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by

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We define nonconforming finite elements of arbitrary order $k \in \mathbf{N}$ over triangles, generalizing the well-known Crouzeix-Raviart element when $k = 1$. To date, finite elements of this sort have only been discussed when $k = 2$ and $k = 3$ in the literature. The elements of this paper satisfy the patch test and, with the right formulation, they provide discretizations with consistency error of the same magnitude as the interpolation error.

1. INTRODUCTION

This paper is devoted to the construction of nonconforming triangular finite elements of arbitrary order $k \in \mathbf{N}$, generalizing the well-known Crouzeix-Raviart [3] element when $k = 1$, with nodes at the middle of the edges. Curiously, even though a cubic element (enriched by three quartics) was already described in [3], the construction of other higher order nonconforming triangular elements has remained very limited. We can only refer to Fortin and Soulie [7] (quadratic + quartic) (see also Fortin [5] for tetrahedra) Farhloul and Fortin [6] (elaborating on [7]) and Knobloch and Tobiska [10] (linear + cubic). Apparently, in spite of some other constructions over rectangles or parallelepipeds, no successful attempt has been made to construct general families.

Originally, nonconforming triangular elements were found to be particularly attractive in the numerical treatment of the Stokes and Navier-Stokes systems, but they have since been used in various other problems; see for

instance Hua and Thomasset [8] (shallow water equations), Arnold and Falk [1] (plate problems), John, Maubach and Tobiska [9] and Matthies and Tobiska [13] (convection-diffusion problems), among others. Furthermore, because of the limited interaction of their basis functions, the linear Crouzeix-Raviart elements have major advantages over their conforming counterparts in devising massively parallel algorithms ([11], [12]). The higher order elements of this paper preserve all the properties relevant to parallelism.

Recall that with nonconforming elements, the error between the continuous and numerical solutions of a variational problem depends upon more than just the local interpolation error and the “nondegeneracy” of the mesh, as it does in the conforming case. In practice, the discontinuities across the inter-element boundaries require calculating the numerical solution by using a discretized variational formulation, different from the formulation of the continuous problem. This introduces a “consistency” error between the two formulations, which must be accounted for in the final error estimate (see Ciarlet [2]). In turn, the consistency error often depends upon the validity of a so-called “patch-test” demanding, for elements of order k , that the jumps across the edges are L^2 -orthogonal to the polynomials of degree up to $k - 1$ along those edges.

While empirical, this patch-test criterion highlights a problem-independent feature that a nonconforming element of order k should possess to be of any broad value. Of course, in a given problem, it should not be taken for granted that the patch test suffices to guarantee the desired consistency error, which also depends upon the choice of the discretized variational formulation and thus must be checked on a case by case basis. In other words, the patch test is a virtually necessary but not always sufficient condition for the existence of discretized variational formulations producing consistency errors comparable with the interpolation error.

Let P_k denote the space of real-valued polynomials on \mathbf{R}^2 . Recall that a set $S \subset \mathbf{R}^2$ is said to be P_k -unisolvent if there is one and only one $p \in P_k$ assuming arbitrarily prescribed values at all points of S . It is well known and trivial that if S is P_k -unisolvent, then $\#S = \dim P_k = \frac{(k+2)(k+1)}{2}$. It is equally trivial that the converse is not true and hence that P_k -unisolvence embodies a geometric property of the set S . Incidentally, this geometric property is by no means self-evident if $k > 1$: A simple glance at a set S of $\frac{(k+2)(k+1)}{2}$ points in \mathbf{R}^2 will generally not reveal whether or not S is P_k -unisolvent.

We now briefly explain our construction of higher-order nonconforming elements. We assume that the given triangular mesh possesses the usual regularity properties involved in finite element discussions. Also, we confine attention to the case when the basis functions of the finite element space

are characterized by nodes in the simplest possible way: Each node defines a unique basis function whose value is 1 at that node and 0 at the others.

The first remark is that, for a piecewise P_k nonconforming finite element space on the triangular mesh, the patch test holds if the nodes lying on the boundary of the elements are *exactly* the Gauss points of order k on each edge. Indeed, the jump across an edge e of a function in the finite element space is a polynomial of degree at most k on e vanishing at the Gauss points of e and hence L^2 -orthogonal to all the polynomials of degree less than k on e . (For the definition and the classical properties of the Gauss points used in this paper, we refer to Engels [4], Stroud and Secrest [14] or Szegő [15].)

The above shows how to specify $3k$ nodes on the boundary of each element T of the triangulation by choosing the Gauss points of order k on each edge. Now, since $\dim P_k - 3k = \frac{(k-1)(k-2)}{2} \geq 0$, the question arises whether these $3k$ nodes can be complemented by a set $\Sigma(T)$ of $\frac{(k-1)(k-2)}{2}$ nodes in $\overset{\circ}{T}$ in such a way that the full set $S(T)$ of nodes is P_k -unisolvent. As is well known, P_k -unisolvence is crucial to obtain optimal interpolation error ([2]). Incidentally, it will be of importance that $\frac{(k-1)(k-2)}{2}$ is exactly the dimension $\dim P_{k-3}$, where $P_{-2} = P_{-1} := \{0\}$ for consistency.

The answer to the above question is obviously yes if $k = 1$ and (a little less obviously) no if $k = 2$. More generally, when the answer is positive for all the triangles T of the triangulation, then the basis functions associated with the set of nodes $\cup_T S(T)$ generate a nonconforming finite element space of order k that passes the patch test.

The key property used in our construction and expressed by a more general criterion than just needed above (Theorem 2.1) is that the answer is always positive if k is odd and always negative if k is even (Corollary 2.1). Furthermore, when k is odd, a suitable set of nodes $\Sigma(T)$ can be obtained by affine equivalence, that is, by affine transformation of the problem to a single reference triangle.

When k is even, the fact that the answer is negative does not mean that nonconforming elements of order k do not exist in this case, but the construction becomes more subtle. In Section 3, we show how a nonconforming space of even order k can be constructed after enriching the space P_k with three suitable polynomials of P_{k+1} .

As an example, we use these new elements in Section 4 to discretize a general second order elliptic problem with Dirichlet boundary condition. The salient point (Theorem 4.1) is that the procedure yields a consistency error of the same magnitude as the interpolation error. This has been corroborated by several numerical tests in special cases, although the detail of these numerical experiments is not reproduced here.

2. A CONSTRUCTION OF UNISOLVENT SETS

Motivated by the discussion in the Introduction, we consider a (nondegenerate) triangle $T \subset \mathbf{R}^2$ with vertices a_i and for each $1 \leq i \leq 3$ a set $\{\xi_{i1}, \dots, \xi_{ik}\}$ of $k \geq 1$ distinct points on the edge e_i of T opposite a_i . In what follows, the points ξ_{ij} are assumed to lie on the relative interior $\overset{\circ}{e}_i$ of e_i , that is, $\xi_{ij} \neq a_\ell$ for $1 \leq i, \ell \leq 3$ and $1 \leq j \leq k$.

In Theorem 2.1 below, we give a very simple necessary and sufficient condition for the set $\Theta_k(\partial T) := \{\xi_{ij} : 1 \leq i \leq 3, 1 \leq j \leq k\}$ to be contained in a P_k -unisolvent subset of \mathbf{R}^2 .

THEOREM 2.1. *For $1 \leq i \leq 3$, let π_i be a nonzero polynomial of degree k on e_i vanishing at the points ξ_{ij} , $1 \leq j \leq k$, so that π_i is unique to within a nonzero constant multiple.*

(i) The set $\Theta_k(\partial T) := \{\xi_{ij} : 1 \leq i \leq 3, 1 \leq j \leq k\}$ is contained in a P_k -unisolvent subset of \mathbf{R}^2 if and only if¹

$$\pi_1(a_2)\pi_2(a_3)\pi_3(a_1) \neq \pi_1(a_3)\pi_2(a_1)\pi_3(a_2). \quad (1)$$

(ii) More precisely, if (1) holds and $\Sigma_{k-3}(\overset{\circ}{T}) \subset \overset{\circ}{T}$ is any P_{k-3} -unisolvent subset², then $\Sigma_{k-3}(\circ T) \cup \Theta_k(\partial T)$ is P_k -unisolvent.

Proof. Clearly, the hypothesis and the conclusion of the theorem are unaffected by affine change of coordinates. Thus, we henceforth assume that T is the “reference” triangle with vertices

$$a_1 = (0, 1), a_2 = (1, 0), a_3 = (0, 0). \quad (2)$$

If so, the points ξ_{ij} have the form

$$\xi_{1j} = (x_j, 0), \xi_{2j} = (0, y_j), \xi_{3j} = (z_j, 1 - z_j), \quad (3)$$

with $x_j, y_j, z_j \in (0, 1)$ and $1 \leq j \leq k$. Also, the polynomials π_1, π_2 and π_3 become polynomials p_1, p_2 and p_3 , respectively, of degree k in one variable and given by

$$p_1(x) := \pi_1(x, 0), p_2(y) = \pi_2(0, y), p_3(z) = \pi_3(z, 1 - z) \quad (4)$$

and hence satisfying

$$p_1(x_j) = p_2(y_j) = p_3(z_j) = 0, \quad 1 \leq j \leq k. \quad (5)$$

¹This condition is obviously unchanged if π_i is replaced by a nonzero constant multiple.

²With once again $P_{-2} = P_{-1} := \{0\}$.

With this, condition (1) becomes

$$p_1(1)p_2(0)p_3(0) \neq p_1(0)p_2(1)p_3(1). \quad (6)$$

For future use, observe that since p_1, p_2 and p_3 are nonzero polynomials of degree k vanishing at k points of $(0, 1)$, we have

$$p_i(0) \neq 0, \quad p_i(1) \neq 0, \quad 1 \leq i \leq 3. \quad (7)$$

The main part of the proof consists in finding a suitable expression for a polynomial $f \in P_k$ vanishing at all the points ξ_{ij} in (3). Specifically, we claim that if (6) holds and $f \in P_k$ satisfies $f(\xi_{ij}) = 0$ for $1 \leq i \leq 3$ and $1 \leq j \leq k$, (i.e. f vanishes on $\Theta_k(\partial T)$), then

$$f(x, y) = xy(x + y - 1)r(x, y), \quad (8)$$

for some polynomial $r \in P_{k-3}$ (hence $r = 0$ if $k = 1$ or 2 since $P_{-2} = P_{-1} = \{0\}$).

To see this, we begin with the remark that since f has degree at most k and $f(x_j, 0) = 0$, $1 \leq j \leq k$, there is a constant $\alpha_1 \in \mathbf{R}$ such that $f(x, 0) = \alpha_1 p_1(x)$ for all $x \in \mathbf{R}$. Hence,

$$f(x, y) = \alpha_1 p_1(x) + yg(x, y), \quad (9)$$

for some $g \in P_{k-1}$.

Next, by using $f(0, y_j) = 0$, $1 \leq j \leq k$, and by the same argument as above, there is a constant $\alpha_2 \in \mathbf{R}$ such that $f(0, y) = \alpha_2 p_2(y)$ for all $y \in \mathbf{R}$. Thus, by letting $x = 0$ in (9), we obtain

$$yg(0, y) = \alpha_2 p_2(y) - \alpha_1 p_1(0) \quad (10)$$

and, in particular,

$$\alpha_2 p_2(0) - \alpha_1 p_1(0) = 0. \quad (11)$$

Since $g(x, y) = g(0, y) + xh(x, y)$ with $h \in P_{k-2}$, it follows from (9) and (10) that

$$f(x, y) = \alpha_1 p_1(x) + \alpha_2 p_2(y) - \alpha_1 p_1(0) + xyh(x, y). \quad (12)$$

By writing $h(x, y) = h(x, 1 - x) + (x + y - 1)r(x, y)$ for some $r \in P_{k-3}$, this becomes

$$f(x, y) = \alpha_1 p_1(x) + \alpha_2 p_2(y) - \alpha_1 p_1(0) + xyh(x, 1 - x) + xy(x + y - 1)r(x, y). \quad (13)$$

Now, we use $f(z_j, 1 - z_j) = 0$ for $1 \leq j \leq k$ to get $f(x, 1 - x) = \alpha_3 p_3(x)$ for some constant $\alpha_3 \in \mathbf{R}$ and all $x \in \mathbf{R}$. Then, by (12),

$$\alpha_3 p_3(x) = \alpha_1 p_1(x) + \alpha_2 p_2(1 - x) - \alpha_1 p_1(0) + x(1 - x)h(x, 1 - x), \quad (14)$$

whence

$$\alpha_3 p_3(0) = \alpha_2 p_2(1) \quad (15)$$

and, for $x \in \mathbf{R}, x \neq 1$,

$$xh(x, 1 - x) = \frac{1}{1 - x} (\alpha_3 p_3(x) - \alpha_1 p_1(x) - \alpha_2 p_2(1 - x) + \alpha_1 p_1(0)) \quad (16)$$

By substitution into (13), we obtain, for $x \neq 1$,

$$\begin{aligned} f(x, y) &= \alpha_1 p_1(x) + \alpha_2 p_2(y) - \alpha_1 p_1(0) + \\ &\frac{y}{1-x} (\alpha_3 p_3(x) - \alpha_1 p_1(x) - \alpha_2 p_2(1 - x) + \alpha_1 p_1(0)) + \quad (17) \\ &xy(x + y - 1)r(x, y). \end{aligned}$$

By (11) and (15) and owing to (7), we find $\alpha_2 = \alpha_1 \frac{p_1(0)}{p_2(0)}$ and $\alpha_3 = \alpha_2 \frac{p_2(1)}{p_3(0)} = \alpha_1 \frac{p_1(0)p_2(1)}{p_2(0)p_3(0)}$. Thus, if $x \neq 1$, it follows from (17) that

$$\begin{aligned} f(x, y) &= \alpha_1 \left(p_1(x) + \frac{p_1(0)}{p_2(0)} p_2(y) - p_1(0) \right) + \\ &\frac{\alpha_1 y}{(1-x)} \left[\frac{p_1(0)p_2(1)}{p_2(0)p_3(0)} p_3(x) - p_1(x) - \frac{p_1(0)}{p_2(0)} p_2(1 - x) + p_1(0) \right] + \quad (18) \\ &xy(x + y - 1)r(x, y). \end{aligned}$$

Since f is a polynomial, the above equality can hold only if $\alpha_1 = 0$ or if the bracketed term in the right-hand side vanishes at $x = 1$. But the latter happens if and only if $p_1(0)p_2(1)p_3(1) - p_1(1)p_2(0)p_3(0) = 0$, in contradiction with (6). Thus, $\alpha_1 = 0$ if (6) holds and, if so, (18) reduces to the desired form (8).

With (8) now established (when (6) holds) for every $f \in P_k$ vanishing on $\Theta_k(\partial T)$, let $\Sigma_{k-3}(\overset{\circ}{T}) \subset \overset{\circ}{T}$ be any P_{k-3} -unisolvent set. If $f \in P_k$ vanishes on $\Sigma_{k-3}(\overset{\circ}{T}) \cup \Theta_k(\partial T)$, then (8) holds and $r \in P_{k-3}$ vanishes on $\Sigma_{k-3}(\overset{\circ}{T})$ since the product $xy(x + y - 1)$ is everywhere nonzero in $\overset{\circ}{T}$. Thus, $r = 0$ since $\Sigma_{k-3}(\overset{\circ}{T})$ is P_{k-3} -unisolvent, whence $f = 0$. Since $\#\Sigma_{k-3}(\overset{\circ}{T}) \cup \Theta_k(\partial T) = \dim P_k$, it follows that $\Sigma_{k-3}(\overset{\circ}{T}) \cup \Theta_k(\partial T)$ is P_k -unisolvent. This proves (ii) of the theorem and also the ‘‘if’’ part in (i).

To complete the proof, we establish the ‘‘only if’’ part in (i). Suppose that (6) does not hold, so that $p_1(1)p_2(0)p_3(0) = p_1(0)p_2(1)p_3(1)$ and hence

the right-hand side of (18) defines a polynomial $f \in P_k$ for every choice of $\alpha_1 \in \mathbf{R}$ and $r \in P_{k-3}$. Furthermore, f vanishes at all the points ξ_{ij} , i.e. at the points of $\Theta_k(\partial T)$.

By contradiction, suppose that there is a P_k -unisolvent set S_k containing $\Theta_k(\partial T)$. The set $S_k \setminus \Theta_k(\partial T)$ has $\frac{(k-1)(k-2)}{2}$ elements and f above depends linearly upon $\alpha_1 \in \mathbf{R}$ and $r \in P_{k-3}$ and hence upon $\frac{(k-1)(k-2)}{2} + 1$ real variables. As a result, there is at least one pair $(\alpha_1, r) \neq (0, 0)$ yielding an f in (18) vanishing at all the points of $S_k \setminus \Theta_k(\partial T)$ and hence at all the points of S_k . But, as we shall verify below, $f \neq 0$, which contradicts the hypothesis that S_k is P_k -unisolvent.

To see that indeed f in (18) is nonzero whenever $(\alpha_1, r) \neq (0, 0)$, suppose first that $\alpha_1 = 0$. Then, $r \neq 0$ and hence $f(x, y) = xy(x+y-1)r(x, y)$ is not the 0 polynomial. If now $\alpha_1 \neq 0$, then, $f(0, 0) = \alpha_1 p_1(0) \neq 0$ by (7), so that once again $f \neq 0$. This completes the proof. ■

COROLLARY 2.1. *For $1 \leq i \leq 3$, let ξ_{ij} denote the Gauss points of order k on e_i . The set $\Theta_k(\partial T) := \{\xi_{ij} : 1 \leq i \leq 3, 1 \leq j \leq k\}$ is contained in a P_k -unisolvent subset of \mathbf{R}^2 if and only if k is odd. More precisely, if $\Sigma_{k-3}(\overset{\circ}{T}) \subset \overset{\circ}{T}$ is any P_{k-3} -unisolvent subset such that $\Sigma_{k-3}(\overset{\circ}{T}) \cap \partial T = \emptyset$, then $\Sigma_{k-3}(\overset{\circ}{T}) \cup \Theta_k(\partial T)$ is P_k -unisolvent.*

Proof. The Gauss points of order k on e_i are obtained from the corresponding Gauss points on $[-1, 1]$ by an affine transformation and hence the polynomials π_1, π_2 and π_3 are all obtained from the k^{th} Legendre polynomial via a linear change of variable.

Since the k^{th} Legendre polynomial is odd (resp. even) when k is odd (resp. even) and does not vanish at ∓ 1 , it follows that $\pi_1(a_2) = -\pi_1(a_3) \neq 0$, $\pi_2(a_1) = -\pi_2(a_3) \neq 0$ and $\pi_3(a_2) = -\pi_3(a_1) \neq 0$ if k is odd while $\pi_1(a_2) = \pi_1(a_3)$, $\pi_2(a_1) = \pi_2(a_3)$ and $\pi_3(a_2) = \pi_3(a_1)$ if k is even. Thus, $\pi_1(a_2)\pi_2(a_3)\pi_3(a_1) = -\pi_1(a_3)\pi_2(a_1)\pi_3(a_2) \neq \pi_1(a_3)\pi_2(a_1)\pi_3(a_2)$ in the first case and $\pi_1(a_2)\pi_2(a_3)\pi_3(a_1) = \pi_1(a_3)\pi_2(a_1)\pi_3(a_2)$ in the second. The conclusion now follows from Theorem 2.1. ■

3. NONCONFORMING FINITE ELEMENTS OF ODD AND EVEN ORDER

Corollary 2.1 makes it obvious how to construct a piecewise P_k non-conforming finite element space passing the patch test when k is an odd integer: Given a regular triangulation \mathcal{T} of some polygonal domain $\Omega \subset \mathbf{R}^2$, choose the boundary nodes to coincide with the Gauss points of order k along all the edges of the triangles and complement this set by choosing

a P_{k-3} -unisolvent subset $\Sigma_{k-3}(\overset{\circ}{T}) \subset \overset{\circ}{T}$ for every triangle $T \in \mathcal{T}$. Then, by Corollary 2.1 the set $S_k(T)$ of nodes in T (that is, $\Sigma_{k-3}(\overset{\circ}{T})$ plus the Gauss points on the edges of T) is P_k -unisolvent. The desired nonconforming space $V_{\mathcal{T}}$ is the space generated by the basis functions (piecewise P_k functions vanishing at all but one node of $\cup_{T \in \mathcal{T}} S_k(T)$).

Note that, if desired, $S_k(T)$ and the restrictions of the basis functions to T can be obtained from a single reference triangle via affine transformation since such a transformation preserves Gauss points on the edges.

Remark 3. 1. When $k = 1$, our construction gives again the Crouzeix-Raviart linear element. When $k = 3$, our nonconforming cubic element is fundamentally different from the one constructed by Crouzeix and Raviart in [3]: The latter has 12 degrees of freedom (instead of 10 in our approach) and requires enriching the pace P_3 with three quartic polynomials.

Corollary 2.1 also shows that the same procedure cannot be used when k is even. For completeness, we now describe a modification of the approach yielding a nonconforming finite element space of even order k .

LEMMA 3.1. *Let $k \geq 2$ be an even integer and let $-1 < \omega_1 < \dots < \omega_{k+1} < 1$ be the Gauss points of order $k+1$ in $[-1, 1]$, so that $\omega_{k+2-j} = -\omega_j$, $1 \leq j \leq k+1$. Let λ_{k+1} denote the corresponding (normalized) Legendre polynomial, that is, $\lambda_{k+1}(x) = \prod_{j=1}^{k+1} (x - \omega_j)$ and set $q_1(x) = \prod_{j=1}^k (x - \omega_j) = \frac{\lambda_{k+1}(x)}{(x - \omega_{k+1})}$ and $q_2(x) = \prod_{j=2}^{k+1} (x - \omega_j) = \frac{\lambda_{k+1}(x)}{(x - \omega_1)}$. Then, for every choice of $\ell_i \in \{1, 2\}$, $1 \leq i \leq 3$, we have $q_{\ell_1}(-1)q_{\ell_2}(-1)q_{\ell_3}(-1) \neq q_{\ell_1}(1)q_{\ell_2}(1)q_{\ell_3}(1)$.*

Proof. By symmetry, it suffices to consider the case $\ell_1 = \ell_2 = \ell_3 = 1$ and the case $\ell_1 = \ell_2 = 1, \ell_3 = 2$.

Case $\ell_1 = \ell_2 = \ell_3 = 1$. Since $k+1$ is odd, $\lambda_{k+1}(-1) = -\lambda_{k+1}(1)$ and the desired result is equivalent to $1 + \omega_{k+1} \neq 1 - \omega_{k+1}$, i.e., to $\omega_{k+1} \neq 0$. But if $\omega_{k+1} = 0$, then also $\omega_1 = -\omega_{k+1} = 0$, so that $k+1 = 1$, i.e., $k = 0$, which contradicts $k \geq 2$.

Case $\ell_1 = \ell_2 = 1, \ell_3 = 2$. Since $k+1$ is odd, $\lambda_{k+1}(-1) = -\lambda_{k+1}(1)$ and the desired result is equivalent to $(1 + \omega_{k+1})^2(1 + \omega_1) \neq (1 - \omega_{k+1})^2(1 - \omega_1)$. But since $1 - \omega_1 = 1 + \omega_{k+1}$ and $1 + \omega_1 = 1 - \omega_{k+1}$, this reduces once again to the condition $\omega_{k+1} \neq 0$ established above. \blacksquare

As in the previous section, let $T \subset \mathbf{R}^2$ be a triangle with vertices a_i and let e_i denote the edge of T opposite a_i , $1 \leq i \leq 3$. Given an even integer $k \geq 2$, choose a *consecutive* numbering of the Gauss points ξ_{ij} , $1 \leq j \leq k+1$ of order $k+1$ on e . In particular, the set $\{\xi_{i1}, \xi_{ik+1}\}$ of the first

and last Gauss points on e_i are unambiguously defined irrespective of the consecutive numbering (although which one is the first point depends of course upon the ordering). For $1 \leq i \leq 3$, we let γ_i denote either ξ_{i1} or ξ_{ik+1} .

Since $k + 1$ is odd, it follows from Corollary 2.1 that the set

$$\Theta_{k+1}(\partial T) := \{\xi_{ij} : 1 \leq i \leq 3, 1 \leq j \leq k + 1\}, \quad (19)$$

can be complemented by a P_{k-2} -unisolvent subset $\Sigma_{k-2}(\overset{\circ}{\circ}T) \subset \overset{\circ}{\circ}T$ in such a way that

$$S_{k+1}(T) := \Sigma_{k-2}(\overset{\circ}{\circ}T) \cup \Theta_{k+1}(\partial T) \quad (20)$$

is P_{k+1} -unisolvent. From now on, we denote by $\phi_i \in P_{k+1}$, $1 \leq i \leq 3$, the corresponding basis function associated with the node γ_i ($= \xi_{i1}$ or ξ_{ik+1}) and the P_{k+1} -unisolvent set $S_{k+1}(T)$ in (20). In other words,

$$\phi_i(\gamma_i) = 1 \text{ and } \phi_i = 0 \text{ on } S_{k+1}(T) \setminus \{\gamma_i\}. \quad (21)$$

A nonzero polynomial π_i of degree k on the edge e_i that vanishes at the k points ξ_{ij} for $1 \leq j \leq k + 1$ and either $j \neq 1$ or $j \neq k + 1$ can be obtained from the polynomial q_1 or q_2 of Lemma 3.1 by a linear change of variable. As a result, we may arrange things so that $\pi_1(a_2) = q_{\ell_1}(-1)$ and $\pi_1(a_3) = q_{\ell_1}(1)$, that $\pi_2(a_3) = q_{\ell_2}(-1)$ and $\pi_2(a_1) = q_{\ell_2}(1)$ and that $\pi_3(a_1) = q_{\ell_3}(-1)$ and $\pi_3(a_3) = q_{\ell_3}(1)$ for some integers $\ell_1, \ell_2, \ell_3 \in \{1, 2\}$. (The choice $\ell_i = 1$ or $\ell_i = 2$ depends upon whether the Gauss point ξ_{ij} not included corresponds to $j = 1$ or to $j = k + 1$.)

From Lemma 3.1, $\pi_1(a_2)\pi_2(a_3)\pi_3(a_1) = q_{\ell_1}(-1)q_{\ell_2}(-1)q_{\ell_3}(-1) \neq q_{\ell_1}(1)q_{\ell_2}(1)q_{\ell_3}(1) = \pi_1(a_3)\pi_2(a_1)\pi_3(a_2)$ and hence, it follows from Theorem 2.1 that, if

$$\Theta_k(\partial T) := \{\xi_{ij} : 1 \leq i \leq 3, 1 \leq j \leq k\} \quad (22)$$

and if $\Sigma_{k-3}(\overset{\circ}{\circ}T) \subset \overset{\circ}{\circ}T$ is a P_{k-3} -unisolvent subset, then

$$S_k(T) := \Sigma_{k-3}(\overset{\circ}{\circ}T) \cup \Theta_k(\partial T) \quad (23)$$

is P_k -unisolvent. Since k and $k + 1$ have different parities, the definitions (22) and (23) are compatible with (19) and (20).

Observe that since $\Sigma_{k-2}(\overset{\circ}{\circ}T)$ is P_{k-2} -unisolvent, it contains at least one P_{k-3} -unisolvent subset (This easily follows from standard properties of

determinants.) Thus, we may and will henceforth assume that $\Sigma_{k-3}(\overset{\circ}{T}) \subset \Sigma_{k-2}(\overset{\circ}{T})$. If so, $S_k(T) \subset S_{k+1}(T) \setminus \{\gamma_i\}$ and it follows from (21) that

$$\phi_i = 0 \text{ on } S_k(T), 1 \leq i \leq 3. \quad (24)$$

As a result, every linear combination of ϕ_1, ϕ_2 and ϕ_3 vanishes on $S_k(T)$, so that $P_k \cap \text{span}\{\phi_1, \phi_2, \phi_3\} = \{0\}$ since $S_k(T)$ is P_k -unisolvent. Therefore, the space

$$\tilde{P}_k := P_k \oplus \text{span}\{\phi_1, \phi_2, \phi_3\} \quad (25)$$

has dimension $\dim \tilde{P}_k = \frac{(k+2)(k+1)}{2} + 3$ equal to the number of points in (compare with $S_k(T)$ in (23))

$$\tilde{S}_k(T) := \Sigma_{k-3}(\overset{\circ}{T}) \cup \Theta_{k+1}(\partial T). \quad (26)$$

LEMMA 3.2. *The set $\tilde{S}_k(T)$ is \tilde{P}_k -unisolvent.*

Proof. Since $\dim \tilde{P}_k = \#\tilde{S}_k(T)$, it suffices to show that if $f \in \tilde{P}_k$ and $f = 0$ on $\tilde{S}_k(T)$, then $f = 0$. Write $f = g + \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3$ with $g \in P_k$ and $\alpha_i \in \mathbf{R}$, $1 \leq i \leq 3$. It follows from (24) and the assumption that $f = 0$ on $\tilde{S}_k(T) \supset S_k(T)$ that g vanishes on $S_k(T)$. Thus, $g = 0$ since $S_k(T)$ is P_k -unisolvent and $f = \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3$. Now, $\phi_\ell(\gamma_i) = \delta_{\ell i}$ (Kronecker delta) for $1 \leq i, \ell \leq 3$ and $f(\gamma_i) = 0$ since $\gamma_i \in \tilde{S}_k(T)$, so that $\alpha_i = 0$, $1 \leq i \leq 3$. ■

It is now a simple matter to define a triangular nonconforming element of even order k -enriched by three polynomials of P_{k+1} on each triangle-based upon the above results. Given a regular triangulation \mathcal{T} of some polygonal domain Ω , plot all the Gauss points of order $k+1$ on the edges of the triangles. For each edge e , choose a Gauss point γ_e which is the first (or last) Gauss point on e relative to some orientation of e . Given any triangle $T \in \mathcal{T}$, the three points γ_e corresponding to the edges of T become the points γ_1, γ_2 and γ_3 above.

The construction of the nonconforming finite element space $V_{\mathcal{T}}$ can then proceed along the same line as in the case when k is odd, just using (see (25) and (26)) the polynomial space \tilde{P}_k and the associated set of nodes $\tilde{S}_k(T)$ over each triangle T instead of P_k and $S_k(T)$. Elementwise, i.e., restricted to an arbitrary triangle $T \in \mathcal{T}$, "the" finite element basis of the space $V_{\mathcal{T}}$ consists of the three polynomials ϕ_1, ϕ_2, ϕ_3 plus $\dim P_k$ polynomials in \tilde{P}_k - hence of degree $k+1$ in general - equal to one at a different point of $S_k(T)$ and 0 at the other points of $\tilde{S}_k(T)$.

Observe that $\tilde{S}_k(T)$ and the corresponding basis functions can be obtained from one of six reference triangles via an affine transformation. The six reference triangles correspond to the six possible diagrams for the location of the “special” Gauss points γ_1, γ_2 and γ_3 . However, it is readily checked that these six diagrams split in two affine equivalence classes, so that only two reference configurations instead of six need to be used in practice.

Remark 3. 2. If the mesh is two-colorable, then every triangle may be oriented (clockwise or counterclockwise) in such a way that any two adjacent triangles have opposite orientation. This produces an orientation of every edge e of the triangulation *independent* of the triangle T containing e . As a result, it may be decided that γ_e above is always the first (or always the last) Gauss point on e relative to this orientation. If so, it is straightforward to check that every triangle T along with the three selected Gauss points on its edges is affine equivalent to *the same* reference configuration independent of T .

Lastly, note that the above finite element space satisfies the desired patch test (and more). Indeed, the jump across an edge e of a function in the finite element space is a polynomial of degree at most $k+1$ on e and vanishes at the Gauss points of order $k+1$ on e . Therefore, this jump is orthogonal to the polynomials of degree at most k (not merely $k-1$) on e . Of course, since $\tilde{P}_k \supset P_k$, the local interpolation error is as good as with polynomials of degree k .

4. AN EXAMPLE

We now discuss the discretization of an elliptic problem using the non-conforming elements of the previous section. Specifically, we consider the problem of finding $u \in H_0^1(\Omega)$ solving

$$-\nabla \cdot (A\nabla u) + b \cdot \nabla u + cu = f \in L^2(\Omega), \quad (27)$$

where $\Omega \subset \mathbf{R}^2$ is a polygonal domain, $A = A(x)$ is a symmetric positive definite 2×2 matrix function, $b = b(x) \in \mathbf{R}^2$ and $c = c(x) \in \mathbf{R}$. We shall assume throughout that A and b are in $C^k(\bar{\Omega})$ for some integer $k \geq 1$, that $c \in L^\infty(\Omega)$ and that

$$c - \frac{1}{2} \nabla \cdot b \geq \alpha > 0 \text{ on } \bar{\Omega}, \quad (28)$$

where $\alpha > 0$ is a constant. Since $A(x)$ is symmetric positive definite for every $x \in \bar{\Omega}$, it is not restrictive to assume, after shrinking $\alpha > 0$ if necessary,

that

$$A(x)\eta \cdot \eta \geq \alpha|\eta|^2, \quad \forall x \in \bar{\Omega}, \forall \eta \in \mathbf{R}^2. \quad (29)$$

The weak formulation of (27) characterizes u as a solution to the variational equation

$$a(u, v) = \ell(v), \quad \forall v \in H_0^1(\Omega), \quad (30)$$

where

$$a(u, v) := \int_{\Omega} A \nabla u \cdot \nabla v + \frac{1}{2} \int_{\Omega} (vb \cdot \nabla u - ub \cdot \nabla v) + \int_{\Omega} (c - \frac{1}{2} \nabla \cdot b) uv \quad (31)$$

and

$$\ell(v) := \int_{\Omega} f v. \quad (32)$$

Observe that (31) is simply obtained by multiplying the left-hand side of (27) by v and by using $\int_{\Omega} vb \cdot \nabla u = -\int_{\Omega} ub \cdot \nabla v - \int_{\Omega} (\nabla \cdot b) uv$ to modify the first order term. While this “skew-symmetrization” is immaterial in the continuous problem and gives only a different formula for $a(u, v)$ without changing its value, it will be essential to derive a discretized variational formulation with a “good” consistency error.

By (28) and (29) and the Lax-Milgram theorem, the variational problem (30) has a unique solution $u \in H_0^1(\Omega)$.

Now, consider a regular triangulation \mathcal{T}_h of Ω depending upon the real parameter $h > 0$, representing the maximum diameter of the triangles. As is customary, we assume that the smallest angle in all the triangles of \mathcal{T}_h is bounded away from below by a constant independent of h .

Assume that k is odd and let V_h (rather than $V_{\mathcal{T}_h}$) denote the finite element space associated with \mathcal{T}_h at the beginning of Section 3. We call V_{h0} the subspace of V_h of those functions vanishing at the nodes lying on $\partial\Omega$ and equip V_h (and V_{h0}) with the norm

$$\|v_h\|_h := \left(\sum_{T \in \mathcal{T}_h} \|v_h\|_{1,T}^2 \right)^{\frac{1}{2}}. \quad (33)$$

In analogy with (31), we now introduce the bilinear form

$$a_h(u, v) := \sum_{T \in \mathcal{T}_h} \int_T A \nabla u \cdot \nabla v + \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_T (vb \cdot \nabla u - ub \cdot \nabla v) + \sum_{T \in \mathcal{T}_h} \int_T (c - \frac{1}{2} \nabla \cdot b) uv, \quad (34)$$

well defined on $(H_0^1(\Omega) + V_{h0})^2$ and satisfying

$$a_h(u, v) = a(u, v), \quad \forall u, v \in H_0^1(\Omega). \quad (35)$$

Furthermore, a_h is continuous and coercive on V_{h0} . More precisely,

$$|a_h(u_h, v_h)| \leq M \|u_h\|_h \|v_h\|_h, \quad \forall u_h, v_h \in V_{h0}, \quad (36)$$

for some constant $M > 0$ depending only upon A, b and c (but not h) and

$$a_h(u_h, u_h) \geq \alpha \|u_h\|_h^2, \quad \forall u_h \in V_{h0}. \quad (37)$$

Once again by the Lax-Milgram theorem (and since the linear form ℓ is well defined on V_{h0}), we obtain the existence and uniqueness of a solution $u_h \in V_{h0}$ of the problem

$$a_h(u_h, v_h) = \ell(v_h), \quad \forall v_h \in V_{h0}. \quad (38)$$

From [2, Theorem 4.2.2, p.210], there is a constant $C > 0$ independent of h such that

$$\|u - u_h\|_h \leq C \left(\inf_{v_h \in V_{h0}} \|u - v_h\|_h + \sup_{v_h \in V_{h0}} \frac{|a_h(u, v_h) - \ell(v_h)|}{\|v_h\|_h} \right). \quad (39)$$

Above, the term $\inf_{v_h \in V_{h0}} \|u - v_h\|_h$ is just the interpolation error. As in the conforming case and by the standard a priori estimates (see e.g. [2]), this term is $O(h^k)$ if $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ and $k \in \mathbb{N}$. The term $\sup_{v_h \in V_{h0}} \frac{|a_h(u, v_h) - \ell(v_h)|}{\|v_h\|_h}$ is the consistency error, typical of nonconforming methods, which we investigate below. The following lemma is useful in this investigation.

LEMMA 4.1. ([3, Lemma 3]) *Given an edge e of the triangulation \mathcal{T}_h , let Π_e^{k-1} denote the orthogonal projection from $L^2(e)$ onto the space of polynomials of degree at most $k-1$ on e . Then, there is a constant $C > 0$ independent of e and of h such that*

$$\int_e (w - \Pi_e^{k-1} w) v \leq Ch^k |w|_{k,T} |v|_{1,T}, \quad (40)$$

for every $T \in \mathcal{T}_h$ containing e , every $v \in H^1(T)$ and every $w \in H^k(T)$.

Although this will not be used, the parity of k is irrelevant in Lemma 4.1, which remains valid if $k \geq 2$ is even.

THEOREM 4.1. *If $u \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$ solves (27), there is a constant $C > 0$ such that*

$$\sup_{v_h \in V_{h0}} \frac{|a_h(u, v_h) - \ell(v_h)|}{\|v_h\|_h} \leq Ch^k. \quad (41)$$

Proof. Since $\ell(v_h) = \int_{\Omega} f v_h = \sum_{T \in \mathcal{T}_h} \int_T f v_h$ and u solves (27), an integration by parts yields

$$\begin{aligned} \ell(v_h) &= \sum_{T \in \mathcal{T}_h} \int_T (-\nabla \cdot (A \nabla u) v_h + v_h b \cdot \nabla u + c u v_h) = \\ &= - \sum_{T \in \mathcal{T}_h} \int_{\partial T} v_h A \nabla u \cdot n_T + \sum_{T \in \mathcal{T}_h} \int_T (A \nabla u \nabla v_h + v_h b \cdot \nabla u + c u v_h), \end{aligned}$$

where n_T denotes the outward unit normal vector along ∂T . The boundary terms are well defined since $u \in H^{k+1}(\Omega)$ and $k \geq 1$. From (34),

$$\begin{aligned} a_h(u, v_h) - \ell(v_h) &= \sum_{T \in \mathcal{T}_h} \int_{\partial T} v_h A \nabla u \cdot n_T \\ &\quad - \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_T (v_h b \cdot \nabla u + u b \cdot \nabla v_h + u v_h \nabla \cdot b). \end{aligned} \quad (42)$$

Now, $\int_T u b \cdot \nabla v_h = - \int_T (v_h b \cdot \nabla u + u v_h \nabla \cdot b) + \int_{\partial T} u v_h b \cdot n_T$, whence

$$a_h(u, v_h) - \ell(v_h) = \sum_{T \in \mathcal{T}_h} \int_{\partial T} v_h \left(A \nabla u - \frac{1}{2} u b \right) \cdot n_T. \quad (43)$$

The right-hand side of (43) can thus be written as a sum of integrals over the edges of the triangulation \mathcal{T}_h . Such an edge e is either common to exactly two triangles T_1 and T_2 or contained in the boundary $\partial\Omega$. In the first case, since $n_{T_2} = -n_{T_1}$, the total contribution of e to the right-hand side of (43) is

$$\int_e [v_h] \left(A \nabla u - \frac{1}{2} u b \right) \cdot n_{T_1}, \quad (44)$$

where $[v_h]$ denotes the jump of v_h across e (from T_1 to T_2). Since $[v_h]$ vanishes at the Gauss points of order k on e , it is L^2 -orthogonal to the polynomials of degree less than k on e and hence

$$\begin{aligned} \left| \int_e [v_h] w \right| &= \left| \int_e [v_h] (w - \Pi_e^{k-1} w) \right| \leq \\ &= \left| \int_e v_h|_{T_1} (w - \Pi_e^{k-1} w) \right| + \left| \int_e v_h|_{T_2} (w - \Pi_e^{k-1} w) \right|, \end{aligned} \quad (45)$$

for every $w \in L^2(e)$ and every $v_h \in V_{h0}$. In particular, it follows from Lemma 4.1 that there is a constant $C > 0$ independent of e and h such

that

$$\left| \int_e [v_h] w \right| \leq Ch^k (|w|_{k,T_1} |v_h|_{1,T_1} + |w|_{k,T_2} |v_h|_{1,T_2}), \quad (46)$$

for every $w \in H^k(\Omega)$ and every $v_h \in V_{h0}$. With the choice $w = (A\nabla u - \frac{1}{2}ub) \cdot n_{T_1}$ and since $n_{T_2} = -n_{T_1}$ and both A and b are in $C^k(\overline{\Omega})$, it is clear that $|w|_{k,T_i} \leq K \|u\|_{k+1,T_i}$ for $i = 1, 2$, where $K > 0$ is a constant depending only upon A, b and k (but independent of h and u). Thus, altogether, after modifying C in (46),

$$\left| \int_e [v_h] (A\nabla u - \frac{1}{2}ub) \cdot n_{T_1} \right| \leq Ch^k (\|u\|_{k+1,T_1} |v_h|_{1,T_1} + \|u\|_{k+1,T_2} |v_h|_{1,T_2}). \quad (47)$$

If now e is an edge contained in $\partial\Omega$, then e is contained in a single triangle $T \in \mathcal{T}_h$ and since v_h vanishes at the Gauss points of e (since $v_h \in V_{h0}$), similar arguments yield

$$\left| \int_e v_h \left(A\nabla u - \frac{1}{2}ub \right) \cdot n_T \right| \leq Ch^k \|u\|_{k+1,T} |v_h|_{1,T}. \quad (48)$$

Clearly, (43), (47) and (48) yield the desired inequality (41) after another modification of C . ■

Theorem 4.1 shows that the consistency error is of the same magnitude as the interpolation error and hence that the nonconforming nature of the finite element space does not alter the accuracy of the approximation.

Virtually nothing has to be changed if $k \in \mathbf{N}$ is even and V_h denotes the corresponding finite element space constructed in Section 3. In fact, since now both the jumps across the edges and the restrictions to $\partial\Omega$ of the functions in V_{h0} vanish at the Gauss points of order $k+1$ (instead of k), not only Theorem 4.1 remains valid but also $\sup_{v_h \in V_{h0}} \frac{|a_h(u, v_h) - \ell(v_h)|}{\|v_h\|_k} \leq Ch^{k+1}$ if $u \in H^{k+2}(\Omega)$. In other words, if $u \in H^{k+2}(\Omega)$, the consistency error is negligible compared with the (theoretical) interpolation error.

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