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Abstract

This report is devoted to the symbolic calculation of the scattering coefficient in diffraction by a circular disk, by use of *Mathematica*. Three diffraction problems are considered: scalar diffraction by an acoustically soft or hard disk, and electromagnetic diffraction by a perfectly conducting disk. In the low-frequency approximation, the solutions of these problems are in the form of expansions in powers of ka, where k is the wave number and a the radius of the disk. The emphasis is on the low-frequency expansion for the scattering coeffcient, of which several terms are determined exactly with the help of *Mathematica*.

Key words. scattering coefficient, circular disk, symbolic calculation, *Mathematica*, diffraction theory, low-frequency approximation

AMS subject classifications. 78A45, 41-04, 45B05

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Contents

1 Introduction

In recent years the symbolic programming language *Mathematica* has become an important tool in the analysis of mathematical problems of which the solution involves extensive analytical calculations. In this report we will use *Mathematica* to calculate the scattering coefficient for low-frequency diffraction by a circular disk. Here it is appropriate ro refer to Hurd [12] for a previous symbolic calculation of the scattering coefficient, as early as 1971 and therefore of limited scope, by use of the programming language FORMAC.

More specific, we consider the diffraction of a normally incident, plane wave by a circular disk of radius a. A harmonic time dependence of the form $\exp(-i\omega t)$, with frequency ω, is assumed and suppressed throughout. Three diffraction problems are distinguished and treated in successive chapters of this report: (i) scalar diffraction by an acoustically soft disk (Chapter 2); (ii) scalar diffraction by an acoustically hard disk (Chapter 3); (iii) electromagnetic diffraction by a perfectly conducting disk (Chapter 4). These diffraction problems have exact solutions in terms of spheroidal wave functions; see [10, Chapter 14] for a survey of methods of solution and results. In this report we are especially interested in low-frequency approximations to the exact solutions, valid when the disk radius a is small compared with the wavelength. In the low-frequency approximation, the solution of the diffraction problem is given by a power-series expansion in powers of $\alpha = ka$, where k is the wave number. Corresponding low-frequency expansions (in powers of α) are obtainable for various field quantities such as the scattered field on the disk, the scattered far field and the scattering coefficient. Here the scattering coefficient is defined as the ratio of the total energy scattered to the energy incident on the disk. The first few terms of these low-frequency expansions can easily be determined and are known from the literature. Evaluation of the higher-order terms involves a considerable amount of work and soon becomes prohibitive with increasing order. However, the calculations are completely systematic and straightforward, and therefore well suited to be carried out by a computer algebra system. To demonstrate this by an example, we will use *Mathematica* to calculate the low-frequency expansion of the scattering coefficient for the three diffraction problems mentioned. In principle the expansion can be evaluated up to arbitrary order; in practice the order is limited by the available computer capacity.

All diffraction problems are solved by two independent methods. In the first method, due to Bazer and Brown [1], Boersma [3], the scattered field is represented by suitable integrals which contain unknown auxiliary functions. The integral representations are designed to satisfy all conditions of the diffraction problem except for the boundary conditions on the circular disk. Imposing the latter conditions leads to Fredholm integral equations of the second kind for the auxiliary functions. The kernel of the integral equations is small of order α, thus permitting a solution of the integral equations by power-series expansion in powers of α or by Picard iteration. The solution obtained is inserted into the expression for the scattering coefficient yielding the desired low-frequency expansion. In the second method, due to Bouwkamp [6, 8, 9], the diffraction problems are formulated in terms of integral equations of the first kind or integro-differential equations for the scattered field on the disk or, in the case of electromagnetic diffraction, the currents induced in the disk. Substitution of low-frequency expansions for the scattered fields or currents and further expansion in powers of ik, leads to a recursive system of integral equations or integro-differential equations. This system is solved by expansion in suitable Legendre polynomials, whereby the expansion coefficients are determined by a recurrence relation. These coefficients are inserted into the expression for the scattering coefficient yielding the desired low-frequency expansion.

The two methods of solution give rise to two different schemes for the calculation of the low-frequency expansion of the scattering coefficient. *Mathematica*-implementations of these schemes are listed in Appendices A–E. The results of the calculations by *Mathematica* are presented in Sections 2.4, 3.4, and 4.4. For the successive diffraction problems we have tabulated the exact values of the first ten coefficients and the numerical values (to six significant digits) of the first twenty coefficients in the low-frequency expansion of the scattering coefficient. It is found that the two different schemes do yield the same results for the scattering coefficient. This provides an excellent check on the correctness of the mathematical analysis and of the *Mathematica*-programmes.

Finally, we present a list of the key equations in our schemes for calculating the scattering coefficient. In Chapter 2 we consider the scalar diffraction of a normally incident, plane wave by an acoustically soft, circular disk. Two independent solutions of the diffraction problem are presented. The solution obtained by the method of Bazer and Brown [1] is expressed in terms of the auxiliary function $g(t)$, which satisfies the integral equation

$$g(t) = \cosh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s}\, g(s)\, ds, \quad -1 \le t \le 1. \tag{1.1}$$

The scattering coefficient σ_2 of the soft circular disk is given by

$$\sigma_2 = -\frac{8}{\pi\alpha}\,\mathrm{Im}\int_{0}^{1} \cosh(\alpha t) g(t)\, dt, \tag{1.2}$$

expressed in terms of the function $g(t)$. In unpublished work of Bouwkamp, referred to in [9, p. 71] and reconstructed in Section 2.3, the solution is described by expansions in Legendre polynomials with expansion coefficients $a_{p,n}$, where $p = 0, 1, 2, \ldots$, $n = 0, 1, \ldots, [p/2]$. These coefficients are determined by the recurrence relation

$$\begin{aligned} a_{p,n} \;=\;& (-1)^{n+1}\left(2n+\tfrac{1}{2}\right)\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\sum_{q=1}^{p}\Gamma^2\left(\tfrac{1}{2}q+\tfrac{1}{2}\right)\sum_{m=0}^{[(p-q)/2]}(-1)^m a_{p-q,m}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} \\ &\cdot \frac{1}{\Gamma\left(\frac{1}{2}q-m-n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}q+m-n+1\right)} \\ &\cdot \frac{1}{\Gamma\left(\frac{1}{2}q-m+n+1\right)\Gamma\left(\frac{1}{2}q+m+n+\frac{3}{2}\right)}, \end{aligned} \tag{1.3}$$

valid for $p = 1, 2, 3, \ldots$, $n = 0, 1, \ldots, [p/2]$, and initiated by $a_{0,0} = 1$. The low-frequency expansion of the scattering coefficient σ_2 is given by

$$\sigma_2 = -\frac{8}{\pi}\sum_{p=0}^{\infty}(-1)^p a_{2p+1,0}\,\alpha^{2p}. \tag{1.4}$$

Chapter 3 deals with the scalar diffraction of a normally incident, plane wave by an acoustically hard, circular disk. Again, two independent solutions of the diffraction problem are considered. The solution obtained by the method of Bazer and Brown [1] is expressed in terms of the auxiliary function $f(t)$, which satisfies the integral equation

$$f(t) = \sinh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} f(s)\,ds, \quad -1 \le t \le 1. \tag{1.5}$$

The scattering coefficient σ_1 of the hard circular disk is given by

$$\sigma_1 = \frac{8}{\pi\alpha}\,\mathrm{Im}\int_0^1 \sinh(\alpha t) f(t)\,dt, \tag{1.6}$$

expressed in terms of the function $f(t)$. The solution due to Bouwkamp [6] is described by expansions in Legendre polynomials with expansion coefficients $b_{p,n}$, where $p = 0, 1, 2, \ldots$, $n = 0, 1, \ldots, N_p$, with $N_p = p/2$ (p even) or $N_p = (p-3)/2$ (p odd). These coefficients are determined by the recurrence relation

$$\begin{aligned} b_{p,n} &= (-1)^{n+1}\left(n+\tfrac{3}{4}\right)\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\sum_{q=2}^{p}\Gamma\left(\tfrac{1}{2}q-\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}q+\tfrac{1}{2}\right)\sum_{m=0}^{N_{p-q}}(-1)^m b_{p-q,m}\frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+1)} \\ &\quad \cdot \frac{1}{\Gamma\left(\frac{1}{2}q-m-n-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}q+m-n+1\right)} \\ &\quad \cdot \frac{1}{\Gamma\left(\frac{1}{2}q-m+n+1\right)\Gamma\left(\frac{1}{2}q+m+n+\frac{5}{2}\right)}, \end{aligned} \tag{1.7}$$

valid for $p = 2, 3, 4, \ldots$, $n = 0, 1, \ldots, N_p$, and initiated by $b_{0,0} = -2/\pi$. The low-frequency expansion of the scattering coefficient σ_1 is given by

$$\sigma_1 = \frac{4}{3}\sum_{p=0}^{\infty}(-1)^p b_{2p+3,0}\,\alpha^{2p+4}. \tag{1.8}$$

It is interesting to note that the integral equations (1.1) and (1.5) may be solved simultaneously. Indeed, by adding (1.1) and (1.5) we obtain the single integral equation

$$h(t) = e^{\alpha t} + \frac{1}{\pi i}\int_{-1}^{1}\frac{\sinh[\alpha(t-s)]}{t-s} h(s)\,ds, \quad -1 \le t \le 1, \tag{1.9}$$

for the function $h(t) = g(t) + f(t)$. Then, $g(t)$ is recovered as the even part of $h(t)$, while $f(t)$ is equal to the odd part of $h(t)$.

In Chapter 4 we consider the electromagnetic diffraction of a normally incident, plane wave by a perfectly conducting, circular disk. Two independent solutions of the diffraction problem are proposed, taken from Boersma [3, Sec 3.3] and Bouwkamp [8].

Boersma's solution is expressed in terms of three auxiliary functions $f_0(t)$, $g_0(t)$, and $g_1(t)$, which satisfy the integral equations

$$f_0(t) = \frac{\sinh(\alpha t)}{\alpha} + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} f_0(s)\,ds, \quad -1 \le t \le 1, \tag{1.10}$$

$$g_0(t) = \cosh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} g_0(s)\,ds, \quad -1 \le t \le 1, \tag{1.11}$$

$$g_1(t) = \frac{t\sinh(\alpha t)}{\alpha} + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} g_1(s)\,ds, \quad -1 \le t \le 1. \tag{1.12}$$

Next, the constants C_0 and C are determined by

$$C_0 = -\frac{g_1(1)}{g_0(1)}, \qquad C = \frac{f_0(1)}{-f_0(1) + g_1'(1) + C_0\, g_0'(1)}. \tag{1.13}$$

The scattering coefficient σ of the conducting circular disk is given by

$$\sigma = \frac{8}{\pi}\,\mathrm{Im}\left((C+1)\int_0^1 \sinh(\alpha t) f_0(t)\,dt\right), \tag{1.14}$$

expressed in terms of the constant C and the function $f_0(t)$. Bouwkamp's solution [8] involves low-frequency expansions with expansion coefficients $a_{n,n-2\nu}$, $b_{n,n-2\nu}$, and p_n, where $n = 1, 2, 3, \ldots$, $\nu = 0, 1, \ldots, [(n+1)/2]$. These coefficients are determined successively by the system of equations

$$\sum_{\tau=1}^{n} \frac{1}{(n-\tau)!} \sum_{\nu=0}^{[(\tau+1)/2]} a_{\tau,\tau-2\nu}\, J(\nu, 0, n-\tau; \rho) = \sum_{\nu=0}^{[(n+1)/2]} p_{n-2\nu} \frac{\rho^{2\nu}}{2^{2\nu}(\nu!)^2}, \tag{1.15}$$

$$\sum_{\tau=1}^{n} \frac{1}{(n-\tau)!} \sum_{\nu=1}^{[(\tau+1)/2]} b_{\tau,\tau-2\nu}\, J(\nu, 1, n-\tau; \rho) = \sum_{\nu=1}^{[(n+1)/2]} p_{n-2\nu} \frac{\rho^{2\nu}}{2^{2\nu}(\nu-1)!\,(\nu+1)!}, \tag{1.16}$$

$$a_{n,n} = \sum_{\nu=1}^{[(n+1)/2]} \left((-1)^{\nu+1} \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\nu+1)} a_{n,n-2\nu} + (-1)^{\nu} \frac{4\,\Gamma(\nu+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\nu)} b_{n,n-2\nu} \right), \tag{1.17}$$

valid for $n = 1, 2, 3, \ldots$, and initiated by $p_{-1} = 1$, $p_0 = 0$. Here the J-functions are polynomials in ρ^2, generally given by

$$J(n, m, \mu; \rho) = \frac{(-1)^{n+m}\,\Gamma^2(\frac{1}{2}\mu+\frac{1}{2})\,\Gamma(n+m+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(m+\frac{1}{2})\,\Gamma(m+1)\,\Gamma(n-m+1)\,\Gamma(\frac{1}{2}\mu-n-m+\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+n-m+1)} \cdot \rho^{2m} F(-\tfrac{1}{2}\mu+n+m+\tfrac{1}{2}, -\tfrac{1}{2}\mu-n+m; 2m+1; \rho^2), \tag{1.18}$$

where F stands for the hypergeometric function. For fixed $n = 1, 2, 3, \ldots$, the equations (1.15)–(1.17) can be reduced to a system of $2[(n+1)/2] + 2$ linear equations for the same number of unknown coefficients $a_{n,n-2\nu}$, $b_{n,n-2\nu}$, and p_n. The low-frequency expansion of the scattering coefficient σ is given by

$$\sigma = \frac{8}{\pi} \sum_{n=1}^{\infty} (-1)^n \, a_{2n,2n} \, \alpha^{2n}, \qquad (1.19)$$

expressed in terms of the coefficients $a_{2n,2n}$ only.

2 Diffraction of a scalar wave by a soft circular disk

2.1 Formulation of the problem

We consider the acoustic diffraction of a normally incident, plane wave by an acoustically soft, circular disk D, of radius a. In terms of Cartesian coordinates x, y, z, the disk D is described by $0 \leq x^2 + y^2 \leq a^2$, $z = 0$. In addition we employ cylindrical coordinates ρ, φ, z, specified by $x = \rho\cos\varphi$, $y = \rho\sin\varphi$. The incident wave is given by $\Phi^i(x, y, z) = \exp(ikz)$, where k is the wave number and a harmonic time dependence $\exp(-i\omega t)$ is suppressed throughout. Thus the primary wave Φ^i is incident from $z < 0$.
Following Bouwkamp [9, pp. 38–39], we express the resulting total field Φ^t as

$$\Phi^t(x, y, z) = \begin{cases} \exp(ikz) - \phi_2(x, y, -z), & z < 0, \\ \exp(ikz) - \phi_2(x, y, z), & z > 0, \end{cases} \tag{2.1}$$

where ϕ_2, which is defined for $z \geq 0$ only, has the following properties:

1) ϕ_2 satisfies the Helmholtz equation

$$\Delta\phi_2 + k^2\phi_2 = 0, \quad \text{when } z > 0; \tag{2.2}$$

2) $$\frac{\partial\phi_2}{\partial z}(x, y, 0) = 0, \quad \text{when } x^2 + y^2 > a^2; \tag{2.3}$$

3) $$\phi_2(x, y, 0) = 1, \quad \text{when } x^2 + y^2 < a^2; \tag{2.4}$$

4) ϕ_2 satisfies the Sommerfeld radiation condition at infinity;

5) ϕ_2 is everywhere finite;

6) $\nabla\phi_2$ is quadratically integrable over any domain in $\mathbb{R}^3$.

In this manner the diffraction problem for a soft circular disk has been reduced to a boundary value problem for ϕ_2. By Babinet's principle [9, p. 39], also the complementary problem of diffraction through a circular aperture in an acoustically hard, infinite, plane screen, can be solved in terms of the function ϕ_2.

Since the diffraction problem is axially symmetric, ϕ_2 is independent of the angle φ and we will use the notation $\phi_2(\rho, z)$ as an alternative to $\phi_2(x, y, z)$. At large distances from the disk, ϕ_2 behaves like an outgoing spherical wave. In terms of spherical coordinates R, θ, φ, specified by

$$\rho = R \sin\theta, \quad z = R\cos\theta, \quad 0 \le \theta \le \pi, \tag{2.5}$$

we have the far-field expansion

$$\phi_2(R\sin\theta, R\cos\theta) = A_2(\theta)\,\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R \to \infty) \tag{2.6}$$

where $A_2(\theta)$ is the far-field amplitude. Of interest is the scattering coefficient σ_2 of the soft circular disk, which is defined as the ratio of the total energy scattered to the energy incident on the disk. Likewise, the transmission coefficient τ_2 of the complementary circular aperture in a hard plane screen is defined as the ratio of the energy transmitted through the aperture to the energy incident on the aperture. According to Levine and Schwinger's cross-section theorem, σ_2 is related to the far-field amplitude of the scattered wave $-\phi_2$ in the direction of incidence ($\theta = 0$),

$$\sigma_2 = -\frac{4}{a^2 k}\,\mathrm{Im}\,A_2(0), \tag{2.7}$$

whereas $\tau_2 = \frac{1}{2}\sigma_2$; see Jones [13, §§ 8.19, 9.4].

In the next sections the boundary value problem for ϕ_2 is solved by methods developed by Bazer and Brown [1] and by Bouwkamp. Both methods are well suited to obtain a low-frequency approximation to the solution, involving low-frequency expansions in powers of $\alpha = ka$ for various field quantities. The emphasis is on the low-frequency expansion for the scattering coefficient σ_2, of which several terms are exactly determined by means of *Mathematica*.

2.2 Bazer and Brown's solution

2.2.1 Reduction to an integral equation

Using the method of Bazer and Brown [1], we represent $\phi_2(\rho, z)$ by the integral (cf. [1, form. (55)])

$$\phi_2(\rho, z) = \int_{-1}^{1} \frac{\exp\left(ik\sqrt{\rho^2 + (z + iat)^2}\right)}{\sqrt{\rho^2 + (z + iat)^2}} f_2(t)\,dt, \tag{2.8}$$

valid for $z \geq 0$. Here, $f_2(t)$ is a yet unknown function which is required to be an even function of t. The branch of the square root is taken such that $\mathrm{Re}\sqrt{\rho^2 + (z + iat)^2} > 0$, when $z > 0$. We then have, for $0 < \rho < a$,

$$\lim_{z \downarrow 0} \sqrt{\rho^2 + (z + iat)^2} = \begin{cases} -i\sqrt{a^2t^2 - \rho^2}, & -1 \leq t < -\rho/a, \\ \sqrt{\rho^2 - a^2t^2}, & -\rho/a < t < \rho/a, \\ i\sqrt{a^2t^2 - \rho^2}, & \rho/a < t \leq 1. \end{cases} \tag{2.9}$$

As shown in [1], the representation (2.8) satisfies all conditions of the boundary value problem for ϕ_2, except for the boundary condition (2.4). By use of (2.9) we have, for $0 < \rho < a$,

$$\phi_2(\rho, 0) = \int_0^{\rho/a} \frac{2\exp\left(ik\sqrt{\rho^2 - a^2t^2}\right)}{\sqrt{\rho^2 - a^2t^2}} f_2(t)\,dt - \int_{\rho/a}^{1} \frac{2\sinh\left(k\sqrt{a^2t^2 - \rho^2}\right)}{i\sqrt{a^2t^2 - \rho^2}} f_2(t)\,dt. \tag{2.10}$$

Imposing the boundary condition (2.4), we obtain the integral equation [1, form. (64)]

$$\int_0^{\rho/a} \frac{\cos\left(\alpha\sqrt{(\rho/a)^2 - t^2}\right)}{\sqrt{(\rho/a)^2 - t^2}} f_2(t)\,dt = \frac{a}{2} + \frac{1}{i}\int_0^{1} \frac{\sinh\left(\alpha\sqrt{t^2 - (\rho/a)^2}\right)}{\sqrt{t^2 - (\rho/a)^2}} f_2(t)\,dt, \tag{2.11}$$

where $\alpha = ka$. As detailed in [1], the integral operator on the left in (2.11) can be inverted by means of Laplace transformation. As a result it is found that equation (2.11) reduces to the Fredholm integral equation [1, form. (67)]

$$f_2(t) = \frac{a}{\pi}\cosh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t - s)]}{t - s} f_2(s)\,ds, \quad -1 \leq t \leq 1. \tag{2.12}$$

By setting $f_2(t) = (a/\pi)\, g(t)$, we are led to the integral equation

$$g(t) = \cosh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s}\, g(s)\, ds, \quad -1 \le t \le 1, \tag{2.13}$$

which is the first key equation in the scheme for calculating the low-frequency expansion of the scattering coefficient σ_2.

2.2.2 Far field and scattering coefficient

At large distance R from the disk the square root $\sqrt{\rho^2 + (z + iat)^2}$ takes the asymptotic form

$$\begin{aligned} \sqrt{\rho^2 + (z+iat)^2} &= \sqrt{R^2 + 2iatR\cos\theta - a^2t^2} \\ &= R + iat\cos\theta + O\left(R^{-1}\right), \quad (R \to \infty). \end{aligned} \tag{2.14}$$

Here, R, θ are spherical coordinates, specified by (2.5). By inserting (2.14) into (2.8), we obtain the far-field expansion

$$\phi_2(R\sin\theta, R\cos\theta) = A_2(\theta)\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R \to \infty) \tag{2.15}$$

with far-field amplitude

$$A_2(\theta) = \int_{-1}^{1} \exp(-\alpha t\cos\theta) f_2(t)\, dt = \frac{2a}{\pi}\int_0^1 \cosh(\alpha t\cos\theta) g(t)\, dt. \tag{2.16}$$

The expression (2.7) for the scattering coefficient σ_2 now becomes

$$\sigma_2 = -\frac{8}{\pi\alpha}\,\mathrm{Im}\int_0^1 \cosh(\alpha t) g(t)\, dt, \tag{2.17}$$

expressed in terms of the function $g(t)$. This is the second key equation in our scheme for calculating the low-frequency expansion of σ_2.

2.2.3 Scheme for calculating the scattering coefficient

Our scheme for calculating the low-frequency expansion of the scattering coefficient σ_2 is based on the two equations (2.13) and (2.17). Two methods are employed for the solution of the integral equation (2.13). In the first method, the kernel of the integral equation is expanded in powers of α and the function $g(t)$ is replaced by the power-series expansion

$$g(t) = \sum_{n=0}^{\infty} g_n(t)\, \alpha^n, \tag{2.18}$$

with coefficients $g_n(t)$ to be determined. By equating terms containing the same power of α, one is led to a recurrence relation for $g_n(t)$ expressed in terms of preceding coefficients. In the second method, the integral equation is solved by Picard iteration, whereby a factor α is gained in each iteration step. The expansion of $g(t)$ thus determined is inserted into (2.17) and the function $\cosh(\alpha t)$ is expanded in powers of α. Then by a straightforward evaluation we find the required low-frequency expansion of σ_2 up to a certain order.

A *Mathematica*-implementation of the scheme using power-series expansion is listed in Appendix A; the function is called `BazerBrown2`. The *Mathematica*-function `Picard2` listed in Appendix B uses Picard iteration. The results of the calculations by *Mathematica* are presented in Section 2.4.

2.3 Bouwkamp's solution

This solution goes back to unpublished work of Bouwkamp, referred to in [9, p. 71]. In this section we will reconstruct the details of Bouwkamp's solution. For convenience we take the radius of the disk D equal to unity.

2.3.1 Reduction to a system of integral equations

Following Bouwkamp [9, form. (2.9)], we represent ϕ_2 by

$$\phi_2(\mathbf{x}) = \frac{1}{2\pi} \iint_D u(\mathbf{x}')\, \frac{\exp(ik|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\, dx'\, dy', \tag{2.19}$$

where $\mathbf{x} = (x, y, z)$, $\mathbf{x}' = (x', y', 0)$ and

$$u(\mathbf{x}') = -\frac{\partial \phi_2}{\partial z'}(x', y', 0). \tag{2.20}$$

By imposing the boundary condition (2.4), we are led to the following integral equation for $u(\mathbf{x}')$:

$$\frac{1}{2\pi}\iint_D u(\mathbf{x}')\,\frac{\exp(ik|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\,dx'\,dy' = 1, \quad \mathbf{x}\in D. \tag{2.21}$$

To solve this equation, we expand the exponential function in powers of ik and we substitute for $u(\mathbf{x}')$ the power-series expansion

$$u(\mathbf{x}') = \frac{2}{\pi}\sum_{p=0}^{\infty} u_p(\mathbf{x}')\,(ik)^p, \tag{2.22}$$

with coefficients $u_p(\mathbf{x}')$ to be determined. As a result we find that equation (2.21) passes into

$$\sum_{p=0}^{\infty}(ik)^p\sum_{q=0}^{p}\frac{1}{q!}\iint_D u_{p-q}(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{q-1}\,dx'\,dy' = \pi^2, \quad \mathbf{x}\in D. \tag{2.23}$$

All terms of the power series on the left have to vanish except the term independent of k, which must be equal to π^2. This leads to the recursive system of integral equations

$$\iint_D \frac{u_p(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,dx'\,dy' = U_p(\mathbf{x}), \quad \mathbf{x}\in D,\ p=0,1,2,\ldots, \tag{2.24}$$

where

$$U_0(\mathbf{x}) = \pi^2, \tag{2.25}$$

$$U_p(\mathbf{x}) = -\sum_{q=1}^{p}\frac{1}{q!}\iint_D u_{p-q}(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{q-1}\,dx'\,dy', \quad p=1,2,3,\ldots \tag{2.26}$$

2.3.2 Solution of the integral equations

We introduce polar coordinates ρ, φ, and ρ', φ', on the disk D, specified by

$$\begin{cases} x=\rho\cos\varphi, & y=\rho\sin\varphi, \quad 0\le\rho\le 1, \quad 0\le\varphi\le 2\pi, \\ x'=\rho'\cos\varphi', & y'=\rho'\sin\varphi', \quad 0\le\rho'\le 1, \quad 0\le\varphi'\le 2\pi. \end{cases} \tag{2.27}$$

In view of the axial symmetry of the diffraction problem, we change the notations $u(\mathbf{x}')$, $u_p(\mathbf{x}')$, into $u(\rho')$, $u_p(\rho')$. Since the integrals in (2.24) and (2.26) are independent of φ, the distance $|\mathbf{x}-\mathbf{x}'| = r$ may be taken as

$$r = \sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\varphi'}, \tag{2.28}$$

and we change the notation $U_p(\mathbf{x})$ into $U_p(\rho)$. Then, in polar coordinates, the system of integral equations (2.24)–(2.26) takes the form

$$\int_0^{2\pi}\int_0^1 \frac{u_p(\rho')}{r}\rho'\,d\rho'\,d\varphi' = U_p(\rho), \quad 0 \le \rho \le 1, \; p = 0, 1, 2, \ldots, \tag{2.29}$$

where

$$U_0(\rho) = \pi^2, \tag{2.30}$$

$$U_p(\rho) = -\sum_{q=1}^{p}\frac{1}{q!}\int_0^{2\pi}\int_0^1 u_{p-q}(\rho')\,r^{q-1}\rho'\,d\rho'\,d\varphi', \quad p = 1, 2, 3, \ldots \tag{2.31}$$

The integral equation (2.29) is solved by expansion of $u_p(\rho')$ in a series of "eigenfunctions" $P_{2m}(\sqrt{1-\rho'^2})/\sqrt{1-\rho'^2}$, $m = 0, 1, 2, \ldots$; here, P_{2m} is the Legendre polynomial of even degree $2m$. Basic in the analysis is the key integral

$$I(m, q; \rho) = \int_0^{2\pi}\int_0^1 \frac{P_{2m}(\sqrt{1-\rho'^2})}{\sqrt{1-\rho'^2}}\,r^{q-1}\,\rho'\,d\rho'\,d\varphi', \quad m = 0, 1, 2, \ldots, \; q = 0, 1, 2, \ldots, \; 0 \le \rho \le 1, \tag{2.32}$$

which is a special case of an integral calculated by Bouwkamp [7]. From [7, form. (2), (4)] we quote the result

$$I(m, q; \rho) = \sum_n c_n(m, q)P_{2n}(\sqrt{1-\rho^2}), \tag{2.33}$$

in which the coefficients c_n are expressible in terms of gamma functions as follows:

$$\begin{aligned} c_n(m, q) = {} & \pi(-1)^{m+n}\left(2n+\tfrac{1}{2}\right)\frac{\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(q+1)\Gamma^2(\frac{1}{2}q+\frac{1}{2})}{\Gamma(m+1)\Gamma(n+1)} \\ & \cdot \frac{1}{\Gamma\left(\frac{1}{2}q-m-n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}q+m-n+1\right)} \\ & \cdot \frac{1}{\Gamma\left(\frac{1}{2}q-m+n+1\right)\Gamma\left(\frac{1}{2}q+m+n+\frac{3}{2}\right)}. \end{aligned} \tag{2.34}$$

In (2.33) the summation over n must be taken according to $\max(0, m-\frac{1}{2}q) \le n \le m+\frac{1}{2}q$ (q even) or $0 \le n \le \frac{1}{2}(q-1)-m$ (q odd). In the latter case, $I(m, q; \rho) = 0$ if $m > \frac{1}{2}(q-1)$. In the special case $q = 0$ we have

$$\begin{aligned} I(m, 0; \rho) &= \int_0^{2\pi}\int_0^1 \frac{P_{2m}(\sqrt{1-\rho'^2})}{\sqrt{1-\rho'^2}}\,\frac{1}{r}\,\rho'\,d\rho'\,d\varphi' \\ &= \pi\,\frac{\Gamma^2(m+\frac{1}{2})}{\Gamma^2(m+1)}\,P_{2m}(\sqrt{1-\rho^2}). \end{aligned} \tag{2.35}$$

This integral relation explains why the functions $P_{2m}(\sqrt{1-\rho'^2})/\sqrt{1-\rho'^2}$ are called "eigenfunctions".

The solution of the integral equation (2.29) now follows in an obvious manner. The right-hand side $U_p(\rho)$ is expanded in a series of Legendre polynomials $P_{2n}(\sqrt{1-\rho^2})$, say

$$U_p(\rho) = \sum_n U_{p,n} P_{2n}(\sqrt{1-\rho^2}). \tag{2.36}$$

Then the solution for $u_p(\rho')$ is given by the "eigenfunction" expansion

$$u_p(\rho') = \frac{1}{\pi} \sum_n \frac{\Gamma^2(n+1)}{\Gamma^2(n+\frac{1}{2})} U_{p,n} \frac{P_{2n}(\sqrt{1-\rho'^2})}{\sqrt{1-\rho'^2}}. \tag{2.37}$$

For $p = 0$ we have $U_0(\rho) = \pi^2 = \pi^2 P_0(\sqrt{1-\rho^2})$, hence, the solution becomes

$$u_0(\rho') = \frac{P_0(\sqrt{1-\rho'^2})}{\sqrt{1-\rho'^2}} = \frac{1}{\sqrt{1-\rho'^2}}. \tag{2.38}$$

Continuing with $p = 1, 2, 3, \ldots$, we find that the expansions (2.36) and (2.37) reduce to finite expansions where the summation over n is from 0 to $[p/2]$; the notation $[p/2]$ stands for the largest integer $\leq p/2$. Accordingly, we make the Ansatz

$$u_p(\rho') = \sum_{n=0}^{[p/2]} a_{p,n} \frac{P_{2n}(\sqrt{1-\rho'^2})}{\sqrt{1-\rho'^2}}, \tag{2.39}$$

in which $a_{0,0} = 1$ by (2.38), and the remaining coefficients $a_{p,n}$ are to be determined recursively. On substitution of (2.39) into (2.31), we obtain the finite expansion

$$\begin{aligned} U_p(\rho) &= -\sum_{q=1}^{p} \frac{1}{q!} \sum_{m=0}^{[(p-q)/2]} a_{p-q,m}\, I(m,q;\rho) \\ &= -\sum_{q=1}^{p} \frac{1}{q!} \sum_{m=0}^{[(p-q)/2]} a_{p-q,m} \sum_n c_n(m,q) P_{2n}(\sqrt{1-\rho^2}), \end{aligned} \tag{2.40}$$

from which the coefficients $U_{p,n}$ can be read off. By identifying the expansions (2.37) and (2.39), we are led to the following recurrence relation for the coefficients $a_{p,n}$:

$$a_{p,n} = -\frac{1}{\pi} \frac{\Gamma^2(n+1)}{\Gamma^2(n+\frac{1}{2})} \sum_{q=1}^{p} \frac{1}{q!} \sum_{m=0}^{[(p-q)/2]} a_{p-q,m}\, c_n(m,q). \tag{2.41}$$

Here we insert the expression (2.34) for $c_n(m, q)$ to find the recurrence relation in its final form

$$\begin{aligned} a_{p,n} &= (-1)^{n+1}\left(2n+\tfrac{1}{2}\right)\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\sum_{q=1}^{p}\Gamma^2\left(\tfrac{1}{2}q+\tfrac{1}{2}\right)\sum_{m=0}^{[(p-q)/2]}(-1)^m a_{p-q,m}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} \\ &\cdot \frac{1}{\Gamma\left(\frac{1}{2}q-m-n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}q+m-n+1\right)} \\ &\cdot \frac{1}{\Gamma\left(\frac{1}{2}q-m+n+1\right)\Gamma\left(\frac{1}{2}q+m+n+\frac{3}{2}\right)}, \end{aligned} \tag{2.42}$$

valid for $p = 1, 2, 3, \ldots, n = 0, 1, \ldots, [p/2]$, and initiated by $a_{0,0} = 1$. This recurrence relation is the first key equation in the scheme for calculating the low-frequency expansion of the scattering coefficient σ_2.

2.3.3 Far field and scattering coefficient

Starting from the representation (2.19) for $\phi_2(\mathbf{x})$, we determine the far field at an observation point $\mathbf{x} = (R\sin\theta, 0, R\cos\theta)$, where R, θ are spherical coordinates. For an integration point $\mathbf{x}' \in D$ with polar coordinates ρ', φ' (see (2.27)), the distance $|\mathbf{x} - \mathbf{x}'|$ takes the asymptotic form

$$\begin{aligned} |\mathbf{x}-\mathbf{x}'| &= \sqrt{R^2 - 2R\rho'\sin\theta\cos\varphi' + \rho'^2} \\ &= R - \rho'\sin\theta\cos\varphi' + O\left(R^{-1}\right), \quad (R\to\infty). \end{aligned} \tag{2.43}$$

Using this result in (2.19), we obtain the far-field expansion

$$\phi_2(R\sin\theta, 0, R\cos\theta) = A_2(\theta)\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R\to\infty) \tag{2.44}$$

with far-field amplitude

$$A_2(\theta) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^1 u(\rho')\exp(-ik\rho'\sin\theta\cos\varphi')\rho'\,d\rho'\,d\varphi'. \tag{2.45}$$

To further evaluate $A_2(\theta)$, we replace $u(\rho')$ by its low-frequency expansion taken from (2.22) and (2.39), viz.

$$u(\rho') = \frac{2}{\pi}\sum_{p=0}^{\infty}(ik)^p\sum_{n=0}^{[p/2]}a_{p,n}\frac{P_{2n}(\sqrt{1-\rho'^2})}{\sqrt{1-\rho'^2}}. \tag{2.46}$$

Furthermore, we need the auxiliary integral

$$\int_0^{2\pi}\int_0^1 \frac{P_{2n}(\sqrt{1-\rho'^2})}{\sqrt{1-\rho'^2}}\exp(-ik\rho'\sin\theta\cos\varphi')\rho'\,d\rho'\,d\varphi'$$
$$= \pi\sqrt{2}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{J_{2n+1/2}(k\sin\theta)}{(k\sin\theta)^{1/2}}, \tag{2.47}$$

quoted from Boersma and Danicki [4, form. (2.6), (2.8)]; here, $J_{2n+1/2}$ is the Bessel function of the first kind of order $2n+\frac{1}{2}$. As a result we have for $A_2(\theta)$ the low-frequency expansion

$$A_2(\theta) = \frac{\sqrt{2}}{\pi}\sum_{p=0}^{\infty}(ik)^p\sum_{n=0}^{[p/2]}a_{p,n}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{J_{2n+1/2}(k\sin\theta)}{(k\sin\theta)^{1/2}}, \tag{2.48}$$

in terms of the coefficients $a_{p,n}$.

According to (2.7), the scattering coefficient σ_2 is related to the far-field amplitude in the direction of incidence. By setting $\theta = 0$ in (2.48), we find the low-frequency expansion

$$\sigma_2 = -\frac{4}{k}\,\mathrm{Im}\,A_2(0) = -\frac{8}{\pi}\sum_{p=0}^{\infty}(-1)^p a_{2p+1,0}k^{2p} \tag{2.49}$$

for the scattering coefficient of a circular disk of unit radius. For a disk of radius a, the low-frequency expansion of σ_2 follows by replacing k with $\alpha = ka$ in (2.49), viz.

$$\sigma_2 = -\frac{8}{\pi}\sum_{p=0}^{\infty}(-1)^p a_{2p+1,0}\alpha^{2p}. \tag{2.50}$$

This is the second key equation in our scheme for calculating the low-frequency expansion of σ_2.

2.3.4 Scheme for calculating the scattering coefficient

The scheme for the calculation of the low-frequency expansion of the scattering coefficient σ_2 is now obvious. Starting from the initial value $a_{0,0} = 1$, we use the recurrence relation (2.42) to determine a number of coefficients $a_{p,n}$, $p = 1, 2, 3, \ldots, n = 0, 1, \ldots, [p/2]$. Only the coefficients $a_{2p+1,0}$, $p = 0, 1, 2, \ldots$, are needed in the low-frequency expansion (2.50) of σ_2.

A *Mathematica*-implementation of this scheme is listed in Appendix C; the function is called `Bouwkamp2`. The results of the calculations by *Mathematica* are presented in the next section.

2.4 Results for the scattering coefficient σ_2

Using the *Mathematica*-packages in Appendices A, B, and C, we can evaluate the low-frequency expansion of the scattering coefficient σ_2 up to arbitrary order. The expansion is in even powers of α and the leading term is found to be $16/\pi^2$. Therefore we set

$$\sigma_2 = \sum_{n=0}^{\infty} \sigma_{2,2n}\,\alpha^{2n} = \frac{16}{\pi^2}\sum_{n=0}^{\infty} \tilde{\sigma}_{2,2n}\,\alpha^{2n}, \tag{2.51}$$

in which $\sigma_{2,0} = 16/\pi^2$, $\tilde{\sigma}_{2,0} = 1$. The expansion (2.51) has been evaluated up to and including terms of order α^{18}. In Table 2.1 we present the exact values of the normalized coefficients $\tilde{\sigma}_{2,2n} = (\pi^2/16)\,\sigma_{2,2n}$ for $n = 0(1)9$. It is observed that $\tilde{\sigma}_{2,2n}$, $n = 0(1)9$, is a polynomial in π^{-2} of degree n, with rational coefficients and leading term $(-1)^n\,2^{2n}\,\pi^{-2n}$; these properties can be proved for general n by induction. In Table 2.2 we present the numerical values, to six significant digits, of the coefficients $\sigma_{2,2n}$ for $n = 0(1)19$. It has been found recently [5] that the expansion (2.51), considered as a power series in α, has a radius of convergence 3.39879, to five decimal places.

The calculations were performed by all three schemes, namely, the two schemes based on Bazer and Brown's method with the integral equation solved by power-series expansion or by Picard iteration (cf. Section 2.2.3), and the scheme based on Bouwkamp's method (cf. Section 2.3.4). All schemes do yield the same results of Table 2.1, which provides an excellent check on the correctness of the mathematical analysis and of the *Mathematica*-programmes. Moreover, the results of Tables 2.1 and 2.2 were also obtained (and extended) by Professor D.S. Jones (Dundee) by an independent *Mathematica* calculation. The *Mathematica*-programmes were executed on a 486DX33 computer with 8 MB internal memory, using *Mathematica* Enhanced Version 2.2 for Windows and Microsoft Windows for Workgroups Version 3.11. For the calculation of ten coefficients as in Table 2.1, the evaluation times for the three programmes were: 224.15 seconds for `BazerBrown2`; 230.25 seconds for `Picard2`; and 937.52 seconds for `Bouwkamp2`.

Our expansion of the scattering coefficient σ_2 agrees with and extends the results of Bouwkamp [9, form. (8.1)] and of Bazer and Brown [1, form. (75)]. In both references the coefficients $\tilde{\sigma}_{2,2n}$ have been determined for $n = 0(1)3$. According to [10, form. (14.50)], the best result available so far is an expansion up to and including terms of order α^{10}, due to Hurd [11]. Later on, Hurd [12] determined two additional terms of the expansion, with coefficients $\tilde{\sigma}_{2,12}$ and $\tilde{\sigma}_{2,14}$, by use of the symbolic programming language FORMAC. The results from [11, Table III] and [12] do agree with our Table 2.1.

In the schemes based on Bazer and Brown's method, the integral equation (2.13) is solved either by power-series expansion or by Picard iteration. The first option leads to the expansion (2.18) of $g(t)$, which is evaluated up to a certain order. In Table 2.3 we have listed the expansion coefficients $g_n(t)$ for $n = 0(1)6$. The corresponding expansion up to order α^7 agrees with that of Bazer and Brown [1, form. (68)]. Note that the coefficient $g_n(t)$ is an even polynomial in t of degree $2[n/2]$. In the solution of (2.13) by Picard iteration, a factor α is gained in each step. Because of the factor $8/\pi\alpha$ in (2.17), we need nineteen iteration steps for the evaluation of (2.51) up to and including terms of order α^{18}.

Table 2.1: Exact values of the normalized coefficients $\tilde{\sigma}_{2,2n} = (\pi^2/16)\,\sigma_{2,2n}$, $n = 0(1)9$, in the expansion (2.51) of σ_2.

$$\tilde{\sigma}_{2,0} = 1$$

$$\tilde{\sigma}_{2,2} = -\frac{4}{\pi^2} + \frac{4}{9}$$

$$\tilde{\sigma}_{2,4} = \frac{16}{\pi^4} - \frac{8}{3\pi^2} + \frac{71}{675}$$

$$\tilde{\sigma}_{2,6} = -\frac{64}{\pi^6} + \frac{128}{9\pi^4} - \frac{1936}{2025\pi^2} + \frac{568}{33075}$$

$$\tilde{\sigma}_{2,8} = \frac{256}{\pi^8} - \frac{640}{9\pi^6} + \frac{304}{45\pi^4} - \frac{43168}{178605\pi^2} + \frac{9523}{4465125}$$

$$\tilde{\sigma}_{2,10} = -\frac{1024}{\pi^{10}} + \frac{1024}{3\pi^8} - \frac{28288}{675\pi^6} + \frac{80704}{35721\pi^4} - \frac{640204}{13395375\pi^2} + \frac{329068}{1620840375}$$

$$\tilde{\sigma}_{2,12} = \frac{4096}{\pi^{12}} - \frac{14336}{9\pi^{10}} + \frac{485632}{2025\pi^8} - \frac{2208512}{127575\pi^6} + \frac{17011712}{28704375\pi^4} - \frac{200408}{25727625\pi^2} + \frac{28561418}{1917454163625}$$

$$\begin{aligned}\tilde{\sigma}_{2,14} = {} & -\frac{16384}{\pi^{14}} + \frac{65536}{9\pi^{12}} - \frac{876544}{675\pi^{10}} + \frac{104992768}{893025\pi^8} - \frac{53558528}{9568125\pi^6} + \frac{9409312768}{72937816875\pi^4} \\ & - \frac{5427789356576}{5033317179515625\pi^2} + \frac{24646112}{28761812454375}\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_{2,16} = {} & \frac{65536}{\pi^{16}} - \frac{32768}{\pi^{14}} + \frac{1519616}{225\pi^{12}} - \frac{219004928}{297675\pi^{10}} + \frac{1004324096}{22325625\pi^8} - \frac{248401408}{165391875\pi^6} \\ & + \frac{3275445271751792}{135899563846921875\pi^4} - \frac{5850372900928}{45299854615640625\pi^2} + \frac{2953662389}{74809474193829375}\end{aligned}$$

$$\begin{aligned}\tilde{\sigma}_{2,18} = {} & -\frac{262144}{\pi^{18}} + \frac{1310720}{9\pi^{16}} - \frac{13795328}{405\pi^{14}} + \frac{774815744}{178605\pi^{12}} - \frac{4345136128}{13395375\pi^{10}} \\ & + \frac{624041153536}{43762690125\pi^8} - \frac{9417147043033088}{27179912769384375\pi^6} + \frac{322152102668096}{81539738308153125\pi^4} \\ & - \frac{7973773660981292}{589124609276406328125\pi^2} + \frac{200771738036}{135031100919862021875}\end{aligned}$$

Table 2.2: Numerical values of the coefficients $\sigma_{2,2n}$, $n = 0(1)19$, in the expansion (2.51) of σ_2.

n	$\sigma_{2,2n}$	n	$\sigma_{2,2n}$
0	$1.62114 \cdot 10^{0}$	10	$-1.25015 \cdot 10^{-11}$
1	$6.34833 \cdot 10^{-2}$	11	$-8.26174 \cdot 10^{-13}$
2	$-1.21411 \cdot 10^{-3}$	12	$1.04976 \cdot 10^{-13}$
3	$-4.21848 \cdot 10^{-4}$	13	$4.75653 \cdot 10^{-15}$
4	$1.49251 \cdot 10^{-5}$	14	$-8.51679 \cdot 10^{-16}$
5	$2.99261 \cdot 10^{-6}$	15	$-2.40050 \cdot 10^{-17}$
6	$-1.53096 \cdot 10^{-7}$	16	$6.71043 \cdot 10^{-18}$
7	$-2.03550 \cdot 10^{-8}$	17	$8.81790 \cdot 10^{-20}$
8	$1.42562 \cdot 10^{-9}$	18	$-5.14921 \cdot 10^{-20}$
9	$1.33048 \cdot 10^{-10}$	19	$4.29350 \cdot 10^{-23}$

Table 2.3: Coefficients $g_n(t)$, $n = 0(1)6$, in the expansion (2.18) of $g(t)$.

$$
\begin{aligned}
g_0(t) &= 1 \\
g_1(t) &= -\frac{2i}{\pi} \\
g_2(t) &= -\frac{4}{\pi^2} + \frac{t^2}{2} \\
g_3(t) &= \frac{8i}{\pi^3} - \frac{4i}{9\pi} - \frac{i t^2}{3\pi} \\
g_4(t) &= \frac{16}{\pi^4} - \frac{4}{3\pi^2} - \frac{2t^2}{3\pi^2} + \frac{t^4}{24} \\
g_5(t) &= -\frac{32i}{\pi^5} + \frac{32i}{9\pi^3} - \frac{4i}{75\pi} + \frac{4it^2}{3\pi^3} - \frac{4it^2}{45\pi} - \frac{it^4}{60\pi} \\
g_6(t) &= -\frac{64}{\pi^6} + \frac{80}{9\pi^4} - \frac{508}{2025\pi^2} + \frac{8t^2}{3\pi^4} - \frac{34t^2}{135\pi^2} - \frac{t^4}{30\pi^2} + \frac{t^6}{720}
\end{aligned}
$$

3 Diffraction of a scalar wave by a hard circular disk

3.1 Formulation of the problem

We consider the acoustic diffraction of a normally incident, plane wave by an acoustically hard, circular disk D, of radius a. The choice of Cartesian coordinates x, y, z, and cylindrical coordinates ρ, φ, z, is the same as in Section 2.1. The disk D is described by $0 \leq x^2 + y^2 \leq a^2$, $z = 0$. The incident wave is given by $\Phi^i(x, y, z) = \exp(ikz)$, with wave number k, and incident from $z < 0$.

Following Bouwkamp [9, pp. 38–39], we express the resulting total field Φ^t as

$$\Phi^t(x, y, z) = \begin{cases} \exp(ikz) + \phi_1(x, y, -z), & z < 0, \\ \exp(ikz) - \phi_1(x, y, z), & z > 0, \end{cases} \tag{3.1}$$

where ϕ_1, which is defined for $z \geq 0$ only, has the following properties:

1) ϕ_1 satisfies the Helmholtz equation

$$\Delta\phi_1 + k^2\phi_1 = 0, \quad \text{when } z > 0; \tag{3.2}$$

2) $$\phi_1(x, y, 0) = 0, \quad \text{when } x^2 + y^2 > a^2; \tag{3.3}$$

3) $$\frac{\partial \phi_1}{\partial z}(x, y, 0) = ik, \quad \text{when } x^2 + y^2 < a^2; \tag{3.4}$$

4) ϕ_1 satisfies the Sommerfeld radiation condition at infinity;

5) ϕ_1 is everywhere finite;

6) $\nabla\phi_1$ is quadratically integrable over any domain in $I\!R^3$.

In this manner the diffraction problem for a hard circular disk has been reduced to a boundary value problem for ϕ_1. Also the radiation problem for a freely vibrating, hard, circular disk and, according to Babinet's principle [9, p. 39], the complementary problem of diffraction through a circular aperture in an acoustically soft, infinite, plane screen, can be solved in terms of the function ϕ_1; see Bouwkamp [6].

Because of the axial symmetry of the diffraction problem, ϕ_1 is independent of the angle φ, and we will use the notation $\phi_1(\rho, z)$ as an alternative to $\phi_1(x, y, z)$. At large distances from the disk we have the far-field expansion

$$\phi_1(R\sin\theta, R\cos\theta) = A_1(\theta)\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R \to \infty) \tag{3.5}$$

with far-field amplitude $A_1(\theta)$; here, R, θ are spherical coordinates specified by (2.5). The scattering coefficient of the hard circular disk and the transmission coefficient of the complementary circular aperture in a soft plane screen are denoted by σ_1 and τ_1, respectively. As in (2.7), σ_1 is related to the far-field amplitude of the scattered wave $-\phi_1$ in the direction of incidence ($\theta = 0$),

$$\sigma_1 = -\frac{4}{a^2k}\,\mathrm{Im}\,A_1(0), \tag{3.6}$$

whereas $\tau_1 = \frac{1}{2}\sigma_1$; see Jones [13, §§ 8.19, 9.4].

In the next sections the boundary value problem for ϕ_1 is solved by methods developed by Bazer and Brown [1] and by Bouwkamp [6]. Both methods are well suited to obtain a low-frequency approximation to the solution, involving low-frequency expansions in powers of $\alpha = ka$ for various field quantities. The emphasis is on the low-frequency expansion for the scattering coefficient σ_1, of which several terms are exactly determined by means of *Mathematica*.

3.2 Bazer and Brown's solution

3.2.1 Reduction to an integral equation

Using the method of Bazer and Brown [1], we represent ϕ_1 by the integral (cf. [1, form. (11)])

$$\phi_1(\rho, z) = \int_{-1}^{1} \frac{\exp\left(ik\sqrt{\rho^2 + (z + iat)^2}\right)}{\sqrt{\rho^2 + (z + iat)^2}}\, f_1(t)\, dt, \tag{3.7}$$

valid for $z \geq 0$. Here, $f_1(t)$ is a yet unknown function which is required to be an odd function of t. The branch of the square root is taken such that $\mathrm{Re}\sqrt{\rho^2 + (z + iat)^2} > 0$, when $z > 0$. As shown in [1], the representation (3.7) satisfies all conditions of the boundary value problem for ϕ_1, except for the boundary condition (3.4). By use of (2.9) we have, for $0 < \rho < a$,

$$\begin{aligned}
\lim_{z\downarrow 0} \frac{\partial\phi_1}{\partial z}(\rho, z) &= \lim_{z\downarrow 0} \frac{1}{\rho}\frac{\partial}{\partial\rho} \int_{-1}^{1} \frac{\exp\left(ik\sqrt{\rho^2+(z+iat)^2}\right)}{\sqrt{\rho^2+(z+iat)^2}} (z+iat)\, f_1(t)\, dt \\
&= \frac{ia}{\rho}\frac{\partial}{\partial\rho} \int_0^{\rho/a} \frac{2\exp\left(ik\sqrt{\rho^2-a^2t^2}\right)}{\sqrt{\rho^2-a^2t^2}} t f_1(t)\, dt \\
&\quad - \frac{a}{\rho}\frac{\partial}{\partial\rho} \int_{\rho/a}^{1} \frac{2\sinh\left(k\sqrt{a^2t^2-\rho^2}\right)}{\sqrt{a^2t^2-\rho^2}} t f_1(t)\, dt. \qquad (3.8)
\end{aligned}$$

Imposing the boundary condition (3.4), we obtain the integro-differential equation [1, form. (33)]

$$\begin{aligned}
&\frac{\partial}{\partial\rho} \int_0^{\rho/a} \frac{\cos\left(\alpha\sqrt{(\rho/a)^2-t^2}\right)}{\sqrt{(\rho/a)^2-t^2}} t f_1(t)\, dt \\
&= \frac{k\rho}{2} + \frac{1}{i}\frac{\partial}{\partial\rho} \int_0^{1} \frac{\sinh\left(\alpha\sqrt{t^2-(\rho/a)^2}\right)}{\sqrt{t^2-(\rho/a)^2}} t f_1(t)\, dt, \qquad (3.9)
\end{aligned}$$

where $\alpha = ka$. As detailed in [1, Sec. IV], both sides of (3.9) are integrated from 0 to ρ, whereupon the integral operator on the left can be inverted by means of Laplace transformation. As a result it is found that equation (3.9) reduces to the Fredholm integral equation [1, form. (49)]

$$t f_1(t) = \frac{at}{\pi}\sinh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1} \left(\frac{\sinh[\alpha(t-s)]}{t-s} - \frac{\cosh(\alpha t)\sinh(\alpha s)}{s} \right) s f_1(s)\, ds, \quad -1 \le t \le 1. \qquad (3.10)$$

We now observe that the latter equation can be simplified through a division by t. Indeed, by setting

$$\frac{s}{t(t-s)} = \frac{1}{t-s} - \frac{1}{t} \qquad (3.11)$$

in the resulting integral on the right of (3.10), this integral can be rewritten as

$$\frac{1}{\pi i}\int_{-1}^{1}\left(\frac{\sinh[\alpha(t-s)]}{t-s}-\frac{\sinh[\alpha(t-s)]+\cosh(\alpha t)\sinh(\alpha s)}{t}\right)f_1(s)\,ds$$

$$= \frac{1}{\pi i}\int_{-1}^{1}\left(\frac{\sinh[\alpha(t-s)]}{t-s}-\frac{\sinh(\alpha t)\cosh(\alpha s)}{t}\right)f_1(s)\,ds$$

$$= \frac{1}{\pi i}\int_{-1}^{1}\frac{\sinh[\alpha(t-s)]}{t-s}f_1(s)\,ds, \tag{3.12}$$

because $f_1(s)$ is an odd function of s. Thus the integral equation (3.10) simplifies to

$$f_1(t) = \frac{a}{\pi}\sinh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1}\frac{\sinh[\alpha(t-s)]}{t-s}f_1(s)\,ds, \quad -1 \le t \le 1. \tag{3.13}$$

This simplification was overlooked in [1]. By setting $f_1(t) = (a/\pi)f(t)$, we are led to the integral equation

$$f(t) = \sinh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1}\frac{\sinh[\alpha(t-s)]}{t-s}f(s)\,ds, \quad -1 \le t \le 1, \tag{3.14}$$

which is the first key equation in the scheme for calculating the low-frequency expansion of the scattering coefficient σ_1.

3.2.2 Far field and scattering coefficient

By inserting (2.14) into (3.7), we obtain the far-field expansion

$$\phi_1(R\sin\theta, R\cos\theta) = A_1(\theta)\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R\to\infty) \tag{3.15}$$

with far-field amplitude

$$A_1(\theta) = \int_{-1}^{1}\exp(-\alpha t\cos\theta)f_1(t)\,dt = -\frac{2a}{\pi}\int_{0}^{1}\sinh(\alpha t\cos\theta)f(t)\,dt. \tag{3.16}$$

The expression (3.6) for the scattering coefficient σ_1 now becomes

$$\sigma_1 = \frac{8}{\pi\alpha}\,\mathrm{Im}\int_{0}^{1}\sinh(\alpha t)f(t)\,dt, \tag{3.17}$$

expressed in terms of the function $f(t)$. This is the second key equation in our scheme for calculating the low-frequency expansion of σ_1.

3.2.3 Scheme for calculating the scattering coefficient

Our scheme for calculating the low-frequency expansion of the scattering coefficient σ_1 is based on the two equations (3.14) and (3.17). Two methods are employed for the solution of the integral equation (3.14). In the first method, the kernel of the integral equation is expanded in powers of α and the function $f(t)$ is replaced by the power-series expansion

$$f(t) = \sum_{n=0}^{\infty} f_n(t)\,\alpha^n, \tag{3.18}$$

with coefficients $f_n(t)$ to be determined. By equating terms containing the same power of α, one is led to a recurrence relation for $f_n(t)$ expressed in terms of preceding coefficients. In the second method, the integral equation is solved by Picard iteration. Here, a factor α^3 is gained in each iteration step, since $f(t)$ is an odd function of t. The expansion of $f(t)$ thus determined is inserted into (3.17) and the function $\sinh(\alpha t)$ is expanded in powers of α. Then by a straightforward evaluation we find the required low-frequency expansion of σ_1 up to a certain order.

A *Mathematica*-implementation of the scheme using power-series expansion is listed in Appendix A; the function is called `BazerBrown1`. The *Mathematica*-function `Picard1` listed in Appendix B uses Picard iteration. The results of the calculations by *Mathematica* are presented in Section 3.4.

3.3 Bouwkamp's solution

3.3.1 Reduction to a system of integro-differential equations

Apart from a change of notation we closely follow the approach of Bouwkamp [6]. For convenience we take the radius of the disk D equal to unity. In accordance with [6, form. (3)], we represent ϕ_1 by

$$\phi_1(\mathbf{x}) = -\frac{1}{2\pi}\frac{\partial}{\partial z}\iint_D v(\mathbf{x}')\,\frac{\exp(ik|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\,dx'\,dy', \tag{3.19}$$

where $\mathbf{x} = (x, y, z)$, $\mathbf{x}' = (x', y', 0)$ and

$$v(\mathbf{x}') = \phi_1(x', y', 0). \tag{3.20}$$

By imposing the boundary condition (3.4), we are led to the following "integro-differential equation" for $v(\mathbf{x}')$:

$$\lim_{z\downarrow 0} -\frac{1}{2\pi}\frac{\partial^2}{\partial z^2}\iint_D v(\mathbf{x}')\,\frac{\exp(ik|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\,dx'\,dy' = ik, \quad x^2+y^2<1. \tag{3.21}$$

To solve this equation, we expand the exponential function in powers of ik and we substitute for $v(\mathbf{x}')$ the power-series expansion

$$v(\mathbf{x}') = ik\sum_{p=0}^{\infty} v_p(\mathbf{x}')\,(ik)^p, \tag{3.22}$$

with coefficients $v_p(\mathbf{x}')$ to be determined. As a result we find that equation (3.21) passes into

$$\sum_{p=0}^{\infty}(ik)^p\left(\lim_{z\downarrow 0} -\frac{1}{2\pi}\frac{\partial^2}{\partial z^2}\sum_{q=0}^{p}\frac{1}{q!}\iint_D v_{p-q}(\mathbf{x}')\,|\mathbf{x}-\mathbf{x}'|^{q-1}\,dx'\,dy'\right) = 1, \tag{3.23}$$
$$x^2+y^2<1.$$

All terms of the power series on the left have to vanish except the term independent of k, which must be equal to unity. This leads to the recursive system of integro-differential equations (cf. [6, form. (10)–(12)])

$$\lim_{z\downarrow 0} -\frac{1}{2\pi}\frac{\partial^2}{\partial z^2}\iint_D \frac{v_p(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,dx'\,dy' = V_p(\mathbf{x}), \quad x^2+y^2<1,\ p=0,1,2,\ldots, \tag{3.24}$$

where

$$V_0(\mathbf{x}) = 1, \tag{3.25}$$

$$V_p(\mathbf{x}) = \lim_{z\downarrow 0}\frac{1}{2\pi}\frac{\partial^2}{\partial z^2}\sum_{q=1}^{p}\frac{1}{q!}\iint_D v_{p-q}(\mathbf{x}')\,|\mathbf{x}-\mathbf{x}'|^{q-1}\,dx'\,dy', \quad p=1,2,3,\ldots \tag{3.26}$$

Note that the derivative

$$\frac{\partial^2}{\partial z^2}\left(|\mathbf{x}-\mathbf{x}'|^{q-1}\right) = (q-1)\,|\mathbf{x}-\mathbf{x}'|^{q-3} + (q-1)(q-3)\,z^2\,|\mathbf{x}-\mathbf{x}'|^{q-5} \tag{3.27}$$

vanishes for $q=1$; hence, $V_1(\mathbf{x})=0$. By taking limits as $z\downarrow 0$, we find that (3.26) simplifies to

$$V_p(\mathbf{x}) = \frac{1}{2\pi}\sum_{q=2}^{p}\frac{q-1}{q!}\iint_D v_{p-q}(\mathbf{x}')|\mathbf{x}-\mathbf{x}'|^{q-3}\,dx'\,dy', \quad p=2,3,4,\ldots \tag{3.28}$$

3.3.2 Solution of the integro-differential equations

We introduce polar coordinates ρ, φ, and ρ', φ', on the disk D, specified by (2.27). In view of the axial symmetry of the diffraction problem, we change the notations $v(\mathbf{x}')$, $v_p(\mathbf{x}')$, into $v(\rho')$, $v_p(\rho')$. Since the integrals in (3.24) and (3.28) are independent of φ, the distance $|\mathbf{x}-\mathbf{x}'| = r$ may be taken as in (2.28) and we change the notation $V_p(\mathbf{x})$ into $V_p(\rho)$. The integral in (3.24) is an axially symmetric potential function, which thus is annihilated by the Laplace operator $\Delta = \partial^2/\partial\rho^2 + \rho^{-1}\,\partial/\partial\rho + \partial^2/\partial z^2$. Hence the left-hand side of (3.24) can be reduced to

$$\begin{aligned}\lim_{z\downarrow 0} -\frac{1}{2\pi}\frac{\partial^2}{\partial z^2}\iint_D \frac{v_p(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,dx'\,dy' &= \lim_{z\downarrow 0}\frac{1}{2\pi}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\iint_D \frac{v_p(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,dx'\,dy' \\ &= \frac{1}{2\pi}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\int_0^{2\pi}\int_0^1 \frac{v_p(\rho')}{r}\,\rho'\,d\rho'\,d\varphi'.\end{aligned} \tag{3.29}$$

Then, in polar coordinates, the system of integro-differential equations (3.24), (3.25), (3.28) takes the form

$$\frac{1}{2\pi}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\int_0^{2\pi}\int_0^1 \frac{v_p(\rho')}{r}\,\rho'\,d\rho'\,d\varphi' = V_p(\rho), \tag{3.30}$$

$$0 \le \rho \le 1, \;\; p = 0, 1, 2, \dots,$$

where

$$V_0(\rho) = 1, \tag{3.31}$$

$$V_1(\rho) = 0, \tag{3.32}$$

$$V_p(\rho) = \frac{1}{2\pi}\sum_{q=2}^{p}\frac{q-1}{q!}\int_0^{2\pi}\int_0^1 v_{p-q}(\rho')\,r^{q-3}\,\rho'\,d\rho'\,d\varphi', \quad p = 2, 3, 4, \dots \tag{3.33}$$

This system is identical to that of Bouwkamp [6, form. (17)].

As detailed in [6, pp. 8–10], the solution of the integro-differential equation (3.30) is most simply expressed in terms of Legendre polynomials $P_{2n+1}(\sqrt{1-\rho'^2})$, $n = 0, 1, 2, \dots$, of odd degree $2n+1$. Explicit results for $v_p(\rho')$, with $p = 0(1)7$, are presented in [6, form. (25)–(32)]. For $p = 0$ and $p = 1$ it is found that

$$v_0(\rho') = -\frac{2}{\pi} P_1(\sqrt{1-\rho'^2}) = -\frac{2}{\pi}\sqrt{1-\rho'^2}, \qquad v_1(\rho') = 0. \tag{3.34}$$

For general p, Bouwkamp makes the Ansatz [6, form. (33)]

$$v_p(\rho') = \sum_{n=0}^{N_p} b_{p,n} P_{2n+1}(\sqrt{1-\rho'^2}), \tag{3.35}$$

in which

$$N_p = \begin{cases} p/2, & p \text{ even}, \\ (p-3)/2, & p \text{ odd}, \end{cases} \tag{3.36}$$

and $b_{0,0} = -2/\pi$ by (3.34). The remaining coefficients $b_{p,n}$ are determined by the recurrence relation established in [6, form. (34)]. We rewrite this relation in a form similar to (2.42), viz.

$$\begin{aligned} b_{p,n} \;=\; & (-1)^{n+1}\left(n+\tfrac{3}{4}\right)\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\sum_{q=2}^{p}\Gamma\left(\tfrac{1}{2}q-\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}q+\tfrac{1}{2}\right)\sum_{m=0}^{N_{p-q}}(-1)^m b_{p-q,m}\frac{\Gamma(m+\frac{3}{2})}{\Gamma(m+1)} \\ & \cdot\frac{1}{\Gamma\left(\frac{1}{2}q-m-n-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}q+m-n+1\right)} \\ & \cdot\frac{1}{\Gamma\left(\frac{1}{2}q-m+n+1\right)\Gamma\left(\frac{1}{2}q+m+n+\frac{5}{2}\right)}, \end{aligned} \tag{3.37}$$

valid for $p = 2, 3, 4, \ldots$, $n = 0, 1, \ldots, N_p$, and initiated by $b_{0,0} = -2/\pi$. This recurrence relation is the first key equation in the scheme for calculating the low-frequency expansion of the scattering coefficient σ_1.

3.3.3 Far field and scattering coefficient

Starting from the representation (3.19) for $\phi_1(\mathbf{x})$, we determine the far field at an observation point $\mathbf{x} = (R\sin\theta, 0, R\cos\theta)$, where R, θ are spherical coordinates. For an integration point $\mathbf{x}' \in D$ with polar coordinates ρ', φ', the distance $|\mathbf{x}-\mathbf{x}'|$ takes the asymptotic form (2.43) as $R \to \infty$. Furthermore we have

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial R} + O\left(R^{-1}\right), \quad (R \to \infty) \tag{3.38}$$

so that

$$\frac{\partial}{\partial z}\left(\frac{\exp\left(ik|\mathbf{x}-\mathbf{x}'|\right)}{|\mathbf{x}-\mathbf{x}'|}\right) = ik\cos\theta\,\frac{e^{ikR}}{R}\,\exp(-ik\rho'\sin\theta\cos\varphi') + O\left(R^{-2}\right), \quad (R\to\infty). \tag{3.39}$$

Using this result in (3.19), we obtain the far-field expansion

$$\phi_1(R\sin\theta, 0, R\cos\theta) = A_1(\theta)\,\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R\to\infty) \tag{3.40}$$

with far-field amplitude

$$A_1(\theta) = -\frac{ik\cos\theta}{2\pi}\int_0^{2\pi}\int_0^1 v(\rho')\exp(-ik\rho'\sin\theta\cos\varphi')\,\rho'\,d\rho'\,d\varphi'. \tag{3.41}$$

By carrying out the integration with respect to φ', which leads to a Bessel function J_0, we find

$$A_1(\theta) = -ik\cos\theta\int_0^1 v(\rho')\,J_0(k\rho'\sin\theta)\,\rho'\,d\rho'. \tag{3.42}$$

To further evaluate $A_1(\theta)$, we replace $v(\rho')$ by its low-frequency expansion taken from (3.22) and (3.35), viz.

$$v(\rho') = ik\sum_{p=0}^{\infty}(ik)^p\sum_{n=0}^{N_p} b_{p,n}\,P_{2n+1}(\sqrt{1-\rho'^2}). \tag{3.43}$$

In addition, we need the auxiliary integral

$$\begin{aligned} I_n &= \int_0^1 P_{2n+1}(\sqrt{1-\rho'^2})\,J_0(k\rho'\sin\theta)\,\rho'\,d\rho' \\ &= (-1)^n\sqrt{\frac{\pi}{2}}\,(2n+1)\,P_{2n}(0)\,\frac{J_{2n+3/2}(k\sin\theta)}{(k\sin\theta)^{3/2}} \\ &= \sqrt{2}\,\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\,\frac{J_{2n+3/2}(k\sin\theta)}{(k\sin\theta)^{3/2}}, \end{aligned} \tag{3.44}$$

quoted from [6, p. 12]. As a result we have for $A_1(\theta)$ the low-frequency expansion

$$A_1(\theta) = \sqrt{2}\,k^2\cos\theta\sum_{p=0}^{\infty}(ik)^p\sum_{n=0}^{N_p} b_{p,n}\,\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\,\frac{J_{2n+3/2}(k\sin\theta)}{(k\sin\theta)^{3/2}}, \tag{3.45}$$

in terms of the coefficients $b_{p,n}$.

According to (3.6), the scattering coefficient σ_1 is related to the far-field amplitude in the direction of incidence. By setting $\theta = 0$ in (3.45), we find the low-frequency expansion

$$\sigma_1 = -\frac{4}{k}\,\mathrm{Im}\,A_1(0) = \frac{4}{3}\sum_{p=0}^{\infty}(-1)^p\, b_{2p+3,0}\, k^{2p+4} \tag{3.46}$$

for the scattering coefficient of a circular disk of unit radius. For a disk of radius a, the low-frequency expansion of σ_1 follows by replacing k with $\alpha = ka$ in (3.46), viz.

$$\sigma_1 = \frac{4}{3}\sum_{p=0}^{\infty}(-1)^p\, b_{2p+3,0}\, \alpha^{2p+4}. \tag{3.47}$$

This is the second key equation in our scheme for calculating the low-frequency expansion of σ_1.

3.3.4 Scheme for calculating the scattering coefficient

The scheme for the calculation of the low-frequency expansion of the scattering coefficient σ_1 is now obvious. Starting from the initial value $b_{0,0} = -2/\pi$, we use the recurrence relation (3.37) to determine a number of coefficients $b_{p,n}$, $p = 2, 3, 4, \ldots, n = 0, 1, \ldots, N_p$. Only the coefficients $b_{2p+3,0}$, $p = 0, 1, 2, \ldots$, are needed in the low-frequency expansion (3.47) of σ_1.

A *Mathematica*-implementation of this scheme is listed in appendix C; the function is called `Bouwkamp1`. The results of the calculations by *Mathematica* are presented in the next section.

3.4 Results for the scattering coefficient σ_1

Using the *Mathematica*-packages in Appendices A, B, and C, we can evaluate the low-frequency expansion of the scattering coefficient σ_1 up to arbitrary order. The expansion is in even powers of α and the leading term is found to be $(16/27\pi^2)\,\alpha^4$. Therefore we set

$$\sigma_1 = \sum_{n=2}^{\infty}\sigma_{1,2n}\,\alpha^{2n} = \frac{16\alpha^4}{27\pi^2}\sum_{n=2}^{\infty}\tilde{\sigma}_{1,2n}\,\alpha^{2n-4}, \tag{3.48}$$

in which $\sigma_{1,4} = 16/27\pi^2$, $\tilde{\sigma}_{1,4} = 1$. The expansion (3.48) has been evaluated up to and including terms of order α^{22}. In Table 3.1 we present the exact values of the normalized coefficients $\tilde{\sigma}_{1,2n} = (27\pi^2/16)\,\sigma_{1,2n}$ for $n = 2(1)11$. From the tabulated values and from additional calculated values of $\tilde{\sigma}_{1,2n}$, not presented here, it appears that $\tilde{\sigma}_{1,2n}$ is a polynomial in π^{-2} of degree $[(n-2)/3]$, with rational coefficients. In Table 3.2 we present the numerical values, to six significant digits, of the coefficients $\sigma_{1,2n}$ for $n = 2(1)21$. It has been found recently [5] that the expansion (3.48), considered as a power series in α, has a radius of convergence 2.12548, to five decimal places.

The calculations were performed by all three schemes, namely, the two schemes based on Bazer and Brown's method with the integral equation solved by power-series expansion or by Picard iteration (cf. Section 3.2.3), and the scheme based on Bouwkamp's method (cf. Section 3.3.4). All schemes do yield the same results of Table 3.1, which provides an excellent check on the correctness of the mathematical analysis and of the *Mathematica*-programmes. The *Mathematica*-programmes were executed on a 486DX33 computer with 8 MB internal memory, using *Mathematica* Enhanced Version 2.2 for Windows and Microsoft Windows for Workgroups Version 3.11. For the calculation of ten coefficients as in Table 3.1, the evaluation times for the three programmes were: 86.73 seconds for `BazerBrown1`; 72.50 seconds for `Picard1`; and 326.53 seconds for `Bouwkamp1`.

Our expansion of the scattering coefficient σ_1 agrees with and extends the results of Bouwkamp [6, form. (44)], [9, form. (8.2)], and of Bazer and Brown [1, form. (54)]; these results are the best available so far, according to [10, form. (14.104)]. In the references mentioned the coefficients $\tilde{\sigma}_{1,2n}$ have been determined for $n = 2(1)5$, corresponding to an expansion up to and including terms of order α^{10}.

In the schemes based on Bazer and Brown's method, the integral equation (3.14) is solved either by power-series expansion or by Picard iteration. The first option leads to the expansion (3.18) of $f(t)$, which is evaluated up to a certain order. In Table 3.3 we have listed the expansion coefficients $f_n(t)$ for $n = 0(1)10$. The corresponding expansion up to order α^{11} agrees with that of Bazer and Brown [1, form. (50)]. Note that the coefficient $f_n(t)$ is an odd polynomial in t of degree $n-3$ or n according as n is even or odd. In the solution of (3.14) by Picard iteration, a factor α^3 is gained in each step. Since the zero-order approximation $\sinh(\alpha t)$ is of order α, we need seven iteration steps for the evaluation of (3.48) up to and including terms of order α^{22}.

Table 3.1: Exact values of the normalized coefficients $\tilde{\sigma}_{1,2n} = (27\pi^2/16)\,\sigma_{1,2n}, n = 2(1)11$, in the expansion (3.48) of σ_1.

$$\tilde{\sigma}_{1,4} = 1$$

$$\tilde{\sigma}_{1,6} = \frac{8}{25}$$

$$\tilde{\sigma}_{1,8} = \frac{311}{6125}$$

$$\tilde{\sigma}_{1,10} = -\frac{4}{81\,\pi^2} + \frac{2612}{496125}$$

$$\tilde{\sigma}_{1,12} = -\frac{56}{2025\,\pi^2} + \frac{166918}{420217875}$$

$$\tilde{\sigma}_{1,14} = -\frac{3872}{496125\,\pi^2} + \frac{4911008}{213050462625}$$

$$\tilde{\sigma}_{1,16} = \frac{16}{6561\,\pi^4} - \frac{2466752}{1674421875\,\pi^2} + \frac{10209259}{9587270818125}$$

$$\tilde{\sigma}_{1,18} = \frac{64}{32805\,\pi^4} - \frac{10344876692}{49638236484375\,\pi^2} + \frac{555982444}{13853606332190625}$$

$$\tilde{\sigma}_{1,20} = \frac{158224}{200930625\,\pi^4} - \frac{17909145896}{762623815078125\,\pi^2} + \frac{207559549214}{165038012235386915625}$$

$$\tilde{\sigma}_{1,22} = -\frac{64}{531441\,\pi^6} + \frac{86777216}{406884515625\,\pi^4} - \frac{218616645875024}{998861794274875781 25\,\pi^2} + \frac{115358087888}{34657982569431 25228125}$$

Table 3.2: Numerical values of the coefficients $\sigma_{1,2n}$, $n = 2(1)21$, in the expansion (3.48) of σ_1.

n	$\sigma_{1,2n}$	n	$\sigma_{1,2n}$
2	$6.00422 \cdot 10^{-2}$	12	$1.79404 \cdot 10^{-8}$
3	$1.92135 \cdot 10^{-2}$	13	$2.11383 \cdot 10^{-10}$
4	$3.04867 \cdot 10^{-3}$	14	$-8.12669 \cdot 10^{-10}$
5	$1.56882 \cdot 10^{-5}$	15	$-2.65462 \cdot 10^{-10}$
6	$-1.44386 \cdot 10^{-4}$	16	$-4.35129 \cdot 10^{-11}$
7	$-4.60949 \cdot 10^{-5}$	17	$-6.52346 \cdot 10^{-13}$
8	$-7.39515 \cdot 10^{-6}$	18	$1.92722 \cdot 10^{-12}$
9	$-6.29012 \cdot 10^{-8}$	19	$6.36935 \cdot 10^{-13}$
10	$3.42594 \cdot 10^{-7}$	20	$1.05513 \cdot 10^{-13}$
11	$1.10625 \cdot 10^{-7}$	21	$1.91370 \cdot 10^{-15}$

Table 3.3: Coefficients $f_n(t)$, $n = 0(1)10$, in the expansion (3.18) of $f(t)$.

$$f_0(t) = 0 \qquad f_1(t) = t \qquad f_2(t) = 0$$

$$f_3(t) = \frac{t^3}{6} \qquad f_4(t) = \frac{2\,i\,t}{9\,\pi} \qquad f_5(t) = \frac{t^5}{120}$$

$$f_6(t) = \frac{8\,i\,t}{225\,\pi} + \frac{i\,t^3}{45\,\pi}$$

$$f_7(t) = -\frac{4\,t}{81\,\pi^2} + \frac{t^7}{5040}$$

$$f_8(t) = \frac{2\,i\,t}{735\,\pi} + \frac{2\,i\,t^3}{525\,\pi} + \frac{i\,t^5}{1260\,\pi}$$

$$f_9(t) = -\frac{28\,t}{2025\,\pi^2} - \frac{2\,t^3}{405\,\pi^2} + \frac{t^9}{362880}$$

$$f_{10}(t) = -\frac{8\,i\,t}{729\,\pi^3} + \frac{16\,i\,t}{127575\,\pi} + \frac{31\,i\,t^3}{99225\,\pi} + \frac{2\,i\,t^5}{14175\,\pi} + \frac{i\,t^7}{68040\,\pi}$$

4 Diffraction of an electromagnetic wave by a conducting circular disk

4.1 Formulation of the problem

We consider the electromagnetic diffraction of a normally incident, plane wave by a perfectly conducting, circular disk D, of radius a. In terms of Cartesian coordinates x, y, z, the disk D is described by $0 \le x^2 + y^2 \le a^2$, $z = 0$. In addition we employ cylindrical coordinates ρ, φ, z, specified by $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, and spherical coordinates R, θ, φ, specified by $\rho = R\sin\theta$, $z = R\cos\theta$. The incident electromagnetic wave, denoted by $(\mathbf{E}^i, \mathbf{H}^i)$, has Cartesian field components

$$\mathbf{E}^i = (1, 0, 0)\exp(ikz), \qquad \mathbf{H}^i = \sqrt{\varepsilon/\mu}\,(0, 1, 0)\exp(ikz), \tag{4.1}$$

where ε and μ are the permittivity and permeability of the medium surrounding the disk. The wave number k is given by $k = \omega\sqrt{\varepsilon\mu}$, and a harmonic time dependence $\exp(-i\omega t)$ is suppressed throughout. Thus the primary wave $(\mathbf{E}^i, \mathbf{H}^i)$ is incident from $z < 0$. The resulting total field $(\mathbf{E}^t, \mathbf{H}^t)$ is expressed as

$$\mathbf{E}^t = \mathbf{E}^i + \mathbf{E}^s, \qquad \mathbf{H}^t = \mathbf{H}^i + \mathbf{H}^s, \tag{4.2}$$

where the scattered field $(\mathbf{E}^s, \mathbf{H}^s)$ is due to the electric currents induced in the disk by the incident wave. This scattered field has the following properties:

1) $(\mathbf{E}^s, \mathbf{H}^s)$ satisfy Maxwell's equations outside the disk D;

2) the tangential electric field components E^t_x, E^t_y, vanish on the perfectly conducting disk, hence

$$E^s_x(x, y, 0) = -1, \qquad E^s_y(x, y, 0) = 0, \qquad \text{when } x^2 + y^2 < a^2; \tag{4.3}$$

3) $(\mathbf{E}^s, \mathbf{H}^s)$ satisfy the radiation condition at infinity;

4) $(\mathbf{E}^s, \mathbf{H}^s)$ satisfy the edge condition formulated as [9, p. 45]: $\mathbf{E}^s$ and $\mathbf{H}^s$ are quadratically integrable over any domain in $I\!R^3$.

In this manner the diffraction problem for a perfectly conducting, circular disk has been reduced to a boundary value problem for $(\mathbf{E}^s, \mathbf{H}^s)$. By Babinet's principle [9, pp. 45–46], also the complementary problem of diffraction through a circular aperture in a perfectly conducting, infinite, plane screen, can be solved in terms of the fields $\mathbf{E}^s, \mathbf{H}^s$.

At large distances from the disk, the scattered field behaves like an outgoing transverse spherical wave. In terms of spherical coordinates R, θ, φ, we have the far-field expansion

$$\mathbf{E}^s(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta) = \mathbf{A}^s(\theta,\varphi)\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R \to \infty) \tag{4.4}$$

where the far-field amplitude $\mathbf{A}^s(\theta, \varphi)$ is perpendicular to the direction of observation. Of interest is the scattering coefficient σ of the conducting circular disk, which is defined as the ratio of the total energy scattered to the energy incident on the disk. Likewise, the transmission coefficient τ of the complementary circular aperture in a conducting plane screen is defined as the ratio of the energy transmitted through the aperture to the energy incident on the aperture. According to Levine and Schwinger's cross-section theorem, σ is related to the x-component A_x^s of the far-field amplitude in the direction of incidence ($\theta = 0$),

$$\sigma = \frac{4}{a^2 k}\,\mathrm{Im}\,A_x^s(0,\varphi), \tag{4.5}$$

whereas $\tau = \frac{1}{2}\sigma$; see Jones [13, §§ 8.19, 9.4]. Here, the component A_x^s appears because the electric field $\mathbf{E}^i$ in (4.1) is parallel to the x-axis and of unit amplitude. The dependence of A_x^s on φ does in fact drop out when $\theta = 0$.

In the next sections the boundary value problem for $(\mathbf{E}^s, \mathbf{H}^s)$ is solved by methods developed by Boersma [3] and by Bouwkamp [8]. Both methods are well suited to obtain a low-frequency approximation to the solution, involving low-frequency expansions in powers of $\alpha = ka$ for various field quantities. The emphasis is on the low-frequency expansion for the scattering coefficient σ, of which several terms are exactly determined by means of *Mathematica*.

4.2 Boersma's solution

4.2.1 Reduction to integral equations

Apart from some minor changes of notation we follow the approach of Boersma [3, Sec. 3.3]. The scattered field $(\mathbf{E}^s, \mathbf{H}^s)$ is represented by

$$\mathbf{E}^s = \nabla \times \mathbf{\Pi}^s, \qquad \mathbf{H}^s = \frac{1}{i\omega\mu} \nabla \times \nabla \times \mathbf{\Pi}^s, \tag{4.6}$$

where the magnetic Hertz vector $\mathbf{\Pi}^s$ satisfies the vector Helmholtz equation

$$\Delta \mathbf{\Pi}^s + k^2 \mathbf{\Pi}^s = 0 \tag{4.7}$$

outside the disk D. According to Lebedev and Skal'skaya [14], the vector $\mathbf{\Pi}^s$ may be taken to have Cartesian components

$$\Pi_x^s = 0, \qquad \Pi_y^s = \Phi(\rho, z), \qquad \Pi_z^s = \Psi(\rho, z) \sin\varphi, \tag{4.8}$$

where the functions Φ and Ψ are odd and even in z, respectively. From (4.6) and (4.8) we determine the electric field components

$$E_x^s = \frac{\partial \Pi_z^s}{\partial y} - \frac{\partial \Pi_y^s}{\partial z} = \frac{\partial \Psi}{\partial \rho} \sin^2\varphi + \frac{\Psi}{\rho} \cos^2\varphi - \frac{\partial \Phi}{\partial z}, \tag{4.9}$$

$$E_y^s = -\frac{\partial \Pi_z^s}{\partial x} = \left(-\frac{\partial \Psi}{\partial \rho} + \frac{\Psi}{\rho} \right) \sin\varphi \cos\varphi. \tag{4.10}$$

By imposing the boundary conditions (4.3) on E_x^s, E_y^s, we find that

$$\frac{\partial \Phi}{\partial z}(\rho, 0) = C + 1, \qquad \Psi(\rho, 0) = C\rho, \qquad \text{when } \rho < a, \tag{4.11}$$

where C is a constant yet to be determined. Summarizing we arrive at the following boundary value problems for the potentials Φ and Ψ (see [3, p. 233]):

1) Φ and Ψ satisfy the Helmholtz equations

$$\frac{\partial^2 \Phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Phi}{\partial \rho} + \frac{\partial^2 \Phi}{\partial z^2} + k^2 \Phi = 0, \tag{4.12}$$

$$\frac{\partial^2 \Psi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \Psi}{\partial \rho} + \frac{\partial^2 \Psi}{\partial z^2} + \left(k^2 - \frac{1}{\rho^2} \right) \Psi = 0; \tag{4.13}$$

2) $\Phi(\rho, 0) = 0, \qquad \dfrac{\partial \Psi}{\partial z}(\rho, 0) = 0, \qquad \text{when } \rho > a;$ (4.14)

3) $\dfrac{\partial \Phi}{\partial z}(\rho, 0) = C + 1, \qquad \Psi(\rho, 0) = C\rho, \qquad \text{when } \rho < a;$ (4.15)

4) Φ and Ψ satisfy the Sommerfeld radiation condition at infinity;

5) Φ and Ψ satisfy the edge condition reformulated as [3, p. 233]: Φ and Ψ remain finite near the edge of the disk D; in a point with coordinates $\rho = a + \delta \cos\gamma$, $z = \delta \sin\gamma$, at a distance $\delta > 0$ from the edge, the expressions

$$\partial\Phi/\partial\gamma - \tfrac{1}{2}\Psi \quad \text{and} \quad \partial\Psi/\partial\gamma + \tfrac{1}{2}\Phi \tag{4.16}$$

have expansions in powers of δ, in which no terms of order $\delta^{1/2}$ occur.

To solve these boundary value problems, we employ the methods of Bazer and Brown [1], Bazer and Hochstadt [2]. Thus we introduce the integral representations (cf. [3, form. (3.39), (3.40)])

$$\Phi(\rho, z) = (C+1) \int_{-1}^{1} \frac{\exp\left(ik\sqrt{\rho^2 + (z + iat)^2}\right)}{\sqrt{\rho^2 + (z+iat)^2}} f(t)\, dt, \tag{4.17}$$

$$\Psi(\rho, z) = C\rho \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\right) \int_{-1}^{1} \frac{\exp\left(ik\sqrt{\rho^2 + (z + iat)^2}\right)}{\sqrt{\rho^2 + (z+iat)^2}} g(t)\, dt, \tag{4.18}$$

valid for $z \geq 0$, whereas for $z \leq 0$ we define $\Phi(\rho, z) = -\Phi(\rho, -z)$, $\Psi(\rho, z) = \Psi(\rho, -z)$. Here the yet unknown functions $f(t)$ and $g(t)$ are required to be odd and even functions of t, respectively; moreover, $g(t)$ must satisfy the condition $g(1) = 0$.

The representations (4.17) and (4.18) satisfy all conditions of the boundary value problems for Φ and Ψ, except for the boundary conditions (4.15) and the second part of the edge condition. Imposing the boundary conditions leads to Fredholm integral equations for the functions $f(t)$ and $g(t)$. The boundary value problem for Φ is basically the same as the problem for ϕ_1, as treated in Section 3.2.1. Therefore the integral equation for $f(t)$ reads, in conformity with (3.13),

$$f(t) = \frac{a^2}{\pi i}\frac{\sinh(\alpha t)}{\alpha} + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} f(s)\, ds, \quad -1 \leq t \leq 1, \tag{4.19}$$

where $\alpha = ka$. The boundary value problem for Ψ is treated by the method of Bazer and Hochstadt [2]. Imposing the boundary condition (4.15) for Ψ, we obtain the integro-differential equation

$$\frac{\partial}{\partial \rho} \int_0^{\rho/a} \frac{\cos\left(\alpha\sqrt{(\rho/a)^2 - t^2}\right)}{\sqrt{(\rho/a)^2 - t^2}} g(t)\, dt = \frac{a\rho}{2} + \frac{1}{i}\frac{\partial}{\partial \rho} \int_0^1 \frac{\sinh\left(\alpha\sqrt{t^2 - (\rho/a)^2}\right)}{\sqrt{t^2 - (\rho/a)^2}} g(t)\, dt. \tag{4.20}$$

Both sides of (4.20) are integrated from 0 to ρ, whereupon the integral operator on the left can be inverted by means of Laplace transformation. As a result we find for $g(t)$ the integral equation

$$g(t) = \frac{a^3}{\pi}\left(\frac{t\sinh(\alpha t)}{\alpha} + C_0\cosh(\alpha t)\right) + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} g(s)\, ds, \qquad -1 \le t \le 1, \tag{4.21}$$

in which C_0 is a constant determined by the condition $g(1) = 0$. The second part of the edge condition applies to the expansions of the expressions (4.16) near the edge in powers of δ. In these expansions the terms of order $\delta^{1/2}$ should vanish. On imposing the latter condition, the constant C is found to be given by (cf. [3, form. (3.43)])

$$C = -\frac{af(1)}{af(1) + ig'(1)}. \tag{4.22}$$

To simplify the presentation we set

$$f(t) = \frac{a^2}{\pi i} f_0(t), \qquad g(t) = \frac{a^3}{\pi}\left[g_1(t) + C_0\, g_0(t)\right]. \tag{4.23}$$

Then the odd function $f_0(t)$, and the even functions $g_0(t)$ and $g_1(t)$, are solutions of the integral equations

$$f_0(t) = \frac{\sinh(\alpha t)}{\alpha} + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} f_0(s)\, ds, \qquad -1 \le t \le 1, \tag{4.24}$$

$$g_0(t) = \cosh(\alpha t) + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} g_0(s)\, ds, \qquad -1 \le t \le 1, \tag{4.25}$$

$$g_1(t) = \frac{t\sinh(\alpha t)}{\alpha} + \frac{1}{\pi i}\int_{-1}^{1} \frac{\sinh[\alpha(t-s)]}{t-s} g_1(s)\, ds, \qquad -1 \le t \le 1. \tag{4.26}$$

Next, the constants C_0 and C are determined by

$$C_0 = -\frac{g_1(1)}{g_0(1)}, \qquad C = \frac{f_0(1)}{-f_0(1) + g_1'(1) + C_0\, g_0'(1)}. \tag{4.27}$$

Equations (4.24)–(4.27) are key equations in our scheme for calculating the low-frequency expansion of the scattering coefficient σ.

4.2.2 Far field and scattering coefficient

By inserting (2.14) into (4.17) and (4.18), we obtain the expansions

$$\Phi(R\sin\theta, R\cos\theta) = A(\theta)\,\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R \to \infty) \tag{4.28}$$

$$\Psi(R\sin\theta, R\cos\theta) = B(\theta)\,\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R \to \infty) \tag{4.29}$$

in which $A(\theta)$ and $B(\theta)$ are given by

$$\begin{aligned} A(\theta) &= (C+1)\int_{-1}^{1} \exp(-\alpha t\cos\theta)\, f(t)\, dt \\ &= -\frac{2a^2}{\pi i}\,(C+1)\int_0^1 \sinh(\alpha t\cos\theta)\, f_0(t)\, dt, \end{aligned} \tag{4.30}$$

$$B(\theta) = 2ikC\,\sin\theta \int_0^1 \cosh(\alpha t\cos\theta)\, g(t)\, dt. \tag{4.31}$$

These results are used in (4.9) to establish the far-field expansion

$$E_x^s(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta) = A_x^s(\theta,\varphi)\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R \to \infty) \tag{4.32}$$

where the x-component A_x^s of the far-field amplitude is found to be

$$A_x^s(\theta,\varphi) = ik\left[B(\theta)\sin\theta\sin^2\varphi - A(\theta)\cos\theta\right]. \tag{4.33}$$

The expression (4.5) for the scattering coefficient σ now becomes

$$\sigma = \frac{4}{a^2 k}\,\mathrm{Im}\,[-ikA(0)] = \frac{8}{\pi}\,\mathrm{Im}\left((C+1)\int_0^1 \sinh(\alpha t) f_0(t)\, dt\right), \tag{4.34}$$

expressed in terms of the constant C and the function $f_0(t)$. This is the final key equation in the scheme for calculating the low-frequency expansion of σ.

4.2.3 Scheme for calculating the scattering coefficient

Our scheme for calculating the low-frequency expansion of the scattering coefficient σ is based on the key equations (4.24)–(4.27) and (4.34). First we solve the integral equations (4.24)–(4.26) by Picard iteration, leading to expansions of the functions $f_0(t)$, $g_0(t)$, $g_1(t)$, in powers of α. In the solution of (4.25) and (4.26) a factor α is gained in each iteration step. As for equation (4.24) there is a gain by a factor α^3, since $f_0(t)$ is an odd function of t. Next the expansions obtained are substituted into (4.27) to determine the constants C_0 and C. Finally, the expansions of C and $f_0(t)$ are inserted into (4.34) and the function $\sinh(\alpha t)$ is expanded in powers of α. Then by a straightforward evaluation we find the required low-frequency expansion of σ up to a certain order.
A *Mathematica*-implementation of this scheme is listed in Appendix D; the function is called `BoersmaEM`. The results of the calculations by *Mathematica* are presented in Section 4.4.

4.3 Bouwkamp's solution

4.3.1 Reduction to a system of integral equations

We closely follow the approach of Bouwkamp [8], except that we use rationalized Giorgi (or m.k.s.) units instead of Gaussian units. For convenience we take the radius of the disk D equal to unity. The scattered field $(\mathbf{E}^s, \mathbf{H}^s)$ is derived from a vector potential $\mathbf{A}$, according to

$$\mathbf{E}^s = ik\mathbf{A} - \frac{1}{ik}\nabla\nabla\cdot\mathbf{A}, \qquad \mathbf{H}^s = \sqrt{\varepsilon/\mu}\,\nabla\times\mathbf{A}. \tag{4.35}$$

The scattered field is due to the electric currents induced in the disk by the incident wave (4.1). Let $\mathbf{I}$ denote the surface-current density, then the vector potential $\mathbf{A}$ may be taken as

$$\mathbf{A}(\mathbf{x}) = \frac{1}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}\int\!\!\int_D \mathbf{I}(\mathbf{x}')\,\frac{\exp(ik|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\,dx'\,dy', \tag{4.36}$$

where $\mathbf{x} = (x, y, z)$ and $\mathbf{x}' = (x', y', 0)$. Clearly, $\mathbf{A}$ has non-zero Cartesian components A_x, A_y, whereas $A_z = 0$. From (4.35) we determine the electric field components

$$E_x^s = ikA_x - \frac{1}{ik}\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right), \tag{4.37}$$

$$E_y^s = ikA_y - \frac{1}{ik}\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}\right), \tag{4.38}$$

to be used in the boundary conditions (4.3) on the disk D. As a result we find that

$$ik\mathcal{A}_x - \frac{1}{ik}\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{A}_x}{\partial x} + \frac{\partial \mathcal{A}_y}{\partial y}\right) = -1, \tag{4.39}$$

$$ik\mathcal{A}_y - \frac{1}{ik}\frac{\partial}{\partial y}\left(\frac{\partial \mathcal{A}_x}{\partial x} + \frac{\partial \mathcal{A}_y}{\partial y}\right) = 0, \tag{4.40}$$

where the notations $\mathcal{A}_x$ and $\mathcal{A}_y$ stand for the x- and y-components of the vector potential on the disk, i.e. $\mathcal{A}_{x,y} = \mathcal{A}_{x,y}(x, y) = A_{x,y}(\mathbf{x})$ with $\mathbf{x} \in D$. It is easily seen that $\mathcal{A}_x$ and $\mathcal{A}_y$ are mutually dependent through

$$\frac{\partial \mathcal{A}_x}{\partial y} = \frac{\partial \mathcal{A}_y}{\partial x}. \tag{4.41}$$

This relation is used to eliminate the mixed derivatives $(\partial^2/\partial x \partial y)$ in (4.39) and (4.40), yielding [8, form. (11), (12)]

$$\frac{\partial^2 \mathcal{A}_x}{\partial x^2} + \frac{\partial^2 \mathcal{A}_x}{\partial y^2} + k^2 \mathcal{A}_x = ik, \tag{4.42}$$

$$\frac{\partial^2 \mathcal{A}_y}{\partial x^2} + \frac{\partial^2 \mathcal{A}_y}{\partial y^2} + k^2 \mathcal{A}_y = 0. \tag{4.43}$$

Consequently, $\mathcal{A}_x$ and $\mathcal{A}_y$ are solutions of the (in)homogeneous two-dimensional Helmholtz equation. Recalling that

$$\mathcal{A}_{x,y}(x, y) = \frac{1}{4\pi}\sqrt{\frac{\mu}{\varepsilon}} \int\!\!\int_D I_{x,y}(\mathbf{x}') \frac{\exp(ik|\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|}\, dx'\, dy', \quad \mathbf{x} = (x, y, 0) \in D, \tag{4.44}$$

we observe that equations (4.41)–(4.43) form a simultaneous system of integro-differential equations for the components of the current density, I_x and I_y.

The solution of the system of integro-differential equations is not unique. To pick out the physically acceptable solution, Bouwkamp [8, Sec. 3] formulates appropriate boundary conditions on the edge of the disk. Presently these conditions go under the name of "edge condition". Formulated in terms of the current density **I**, the edge condition reads [8, form. (20)]

$$I_n = O\left(\delta^{1/2}\right), \qquad I_t = O\left(\delta^{-1/2}\right), \qquad (\delta \to 0) \tag{4.45}$$

where the subscripts n and t refer to the components normal and tangential to the edge, respectively, and δ measures the distance from the field point to the edge.
We now introduce polar coordinates ρ, φ, and ρ', φ', on the disk D, specified by

$$\begin{cases} x = \rho\cos\varphi, & y = \rho\sin\varphi, \quad 0 \le \rho \le 1, \quad 0 \le \varphi \le 2\pi, \\ x' = \rho'\cos\varphi', & y' = \rho'\sin\varphi', \quad 0 \le \rho' \le 1, \quad 0 \le \varphi' \le 2\pi. \end{cases} \tag{4.46}$$

In (4.44) we change the notations $\mathcal{A}_{x,y}(x,y)$, $I_{x,y}(\mathbf{x}')$, into $\mathcal{A}_{x,y}(\rho,\varphi)$, $I_{x,y}(\rho',\varphi')$, while the distance $|\mathbf{x}-\mathbf{x}'| = r$ becomes

$$r = \sqrt{\rho^2 + \rho'^2 - 2\rho\rho'\cos(\varphi - \varphi')}. \tag{4.47}$$

In view of the edge condition (4.45) and of the axial symmetry of the diffraction problem, Bouwkamp infers that the components I_x and I_y of the current density are of the form [8, form. (23), (24)]

$$\frac{1}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}\, I_x(\rho',\varphi') = \frac{A(\rho') + B(\rho')\cos(2\varphi')}{\pi^2\sqrt{1-\rho'^2}}, \tag{4.48}$$

$$\frac{1}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}\, I_y(\rho',\varphi') = \frac{B(\rho')\sin(2\varphi')}{\pi^2\sqrt{1-\rho'^2}}, \tag{4.49}$$

where $A(\rho')$ and $B(\rho')$ are uniformly bounded for $0 \le \rho' \le 1$, and

$$A(1) + B(1) = 0, \qquad B(0) = 0. \tag{4.50}$$

Substitution of (4.48), (4.49), into (4.44) yields the vector potential on the disk showing the same typical behaviour with respect to the angle φ:

$$\mathcal{A}_x(\rho,\varphi) = C(\rho) + D(\rho)\cos(2\varphi), \qquad \mathcal{A}_y(\rho,\varphi) = D(\rho)\sin(2\varphi), \tag{4.51}$$

where $C(\rho)$ and $D(\rho)$ are expressible in terms of $A(\rho')$ and $B(\rho')$ by means of certain surface integrals over the disk. The expressions (4.51) are inserted into (4.41) and into the Helmholtz equations (4.42) and (4.43), transformed in polar coordinates. As a result it is found that $C(\rho)$ and $D(\rho)$ satisfy the ordinary differential equations

$$C' = D' + \frac{2}{\rho} D, \tag{4.52}$$

and

$$C'' + \frac{1}{\rho} C' + k^2 C = ik, \qquad D'' + \frac{1}{\rho} D' + \left(k^2 - \frac{4}{\rho^2} \right) D = 0. \tag{4.53}$$

The solutions of the latter equations are expressible in terms of Bessel functions $J_0(k\rho)$ and $J_2(k\rho)$, respectively. Thus we obtain

$$C(\rho) = -\frac{1}{ik} - pJ_0(k\rho), \qquad D(\rho) = pJ_2(k\rho), \tag{4.54}$$

with a common integration constant p, in virtue of (4.52).

By use of (4.51), (4.54), in (4.44), the problem of calculating the surface-current density (4.48)–(4.49) reduces to solving the integral equations (cf. [8, form. (33)])

$$\begin{aligned} &\int_0^{2\pi} \int_0^1 \frac{A(\rho') + B(\rho') \cos(2\varphi')}{\pi^2 \sqrt{1 - \rho'^2}} \frac{e^{ikr}}{r} \rho' \, d\rho' \, d\varphi' \\ &= -\frac{1}{ik} - pJ_0(k\rho) + pJ_2(k\rho) \cos(2\varphi), \qquad 0 \le \rho \le 1, \ 0 \le \varphi \le 2\pi, \end{aligned} \tag{4.55}$$

$$\begin{aligned} &\int_0^{2\pi} \int_0^1 \frac{B(\rho') \sin(2\varphi')}{\pi^2 \sqrt{1 - \rho'^2}} \frac{e^{ikr}}{r} \rho' \, d\rho' \, d\varphi' \\ &= pJ_2(k\rho) \sin(2\varphi), \qquad 0 \le \rho \le 1, \ 0 \le \varphi \le 2\pi, \end{aligned} \tag{4.56}$$

for the three unknowns p, $A(\rho')$ and $B(\rho')$. Here, r is given by (4.47), while $A(\rho')$ and $B(\rho')$ must satisfy the further conditions (4.50). It is easily seen that equation (4.55) can be split up into two separate integral equations involving $A(\rho')$ or $B(\rho')$ only. The integral equation for $B(\rho')$ can be shown to be equivalent to (4.56).

To solve the integral equation (4.55), we expand the functions $\exp(ikr)$, $J_0(k\rho)$, and $J_2(k\rho)$ in powers of ik and we substitute for $A(\rho')$ and $B(\rho')$ the power-series expansions

$$A(\rho') = \sum_{n=1}^{\infty} A_n(\rho')\,(ik)^n, \qquad B(\rho') = \sum_{n=1}^{\infty} B_n(\rho')\,(ik)^n, \tag{4.57}$$

with coefficients $A_n(\rho')$ and $B_n(\rho')$ to be determined. In view of (4.50), these coefficients must satisfy

$$A_n(1) + B_n(1) = 0, \qquad B_n(0) = 0. \tag{4.58}$$

The constant p in (4.55), as a function of ik, has a simple pole at $ik = 0$ with residue -1, because $C(\rho)$ in (4.54) is finite at $\rho = 0$. Therefore we put

$$-p = \sum_{n=-1}^{\infty} p_n\,(ik)^n, \tag{4.59}$$

with $p_{-1} = 1$. In this manner we expand both sides of (4.55) in powers of ik. By equating the terms of order $(ik)^n$ on the left and on the right of (4.55), we are led to the recursive system of integral equations [8, form. (42), (43)]

$$\sum_{\tau=1}^{n} \frac{1}{(n-\tau)!} \int_0^{2\pi} \int_0^1 \frac{A_\tau(\rho') + B_\tau(\rho')\cos(2\varphi')}{\pi^2\sqrt{1-\rho'^2}}\, r^{n-\tau-1}\, \rho' d\rho'\, d\varphi'$$

$$= \sum_{\nu=0}^{[(n+1)/2]} p_{n-2\nu} \frac{\rho^{2\nu}}{2^{2\nu}\,(\nu!)^2} + \cos(2\varphi) \sum_{\nu=1}^{[(n+1)/2]} p_{n-2\nu} \frac{\rho^{2\nu}}{2^{2\nu}\,(\nu-1)!\,(\nu+1)!}, \tag{4.60}$$

$$0 \le \rho \le 1,\ 0 \le \varphi \le 2\pi,\ n = 1, 2, 3, \ldots$$

For $n = 0$ the left-hand side of (4.60) is zero, so that $p_0 = 0$.

4.3.2 Solution of the integral equations

The integral equations (4.60) are solved through expansion of $A_n(\rho')$ in Legendre polynomials $P_{2\nu}(\sqrt{1-\rho'^2})$ of even degree 2ν, while $B_n(\rho')$ is expanded in associated Legendre polynomials $P_{2\nu}^2(\sqrt{1-\rho'^2})$ of the second order. Following Bouwkamp [8, form. (44)], we make the Ansatz

$$A_n(\rho') = \sum_{\nu=0}^{[(n+1)/2]} a_{n,n-2\nu} P_{2\nu}(\sqrt{1-\rho'^2}), \quad B_n(\rho') = \sum_{\nu=1}^{[(n+1)/2]} b_{n,n-2\nu} P_{2\nu}^2(\sqrt{1-\rho'^2}), \tag{4.61}$$

in which the coefficients $a_{n,n-2\nu}$ and $b_{n,n-2\nu}$ are to be determined recursively. Notice that the second condition in (4.58) is automatically satisfied, since $P_{2\nu}^2(1) = 0$. In the further analysis we need the auxiliary integral

$$\begin{aligned} &\frac{1}{\pi^2}\int_0^{2\pi}\int_0^1 \frac{P_{2n}^{2m}(\sqrt{1-\rho'^2})}{\sqrt{1-\rho'^2}} \cos(2m\varphi')\, r^{\mu-1}\, \rho'\, d\rho'\, d\varphi' \\ &= \; J(n,m,\mu;\rho)\cos(2m\varphi), \qquad\qquad (4.62) \\ &\qquad n=0,1,2,\ldots,\; m=0,1,\ldots,n,\; \mu=0,1,2,\ldots,\; 0\le\rho\le 1,\; 0\le\varphi\le 2\pi, \end{aligned}$$

which has been calculated by Bouwkamp [7]. From [7, form. (3), (5)] we quote the result

$$\begin{aligned} &J(n,m,\mu;\rho) \\ &= \frac{(-1)^{n+m}\,\Gamma^2(\frac{1}{2}\mu+\frac{1}{2})\,\Gamma(n+m+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(m+\frac{1}{2})\,\Gamma(m+1)\,\Gamma(n-m+1)\,\Gamma(\frac{1}{2}\mu-n-m+\frac{1}{2})\,\Gamma(\frac{1}{2}\mu+n-m+1)} \\ &\quad \cdot \rho^{2m} F(-\tfrac{1}{2}\mu+n+m+\tfrac{1}{2}, -\tfrac{1}{2}\mu-n+m; 2m+1; \rho^2), \qquad (4.63) \end{aligned}$$

where F stands for the hypergeometric function. It is easily verified that $J(n,m,\mu;\rho)$ is a polynomial in ρ^2 of degree $n+\frac{1}{2}\mu$ (μ even) or $\frac{1}{2}(\mu-1)-n$ (μ odd); in the latter case, $J(n,m,\mu;\rho) = 0$ if $m+n > \frac{1}{2}(\mu-1)$.
On substitution of (4.61) into (4.60), we find the following two identities in ρ [8, form. (47), (48)]:

$$\sum_{\tau=1}^{n} \frac{1}{(n-\tau)!} \sum_{\nu=0}^{[(\tau+1)/2]} a_{\tau,\tau-2\nu}\, J(\nu,0,n-\tau;\rho) = \sum_{\nu=0}^{[(n+1)/2]} p_{n-2\nu} \frac{\rho^{2\nu}}{2^{2\nu}(\nu!)^2}, \tag{4.64}$$

$$\sum_{\tau=1}^{n} \frac{1}{(n-\tau)!} \sum_{\nu=1}^{[(\tau+1)/2]} b_{\tau,\tau-2\nu}\, J(\nu, 1, n-\tau; \rho) = \sum_{\nu=1}^{[(n+1)/2]} p_{n-2\nu} \frac{\rho^{2\nu}}{2^{2\nu}\,(\nu-1)!\,(\nu+1)!}. \tag{4.65}$$

Furthermore, substitution of (4.61) into the first condition in (4.58) leads to the equation

$$a_{n,n} = \sum_{\nu=1}^{[(n+1)/2]} \left((-1)^{\nu+1} \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\nu+1)} a_{n,n-2\nu} + (-1)^{\nu} \frac{4\,\Gamma(\nu+\frac{3}{2})}{\Gamma(\frac{1}{2})\,\Gamma(\nu)} b_{n,n-2\nu} \right); \tag{4.66}$$

see [8, form. (49)] with the second factor $\Gamma(\nu+1)$ corrected into $\Gamma(\nu)$. The system (4.64)–(4.66) suffices to determine successively all coefficients a, b, and p. To show this we proceed by induction. For fixed $n = 1, 2, 3, \ldots$, we rewrite (4.64) and (4.65) as

$$\sum_{\nu=0}^{[(n+1)/2]} a_{n,n-2\nu}\, J(\nu, 0, 0; \rho) - p_n = \sum_{\nu=1}^{[(n+1)/2]} p_{n-2\nu} \frac{\rho^{2\nu}}{2^{2\nu}\,(\nu!)^2} - \sum_{\tau=1}^{n-1} \frac{1}{(n-\tau)!} \sum_{\nu=0}^{[(\tau+1)/2]} a_{\tau,\tau-2\nu}\, J(\nu, 0, n-\tau; \rho), \tag{4.67}$$

$$\sum_{\nu=1}^{[(n+1)/2]} b_{n,n-2\nu}\, J(\nu, 1, 0; \rho) = \sum_{\nu=1}^{[(n+1)/2]} p_{n-2\nu} \frac{\rho^{2\nu}}{2^{2\nu}\,(\nu-1)!\,(\nu+1)!} - \sum_{\tau=1}^{n-1} \frac{1}{(n-\tau)!} \sum_{\nu=1}^{[(\tau+1)/2]} b_{\tau,\tau-2\nu}\, J(\nu, 1, n-\tau; \rho), \tag{4.68}$$

in which the right-hand sides are supposed to be known. These equations are identities for polynomials in ρ^2 of degree $[(n+1)/2]$. By equating the coefficients of $\rho^{2\nu}$ ($\nu = (0), 1, 2, \ldots, [(n+1)/2]$) on the left and on the right of (4.67) and (4.68), we are led to a system of linear algebraic equations for the coefficients $a_{n,n-2\nu}$, $b_{n,n-2\nu}$, p_n, of the nth approximation, expressed in terms of preceding coefficients for the approximations of order $n-1, n-2, \ldots, 1$. Here we also need the initial values $p_{-1} = 1$, $p_0 = 0$. Together with equation (4.66), we now have a system of $2[(n+1)/2] + 2$ linear equations for the same number of unknown coefficients $a_{n,n-2\nu}$, $b_{n,n-2\nu}$, and p_n. This system has a unique solution.

Equations (4.64)–(4.66) are key equations in our scheme for calculating the low-frequency expansion of the scattering coefficient σ.

4.3.3 Far field and scattering coefficient

Starting from the representation (4.36) for $\mathbf{A}(\mathbf{x})$, we determine the far field at an observation point $\mathbf{x} = (R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)$, where R, θ, φ are spherical coordinates. For an integration point $\mathbf{x}' \in D$ with polar coordinates ρ', φ' (see (4.46)), the distance $|\mathbf{x}-\mathbf{x}'|$ takes the asymptotic form

$$\begin{aligned} |\mathbf{x}-\mathbf{x}'| &= \sqrt{R^2 - 2R\rho'\sin\theta\cos(\varphi-\varphi') + \rho'^2} \\ &= R - \rho'\sin\theta\cos(\varphi-\varphi') + O\left(R^{-1}\right), \quad (R\to\infty). \end{aligned} \tag{4.69}$$

Using this result in (4.36), we obtain the expansion

$$\begin{aligned} &\mathbf{A}(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta) \\ &= \frac{1}{4\pi}\sqrt{\frac{\mu}{\varepsilon}}\int_0^{2\pi}\int_0^1 \mathbf{I}(\rho',\varphi')\exp(-ik\rho'\sin\theta\cos(\varphi-\varphi'))\,\rho'\,d\rho'\,d\varphi'\,\frac{e^{ikR}}{R} \\ &\qquad + O\left(R^{-2}\right), \quad (R\to\infty). \end{aligned} \tag{4.70}$$

Next we substitute the expressions (4.48)–(4.49) for the currents, leading to

$$\begin{aligned} &A_x(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta) \\ &= [F(\theta) + G(\theta)\cos(2\varphi)]\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R\to\infty) \end{aligned} \tag{4.71}$$

$$\begin{aligned} &A_y(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta) \\ &= G(\theta)\sin(2\varphi)\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R\to\infty) \end{aligned} \tag{4.72}$$

in which $F(\theta)$ and $G(\theta)$ are given by

$$F(\theta) = \int_0^{2\pi}\int_0^1 \frac{A(\rho')}{\pi^2\sqrt{1-\rho'^2}}\exp(-ik\rho'\sin\theta\cos\varphi')\,\rho'\,d\rho'\,d\varphi', \tag{4.73}$$

$$G(\theta) = \int_0^{2\pi}\int_0^1 \frac{B(\rho')\cos(2\varphi')}{\pi^2\sqrt{1-\rho'^2}}\exp(-ik\rho'\sin\theta\cos\varphi')\,\rho'\,d\rho'\,d\varphi'. \tag{4.74}$$

To further evaluate $F(\theta)$ and $G(\theta)$, we replace $A(\rho')$ and $B(\rho')$ by their low-frequency expansions taken from (4.57) and (4.61), viz.

$$A(\rho') = \sum_{n=1}^{\infty}(ik)^n \sum_{\nu=0}^{[(n+1)/2]} a_{n,n-2\nu}\,P_{2\nu}(\sqrt{1-\rho'^2}), \tag{4.75}$$

$$B(\rho') = \sum_{n=1}^{\infty}(ik)^n \sum_{\nu=1}^{[(n+1)/2]} b_{n,n-2\nu}\, P_{2\nu}^2(\sqrt{1-\rho'^2}). \tag{4.76}$$

In addition, we need the auxiliary integrals

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 \frac{P_{2\nu}(\sqrt{1-\rho'^2})}{\sqrt{1-\rho'^2}} \exp(-ik\rho'\sin\theta\cos\varphi')\,\rho'\,d\rho'\,d\varphi' \\
&= \pi\sqrt{2}\,\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}\,\frac{J_{2\nu+1/2}(k\sin\theta)}{(k\sin\theta)^{1/2}},
\end{aligned} \tag{4.77}$$

equal to (2.47), and

$$\begin{aligned}
&\int_0^{2\pi}\int_0^1 \frac{P_{2\nu}^2(\sqrt{1-\rho'^2})}{\sqrt{1-\rho'^2}} \cos(2\varphi')\,\exp(-ik\rho'\sin\theta\cos\varphi')\,\rho'\,d\rho'\,d\varphi' \\
&= -4\pi\sqrt{2}\,\frac{\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu)}\,\frac{J_{2\nu+1/2}(k\sin\theta)}{(k\sin\theta)^{1/2}},
\end{aligned} \tag{4.78}$$

obtainable from [4, form. (2.6), (2.8)]. As a result we have for $F(\theta)$ and $G(\theta)$ the low-frequency expansions

$$F(\theta) = \frac{\sqrt{2}}{\pi}\sum_{n=1}^{\infty}(ik)^n \sum_{\nu=0}^{[(n+1)/2]} a_{n,n-2\nu}\,\frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu+1)}\,\frac{J_{2\nu+1/2}(k\sin\theta)}{(k\sin\theta)^{1/2}}, \tag{4.79}$$

$$G(\theta) = -\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}(ik)^n \sum_{\nu=1}^{[(n+1)/2]} b_{n,n-2\nu}\,\frac{\Gamma(\nu+\frac{3}{2})}{\Gamma(\nu)}\,\frac{J_{2\nu+1/2}(k\sin\theta)}{(k\sin\theta)^{1/2}}, \tag{4.80}$$

in terms of the coefficients $a_{n,n-2\nu}$ and $b_{n,n-2\nu}$.

By inserting (4.71) and (4.72) into (4.37), we establish the far-field expansion

$$E_x^s(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta) = A_x^s(\theta,\varphi)\,\frac{e^{ikR}}{R} + O\left(R^{-2}\right), \quad (R\to\infty) \tag{4.81}$$

where the x-component A_x^s of the far-field amplitude is found to be

$$A_x^s(\theta,\varphi) = ik\left[F(\theta) + G(\theta)\cos(2\varphi) - \{F(\theta)+G(\theta)\}\sin^2\theta\cos^2\varphi\right]. \tag{4.82}$$

According to (4.5), the scattering coefficient σ is related to A_x^s in the direction of incidence. By setting $\theta = 0$ in (4.82), we find the low-frequency expansion

$$\sigma = \frac{4}{k}\,\mathrm{Im}\,A_x^s(0,\varphi) = \frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^n a_{2n,2n}\,k^{2n} \tag{4.83}$$

for the scattering coefficient of a circular disk of unit radius. For a disk of radius a, the low-frequency expansion of σ follows by replacing k with $\alpha = ka$ in (4.83), viz.

$$\sigma = \frac{8}{\pi} \sum_{n=1}^{\infty} (-1)^n \, a_{2n,2n} \, \alpha^{2n}. \tag{4.84}$$

This is the final key equation in the scheme for calculating the low-frequency expansion of σ.

4.3.4 Scheme for calculating the scattering coefficient

Our scheme for the calculation of the low-frequency expansion of the scattering coefficient σ is based on the key equations (4.64)–(4.66) and (4.84). Starting from the initial values $p_{-1} = 1$, $p_0 = 0$, we solve the system (4.64)–(4.66) successively for $n = 1, 2, 3, \ldots$ Thus we determine a number of coefficients $a_{n,n-2\nu}$, $b_{n,n-2\nu}$, and p_n, $n = 1, 2, 3, \ldots$, $\nu = (0), 1, 2, \ldots, [(n+1)/2]$. Only the coefficients $a_{2n,2n}$, $n = 1, 2, 3, \ldots$, are needed in the low-frequency expansion (4.84) of σ.

A *Mathematica*-implementation of this scheme is listed in Appendix E; the function is called `BouwkampEM`. The results of the calculations by *Mathematica* are presented in the next section.

4.4 Results for the scattering coefficient σ

Using the *Mathematica*-packages in Appendices D and E, we can evaluate the low-frequency expansion of the scattering coefficient σ up to arbitrary order. The expansion is in even powers of α and the leading term is found to be $(128/27\pi^2)\,\alpha^4$. Therefore we set

$$\sigma = \sum_{n=2}^{\infty} \sigma_{2n} \, \alpha^{2n} = \frac{128\,\alpha^4}{27\pi^2} \sum_{n=2}^{\infty} \tilde{\sigma}_{2n} \, \alpha^{2n-4}, \tag{4.85}$$

in which $\sigma_4 = 128/27\pi^2$, $\tilde{\sigma}_4 = 1$. The expansion (4.85) has been evaluated up to and including terms of order α^{22}. In Table 4.1 we present the exact values of the normalized coefficients $\tilde{\sigma}_{2n} = (27\pi^2/128)\,\sigma_{2n}$ for $n = 2(1)11$. From the tabulated values and from additional calculated values of $\tilde{\sigma}_{2n}$, not presented here, it appears that $\tilde{\sigma}_{2n}$ is a polynomial in π^{-2} of degree $[(n-2)/3]$, with rational coefficients. In Table 4.2 we present the numerical values, to six significant digits, of the coefficients σ_{2n} for $n = 2(1)21$. It has been found recently [5] that the expansion (4.85), considered as a power series in α, has a radius of convergence 1.32335, to five decimal places.

The calculations were performed by the scheme based on Boersma's method with the integral equations solved by Picard iteration (cf. Section 4.2.3), and by the scheme based on Bouwkamp's method (cf. Section 4.3.4). Both schemes do yield the same results of Table 4.1, which provides an excellent check on the correctness of the mathematical analysis and of the *Mathematica*-programmes. The *Mathematica*-programmes were executed on a 486DX33 computer with 8 MB internal memory, using *Mathematica* Enhanced Version 2.2 for Windows and Microsoft Windows for Workgroups Version 3.11. For the calculation of ten coefficients as in Table 4.1, the evaluation times for the two programmes were: 2224.90 seconds for `BoersmaEM`; and 10383.85 seconds for `BouwkampEM`.

Our expansion of the scattering coefficient σ agrees with and includes the three-term expansion due to Bouwkamp [8, form. (63)], and the five-term expansion due to Boersma [3, form. (3.54)]. Boersma's expansion up to and including terms of order α^{12} is the best result available so far, according to [10, form. (14.277)].

Table 4.1: Exact values of the normalized coefficients $\tilde{\sigma}_{2n} = (27\pi^2/128)\,\sigma_{2n}$, $n = 2(1)11$, in the expansion (4.85) of σ.

$$\tilde{\sigma}_4 = 1$$

$$\tilde{\sigma}_6 = \frac{22}{25}$$

$$\tilde{\sigma}_8 = \frac{7312}{18375}$$

$$\tilde{\sigma}_{10} = -\frac{64}{81\,\pi^2} + \frac{60224}{496125}$$

$$\tilde{\sigma}_{12} = -\frac{2464}{2025\,\pi^2} + \frac{35048192}{1260653625}$$

$$\tilde{\sigma}_{14} = -\frac{477152}{496125\,\pi^2} + \frac{1074505984}{213050462625}$$

$$\tilde{\sigma}_{16} = \frac{4096}{6561\,\pi^4} - \frac{866102528}{1674421875\,\pi^2} + \frac{7165401088}{9587270818125}$$

$$\tilde{\sigma}_{18} = \frac{45056}{32805\,\pi^4} - \frac{10518751249408}{49638236484375\,\pi^2} + \frac{3838104543232}{41560818996571875}$$

$$\tilde{\sigma}_{20} = \frac{310123264}{200930625\,\pi^4} - \frac{1623120103424}{23109812578125\,\pi^2} + \frac{1594679901356032}{165038012235386915625}$$

$$\tilde{\sigma}_{22} = -\frac{262144}{531441\,\pi^6} + \frac{479139075584}{406884515625\,\pi^4} - \frac{1952093212715159552}{99886179427487578125\,\pi^2} + \frac{600919849172992}{693159651388625045625}$$

Table 4.2: Numerical values of the coefficients σ_{2n}, $n = 2(1)21$, in the expansion (4.85) of σ.

n	σ_{2n}	n	σ_{2n}
2	$4.80337 \cdot 10^{-1}$	12	$2.44195 \cdot 10^{-3}$
3	$4.22697 \cdot 10^{-1}$	13	$5.15600 \cdot 10^{-4}$
4	$1.91142 \cdot 10^{-1}$	14	$-3.70006 \cdot 10^{-4}$
5	$1.98536 \cdot 10^{-2}$	15	$-4.73979 \cdot 10^{-4}$
6	$-4.58650 \cdot 10^{-2}$	16	$-2.71167 \cdot 10^{-4}$
7	$-4.43846 \cdot 10^{-2}$	17	$-6.96125 \cdot 10^{-5}$
8	$-2.17365 \cdot 10^{-2}$	18	$3.08721 \cdot 10^{-5}$
9	$-3.49620 \cdot 10^{-3}$	19	$4.82181 \cdot 10^{-5}$
10	$4.19730 \cdot 10^{-3}$	20	$2.97929 \cdot 10^{-5}$
11	$4.60964 \cdot 10^{-3}$	21	$8.90612 \cdot 10^{-6}$

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Appendix A Package for the scalar diffraction problem solved by Bazer and Brown's method with power-series expansion

```
(* Summary Package BazerBrown:
   Implementation of the scheme based on Bazer and Brown's
   method for the problem of the acoustic diffraction of a
   normally incident, plane wave by a circular disk; solution
   of the integral equation by expansion in powers of alpha.    *)

BeginPackage["BazerBrown`"]

BazerBrown1::usage = "This function calculates the low-frequency
  expansion of the scattering coefficient sigma_1 up to order alpha^n,
  where n is the first argument of the function. The second, optional,
  argument can be set to either True or False. Default value is False;
  if it is set to True, intermediate results are shown. The function
  uses the scheme based on Bazer and Brown's method for the problem of
  the acoustic diffraction of a normally incident, plane wave by an
  acoustically hard, circular disk. Power-series expansion is used
  to solve the integral equation."

BazerBrown2::usage = "This function calculates the low-frequency
  expansion of the scattering coefficient sigma_2 up to order alpha^n,
  where n is the first argument of the function. The second, optional,
  argument can be set to either True or False. Default value is False;
  if it is set to True, intermediate results are shown. The function
  uses the scheme based on Bazer and Brown's method for the problem of
  the acoustic diffraction of a normally incident, plane wave by an
  acoustically soft, circular disk. Power-series expansion is used
  to solve the integral equation."

BazerBrownTogether::usage = "This function calculates the
```

```
  low-frequency expansions of both scattering coefficients sigma_1
  and sigma_2 up to order alpha^n, where n is the first argument of
  the function. The second, optional, argument can be set to either
  True or False. Default value is False; if it is set to True,
  intermediate results are shown. The function uses the scheme based
  on Bazer and Brown's method for the problem of the acoustic
  diffraction of a normally incident, plane wave by a circular disk.
  Power-series expansion is used to solve the integral equation."

alpha::usage = "The expansion for sigma_1 resp. sigma_2 is in powers
  of alpha = k a, where k is the wave number and a is the radius
  of the disk."

g::usage = "The functions g[t, i] are the coefficients in the
  expansion of the function F[t], G[t] or H[t] in powers of
  alpha, depending on which Mathematica function is called."

t::usage = "The independent variable of the functions F[t], G[t] and H[t]."

Begin["`Private`"]

IntegratePol[pol_, {x_, a_, b_}] :=
  Module[ {f},
          f = Expand[ pol ];
          f = f /. x^m_. -> x^(m+1)/(m+1);
          (f /. (x -> b)) - (f /. (x -> a)) + (b-a) (pol /. x -> 0)
        ]

BazerBrownCommonPart[n_, verbose_, choice_] :=
  Module[ {G, sinhseries, integrand, integral,
             integralequation, system},
          G[t_] := Sum[ g[t,i] alpha^i, {i,0,n} ] + O[alpha]^(n+1);
          sinhseries = Series[ Sinh[ alpha (t-s) ] / (t-s),
            {alpha,0,n} ];
```

```
        sinhseries = sinhseries /. s -> -x;
        sinhseries = sinhseries /. (k_ t + k_ x)^m_. -> k^m (t + x)^m;
        sinhseries = sinhseries /. x -> -s;
        integrand  = sinhseries G[s];
        integral = SeriesData[ integrand[[1]], integrand[[2]],
          Map[ Hold[ IntegratePol[ #, {s,-1,1} ]] &,
          integrand[[3]] ], integrand[[4]], integrand[[5]],
          integrand[[6]] ];
        integralequation =
          Switch[ choice,
                  1, G[t] == Sinh[alpha t] + 1/(Pi I) integral,
                  2, G[t] == Cosh[alpha t] + 1/(Pi I) integral,
                  3, G[t] ==  Exp[alpha t] + 1/(Pi I) integral
                ];
        system = Apply[ List, LogicalExpand[integralequation] ];
        If[ verbose, calctime = TimeUsed[] ];
        Do[ equation = ReleaseHold[ system[[1]] ];
            sol = Simplify[ Solve[ equation, g[t,i-1] ] ];
            func = Function[ t, g[t,i-1] /. sol[[1]] ][t];
            substrule = g[t_,i-1] -> func;
            system = Drop[ system, 1 ] /. substrule;
            G[t] = G[t] /. substrule;
            If[ verbose,
                Print["Step ", i-1, " took ",
                  TimeUsed[] - calctime, " seconds; result: "];
                calctime = TimeUsed[];
                Print[substrule]
              ],
            {i, Length[system]}
          ];
        G[t]
      ]

CalculateSigma1[F_] :=
```

```
  Module[ {integrand, integral, sigma1},
          integrand = Sinh[alpha t] F;
          integral = SeriesData[ integrand[[1]], integrand[[2]],
            Map[ IntegratePol[ #, {t,0,1} ] &, integrand[[3]] ],
            integrand[[4]], integrand[[5]], integrand[[6]] ];
          sigma1 = 8/(Pi alpha) ComplexExpand[ Im[ Normal[integral] ] ] +
            O[alpha]^(integral[[5]]-1);
          sigma1 = Simplify[sigma1]
        ]

CalculateSigma2[G_] :=
  Module[ {integrand, integral, sigma2},
          integrand = Cosh[alpha t] G;
          integral = SeriesData[ integrand[[1]], integrand[[2]],
            Map[ IntegratePol[ #, {t,0,1} ] &, integrand[[3]] ],
            integrand[[4]], integrand[[5]], integrand[[6]] ];
          sigma2 = -8/(Pi alpha) ComplexExpand[ Im[ Normal[integral] ] ] +
            O[alpha]^(integral[[5]]-1);
          sigma2 = Simplify[sigma2]
        ]

BazerBrown1[n_, verbose_:False] :=
  Module[ {F},
          If[ verbose, Print["Calculating F[t]"] ];
          F = BazerBrownCommonPart[ n, verbose, 1 ];
          If[ verbose, Print["Calculating sigma_1"] ];
          CalculateSigma1[ F ]
        ]

BazerBrown2[n_, verbose_:False] :=
  Module[ {G},
          If[ verbose, Print["Calculating G[t]"] ];
          G = BazerBrownCommonPart[ n, verbose, 2 ];
          If[ verbose, Print["Calculating sigma_2"] ];
```

```
          CalculateSigma2[ G ]
        ]

BazerBrownTogether[n_, verbose_:False] :=
  Module[ {H},
          If[ verbose, Print["Calculating H[t]"] ];
          H = BazerBrownCommonPart[ n, verbose, 3 ];
          F = (H - (H /. t -> -t)) / 2;
          G = (H + (H /. t -> -t)) / 2;
          If[ verbose, Print["Calculating sigma_1 and sigma_2"] ];
          {CalculateSigma1[ F ], CalculateSigma2[ G ]}
        ]

End[]

EndPackage[]
```

Appendix B Package for the scalar diffraction problem solved by Bazer and Brown's method with Picard iteration

```
(* Summary Package PicardIteration:
   Implementation of the scheme based on Bazer and Brown's
   method for the problem of the acoustic diffraction of a
   normally incident, plane wave by a circular disk; solution
   of the integral equation by Picard iteration.              *)

BeginPackage["PicardIteration`"]

Picard1::usage = "This function calculates the low-frequency
  expansion of the scattering coefficient sigma_1 up to order alpha^n,
  where n is the first argument of the function. The second, optional,
  argument can be set to either True or False. Default value is False;
  if it is set to True, intermediate results are shown. The function
  uses the scheme based on Bazer and Brown's method for the problem of
  the acoustic diffraction of a normally incident, plane wave by an
  acoustically hard, circular disk. Picard iteration is used to solve
  the integral equation."

Picard2::usage = "This function calculates the low-frequency
  expansion of the scattering coefficient sigma_2 up to order alpha^n,
  where n is the first argument of the function. The second, optional,
  argument can be set to either True or False. Default value is False;
  if it is set to True, intermediate results are shown. The function
  uses the scheme based on Bazer and Brown's method for the problem of
  the acoustic diffraction of a normally incident, plane wave by an
  acoustically soft, circular disk. Picard iteration is used to solve
  the integral equation."

PicardTogether::usage = "This function calculates the low-frequency
  expansions of both scattering coefficients sigma_1 and sigma_2 up
  to order alpha^n, where n is the first argument of the function.
```

```
  The second, optional, argument can be set to either True or False.
  Default value is False; if it is set to True, intermediate results
  are shown. The function uses the scheme based on Bazer and Brown's
  method for the problem of the acoustic diffraction of a normally
  incident, plane wave by a circular disk. Picard iteration is used
  to solve the integral equation."

alpha::usage = "The expansion for sigma_1 resp. sigma_2 is in powers
  of alpha = k a, where k is the wave number and a is the radius
  of the disk."

t::usage = "The independent variable of the functions F[t], G[t] and H[t]."

Begin["`Private`"]

IntegratePol[pol_, {x_, a_, b_}] :=
  Module[ {f},
          f = Expand[ pol ];
          f = f /. x^m_. -> x^(m+1)/(m+1);
          (f /. (x -> b)) - (f /. (x -> a)) + (b-a) (pol /. x -> 0)
        ]

PicardCommonPart[n_, verbose_, choice_] :=
  Module[ {v, G, GOld, ser, i, integrand, integral},
          v[u_] =
            Switch[ choice,
                    1, Series[Sinh[alpha u], {alpha,0,(n+1)}],
                    2, Series[Cosh[alpha u], {alpha,0,(n+1)}],
                    3, Series[ Exp[alpha u], {alpha,0,(n+1)}]
                  ];
          G[t_] = v[t];
          ser = Series[ Sinh[ alpha (t-s) ] / (t-s), {alpha,0,(n+1)} ];
          ser = ser /. s -> -x;
          ser = ser /. (k_ t + k_ x)^m_. -> k^m (t + x)^m;
```

```
        ser = ser /. x -> -s;
        If[ verbose, i = 1; calctime = TimeUsed[] ];
        GOld = 0;
        While[ Not[ G[t] === GOld ],
                integrand = ser v[s];
                integral = SeriesData[ integrand[[1]],
                  integrand[[2]], Map[ IntegratePol[ #, {s,-1,1} ] &,
                  integrand[[3]] ], integrand[[4]],
                  integrand[[5]], integrand[[6]] ];
                v[u_] = Simplify[ 1/(Pi I) integral +
                  O[alpha]^(n+1)] /. t -> u;
                GOld = G[t];
                G[t_] = G[t] + v[t];
                If[ verbose,
                    Print["Step ", i-1, " took ", TimeUsed[] - calctime,
                      " seconds; result: "];
                    i = i + 1; calctime = TimeUsed[];
                    Print[ InputForm[G[t]] ]
                  ]
              ];
        G[t]
      ]

CalculateSigma1[F_] :=
  Module[ {integrand, integral, sigma1},
        integrand = Sinh[alpha t] F;
        integral = SeriesData[ integrand[[1]], integrand[[2]],
          Map[ IntegratePol[ #, {t,0,1} ] &, integrand[[3]] ],
          integrand[[4]], integrand[[5]], integrand[[6]] ];
        sigma1 = 8/(Pi alpha) ComplexExpand[ Im[ Normal[integral] ] ] +
          O[alpha]^(integral[[5]]-1);
        sigma1 = Simplify[sigma1]
      ]
```

```
CalculateSigma2[G_] :=
  Module[ {integrand, integral, sigma2},
          integrand = Cosh[alpha t] G;
          integral = SeriesData[ integrand[[1]], integrand[[2]],
            Map[ IntegratePol[ #, {t,0,1} ] &, integrand[[3]] ],
            integrand[[4]], integrand[[5]], integrand[[6]] ];
          sigma2 = -8/(Pi alpha) ComplexExpand[ Im[ Normal[integral] ] ] +
            O[alpha]^(integral[[5]]-1);
          sigma2 = Simplify[sigma2]
        ]

Picard1[n_, verbose_:False] :=
  Module[ {F},
          If[ verbose, Print["Calculating F[t]"] ];
          F = PicardCommonPart[ n, verbose, 1 ];
          If[ verbose, Print["Calculating sigma_1"] ];
          CalculateSigma1[ F ]
        ]

Picard2[n_, verbose_:False] :=
  Module[ {G},
          If[ verbose, Print["Calculating G[t]"] ];
          G = PicardCommonPart[ n, verbose, 2 ];
          If[ verbose, Print["Calculating sigma_2"] ];
          CalculateSigma2[ G ]
        ]

PicardTogether[n_, verbose_:False] :=
  Module[ {H},
          If[ verbose, Print["Calculating H[t]"] ];
          H = PicardCommonPart[ n, verbose, 3 ];
          F = (H - (H /. t -> -t)) / 2;
          G = (H + (H /. t -> -t)) / 2;
          If[ verbose, Print["Calculating sigma_1 and sigma_2"] ];
```

```
    {CalculateSigma1[ F ], CalculateSigma2[ G ]}
  ]

End[]

EndPackage[]
```

Appendix C Package for the scalar diffraction problem solved by Bouwkamp's method

```
(* Summary Package Bouwkamp:
   Implementation of the scheme based on Bouwkamp's method
   for the problem of the acoustic diffraction of a normally
   incident, plane wave by a circular disk.                    *)

BeginPackage["Bouwkamp`"]

Bouwkamp1::usage = "This function calculates the low-frequency
  expansion of the scattering coefficient sigma_1 up to order alpha^n,
  where n is the single argument of the function. The function uses
  the scheme based on Bouwkamp's method for the problem of the acoustic
  diffraction of a normally incident, plane wave by an acoustically
  hard, circular disk."

Bouwkamp2::usage = "This function calculates the low-frequency
  expansion of the scattering coefficient sigma_2 up to order alpha^n,
  where n is the single argument of the function. The function uses
  the scheme based on Bouwkamp's method for the problem of the acoustic
  diffraction of a normally incident, plane wave by an acoustically
  soft, circular disk."

alpha::usage = "The expansion for sigma_1 resp. sigma_2 is in powers
  of alpha = k a, where k is the wave number and a is the radius
  of the disk."

Begin["`Private`"]

b[p_, n_] :=
  b[p, n] = Simplify[ (-1)^(n+1) (n+3/4) *
  Gamma[n+1] / Gamma[n+3/2] *
  Sum[ Gamma[(q-1)/2] Gamma[(q+1)/2] (-1)^m b[p-q, m] Gamma[m+3/2] / Gamma[m+1] *
```

```
        1 / (Gamma[1/2q-m-n-1/2] * Gamma[1/2q+m-n+1] *
             Gamma[1/2q-m+n+1] * Gamma[1/2q+m+n+5/2]),
        {q,2,p}, {m,0,If[EvenQ[p-q],(p-q)/2,(p-q-3)/2]}
      ] ]

b[0,0] = -2/Pi

Bouwkamp1[n_] := Simplify[ 4/3 Sum[ (-1)^p b[2p+3, 0] alpha^(2p+4),
   {p,0,Max[0, Ceiling[n/2 - 3]]} ] +
   O[alpha]^(2 Max[0, Ceiling[n/2 - 3]] + 6) ]

a[p_, n_] :=
   a[p, n] = Simplify[ (-1)^(n+1) (2n+1/2) *
   Gamma[n+1] / Gamma[n+1/2] *
   Sum[ Gamma[(q+1)/2]^2 (-1)^m a[p-q, m] Gamma[m+1/2] / Gamma[m+1] *
        1 / (Gamma[1/2q-m-n+1/2] * Gamma[1/2q+m-n+1] *
             Gamma[1/2q-m+n+1] * Gamma[1/2q+m+n+3/2]),
        {q,1,p}, {m,0,Floor[ (p-q)/2 ]}
      ] ]

a[0,0] = 1

Bouwkamp2[n_] := Simplify[ -8/Pi Sum[ (-1)^p a[2p+1,0] alpha^(2p),
   {p,0,Max[0, Ceiling[n/2 - 1]]} ] +
   O[alpha]^(2 Max[0, Ceiling[n/2 - 1]] + 2) ]

End[]

EndPackage[]
```

Appendix D Package for the electromagnetic diffraction problem solved by Boersma's method

```
(* Summary Package BoersmaEM:
   Implementation of the scheme based on Boersma's method
   for the problem of the electromagnetic diffraction of a
   normally incident, plane wave by a perfectly conducting,
   circular disk.                                             *)

BeginPackage["BoersmaEM`"]

BoersmaEM::usage = "This function calculates the low-frequency
  expansion of the scattering coefficient sigma up to order alpha^n,
  where n is the single argument of the function. The function uses
  the scheme based on Boersma's method for the problem of the
  electromagnetic diffraction of a normally incident, plane wave
  by a perfectly conducting, circular disk."

alpha::usage = "The expansion for sigma is in powers of alpha = k a,
  where k is the wave number and a is the radius of the disk."

Begin["`Private`"]

IntegratePol[ pol_, {x_, a_, b_} ] :=
  Module[ {f},
          f = Expand[ pol ];
          f = f /. x^m_. -> x^(m+1)/(m+1);
          (f /. (x -> b)) - (f /. (x -> a)) + (b-a) (pol /. x -> 0)
        ]

PicardIteration[n_, choice_] :=
  Module[ {v, G, GOld, ser, integrand, integral},
          v[u_] =
            Switch[ choice,
```

```
                1, Series[  Sinh[alpha u] / alpha, {alpha,0,n}],
                2, Series[          Cosh[alpha u], {alpha,0,n}],
                3, Series[u Sinh[alpha u] / alpha, {alpha,0,n}]
              ];
        G[t_] = v[t];
        ser = Series[ Sinh[ alpha (t-s) ] / (t-s), {alpha,0,n} ];
        ser = ser /. s -> -x;
        ser = ser /. (k_ t + k_ x)^m_. -> k^m (t + x)^m;
        ser = ser /. x -> -s;
        GOld = 0;
        While[ Not[ G[t] === GOld ],
               integrand = ser v[s];
               integral = SeriesData[ integrand[[1]],
                 integrand[[2]], Map[ IntegratePol[ #, {s,-1,1} ] &,
                 integrand[[3]] ], integrand[[4]],
                 integrand[[5]], integrand[[6]] ];
               v[u_] = Simplify[ 1/(Pi I) integral +
                 O[alpha]^n] /. t -> u;
               GOld = G[t];
               G[t_] = G[t] + v[t]
             ];
        G[t]
      ]


BoersmaEM[n_] :=
 Module[ {F0, G0, G1, C0, c, integrand, integral, ser, sigma},
        F0 = PicardIteration[n, 1];
        G0 = PicardIteration[n, 2];
        G1 = PicardIteration[n, 3];
        C0 = - Simplify[G1 / G0 /. t->1];
        c  =   Simplify[F0 / (-F0 + D[G1, t] + C0 D[G0, t]) /. t->1];
        integrand = Simplify[ Sinh[alpha t] F0 ];
        integral  = SeriesData[ integrand[[1]], integrand[[2]],
                      Map[ Integrate[ #, {t,0,1} ] &, integrand[[3]] ],
```

```
                    integrand[[4]], integrand[[5]], integrand[[6]] ];
        ser = (c+1) integral;
        sigma = 8/Pi ComplexExpand[ Im[ Normal[ ser ] ] ] +
          O[alpha]^ser[[5]];
        sigma = Simplify[ sigma ]
      ]

End[]

EndPackage[]
```

Appendix E Package for the electromagnetic diffraction problem solved by Bouwkamp's method

```
(* Summary Package BouwkampEM:
   Implementation of the scheme based on Bouwkamp's method
   for the problem of the electromagnetic diffraction of a
   normally incident, plane wave by a perfectly conducting,
   circular disk.                                          *)

BeginPackage["BouwkampEM`"]

BouwkampEM::usage = "This function calculates the low-frequency
  expansion of the scattering coefficient sigma up to order alpha^n,
  where n is the single argument of the function. The argument n should
  be an even positive integer. The function uses the scheme based on
  Bouwkamp's method for the problem of the electromagnetic diffraction of
  a normally incident, plane wave by a perfectly conducting, circular disk."

alpha::usage = "The expansion for sigma is in powers of alpha = k a,
  where k is the wave number and a is the radius of the disk."

Begin["`Private`"]

(* The system of equations for the coefficients a_{n, n-2nu},
   b_{n, n-2nu}, and p_n                                   *)

eq1 = Sum[ 1/(n-tau)! a[tau, tau-2nu] J[nu, 0, n-tau, rho],
          {tau, 1, n},
          {nu, 0, Floor[ (tau+1)/2 ]}
        ] ==
      Sum[ p[n-2nu] rho^(2nu)/(2^(2nu) (nu!)^2),
          {nu, 0, Floor[ (n+1)/2 ]}
        ]
```

```
eq2 = Sum[ 1/(n-tau)! b[tau, tau-2nu] J[nu, 1, n-tau, rho],
           {tau, 1, n},
           {nu, 1, Floor[ (tau+1)/2 ]}
         ] ==
      Sum[ p[n-2nu] rho^(2nu)/(2^(2nu) (nu-1)! (nu+1)!),
           {nu, 1, Floor[ (n+1)/2 ]}
         ]

eq3 = a[n, n] == Sum[ (-1)^(nu+1) Gamma[nu+1/2] /
                      (Gamma[1/2] Gamma[nu+1]) a[n, n-2nu] +
                      (-1)^nu 4 Gamma[nu+3/2] /
                      (Gamma[1/2] Gamma[nu]) b[n, n-2nu],
                      {nu, 1, Floor[ (n+1)/2 ]}
                    ]

(* Definition of the functions J(n, m, mu, rho)                      *)

J[n_, m_, mu_, rho_] := (-1)^(n+m) Gamma[1/2mu+1/2]^2 *
  Gamma[n+m+1/2] / (Sqrt[Pi] Gamma[m+1/2] Gamma[m+1] Gamma[n-m+1] *
  Gamma[1/2mu-n-m+1/2] Gamma[1/2mu+n-m+1]) *
  rho^(2m) * Hypergeometric2F1[ -1/2mu+n+m+1/2, -1/2mu-n+m, 2m+1, rho^2 ]

(* Solve the system of linear equations and calculate
   the expansion of sigma                                             *)

BouwkampEM[maxorder_] :=
  Module[ {solved = {p[-1] -> 1, p[0] -> 0}, unknowns, system, dummy, coeff},
          Do[ unknowns = Union[ Table[ a[n, n-2nu],
                                       {nu, 0, Floor[(n+1)/2]} ],
                                {p[n]},
                                Table[ b[n, n-2nu],
                                       {nu, 1, Floor[(n+1)/2]} ]
                              ];
              system = {eq1[[1]] + O[rho]^(n+2) == eq1[[2]],
```

```
                eq2[[1]] + O[rho]^(n+2) == eq2[[2]],
                eq3} //. solved;
      dummy  = Solve[ system, unknowns ][[1]];
      solved = Union[ solved, dummy ];
      If[EvenQ[n], Print["Coefficient alpha^", n, " equals "];
                  coeff[n/2] = Simplify[ 8/Pi (-1)^(n/2)
                   (a[n, n] /. solved) ];
                  Print[coeff[n/2]]
        ],
      {n, 1, maxorder}
    ];
   Sum[ coeff[n/2] alpha^n, {n, 2, maxorder, 2} ] +
    O[alpha]^(maxorder+2)
  ]

End[]

EndPackage[]
```