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Abstract

In this paper, we consider a single machine at which two types of orders with deterministic processing times arrive according to independent Bernoulli processes. When the machine switches from producing one type of order to the other, fixed switch-over times are incurred. The machine switches according to a periodic switching strategy, i.e., switches at predetermined and periodic time instants. We consider two simple possibilities to improve the performance of this periodic switching strategy: working overtime and producing to stock. We show for the two models that the queue-length distribution for both types of orders can be determined efficiently. From the queue-length distribution, we can determine the delivery-time distribution of orders for investigating how much the performance of the periodic switching strategy may be improved by introducing either working overtime or producing to stock.

1 Introduction

Machines in production centers often (partly) manufacture different types of jobs. Whenever such a machine switches from manufacturing one type of job to another, switch-over times are incurred since, e.g., the machine has to be cleaned or tuned up. For these production centers, the average delivery times of orders and the fraction of orders that is handled in time are important performance measures, which are obviously affected by the switching policy of the machines. In a previous paper (cf. Van Eenige et al. [1]), we compared two switching strategies, which naturally appear in practice: a periodic and an exhaustive strategy. For a periodic switching strategy, the machine switches at predetermined and fixed periodic time epochs, so that for each type of job time can be divided into intervals of fixed length in which the machine is alternately available for producing this type of job (on periods) and not available (off periods). For an exhaustive service policy, each type of order is handled exhaustively before the machine switches to handling another type of order. It turned out that the performance, with respect to the delivery times of orders, of a periodic switching strategy is considerably worse than that of an exhaustive service policy.

A periodic switching strategy, however, has various practical advantages with regard to the exhaustive service policy. For instance, it is easier to control the inventory levels of materials, it is possible to give accurate estimates for delivery times of orders, and it is easier to plan and schedule personnel and operators. Two simple possibilities to improve the performance of the periodic switching strategy are to allow working overtime and production to stock. The aim of this paper is to get insight at what rate this performance can be improved when overtime or producing to stock is incorporated, so that the practical advantages of the periodic switching strategy are still maintained, while at the same time the performance is (substantially) improved. We are aware of the fact that these possibilities are usually implemented at higher costs, since a production center, e.g., has to pay higher wages for hours worked in overtime, and has to charge holding costs for jobs on stock. Nevertheless, we only concentrate on the improvement of the performance, particularly the delivery times of orders, if either of these features is implemented for a periodic switching strategy.

For a periodic switching strategy with working overtime, we consider a so-called threshold policy, i.e., if at the end of an on period the amount of work at the machine of the corresponding type of job exceeds a certain threshold, the machine works a specific amount of overtime. We assume that working overtime does not affect the future switching epochs. For example, working overtime is done outside the normal working hours by another firm or by hiring some additional capacity or machines.

For a periodic switching strategy with producing to stock, the production center can have for each type of job at most a certain number of finished jobs on stock. If during an on period there are no orders for the corresponding type of job and the number of finished jobs of this type on stock is less than the maximum number of these jobs allowed on stock, then the machine produces another job to stock. Orders that find their requested job on stock are delivered immediately from stock.

Clearly, the performance of the basic periodic switching strategy will be improved when either of these two possibilities is implemented. Whether to incorporate overtime or producing to stock will depend on the situation at the production center. If customers place orders for jobs which are rather customer specific, then producing to stock is not a serious option, but overtime is. Otherwise, production to stock may be preferred, since a more efficient use of the machine's capacity is made.

As a first attempt to get insight in how much the performance improves by introducing these two additional features, we consider in this paper a discrete-time queueing system with one machine, which manufactures two types of jobs. Orders for each type of job arrive according to a Bernoulli process and have fixed service times. When the machine switches from handling one type of order to another, fixed switch-over times are incurred. We show that these models with working overtime and producing to stock can be analysed efficiently by exploiting the geometric tail behaviour of the stationary queue-length distribution. From this stationary distribution, we can determine the delivery-time distribution for both types of orders, so that we are able to compute the average delivery time of orders and the fraction of orders that is handled in time.

The outline of the paper is as follows. The models are described in detail in Section 2. The analysis of the queue-length processes is the topic of Section 3. Numerical results will be given in Section 4, and the paper is concluded with a summary in Section 5.

2 The models

In this section, we describe the models in detail. We begin with the description of their common characteristics, followed by the specific ones.

Orders for two types of jobs, each with its own queue, are served by a common server. Whenever the server switches from one queue to the other, switch-over times are incurred. We divide time into intervals of equal length, in the sequel called slots. During a slot, the server is either serving, idling, or switching.

Orders for jobs of type i, $i = 1, 2$, arrive according to a Bernoulli process with parameter p_i and have fixed service times equal to b_i slots. The service discipline at each queue is FCFS and the arrival processes are mutually independent. The switch-over times from queue 1 to queue 2 (from queue 2 to queue 1) are fixed and equal to s_{12} slots (s_{21} slots). Note that only the arrival processes are random.

The server is assigned to the queues according to a fixed periodic pattern. More precisely, the server is first assigned to queue 1 for a_1 consecutive slots. Then, it switches to queue 2 (which takes s_{12} slots), after which it is assigned to queue 2 for a_2 consecutive slots. Finally, the server switches back to queue 1 (which takes s_{21} slots), after which the periodic switching pattern starts over again. The time interval between two successive departures of the server from queue 1 is called a cycle (note that each cycle contains exactly one on period for each type of order). Clearly, the length of a cycle is $a_1 + s_{12} + a_2 + s_{21}$ slots, and is denoted by N. The slots in a cycle are numbered $1, 2, \ldots, N$. The period in a cycle during which the server is assigned to queue i is called an on period for queue i. If, due to the periodic switching pattern, the server has to interrupt servicing, upon return of the server this service is resumed where it was interrupted.

Order arrivals as well as the start and completion of slots of service (and thus, also the start and completion of a job) occur at slot boundaries. Since our main concern is the delivery-time distribution of orders, we assume that order arrivals and the start of slots of service occur *just after* slot boundaries, and that the completion of slots of service occurs *just before* slot boundaries. In the sequel, an order arriving at the slot boundary between slot $n-1$ and slot n (i.e., the n-th slot boundary) is said to be arriving in slot n. Finally, the service of an order that arrives at the queue at which the server is idling begins instantaneously.

Let the model with the opportunity of working overtime be denoted by the OVT-model. If, for this model, the number of slots work at queue i at the end of the last slot of its on period (after a completion of a slot of service) is at least c_i, then the server works d_i slots overtime (or the number of slots work at this instant if this number is less than d_i). Working overtime occurs at the slot boundary between the last slot of the corresponding on period and the first slot switching. So, working overtime is assumed not to affect the periodic switching pattern, i.e., it neither lengthens the corresponding on period nor affects the future switching epochs.

We denote with the PTS-model the model with the possibility of producing to stock. If, for this model, the attended queue i is empty just after possible arrival instants and the number of jobs of type i on stock is less than S_i, then the server continues or starts producing type i jobs to stock. Arriving orders that find their requested jobs on stock are delivered immediately from stock (independent of whether the server is attending their queue or not). We note that an order arriving at an empty queue at which the server is already producing the first job to stock has a reduced service time.

3 The queue-length processes

In this section, we analyse the queue-length processes of the two models described in the previous section. For both models, the two queue-length processes do not affect each other so that they can be analysed separately. Therefore, we consider for both models only one of the queues. For notational convenience, we simply write b for the service times of orders of the queue to be analysed, p for the arrival rate of these orders, and a for the length of the on period for this queue, respectively. We assume without loss of generality that these a slots are the last slots in the cycle.

For both models, we will use an embedded Markov chain approach, similar to the one in [1], for analysing the queue-length processes. We will conclude that the asymptotic behaviour of the stationary queue-length distribution is geometric. This asymptotic behaviour will be exploited for numerical purposes.

3.1 The OVT-model

Let c denote the threshold for working overtime and d the maximum number of slots working overtime of the queue to be analysed. Consider the queueing system embedded just before the possible arrival instant in the first slot of the cycle (i.e., just after working overtime), and let X_k denote the number of slots work in this queue at this instant in the k-th cycle. The process $\{X_k\}_{k\in\mathbb{N}_0}$ is a discrete-time parameter Markov chain with state space the nonnegative integers $\{j|j=0,1,2,\dots\}$. We assume that this chain is ergodic (i.e., $Npb < a+d$), and hence, that the stationary distribution $\{\pi_j, j=0,1,2,\dots\}$ exists. This stationary distribution is the unique normalized solution of the one-cycle equilibrium equations

$$\pi_j = q_{0,j}\pi_0 + q_{1,j}\pi_1 + \cdots + q_{j+a+d,j}\pi_{j+a+d}, \qquad j=0,1,2,\dots, \tag{1}$$

where $q_{i,j}$ denotes the one-cycle transition probability from state i to state j, i.e.,

$$q_{i,j} := \Pr\{X_1 = j|X_0 = i\}.$$

If $a+d \geq Nb$, then the states $j > c$ are transient. The reason is that in this case the number of slots handled in a cycle (including the slots handled in overtime) is with probability one greater than or equal to the maximum number of slots work that arrives during a cycle. Thus, $\pi_j = 0$ for $j > c$ (if $a+d > Nb$, then state c is transient as well), so that solving the finite set of equilibrium equations for the recurrent states and the normalization equation yields the stationary distribution. Henceforth, we assume that $a+d < Nb$.

We will first prove that for $j \geq c+Nb-a-d$ the transition probabilities $q_{i,j}$ satisfy

$$q_{i,j} = \begin{cases} \beta_m & \text{if } i = j-mb+a+d \text{ for some } m=0,1,\dots,N, \\ 0 & \text{elsewhere,} \end{cases} \tag{2}$$

where β_m denotes the probability of m customers arriving in a cycle consisting of N slots, i.e.,

$$\beta_m = \binom{N}{m} p^m (1-p)^{N-m}.$$

Hence, for $j \geq c+Nb-a-d$, the equilibrium equations (1) reduce to

$$\pi_j = \beta_N \pi_{j-(Nb-a-d)} + \beta_{N-1}\pi_{j-((N-1)b-a-d)} + \cdots + \beta_0 \pi_{j+a+d}.$$

Later on, the structure of these equations is used to determine the stationary distribution.

To prove (2), let $j \geq c + Nb - a - d$. Since $j \geq c$, the server had to work overtime at the end of the cycle, so that the number of slots work just before working overtime was equal to $j + d$. If the server is idle in a slot between $N - a$ and N, then, at the end of the on period but just before working overtime, there are at most $(a-1)(b-1) < N(b-1) < Nb - a \leq j + d$ slots of work. So, if at the beginning of the next cycle just after working overtime the number of slots work equals j, the server cannot have been idle in the cycle. Hence, $q_{i,j} = 0$ for $i \neq j - mb + a + d$ for any $m = 0, 1, \dots, N$. If $i = j - mb + a + d$ for some $m = 0, 1, \dots, N$, then there should have been exactly m arrivals in the cycle. Even if these m arrivals take place in the last m slots, the server cannot have been idle as there are $i = j - mb + a + d \geq (N-m)b \geq N - m$ slots of work at the beginning of the cycle. Thus, any pattern of m arrivals prevents the server from becoming idle, and leads to j slots of work at the beginning of the next cycle. Hence, $q_{i,j} = \beta_m$ if $i = j - mb + a + d$ for some $m = 0, 1, \dots, N$.

So, the equilibrium equations (1) can be partitioned as follows

$$\pi_j = q_{0,j}\pi_0 + q_{1,j}\pi_1 + \cdots + q_{j+a+d,j}\pi_{j+a+d}, \qquad 0 \leq j < c + Nb - a - d, \quad (3)$$
$$\pi_j = \beta_N \pi_{j-(Nb-a-d)} + \beta_{N-1}\pi_{j-((N-1)b-a-d)} + \cdots + \beta_0 \pi_{j+a+d}, \qquad j \geq c + Nb - a - d, \quad (4)$$

Equations (4) are called inner conditions and equations (3) are called boundary conditions. In the boundary conditions, the transition probabilities $q_{i,j}$ with $i \geq a + \max\{c, d\}$ are also given by (2), because the server cannot become idle in the cycle and the server has to work the maximum amount of overtime. The transition probabilities $q_{i,j}$ for $i < a + \max\{c, d\}$ can be obtained iteratively from the one-slot transition probabilities. These one-slot transition probabilities can easily be obtained, because in each slot at most one order arrives and at most one slot of work is handled. Only for the one-slot transition probability from the last slot of the cycle to the first slot of the next cycle, the server may effectively handle more than one slot due to working overtime.

As aforementioned, the stationary probability distribution is the unique normalized solution of the boundary and inner conditions. We will first seek solutions of the inner conditions of the form $\pi_j = z^j$, and then construct a linear combination of these solutions for solving the boundary conditions and the normalization equation.

Inserting z^j into the equations (4) and dividing by $z^{j-(Nb-a-d)}$ yields

$$z^{Nb-a-d} = \beta_0 z^{Nb} + \beta_1 z^{(N-1)b} + \cdots + \beta_N. \quad (5)$$

Clearly, we are only interested in solutions z with $|z| < 1$. The next lemma states that there are $Nb - a - d$ solutions with $|z| < 1$, and that exactly one of these solutions is positive and real, which is the largest solution in absolute value. The proof of this lemma can be found in [1].

Lemma 1 *Let M be a positive integer and consider the following equation*

$$z^M = \beta_0 z^{Nb} + \beta_1 z^{(N-1)b} + \cdots + \beta_N.$$

Then, provided that $M < (1-p)Nb$, this equation has exactly M roots $z_1, z_2, \dots, z_M$ inside the unit circle, and exactly one of these roots, z_1, is in the interval $(0, 1)$. Furthermore, if M is not a multiple of b, then $|z_i| < |z_1|$ for $i = 2, 3, \dots, M$. Otherwise, there are $b - 1$ roots, $z_2, z_3, \dots, z_b$, with $|z_i| = |z_1|$, and $M - b$ roots, $z_{b+1}, z_{b+2}, \dots, z_M$, with $|z_i| < |z_1|$.

For convenience, we assume that the $Nb - a - d$ roots of equation (5) are distinct, so that we have $Nb - a - d$ solutions z^j satisfying the inner conditions. Hence, the inner conditions are also satisfied by a linear combination of these basic solutions, i.e., by solutions of the form

$$\pi_j = \sum_{i=1}^{Nb-a-d} \alpha_i z_i^j, \tag{6}$$

where the α_i's denote the coefficients of the linear combination. The assumption that these $Nb-a-d$ roots are distinct is not restrictive, because if, for instance, roots z_i and z_{i+1} are identical, z_{i+1}^j should be replaced in (6) by jz_i^j, so that the number of basic solutions is still $Nb-a-d$. Similarly, if there are more roots equal to z_i, then higher powers of j should be used. It now remains to solve the boundary conditions and the normalization.

Since the inner conditions do not contain the stationary probabilities π_j for states $j = 0, 1, \dots, c-1$, we seek coefficients α_i such that

$$\pi_j = \sum_{i=1}^{Nb-a-d} \alpha_i z_i^j, \quad j = c, c+1, \dots,$$

satisfies the boundary conditions and the normalization equation, and treat the stationary probabilities $\pi_0, \pi_1, \dots, \pi_{c-1}$ as unknowns. Because the equilibrium equations are dependent, one of the boundary conditions in (3) may be omitted, the boundary condition for $j = 0$, say. Substituting the representation (6) and the unknowns $\pi_0, \pi_1, \dots, \pi_{c-1}$ into the reduced set of boundary conditions and the normalization equation yields a set of $c + Nb - a - d$ linearly independent equations with $c + Nb - a - d$ unknowns, which are the coefficients α_i, $i = 1, 2, \dots, Nb - a - d$, and the probabilities π_k, $k = 0, 1, \dots, c-1$. Thus, this set of equations has a unique solution, yielding the stationary distribution.

To use solution (6), we have to find $Nb - a - d$ roots of equation (5), and to solve the boundary conditions and the normalization equation for the coefficients α_i and the probabilities $\pi_0, \pi_1, \dots, \pi_{c-1}$. The single positive root can easily be determined numerically, e.g., by using bisection. Finding all roots numerically, however, can be very difficult. These difficulties especially occur when $Nb - a - d$ is large or when (some of) the roots are closely clustered inside the unit circle. Moreover, even if we are able to compute all roots accurately, the reduced set of boundary conditions and the normalization equations is nearly singular if the roots are closely clustered (see, e.g., Section 1.6 in Neuts [2]). Since the largest root(s) obviously determine(s) the asymptotic behaviour of the stationary distribution, we will exploit this behaviour for approximating the stationary distribution numerically.

By Lemma 1, the largest root of equation (5) is unique if $Nb - a - d$ is not a multiple of b, and there are b such roots otherwise. In the former case, we have

$$\lim_{j\to\infty} \frac{\pi_{j+1}}{\pi_j} = z_1, \tag{7}$$

so that the stationary distribution has a geometric tail. In the latter case, since this limit does not exist, we consider the quotient π_{j+b}/π_j. By using $z_i^b = z_1^b$ for $i = 2, 3, \dots, b$, one easily verifies that

$$\lim_{j\to\infty} \frac{\pi_{j+b}}{\pi_j} = z_1^b. \tag{8}$$

The fact that the single positive root completely determines the asymptotic behaviour of the stationary distribution will be exploited numerically by using an adapted version of the numerical approach of Tijms and Van de Coevering [3].

The starting point in [3] is the assumption that the limit (7) exists. Since this limit does not exist if b is a divisor of $Nb - a - d$, their approach is slightly adapted for the present problem by exploiting the behaviour of π_{j+b}/π_j in stead of π_{j+1}/π_j. We choose a positive integer J, and set

$$\pi_j = z_1^b \pi_{j-b}, \quad j \geq J+1, \tag{9}$$

so that we implicitly assume that z_1^b is a (fairly) good approximation for the quotient π_{j+b}/π_j for $j > J$. The probabilities $\pi_0, \pi_1, \ldots, \pi_J$ are now obtained by solving the equilibrium equations for $j = 0, 1, \ldots, J$ (in which the approximation (9) is substituted), and the normalization equation, which reads as follows

$$\sum_{j=0}^{J-b} \pi_j + \sum_{i=1}^{b} \frac{1}{1 - z_1^b} \pi_{J-b+i} = 1.$$

The value of the integer J depends of course on the required accuracy, and has to be determined experimentally.

As suggested in [3] and already confirmed by the numerical examples in [1], the asymptotic behaviour appears rather quickly, i.e., for already small values of J. In particular, these values are smaller than $c + Nb - a - d$, i.e., smaller than the number of boundary conditions.

From this approximated stationary queue-length distribution, we easily compute the stationary queue-length distribution for the other slots in the cycle. From these distributions, one can straightforwardly compute the delivery-time distribution of an arbitrary order.

3.2 The PTS-model

The analysis of the queue-length process for the PTS-model is similar to the one for the OVT-model. Let V denote the maximum number of jobs on stock expressed in number of slots work (i.e., $S \cdot b$, with S the maximum number of jobs on stock) for the queue to be analysed. Again, consider the queueing system embedded just before the possible arrival instant in the first slot of the cycle, and let Y_k denote the number of slots work in this queue at this instant in the k-th cycle. A negative value of Y_k denotes the number of slots work already on stock. The process $\{Y_k\}_{k \in \mathbb{N}_0}$ is a discrete-time parameter Markov chain with state space $\{j | j = -V, -(V-1), \ldots, -1, 0, 1, \ldots\}$. When assuming that this chain is ergodic (i.e., $Npb < a$), the stationary distribution of this chain $\{\pi_j, j = -V, -(V-1), \ldots, -1, 0, 1, \ldots\}$ exists and it is the unique normalized solution of the one-cycle equilibrium equations

$$\pi_j = r_{-V,j}\pi_{-V} + r_{-(V-1),j}\pi_{-(V-1)} + \cdots + r_{j+a,j}\pi_{j+a}, \qquad j = -V, -(V-1), \ldots, \tag{10}$$

where $r_{i,j}$ denotes the one-cycle transition probability from state i to state j, i.e.,

$$r_{i,j} := \Pr\{Y_1 = j | Y_0 = i\}.$$

By similar arguments as for the transition probabilities $q_{i,j}$, we can show (see also [1]) that for $j \geq Nb - a$

$$r_{i,j} = \begin{cases} \beta_m & \text{if} = j - mb + a \text{ for some } m = 0, 1, 2 \ldots, N, \\ 0 & \text{otherwise}, \end{cases} \tag{11}$$

where β_m again denotes the probability of m arrivals in a cycle consisting of N slots. Then, the equilibrium equations (10) can be partitioned as follows

$$\pi_j = r_{-V,j}\pi_{-V} + r_{-V+1,j}\pi_1 + \cdots + r_{j+a,j}\pi_{j+a}, \qquad -V \leq j < Nb - a, \tag{12}$$

$$\pi_j = \beta_N \pi_{j-(Nb-a)} + \beta_{N-1}\pi_{j-((N-1)b-a)} + \cdots + \beta_0 \pi_{j+a}, \qquad j \geq Nb - a. \tag{13}$$

The inner conditions (13) are identical to the inner conditions for the model without producing to stock (see [1]). Only the boundary conditions (12) and the number of these boundary conditions differ from the model without producing to stock. For $i \geq a$, the transition probabilities $r_{i,j}$ are also equal to (11). The transition probabilities $r_{i,j}$ for $i < a$ can easily be obtained from the one-slot transition probabilities. Like for the OVT-model, the structure of the inner conditions (13) is used for determining the stationary queue-length distribution.

Since the inner conditions are identical to the inner conditions for the model without producing to stock, we have from [1] that there are exactly $Nb - a$ solutions of the form $\pi_j = z^j$ for the inner conditions with $|z| < 1$, $z_1, z_2, \ldots, z_{Nb-a}$, and that exactly one of these roots, z_1, is in the interval $(0, 1)$, which is the largest root in absolute value.

Like for the analysis of the OVT-model, the number of basic solutions is smaller than the number of boundary conditions. Furthermore, since the stationary probabilities π_j, $j = -V, -(V-1), \ldots, -1$ do not appear in the inner conditions, we represent π_j for $j \geq 0$ as

$$\pi_j = \sum_{i=1}^{Nb-a} \gamma_i z_i^j,$$

and consider the remaining stationary probabilities $\pi_{-V}, \pi_{-(V-1)}, \ldots, \pi_{-1}$ as unknowns. We substitute this representation into the boundary conditions and the normalization equation, so that (by arguments given in the preceding subsection) the solution of these equations uniquely yields the coefficients γ_i and the stationary probabilities $\pi_{-V}, \pi_{-(V-1)}, \ldots, \pi_{-1}$.

Again, the tail of the stationary queue-length distribution is geometric. Due to reasons mentioned for the analysis of the OVT-model, we will use the adapted approach of [3] for computing the stationary queue-length distribution. Since from the state of the system the sojourn time of a customer is completely known, the sojourn-time distribution of a customer can easily be determined from the stationary queue-length distribution.

4 Numerical results

The delivery time of an order is an important performance measure for production centers. In this section, we give numerical results for the delivery times of orders for the OVT-model and PTS-model. These results are compared with the results for the periodic switching strategy with neither working overtime nor producing to stock, so that the rate at which working overtime or producing to stock improves the performance can be determined. Furthermore, we compare these results with the exhaustive service policy, as analysed in [1], which is denoted by the EXH-model. In the latter case, each type of order is handled exhaustively before the machine switches to the other type of order, and if there are no orders at the machine at all, then the machine waits for the next arrival before switching. The exhaustive service policy often minimizes the number of orders in the system if no additional features, like working overtime or producing to stock, are allowed (see [1]). The presented results are based on the following basic example. In a production center, a machine is used to produce two types of products. This production center can run 8 hours a day for 5 days a week. In each of these hours (i.e., in each slot) at most one order arrives for a product of type 1 and at most one order for a product of type 2. The probability of an arrival of these orders is p_1 and p_2, respectively. The time for producing one product is equal to 2 hours for products of type 1, and 3 hours for products of type 2. When the production center changes from handling orders of type 1 to orders of type 2 or vice versa, the machine has to be cleaned thoroughly. The cleaning of the machine takes 6 hours. We define $\rho = 2p_1 + 3p_2$, i.e., the average amount of work in hours arriving at the machine per hour.

For the numerical results, the lengths of the on periods (i.e., a_1 and a_2) are chosen such that the number of orders at the machine for the model with neither working overtime nor producing to stock is minimal. For our numerical examples, these values are presented in Table 1 (see [1]). Furthermore, for the OVT-model, we have set the thresholds for working overtime (i.e., the c_i's) at zero, so that optimal usage can be made of the opportunity of working overtime.

ρ	p_1	p_2	Type 1	Type 2
0.50	1/10	1/10	14	21
	1/8	1/12	18	18
	1/16	1/8	10	27
0.75	3/20	3/20	32	48
	3/16	1/8	42	42
	3/32	3/16	24	66
0.90	9/50	9/50	86	128
	9/40	3/20	108	108
	9/80	9/40	58	168

Table 1: The length of the on periods in hours.

In Table 2, we give the average delivery times of orders for the OVT-model, the PTS-model, and the EXH-model. Furthermore, we give between brackets the average number of hours working overtime for the OVT-model and the average number of finished products on stock for the PTS-model. The case $d_i = \infty$ denotes the situation for which all orders that arrive during a cycle are handled in the same cycle. Except for the case $\rho = 0.5$, the last row for each of the examples for the PTS-model shows the minimum S_i for which the average delivery time of an order is lower than the average delivery time of an order for the EXH-model.

Clearly, the opportunities of working overtime and producing to stock improve the performance compared to the standard periodic switching strategy (i.e., the case $d_1 = d_2 = 0$ in the OVT-model or the case $S_1 = S_2 = 0$ in the PTS-model). However, whether working overtime or producing to stock is more effective depends on the work load.

If the load of the system is low, then there are many opportunities for producing to stock, while the opportunity of working overtime will not be used often (compare $d_i = b_i$ and $d_i = \infty$ for $\rho = 0.50$). Therefore, producing to stock is more effective than working overtime for these situations. In fact, allowing only a few products on stock already yields a lower average delivery time for the PTS-model than for the EXH-model. For these work loads, the improvement of producing only one product in overtime practically equals the maximum improvement that can be obtained by working overtime (i.e., $d_i = \infty$).

If, on the other hand, the work load is high, there will often be (some) orders at the machine at the end of an on period, so that the machine will work overtime quite often (compare $d_i = b_i$ and $d_i = \infty$ for $\rho = 0.90$). For producing to stock, allowing only a small number of jobs on stock is not quite effective. But, if the maximum number of jobs allowed on stock increases, the average delivery time steadily decreases. In fact, if this maximum number goes to infinity, the average delivery time of an order will even tend to zero. This is in contrast with the situation with working overtime. For this latter case, an order that arrives during an off period has to wait at least the residual length of this off period before being served, so that the average waiting time for an order has a strictly positive lower bound. Finally, although, the maximum number of jobs allowed on stock increases in order to obtain a similar performance as for the EXH-model, the average number of finished jobs on stock remains rather small.

			Type 1 product							Type 2 product						
ρ	p_1	p_2	d_1	OVT		S_1	PTS		EXH	d_2	OVT		S_2	PTS		EXH
0.50	$\frac{1}{10}$	$\frac{1}{10}$	0	19.97		0	19.97		11.60	0	17.55		0	17.55		11.23
			2	17.66	(0.30)	1	12.40	(0.24)		3	14.65	(0.62)	1	10.25	(0.27)	
			4	17.16	(0.38)	2	7.14	(0.69)		6	14.05	(0.75)	2	5.53	(0.77)	
			∞	16.89	(0.42)	3	3.79	(1.31)		∞	13.78	(0.81)	3	2.74	(1.43)	
	$\frac{1}{8}$	$\frac{1}{12}$	0	17.02		0	17.02		10.72	0	20.54		0	20.54		11.93
			2	15.44	(0.29)	1	10.92	(0.24)		3	16.97	(0.58)	1	11.85	(0.79)	
			4	15.14	(0.35)	2	6.56	(0.66)		6	16.24	(0.70)	2	6.29	(0.79)	
			∞	14.97	(0.39)	3	3.64	(1.23)		∞	15.91	(0.76)	3	3.07	(1.47)	
	$\frac{1}{16}$	$\frac{1}{8}$	0	23.95		0	23.95		12.66	0	14.47		0	14.47		9.86
			2	20.97	(0.24)	1	12.96	(0.31)		3	12.26	(0.70)	1	8.60	(0.26)	
			4	20.36	(0.29)	2	6.33	(0.88)		6	11.78	(0.84)	2	4.76	(0.75)	
			∞	20.10	(0.32)	3	2.81	(1.62)		∞	11.57	(0.91)	3	2.43	(1.39)	
0.75	$\frac{3}{20}$	$\frac{3}{20}$	0	39.19		0	39.19		22.05	0	32.39		0	32.39		18.23
			2	34.05	(0.66)	1	33.07	(0.08)		3	26.66	(1.23)	1	26.38	(0.10)	
			4	32.19	(0.95)	2	27.51	(0.23)		6	24.65	(1.72)	2	21.13	(0.29)	
			∞	30.35	(1.26)	4	18.05	(0.74)		∞	22.82	(2.21)	3	16.56	(0.57)	
	$\frac{3}{16}$	$\frac{1}{8}$	0	32.39		0	32.39		19.51	0	39.26		0	39.26		20.27
			2	29.25	(0.63)	1	27.53	(0.09)		3	32.28	(1.15)	1	32.04	(0.10)	
			4	28.11	(0.89)	2	23.14	(0.25)		6	29.10	(1.60)	2	25.67	(0.29)	
			∞	26.82	(1.20)	3	19.15	(0.46)		∞	27.81	(2.04)	3	20.12	(0.57)	
	$\frac{3}{32}$	$\frac{3}{16}$	0	46.76		0	46.76		26.24	0	25.90		0	25.90		15.23
			2	42.09	(0.46)	1	37.30	(0.11)		3	21.79	(1.38)	1	21.11	(0.10)	
			4	40.48	(0.64)	2	29.04	(0.33)		6	20.20	(1.94)	2	16.98	(0.30)	
			∞	39.09	(0.81)	3	21.96	(0.64)		∞	18.63	(2.54)	3	13.40	(0.59)	
0.90	$\frac{9}{50}$	$\frac{9}{50}$	0	95.42		0	95.42		51.62	0	82.59		0	82.59		37.87
			2	83.76	(1.02)	1	90.02	(0.03)		3	66.94	(1.86)	1	77.21	(0.03)	
			4	78.41	(1.62)	2	84.76	(0.08)		6	60.08	(2.94)	2	73.03	(0.09)	
			6	75.39	(2.01)	3	79.63	(0.14)		9	56.39	(3.62)	3	67.00	(0.17)	
			∞	70.13	(2.85)	10	47.72	(1.20)		∞	49.94	(5.05)	10	36.76	(1.49)	
	$\frac{9}{40}$	$\frac{3}{20}$	0	80.00		0	80.00		43.69	0	98.57		0	98.57		44.29
			2	71.27	(1.05)	1	75.69	(0.03)		3	79.35	(1.76)	1	92.10	(0.03)	
			4	67.08	(1.67)	2	71.50	(0.08)		6	71.45	(2.76)	2	85.85	(0.09)	
			6	64.62	(2.09)	3	67.42	(0.14)		9	67.37	(3.39)	3	79.78	(0.17)	
			∞	59.90	(3.06)	10	41.76	(1.12)		∞	60.92	(4.61)	10	43.34	(1.50)	
	$\frac{9}{80}$	$\frac{9}{40}$	0	118.33		0	118.33		66.07	0	61.48		0	61.48		28.69
			2	104.57	(0.86)	1	109.74	(0.03)		3	50.06	(2.04)	1	57.18	(0.03)	
			4	98.47	(1.35)	2	101.43	(0.09)		6	44.59	(3.27)	2	53.09	(0.10)	
			6	95.14	(1.65)	3	93.29	(0.18)		9	41.58	(4.06)	3	49.17	(0.20)	
			∞	90.13	(2.19)	7	64.30	(0.84)		∞	36.08	(5.80)	10	26.03	(1.64)	

Table 2: Average delivery times in hours for the OVT-model, PTS-model and EXH-model, and between brackets the average number of hours working overtime for the OVT-model and the average number of jobs on stock for the PTS-model.

In Table 3, we show results for the percentiles of the delivery-time distribution, i.e., the smallest value of the delivery time such that a fraction α of the orders have a delivery time less than or equal to this value. Hence, the values in this table can be interpreted as the delivery time given by the production center independent of the state of this center upon arrival such that at least a fraction α of the orders are handled before this time.

Again, if the work loads are low, producing to stock is more effective than working overtime. Furthermore, the improvement of the performance by producing one product in overtime almost equals the maximum improvement that can be obtained by working overtime ($d_i = \infty$).

For high work loads, similar remarks as for Table 2 can be made as well. However, although the average delivery time for the PTS-model is lower than for the EXH-model in the last row of $\rho = 0.90$, this does not imply that the fraction of customers served in time for a given delivery time is also less than for the EXH-model. Of course, if the maximum number of products in stock increases, the PTS-model will outperform the EXH-model with respect to this performance measure as well.

OVT-model												
			Type 1 product					Type 2 product				
ρ	p_1	p_2	d_1	α: 0.70	0.80	0.90	0.95	d_2	α: 0.70	0.80	0.90	0.95
0.50	$\frac{1}{10}$	$\frac{1}{10}$	0	28	31	35	39	0	24	28	30	35
			2	25	29	33	35	3	21	24	28	29
			4	25	28	32	34	6	20	24	27	29
			∞	24	28	32	34	∞	20	23	27	28
0.75	$\frac{3}{20}$	$\frac{3}{20}$	0	52	59	68	80	0	42	47	58	70
			2	47	54	60	65	3	37	42	47	55
			4	45	52	58	61	6	35	40	45	48
			∞	43	50	56	59	∞	33	38	43	46
0.90	$\frac{9}{50}$	$\frac{9}{50}$	0	124	139	165	192	0	101	119	151	182
			2	113	127	142	163	3	88	98	119	141
			4	108	122	137	147	6	81	92	103	122
			6	105	119	134	141	9	78	88	99	109
			∞	99	114	128	136	∞	71	82	92	97

PTS-model												
			Type 1 product					Type 2 product				
ρ	p_1	p_2	S_1	α: 0.70	0.80	0.90	0.95	S_2	α: 0.70	0.80	0.90	0.95
0.50	$\frac{1}{10}$	$\frac{1}{10}$	0	28	31	35	39	0	24	28	30	35
			1	18	23	28	32	1	14	18	24	28
			2	9	15	21	26	2	5	10	16	21
			3	0	6	14	20	3	0	2	9	15
0.75	$\frac{3}{20}$	$\frac{3}{20}$	0	52	59	68	80	0	42	47	58	70
			1	45	52	61	73	1	34	41	50	63
			2	38	46	56	67	2	28	35	44	56
			4	27	35	46	57	3	22	29	39	51
0.90	$\frac{9}{50}$	$\frac{9}{50}$	0	124	139	165	192	0	101	119	151	182
			1	117	133	158	185	1	94	112	144	175
			2	112	127	153	180	2	89	106	138	170
			3	106	122	148	175	3	84	101	133	164
			10	69	86	112	139	10	48	66	98	129

EXH-model										
			Type 1 product				Type 2 product			
ρ	p_1	p_2	α: 0.70	0.80	0.90	0.95	α: 0.70	0.80	0.90	0.95
0.50	$\frac{1}{10}$	$\frac{1}{10}$	15	18	22	26	14	16	19	22
0.75	$\frac{3}{20}$	$\frac{3}{20}$	27	34	43	53	23	27	33	39
0.90	$\frac{9}{50}$	$\frac{9}{50}$	65	81	106	130	47	58	74	90

Table 3: Percentiles for the delivery-time distribution of orders in hours for the OVT-model, the PTS-model, and the EXH-model.

5 Conclusions

In this paper, we have considered a discrete-time queueing system with a single server which serves two types of jobs. Orders for these jobs arrive according to a Bernoulli process with deterministic service times. The server switches at predetermined and periodic time epochs from serving one type of order to another, and switching takes a fixed amount of time. In addition, we considered the opportunity of working overtime and the opportunity of producing to stock.

We have presented a technique for efficiently determining the stationary queue-length distribution for both types of customers. From this distribution, we computed the average delivery time, the average amount of overtime, the average number of finished jobs on stock, and the fraction of orders that is handled in time.

If the load of the system is low, allowing only a few products on stock already yields a better performance than the exhaustive service policy. For these loads of the system, the effect of working overtime is small. If the load of the system is high, much more products have to be allowed on stock for outperforming the exhaustive service policy. However, the average number of finished products on stock is rather small. As expected, for these loads of the system, the effect of working overtime is more apparent than for low loads of the system.

The technique for efficiently determining the stationary queue-length distribution can easily be extended to the case with more than two types of jobs. Furthermore, we can extend this technique to service-time distributions which are phase-type, like the geometric and negative binomial distribution.

References

[1] M.J.A. van Eenige, I.J.B.F. Adan, J.A.C. Resing, & J. van der Wal (1995). Periodic versus exhaustive service in a multi-product production center, Memorandum COSOR 95-01, Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven, The Netherlands.

[2] M.F. Neuts (1981). *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, The Johns Hopkins University Press, Baltimore.

[3] H.C. Tijms & M.C.T. van de Coevering (1991). A simple numerical approach for infinite-state Markov chains, *Probability in the Engineering and Informational Sciences* **5**, pp. 285-295.