin{equation*}
\partial_{t} u=\left(D_{11} \partial_{\theta}^{2}+D_{22}\left(\partial_{\xi}^{2}+\partial_{\eta}^{2}\right)\right) u=\left(D_{11} \partial_{\theta}^{2}+D_{22}\left(\partial_{x}^{2}+\partial_{y}^{2}\right)\right) u \tag{11}
\end{equation*}
$$

Since the operators $\partial_{\theta}, \partial_{x}$, and $\partial_{y}$ commute, this equation the same as the diffusion equation in $\mathbb{R}^{3}$. The Green's function is a Gaussian $\frac{1}{8 \sqrt{\pi^{3} t_{s}^{2} t_{o}}} e^{-\frac{x^{2}+y^{2}}{4 t_{s}}}-\frac{\theta^{2}}{4 t_{o}}$ where $t_{o}=t D_{11}$ and $t_{s}=t D_{22}$. In this special case we can use standard (separable) implementations of Gaussian derivatives, but we have to be careful because of the non-commuting operators. A normal $(i, j, k)$ th order Gaussian derivative implementation for a 3D image $f$ is

$$
\begin{equation*}
\partial_{x}^{i} \partial_{y}^{j} \partial_{z}^{k} e^{t\left(\partial_{x}^{2}+\partial_{y}^{2}+\partial_{z}^{2}\right)} f=\partial_{x}^{i} e^{t \partial_{x}^{2}} \partial_{y}^{j} e^{t} \partial_{y}^{2} \partial_{z}^{k} e^{t \partial_{z}^{2}} f \tag{12}
\end{equation*}
$$

where the equality between the left and right side is essential, since it implies separability along the three dimensions. We want to use the same implementations to construct Gaussian derivatives in the orientation scores, meaning that we have to ensure that the same permutation of differential operators and regularization operators is allowed. By noting that

$$
\begin{align*}
& \partial_{\xi}^{i} \partial_{\eta}^{j} \partial_{\theta}^{k} e^{t_{o} \partial_{\theta}^{2}+t_{s}\left(\partial_{\xi}^{2}+\partial_{\eta}^{2}\right)}=\partial_{\xi}^{i} \partial_{\eta}^{j} e^{t_{s}\left(\partial_{x}^{2}+\partial_{y}^{2}\right)} \partial_{\theta}^{k} e^{t_{o} \partial_{\theta}^{2}} \\
& \partial_{\theta}^{k} \partial_{\xi}^{i} \partial_{\eta}^{j} e^{t_{o} \partial_{\theta}^{2}+t_{s}\left(\partial_{\xi}^{2}+\partial_{\eta}^{2}\right)} \neq \partial_{\theta}^{k} e^{t_{o} \partial_{\theta}^{2}} \partial_{\xi}^{i} \partial_{\eta}^{j} e^{t_{s}\left(\partial_{x}^{2}+\partial_{y}^{2}\right)}, \tag{13}
\end{align*}
$$

we conclude that we always should ensure a certain ordering of the derivative operators, i.e. one should first calculate the orientational derivative $\partial_{\theta}$ and then the commuting spatial derivatives $\left\{\partial_{\xi}, \partial_{\eta}\right\}$, which are calculated from the Cartesian derivatives $\left\{\partial_{x}, \partial_{y}\right\}$ using (6). The commutator relations of (8) allow to rewrite the derivatives in this canonical order. For instance, the derivative $\partial_{\xi} \partial_{\theta}$ can be calculated directly with Gaussian derivatives, while $\partial_{\theta} \partial_{\xi}$ should be operationalized as $\partial_{\xi} \partial_{\theta}+\partial_{\eta}$.

Note that one has to be careful with the sampled $\theta$ dimensions of the orientation score. One should ensure to make both the scale $t_{s}$ and the derivatives $\partial_{\theta}^{k}$ dimensionless and consequently independent on the sampling step $s_{\theta}$.

## 5 Curvature Estimation in Orientation Scores

Before turning to nonlinear diffusion, we first discuss how to estimate curvature from orientation scores. Our procedure to measure this is inspired by van Ginkel [12]. Suppose we have at position $g_{0}$ a tangent vector $\mathbf{v}\left(g_{0}\right)=v_{\theta} \mathbf{e}_{\theta}+v_{\xi} \mathbf{e}_{\xi}+v_{\eta} \mathbf{e}_{\eta}$. Similar to the concept of a tangent line in $\mathbb{R}^{3}$ we can define a "tangent spiral" in an orientation score by means of the exponential map. The parametrization $\mathbf{h}: \mathbb{R} \rightarrow \mathbb{R}^{2} \rtimes \mathbb{T}$ of this spiral at $g_{0}=\left(\mathbf{x}_{0}, \theta_{0}\right)$ with tangent vector $\mathbf{v}\left(g_{0}\right)$ is given by (if $v_{\theta} \neq 0$ )

$$
\begin{equation*}
\mathbf{h}(t)=e^{t \mathbf{v}\left(g_{0}\right)}=\left(\mathbf{x}_{0}+\frac{1}{v_{\theta}}\left(\mathbf{R}_{v_{\theta} t+\theta_{0}-\pi / 2}-\mathbf{R}_{\theta_{0}-\pi / 2}\right)\binom{v_{\xi}}{v_{\eta}}, v_{\theta} t+\theta_{0}\right) \tag{14}
\end{equation*}
$$

We are interested in the curvature on the spatial plane, so we project $\mathbf{h}(t)$ to the $\mathbb{R}^{2}$ plane $\mathbf{x}(t)=\mathbb{P}_{\mathbb{R}^{2}} \mathbf{h}(t)$. The curvature in this plane is given by

$$
\begin{equation*}
\boldsymbol{\kappa}(t)=\frac{d^{2}}{d s^{2}} \mathbf{x}(t(s))=\frac{v_{\theta}}{v_{\eta}^{2}+v_{\xi}^{2}}\binom{v_{\eta} \cos \left(t v_{\theta}+\theta_{0}\right)+v_{\xi} \sin \left(t v_{\theta}+\theta_{0}\right)}{v_{\xi} \cos \left(t v_{\theta}+\theta_{0}\right)-v_{\eta} \sin \left(t v_{\theta}+\theta_{0}\right)} \tag{15}
\end{equation*}
$$

where $s$ is the parameterization such that $\left\|\frac{d}{d s} x(t(s))\right\|=1$. The signed norm of the curvature vector is

$$
\begin{equation*}
\kappa=\|\boldsymbol{\kappa}\| \operatorname{sign}\left(\boldsymbol{\kappa} \cdot \mathbf{e}_{\eta}\right)=\frac{-v_{\theta}}{\sqrt{v_{\eta}^{2}+v_{\xi}^{2}}} \tag{16}
\end{equation*}
$$

This result has an intuitive interpretation: the curvature is equal to the slope at which the curve in the orientation score meets the spatial plane spanned by $\left\{\mathbf{e}_{\xi}, \mathbf{e}_{\eta}\right\}$.

Ideally, $v_{\eta}=0$ because by construction oriented structures are orthogonal to $\mathbf{e}_{\eta}$. In practice, however, assuming $v_{\eta}=0$ leads to a biased curvature estimate if the orientation $\theta$ deviates from the true orientation of an oriented structure, which occurs frequently since an oriented structure will always cause a response within a certain range of orientations.

How to find the vector field $\mathbf{v}$ from an orientation score $u$ ? A curve or oriented pattern appears in the phase-invariant representation of the orientation score cf. Section 2.1 as a ridge. Therefore we calculate the Hessian, which is defined by

$$
\mathbf{H}(u)=\left(\begin{array}{cc}
\partial_{\theta}^{2}|u| & \partial_{\xi} \partial_{\theta}|u|  \tag{17}\\
\partial_{\theta} \partial_{\xi}|u| & \partial_{\eta}^{2}|u| \\
\partial_{\theta}|u| \\
\partial_{\theta} \partial_{\eta}|u| & \partial_{\xi} \partial_{\eta}|u| \\
\partial_{\xi}|u| \\
\partial_{\eta}^{2}|u|
\end{array}\right)=\left(\begin{array}{cc}
\partial_{\theta}^{2}|u| & \partial_{\xi} \partial_{\theta}|u| \\
\partial_{\eta} \partial_{\theta}|u| \\
\partial_{\xi} \partial_{\theta}|u|+\partial_{\eta}|u| & \partial_{\xi}^{2}|u| \\
\partial_{\eta} \partial_{\theta}|u|-\partial_{\xi}|u| \partial_{\xi}|u| \\
\partial_{\eta}|u| & \partial_{\eta}^{2}|u|
\end{array}\right)
$$

where Gaussian derivatives are used with scales $t_{s}$ and $t_{o}$ using the canonical ordering in the expression on the right. Note that the Hessian matrix is not symmetric because of the torsion of the space, implying that we can get complexvalued eigenvalues and eigenvectors. However, we can still find local and global extrema of $\left\{\|\mathbf{H} \mathbf{a}\|^{2} \mid\|\mathbf{a}\|^{2}=1\right\}$ with $\mathbf{a}=(x, y, \theta)$. Now by Lagrange these extrema satisfy $\nabla_{\mathbf{a}}\|\mathbf{H} \mathbf{a}\|^{2}=\nabla_{\mathbf{a}}\left(\mathbf{a}^{\mathrm{T}} \mathbf{H}^{\mathrm{T}} \mathbf{H} \mathbf{a}\right)=2 \mathbf{H}^{\mathrm{T}} \mathbf{H} \mathbf{a}=2 \lambda\|\mathbf{a}\|$. Therefore we apply eigen analysis on $\mathbf{H}^{\mathrm{T}} \mathbf{H}$ rather than $\mathbf{H}$. An oriented structure will lie approximately within the 2D plane spanned by $\left\{\mathbf{e}_{\xi}, \mathbf{e}_{\theta}\right\}$, so the two eigenvectors of $\mathbf{H}^{\mathrm{T}} \mathbf{H}$ that are closest to the plane are selected by leaving out the eigenvector with the largest $\mathbf{e}_{\eta}$ component. From these two eigenvectors the one with the smallest eigenvalue is tangent to the oriented structure and is used to estimate the curvature with (16).

## 6 Conductivity Functions for Nonlinear Diffusion

At positions in the orientation score with a strongly oriented structure we only want to diffuse tangent to this structure, i.e. $D_{22}$ should be large, $D_{11}^{\prime}$ and $D_{33}$ should be small, and the curvature measurement of the previous section should
be taken into account. If there is no strong orientation, the diffusion should be isotropic in the spatial plane, i.e. $D_{22}=D_{33}$ should be large as well as $D_{11}^{\prime}$. Curvature is not defined at such positions so $\kappa=0$.

If an oriented structure is present at a position in the orientation score, one eigenvalue of the Hessian of $|u|$ will have a large negative real part. Therefore we propose as measure for the presence of oriented structures

$$
\begin{equation*}
s(\mathbf{x}, \theta)=\operatorname{Max}\left(-\operatorname{Re}\left(\lambda_{1}(\mathbf{x}, \theta)\right), 0\right) \tag{18}
\end{equation*}
$$

where $\lambda_{1}$ denotes the largest eigenvalue of the Hessian at every position. In the equation for the Hessian (17) we substitute $\partial_{\theta} \leftarrow \gamma \partial_{\theta}$ where $\gamma$ is a parameter with unit 1 /pixel that is necessary to make the units of all Hessian components $1 /$ pixel $^{2}$. For the conductivity functions we propose

$$
\begin{array}{r}
D_{33}(\mathbf{x}, \theta)=\exp \left(-\frac{s(\mathbf{x}, \theta)}{c}\right) ; \quad D_{11}^{\prime}(\mathbf{x}, \theta)=\epsilon_{11} D_{33}(\mathbf{x}, \theta) \\
\kappa(\mathbf{x}, \theta)=\left(1-\exp \left(-\left(\frac{d_{\kappa}}{D_{33}(\mathbf{x}, \theta)}\right)^{4}\right)\right) \kappa_{\mathrm{est}}(\mathbf{x}, \theta) ; \quad D_{22}(\mathbf{x}, \theta)=1 \tag{19}
\end{array}
$$

where the nonlinear function for $D_{33}$ makes the separation between isotropic and oriented regions stronger. The function is chosen such that the result is always between 0 and 1 for $s \geq 0$. The nonlinear function for $\kappa$ is chosen such that it puts a soft threshold determining whether to include the curvature estimate $\kappa_{\text {est }}$ depending on the value of $D_{33}$. There are six parameters involved: $c$ controls the behavior of the nonlinear $e$-power, $\gamma$ controls the weight factor of the $\theta$ derivatives, $\epsilon_{11}$ controls the strength of the diffusion in $\theta$ direction in isotropic regions, $d_{\kappa}$ determines the soft threshold on including curvature, and $t_{s}$ and $t_{o}$ are the two scale parameters.

## 7 Numerical Scheme

We propose an explicit finite difference scheme to solve diffusion equation (10). Since the PDE on the orientation score is highly anisotropic we require good rotational invariance. Many efficient numerical schemes proposed in literature, e.g. the AOS (additive operator splitting) scheme [4] are therefore discarded since they show poor rotation invariance. The LSAS scheme [13] has good rotational invariance, but it is not straightforward to make a 3 D version. The scheme in [14] suffers from checkerboard artefacts.

An important property of the differential operators $\partial_{\xi}, \partial_{\eta}$, and $\partial_{\theta}$ is their left-invariance. The performance of a numerical scheme will therefore be more optimal if this left-invariance is carried over to the finite differences that are used. To achieve this we should define the spatial finite differences in the directions defined by the left-invariant $\mathbf{e}_{\xi}, \mathbf{e}_{\eta}$ tangent basis vectors, instead of the sampled $\mathbf{e}_{x}, \mathbf{e}_{y}$ grid. In effect, the principal axes of diffusion in the spatial plane are always aligned with the finite differences.


$$
\begin{aligned}
& \partial_{\theta} u \approx \frac{1}{2 s_{\theta}}(u(\mathbf{x}, l+1)-u(\mathbf{x}, l-1)) \\
& \partial_{\theta}^{2} u \approx \frac{1}{s_{\theta}^{2}}(u(\mathbf{x}, l+1)-2 u(\mathbf{x}, l)+u(\mathbf{x}, l-1)) \\
& \partial_{\xi} u \approx \frac{1}{2}\left(u\left(\mathbf{x}+\mathbf{e}_{\xi}^{l}, l\right)-u\left(\mathbf{x}-\mathbf{e}_{\xi}^{l}, l\right)\right) \\
& \partial_{\xi}^{2} u \approx u\left(\mathbf{x}+\mathbf{e}_{\xi}^{l}, l\right)-2 u(\mathbf{x}, l)+u\left(\mathbf{x}-\mathbf{e}_{\xi}^{l}, l\right) \\
& \partial_{\eta} u \approx \frac{1}{2}\left(u\left(\mathbf{x}+\mathbf{e}_{\eta}^{l}, l\right)-u\left(\mathbf{x}-\mathbf{e}_{\eta}^{l}, l\right)\right) \\
& \partial_{\eta}^{2} u \approx u\left(\mathbf{x}+\mathbf{e}_{\eta}^{l}, l\right)-2 u(\mathbf{x}, l)+u\left(\mathbf{x}-\mathbf{e}_{\eta}^{l}, l\right)
\end{aligned}
$$

$$
\partial_{\xi} \partial_{\theta} u \approx \frac{1}{4 s_{\theta}}\left(u\left(\mathbf{x}+\mathbf{e}_{\xi}^{l}, l+1\right)-u\left(\mathbf{x}+\mathbf{e}_{\xi}^{l}, l-1\right)-u\left(\mathbf{x}-\mathbf{e}_{\xi}^{l}, l+1\right)+u\left(\mathbf{x}-\mathbf{e}_{\xi}^{l}, l-1\right)\right)
$$

$$
\partial_{\theta} \partial_{\xi} u \approx \frac{1}{4 s_{\theta}}\left(u\left(\mathbf{x}+\mathbf{e}_{\xi}^{l+1}, l+1\right)-u\left(\mathbf{x}+\mathbf{e}_{\xi}^{l+1}, l-1\right)-u\left(\mathbf{x}-\mathbf{e}_{\xi}^{l-1}, l+1\right)+u\left(\mathbf{x}-\mathbf{e}_{\xi}^{l-1}, l-1\right)\right)
$$

Fig. 3. Illustration of the spatial part of the stencil of the numerical scheme. The horizontal and vertical dashed lines indicate the sampling grid, which is aligned with $\left\{\mathbf{e}_{x}, \mathbf{e}_{y}\right\}$. The stencil points, indicated by the black dots, are aligned with the rotated coordinate system cf. (7) with $\theta=l s_{\theta}$.

For our numerical scheme we apply the chain rule on the right-hand side of the PDE (10) (i.e., analog to 1D: $\partial_{x} D \partial_{x} u=D \partial_{x}^{2} u+\left(\partial_{x} D\right)\left(\partial_{x} u\right)$ ) and the derivatives are replaced by the finite differences defined in Figure3. In time direction we use the first order forward finite difference, i.e. $\left(u^{k+1}-u^{k}\right) / \tau$ where $k$ is the discrete time and $\tau$ the time step. Interpolation is required at spatial positions $\mathbf{x} \pm \mathbf{e}_{\xi}$ and $\mathbf{x} \pm \mathbf{e}_{\eta}$. For this purpose we use the algorithms for B-spline interpolation proposed by Unser et al. [15] with B-spline order 3. This interpolation algorithm consists of a prefiltering step with a separable IIR filter to determine the B-spline coefficients. The interpolation images such as $u^{k}\left(\mathbf{x} \pm \mathbf{e}_{\xi}\right)$ can then be calculated by a separable convolution with a shifted B-spline. The examples in Figure 2 and all experiments in the next section are obtained with this numerical scheme.

The drawback of this explicit scheme is the numerical stability. An analysis on stability is difficult due to the interpolation step. From experiments, we conclude that one should choose $\tau \leq 0.25$ to ensure numerical stability.

## 8 Experiments

In this section we compare the results of coherence enhancing diffusion in the orientation score (CED-OS) with results obtained by the normal coherence enhancing diffusion (CED) approach [4] where we use the LSAS numerical scheme with [13] since this has particularly good rotation invariance.

In all experiments we construct orientation scores with period $2 \pi$ with $n_{\theta}=$ 64. The following parameters are used for the orientation score transformation (Section 2.1): $k=2, q=8, t=1000$, and $s=200$. These parameters are chosen such that the reconstruction is visually indistinguishable from the original. Since computational speed was not our main concern, we use a small time step of $\tau=0.1$ to ensure numerical stability. The parameters for the nonlinear diffusion


Fig. 4. Shows the effect of including curvature on a noisy test image in CED-OS. At $t=5$ the results with and without curvature are visually indistinguishable. At $t=20$ the effect is visible: higher curvatures are better preserved when curvature is included.
in $S E(2)$ for all experiments are: $\epsilon_{11}=0.001, t_{s}=12, t_{o}=0.04, \gamma=0.05$, $c=0.08$, and $d_{\kappa}=0.13$. Note that the resulting images we will show of CED-OS do not represent the evolving orientation score, but only the reconstructed image (i.e. after summation over all orientations).

The parameters that we used for CED are (see [4]): $\sigma=1, \rho=1$ (artificial images) or $\rho=6$ (medical image), $C=1$, and $\alpha=0.001$. The artificial images all have a size of $56 \times 56$ and a range of 0 to 255 .

Figure 4 shows CED-OS with and without including curvature. As expected, the noise is removed while the line structures are well-preserved. At time $t=5$ no visible differences are observed in the resulting image reconstructions so only the result with curvature is shown. At $t=20$, however, the difference is visible: when curvature is included the preservation of the high-curvature inner circles is better. Still, in all cases the smallest circles are blurred isotropically. This is


Fig. 5. Shows the typical different behavior of CED-OS compared to CED. In CED-OS crossing structures are better preserved.


Fig. 6. Shows results on an image constructed from two rotated 2-photon images of collagen tissue in the heart. At $t=2$ we obtain a nice enhancement of the image. Comparing with $t=30$ a nonlinear scale-space behavior can be seen. For comparison, the right column shows the behavior of CED.
due to smaller response of the Hessian on curved lines, causing the value for $D_{33}$ to be larger on high-curvature circles.

Note that CED will also perform good on the image in Figure4 The difference in behavior becomes apparent if we consider images with crossing line structures. This is shown in Figure 5. The upper image shows an additive superimposition of two images with concentric circles. Our method is able to preserve this structure, while CED can not. The same holds for the lower image with crossing straight lines, where it should be noted that our method leads to amplification of the crossings, which is because the lines in the original image are not superimposed linearly.

Figure 6 shows the results on an image of collagen fibres obtained using 2-photon microscopy. These kind of images are acquired in tissue engineering research, where the goal is to create artificial heart valves. The image shows an artificial superposition of the same image with two different rotations, for the purpose of this experiment. This is not entirely artificial, since there exist collagen structures with this kind of properties. The parameters during these experiments were set the same as the artificial images. The image size is $160 \times 160$.

## 9 Conclusions

In this paper we introduced nonlinear diffusion on the Euclidean motion group. Starting from a 2D image, we constructed a three-dimensional orientation score using rotated versions of a directional quadrature filter. We considered the orientation score as a function on the Euclidean motion group and defined the left-invariant diffusion equation. We showed how one can use normal Gaussian derivatives to calculate regularized derivatives in the orientation score. The nonlinear diffusion is steered by estimates for oriented feature strength and curvature that are obtained from Gaussian derivatives. Furthermore, we proposed to use finite differences that approximate the left-invariance of the derivative operators.

The experiments show that we are indeed able to enhance elongated patterns in images and that including curvature helps to enhance lines with large
curvature. Especially at crossings our method renders a more natural result than coherence enhancing diffusion. The diffusion shows the typical nonlinear scalespace behavior when increasing time: blurring occurs, but the important features of images are preserved over a longer range of time. Furthermore we showed that including curvature renders better results on curved line structures.

Some problems should still be addressed in future work. The numerical algorithm is currently computationally expensive due to the small time step and interpolation. Furthermore, embedding the nonlinear diffusion in orientation scores in the variational framework may lead to better control on the behavior of the evolution equations. Finally, it would be interesting to extend this approach to the similitude group, i.e. to use multi-scale and multi-orientation simultaneously to resolve the problem of selecting the appropriate scale.

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[^1]:    ${ }^{1}$ The space of orientation scores of images $V=\left\{U_{f} \mid f \in \mathbb{L}_{2}\left(\mathbb{R}^{2}\right)\right\}$ is a vector subspace of $\mathbb{L}_{2}(S E(2))$. Note that the operations in $\mathbb{L}_{2}(S E(2))$ described in the rest of this paper do not leave $V$ invariant. From a practical point of view, however, this is not a prolbem since the inverse transformation implicitely projects on $V$. For mathematical details, see [8] where $V=\mathbb{C}_{K}^{G}$.

