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by

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# DRK methods for time-domain oscillator simulation

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**Summary.** This paper presents a new Runge-Kutta type integration method that is well-suited for time-domain simulation of oscillators. A unique property of the new method is that its damping characteristics can be controlled by a continuous parameter.

## Introduction

In the case of weakly non-linear circuit behaviour, oscillators can be simulated in the frequency domain, using e.g. the Harmonic Balance method. In the case of strongly non-linear circuit behaviour, they are to be simulated in the time domain, using e.g. the BDF-methods or the Trapezoidal Rule (TR). If the start-up behaviour of an oscillator is to be observed, a time domain method is even mandatory. However, the BDF methods exert a considerable damping on an oscillatory solution of the circuit equations. The TR method, when used on oscillators, does not exert any damping at all for all frequencies (which is also not wanted). To remedy this situation, an integration method would be preferred that has some damping to avoid numerical instability, but still so small that its effect on the oscillation can be neglected. In the next sections, DRK methods will be investigated as potential candidates for such methods.

## 1 DRK methods

We apply a Diagonal Runge-Kutta (DRK) method to a general DAE of the form

$$g(t, \dot{x}, x) = 0. \quad (1)$$

Given a step-size  $h$  and an initial value  $x_0$ , the DRK method computes a sequence  $\{x_n\}$ , where  $x_n$  is an approximation to the solution at  $t = nh$ . Given  $(a_{ii})$  and  $(b_i)$ ,  $x_{n+1}$  is computed from  $x_n$  as follows

$$g\left(t_n + ha_{ii}, \mathbf{X}_n^{(i)}, \mathbf{x}_n + ha_{ii}\mathbf{X}_n^{(i)}\right) = \mathbf{0}, \quad (2a)$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h \sum_{i=1}^s b_i \mathbf{X}_n^{(i)}. \quad (2b)$$

The quantities  $\mathbf{X}_n^{(i)}$  are called the stages of the DRK method, and  $s$  is called the stage count. Note that (2a) constitutes an Implicit Euler step with step-size  $ha_{ii}$ . So in essence, a DRK method is a linear combination of Implicit Euler steps.

### 1.1 Order conditions

For the ODE  $\dot{x} = f(x)$ ,  $f \in C^\infty(\mathbb{R}, \mathbb{R})$  we have  $g(t, \dot{x}, x) = \dot{x} - f(x) = 0$ , leading to the following DRK procedure:

$$\mathbf{X}_n^{(i)} = f(\mathbf{x}_n + ha_{ii}\mathbf{X}_n^{(i)}), \quad \text{for } i = 1 \dots s, \quad (3a)$$

$$\mathbf{x}_{n+1} = \mathbf{x}_n + h \sum_{i=1}^s b_i \mathbf{X}_n^{(i)}. \quad (3b)$$

The order conditions up to order  $k$  are now found by equating, for arbitrary  $f$ , the following terms at the point  $h = 0$  (see [2]):

$$\frac{d^j \mathbf{x}_{n+1}}{dh^j} = \frac{d^j \mathbf{x}(t_n + h)}{dh^j}, \quad \text{for } j = 0 \dots k. \quad (4)$$

In [4] it is shown that this is only possible up to order 2. Then, the following order conditions should be satisfied:

$$\sum_{i=1}^s b_i = 1, \quad \sum_{i=1}^s b_i a_{ii} = \frac{1}{2}. \quad (5)$$

The fact that DRK methods are limited to such low orders, appears to make them quite unappealing. In contrast to the common approach in Runge-Kutta theory, as presented in e.g. [2], we do not aim for maximising of the order of the method. Rather we balance the desire for a high order against the goal of obtaining a method that does not damp out oscillations.

### 1.2 Stability conditions

To study stability we apply the DRK-methods to the Dahlquist test equation

$$\dot{x} = \lambda x, \quad \lambda \in \mathbb{C}$$

Let  $x_{n+1}$  be computed from  $x_n$  with the DRK method using stepsize  $h > 0$ . Then

$$\zeta(h\lambda) := \frac{x_{n+1}}{x_n} = 1 + \sum_{i=1}^s \frac{b_i h\lambda}{1 - a_{ii} h\lambda} \quad (6)$$

defines the amplification factor. For a DRK method to be usable as an integration method for oscillator problems,  $\zeta(h\lambda)$  should satisfy the following conditions:

$$|\zeta(j\omega)| \lesssim 1, \quad \omega \in \mathbb{R}, \quad (7a)$$

$$|\zeta(z)| < 1 \text{ for } \operatorname{Re}(z) < 0, \quad (7b)$$

$$\lim_{\operatorname{Re}(z) \rightarrow -\infty} \zeta(z) = 0, \quad (7c)$$

It can be shown that, once condition (7a) is satisfied, and  $\zeta(z)$  is analytic in the left half of the complex  $z$ -plane, then also condition (7b) is satisfied (see [3]). Assuming for the moment that condition (7a) is satisfied, then to also satisfy condition (7b) the poles of  $\zeta(z)$  need to be in the right half of the complex  $z$ -plane. This leads to the following restriction on the coefficients  $a_{ii}$ :

$$a_{ii} > 0 \quad \text{for } i = 1, \dots, s \quad (8)$$

Note that even with this restriction satisfied, we still need to check on any proposed set of coefficients whether (7a) is satisfied. Applying condition (7c) to (6) leads to:

$$\sum_{i=1}^s \frac{b_i}{a_{ii}} = 1 \quad (9)$$

which embodies another restriction on the DRK coefficients.

## 2 Two-stage example

To have a DRK method suitable for use in oscillator simulation, the coefficients  $a_{ii}$  and  $b_i$  should satisfy order and stability conditions as derived in the preceding sections. For two stages already solutions with one degree of freedom exist. For this particular case, the set of equations to be solved is:

$$b_1 + b_2 = 1, \quad b_1 a_{11} + b_2 a_{22} = \frac{1}{2}, \quad \frac{b_1}{a_{11}} + \frac{b_2}{a_{22}} = 1, \quad (10a)$$

$$a_{11} > 0, \quad a_{22} > 0. \quad (10b)$$

In the sequel, we denote  $\gamma := a_{22}$  as the degree of freedom. We then find the following solution to (10):

$$b_1 = \frac{2\gamma^2 - 3\gamma + 1}{2\gamma^2 - 4\gamma + 1}, \quad b_2 = \frac{-\gamma}{2\gamma^2 - 4\gamma + 1}, \quad a_{11} = \frac{2\gamma - 1}{2\gamma - 2}, \quad a_{22} = \gamma, \quad (11a)$$

$$\gamma \in (0, \frac{1}{2}) \cup (1, \infty), \quad \gamma \neq \frac{1}{2 \pm \sqrt{2}}. \quad (11b)$$

It remains to be checked if this set of coefficients satisfies the condition (7a). To that end, we investigate

$$\zeta(j\omega) = \frac{b_1/a_{11}}{1 - ja_{11}\omega} + \frac{b_2/a_{22}}{1 - ja_{22}\omega}, \quad (12)$$

from which we find (see [4])

$$|\zeta(j\omega)|^2 = 1 - \frac{\omega^4 \gamma^2 (1 - 2\gamma)^2}{(1 + \gamma^2 \omega^2)[4(1 - \gamma)^2 + \omega^2(1 - 2\gamma)^2]}, \quad (13)$$

showing that  $|\zeta(j\omega)| \leq 1$ , thereby satisfying condition (7a). Note that by making  $\gamma$  small enough,  $|\zeta(j\omega)|$  can be brought as close to 1 as we want. For  $\gamma \rightarrow 0$  the method approaches the midpoint rule.

The stability diagrams for various values of  $\gamma$  are shown in Figure 1. This figure clearly illustrates the fact that  $\gamma$  can be used to control the amount of damping on the imaginary axis. Furthermore, it shows that by sufficiently decreasing the value of  $\gamma$  the amount of damping can be brought as close to zero as we require.

### 3 Alternative formulation

Using the transformation  $\widetilde{\mathbf{X}}_n^{(i)} = \mathbf{x}_n + ha_{ii}\mathbf{X}_n^{(i)}$  for  $i = 1, \dots, s$  and assuming that (9) holds, we obtain the following alternative formulation of the DRK method:

$$g \left( t_n + ha_{ii}, \frac{\widetilde{\mathbf{X}}_n^{(i)} - \mathbf{x}_n}{ha_{ii}}, \widetilde{\mathbf{X}}_n^{(i)} \right) = \mathbf{0}, \quad (14a)$$

$$\mathbf{x}_{n+1} = \sum_{i=1}^s \frac{b_i}{a_{ii}} \widetilde{\mathbf{X}}_n^{(i)}, \quad (14b)$$

The numerical robustness of this alternative formulation is better than the one of the standard formulation, as it avoids the summation of relatively small quantities to the current approximation in the update equation (see (2b)). It thereby circumvents the unnecessary loss of accuracy. For the two-stage case, considered in the previous section, the coefficients (11) satisfy condition (9). So (14) holds for this case, with the following expressions for its coefficients:

$$\frac{b_1}{a_{11}} = \frac{2(\gamma - 1)^2}{2\gamma^2 - 4\gamma + 1}, \quad \frac{b_2}{a_{22}} = \frac{-1}{2\gamma^2 - 4\gamma + 1}, \quad (15)$$

with the coefficients  $a_{11}$ ,  $a_{22}$  and the restrictions on  $\gamma$  the same as in (11).

## 4 Conclusions

We developed a Diagonal Runge-Kutta algorithm that is particularly suited for transient simulation of oscillators, in the sense that it does not damp out any oscillation present in the solution of the circuit equations. In fact it has been shown that its damping characteristics can be controlled by a dedicated parameter. The new algorithm allows designers to better simulate oscillators, or to detect unwanted oscillation earlier than would be the case with standard integration methods.

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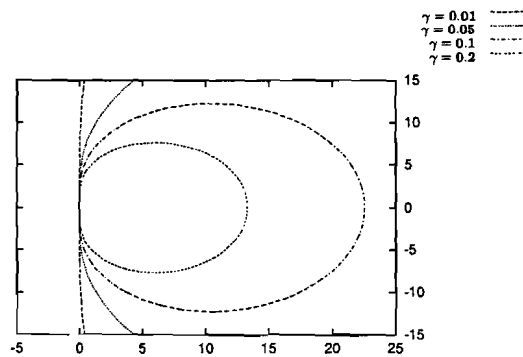


Fig. 1. Stability diagram of the DRK method in the complex  $h\lambda$ -plane.