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System Analysis, Modelling and Control with Polytopic Linear Models

PROEFSCHRIFT

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Georgo Z. Angelis

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Contents

Summary

This research investigates the suitability of Polytopic Linear Models (PLMs) for the analysis, modelling and control of a class of nonlinear dynamical systems. The PLM structure is introduced as an approximate and alternative description of nonlinear dynamical systems for the benefit of system analysis and controller design. The model structure possesses three properties that we would like to exploit.

Firstly, a PLM is build upon a number of linear models, each one of which describes the system locally within a so-called operating regime. If these models are combined in an appropriate way, that is by taking operating point dependent convex combinations of parameter values that belong to the different linear models, then a PLM will result. Consequently, the parameter values of a PLM vary within a polytope, and the vertices of this polytope are the parameter values that belong to the different linear models. A PLM owes its name to this feature. Accordingly, a PLM can be interpreted on the basis of a regime decomposition. Secondly, since a PLM is based on several linear models, it is possible to describe the nonlinear system more globally compared to only a single linear model. Thirdly, it is demonstrated that, under the appropriate conditions, nonlinear systems can be approximated arbitrary close by a PLM, parametrized with a finite number of parameters. There will be given an upper bound for the number of required parameters, that is sufficient to achieve the prescribed desired accuracy of the approximation.

An important motivation for considering PLMs rests on its structural similarities with linear models. Linear systems are well understood, and the accompanying system and control theory is well developed. Whether or not the control related system properties such as stability, controllability etcetera, are fulfilled, can be demonstrated by means of (often relatively simple) mathematical manipulations on the linear system's parameterization. Controller design can often be automated and founded on the parameterization and the control objective. Think of control laws based on stability, optimality and so on. For nonlinear systems this is only partly the case, and therefore further development of system and control theory is of major importance. In view of the similarities between a linear model and a PLM, the expectation exists

that one can benefit from (results and concepts of) the well developed linear system and control theory. This hypothesis is partly confirmed by the results of this study.

Under the appropriate conditions, and through a simple analysis of the parametrization of a PLM, it is possible to establish from a control perspective relevant system properties. One of these properties is stability. Under the appropriate conditions stability of the PLM implies stability of the system. Moreover, a few easy to check conditions are derived concerning the notion of controllability and observability. It has to be noticed however, that these conditions apply to a class of PLMs of which the structure is further restricted.

The determination of system properties from a PLM is done with the intention to derive a suitable model, and in particular to design a model based controller. This study describes several constructive methods that aim at building a PLM representation of the real system.

On the basis of a PLM several control laws are formulated. The main objective of these control laws is to stabilize the system in a desired operating point. A few computerized stabilizing control designs, that additionally aim at optimality or robustness, are the outcome of this research.

The entire route of representing a system with an approximate PLM, subsequently analyzing the PLM, and finally controlling the system by a PLM based control design is illustrated by means of several examples. These examples include experimental as well as simulation studies, and nonlinear dynamic (mechanical) systems are the subject of research.

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Georgo Angelis
Eindhoven, November 2000.

Chapter 1

Introduction

This thesis describes a systematic and creative approach towards the analysis, modelling and control of a broad class of nonlinear systems. Polytopic linear models are introduced to represent the system under investigation, and the potential that these models offer for system analysis, modelling and control is explored in detail. In this introductory chapter we give a motivation for this thesis, outline the major themes to be explored and developed in subsequent chapters, and it describes how the study is organized.

1.1 Motivation

Mathematical models, abstractions of real life systems, are of great importance for the understanding of system behavior. From a control engineering point of view it is of great interest how to choose the system inputs, such that desired control objectives are met. The intended application of a model, in our case control, will impose restrictions on its structure and complexity. There is a trade-off, since models that can be analyzed and dealt with within a control scheme are often inaccurate, and models that reflect system behavior more accurately are often too complex to analyze and deal with within a control scheme.

Model paradigms

Models that relate observed variables (inputs and outputs) by means of a state-space description are common in systems and control theory. According to such a model, at each time instant, the state variables summarize all of the information needed in order to predict together with the future inputs, the future evolution of the system in time. The dynamics of a large class of nonlinear dynamical systems may be cast by a set of nonlinear first order

differential equations together with a nonlinear algebraic output equation as follows:

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{1.1}$$

with the state $x \in X \subseteq \mathbb{R}^n$, the manipulated input $u \in U \subseteq \mathbb{R}^m$, and the output $y \in Y \subseteq \mathbb{R}^p$.

It is clear that systems as described by (1.1) are more general than their linear time invariant (LTI) counterparts:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1.2}$$

However a large part of the control literature is devoted to linear systems and many linear system theoretic properties and control problems have been satisfactory dealt with in the literature [67], [41]. There are two main reasons for studying linear systems for the purpose of control. Firstly, linear systems are parametrized, and system properties can be revealed easily by analysis of its representation (A, B, C, D). Furthermore, for linear systems the available analysis tools enable strong results on system and control theoretic properties such as stability, controllability, observability, optimality, robustness etc.. Secondly, first order approximations are in many cases sufficient to characterize the local behavior of the nonlinear system. This means that often analysis based on linearizations reveals properties of the system locally, and designs based on linearizations often work locally for the original system [67]. This 'linearization principle'[1] restricts the applicability of the linear model since desirable and expected behavior of the system can only be guaranteed for operating conditions that are close to the point of linearization. This fact, together with the problem of characterizing system properties, for which the analysis based on its linearization fails, motivated researchers to study nonlinear models, with the intention to derive stronger results, confer [57], [65]. Much research effort is directed towards that goal, and since it is not just a matter of extending the linear theory, new concepts are needed.

Instead of studying the general nonlinear model (1.1) to enlarge the applicability compared to the linear model we will follow another approach based on the linearization principle mentioned above. The idea behind the approach is simple and several researchers and application oriented engineers have come up with similar ideas, confer [55], [34], [71].

[1] The linearization principle is made precise in [67]. Sufficient conditions for local stability, controllability and observability of the system (1.1) are derived based on the parametrization (A, B, C, D) of the linearization (1.2) of the system (1.1).

The basic idea is to linearize (1.1) at several specific operating conditions. The operating conditions are chosen in such a way that the corresponding linear models reflect the qualitatively different behavior in these operating regimes. The operating regimes cover the full operating space. To mimic the behavior of the general nonlinear model, the different linear models obtained for different operating conditions are scheduled in the operating-space, as behavior changes qualitatively under varying operating conditions. Scheduling in this case means forming convex combinations of locally valid linear models. In this way a more global model is obtained that approximately reflects the behavior of the general model for the full range of operation. The resulting model is represented by a finite number N_m of linear models $\{(A_i, B_i, a_i, C_i, D_i, c_i)\}$ together with corresponding scheduling functions $\{w_i(x,u)\}$ as follows

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{N_m} w_i(x,u)\{A_i x + B_i u + a_i\} \\ y &= \sum_{i=1}^{N_m} w_i(x,u)\{C_i x + D_i u + c_i\} \end{aligned} \tag{PLM}$$

with

$$\sum_{i=1}^{N_m} \omega_i(x,u) = 1 \text{ and } \omega_i(x,u) \geq 0 \text{ for all } (x,u) \in X \times U$$

These nonlinear models frequently occur in literature and although of an equivalent mathematical structure they are given different names such as Fuzzy Models [69], [86], Multi-Models [55] or Local Model Networks [38]. The model structure has several desirable attributes that we would like to exploit. First of all, the model class is rich since a large class of nonlinear systems can be approximated arbitrarily close with the proposed model structure [86], [38]. Secondly, the model is interpretable on the basis of a regime decomposition [69], [38]. Also the model has an a priori fixed structure, that shows similarities with linear systems. The obtained model is called a Polytopic Linear Model (PLM) since the model consists of a set of representations of linear models that define a polytope in the models parameter-space, which will become clear later on.

Trade-off

Especially within the area of systems and control engineering, the diversity of the intended model applications, such as system analysis, controller synthesis and simulation puts high demands on the model. As mentioned earlier, representing a system with an accurate mathematical description of its dynamical

properties that is also simple to use for the intended application at hand, results in conflicting demands. On one hand, general nonlinear models can be very accurate but as a result of their complexity difficult to build from first principles and data. Furthermore, these models are difficult to analyze and to apply in model based control schemes. So, these models are not simple. On the other hand LTI systems are well understood, but are at most locally valid in representing an actual system, i.e. incorrect conclusions may be drawn from it, resulting in an unsuccessful control strategy. In the ideal situation, one would like to build a model that is as accurate as the most detailed mathematical model and as simple as a LTI system in the sense as described before. This situation is sketched in Figure 1.1.

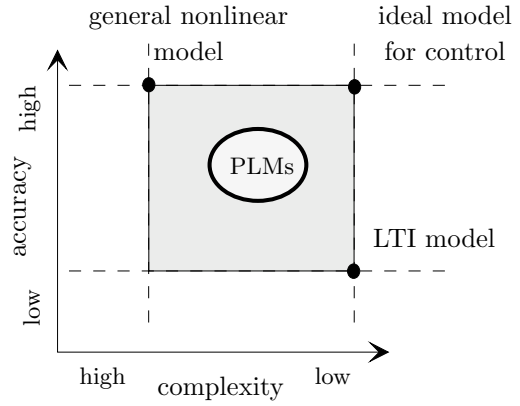


Figure 1.1: *The PLM is an alternative compromise between the conflicting demands, accuracy and complexity. Every candidate model is within the grey region. The best candidate model is the one closest to the ideal model.*

The ideal model as indicated in this figure does of course not exist. The choice for a general nonlinear model or a LTI model to represent the system for the intended application in mind are just two possibilities out of numerous possible compromises between complexity and accuracy. The PLM is an alternative compromise between these two conflicting demands that lies within the shaded region in Figure 1.1. The intended application of the model, in our case control, imposes demands on the model to be selected. The choice for a fixed and flexible model structure that locally shows similarities with LTI systems, hopefully enlarges the possibility to develop systems and control theory, and model building methodologies. This hypothesis motivates to explore the suitability of the PLM as a candidate model structure for model based control application. Suitability involves interpretability, representation capacity, and the analysis and controller synthesis abilities as explored in this thesis. As a consequence, it is to be expected that as a result of further research, the

distance between the ideal model and the PLM will decrease further in the future. This will probably increase the number of applications for which the PLM is the minimizer of the distance between the ideal model and a member of the set of all candidate model structures.

1.2 Overview of the Study

This thesis describes a systematic approach for the analysis, modelling and control of a broad class of nonlinear systems, and consists of three main parts.

The first part, Part I, explores the system analysis potential of PLMs that are relevant for model based control application. In Chapter 3, the PLM structure will be introduced and its interpretations are explained. In Chapter 4 the approximation properties of PLMs are investigated. After that, in Chapter 5, properties will be explored that are relevant for system analysis and control purpose, such as stability, controllability and observability of a PLM.

The second part, Part II, investigates the modelling power of PLMs and describes in detail, how to construct a systems model with the desired PLM structure. In Chapter 7, a model based modelling method is considered. It is assumed that a nonlinear model of the system is available. In Chapter 8, two data based modelling methods are considered. It is assumed that measurements have been obtained from the system.

The last part, Part III, examines the model based control capabilities of PLMs. The acquired knowledge from the previous parts, will be utilized in four experimental and simulation case-studies, to design controllers for nonlinear systems that meet pre-imposed control objectives. In Chapter 10, a friction compensator design for a real rotational robotic manipulator system is reported. In Chapter 11, an optimal regulator design is reported. In Chapter 12, a model based controller design will be proposed with the objective to perform servo tasks on mechanical systems that exhibit friction. Again the control of a real rotational robotic manipulator system is considered. In Chapter 13, a stabilizing controller is designed for a family of (nonlinear) systems. The controller is robust against parametric uncertainty of the system.

The final chapter, Chapter 14, contains the main conclusions, summarizes the main contributions, and gives some suggestions for further research.

Part I

ANALYSIS

Chapter 2

Introduction to Analysis

In this part the analysis of PLMs, that are relevant for model based control applications, are explored in detail.

In Chapter 3, the PLM structure is introduced and its interpretations are explained. In Chapter 4, the approximation properties of PLMs are investigated. After that, in Chapter 5, properties are explored that are relevant for system analysis and control purpose, such as stability, controllability and observability of a PLM.

Chapter 3

Model and Interpretations

3.1 Introduction

In this chapter the PLM is introduced and its interpretations are explained in detail. We will start building on the results obtained in [34], where the idea of patching together locally valid models was formalized, and was given a theoretical foundation.

In Section 3.2, the polytopic model structure will be introduced on the basis of a regime decomposition of the operating space of a nonlinear system. After that, in Section 3.3, it is shown that the proposed PLM structure is optimal in an appealing sense. In Section 3.4, the model based and knowledge based interpretation of the PLM is introduced. In Section 3.5, a more in depth description of the several model based interpretations is given. In Section 3.6, the knowledge based interpretation of the PLM is clarified. Finally, in Section 3.7, some notes and comments are made regarding the presented model structure and interpretation.

3.2 Polytopic Model Structure

Suppose we want to characterize the behavior of

$$\begin{aligned} \dot{x} &= f(x, u, v) \\ y &= h(x, u, v) \end{aligned} \tag{3.1}$$

with the state $x \in X \subseteq \mathbb{R}^n$, the manipulated input $u \in U \subseteq \mathbb{R}^m$, the observed auxiliary variable $v \in V \subseteq \mathbb{R}^l$ and the output $y \in Y \subseteq \mathbb{R}^p$. Then within a sufficient small region $X_i \times U_i \times V_i$ around (x_i, u_i, v_i), a first order approximation, a linear model M_i obtained by a Taylor linearization of (3.1) in (x_i, u_i, v_i), that

is

$$\begin{aligned}\dot{x} &= f_i(x,u,v) \\ &= f(x_i,u_i,v_i) + \frac{\partial f}{\partial x}(x_i,u_i,v_i)(x-x_{0i}) + \frac{\partial f}{\partial u}(x_i,u_i,v_i)(u-u_{0i}) \\ &+ \frac{\partial f}{\partial v}(x_i,u_i,v_i)(v-v_{0i}) \\ y &= h_i(x,u,v) \\ &= h(x_i,u_i,v_i) + \frac{\partial h}{\partial x}(x_i,u_i,v_i)(x-x_{0i}) + \frac{\partial h}{\partial u}(x_i,u_i,v_i)(u-u_{0i}) \\ &+ \frac{\partial h}{\partial v}(x_i,u_i,v_i)(v-v_{0i})\end{aligned} \tag{3.2}$$

will give an approximate description of the system, provided the system is at least differentiable with respect to its arguments. At this stage the operating space $\Psi = X \times U \times V$ with operating vector $\psi \in \Psi$ is introduced, and it is assumed that M_i has a range of validity $\Psi_i \subset \Psi$.

A model (3.2) that has a range of validity[1] less than the desired range of validity will be called a local model, as opposed to a global model that is valid in the full range of operation. If now the models are chosen in such a way that $\Psi \subseteq (\cup_i \Psi_i)$ then the models can be combined to form a globally valid model.

Often it is not necessary to characterize a validity region for a model within the (high dimensional) operating space. Therefore the scheduling space Z, consisting of the set of scheduling variables $z \in Z$ is introduced. A scheduling vector z, consists of variables that schedule the models, that means influences validity of M_i. In many cases there will exist a function $s : \Psi = X \times U \times V \to Z$ that projects ψ onto a lower dimensional scheduling space Z, thus $z = s(\psi)$ with $\dim(z) < \dim(\psi)$. The region in which a model is valid is called an operating regime, that is $\Psi_i = \{\psi \in \Psi \mid s(\psi) \in Z_i\}$. The corresponding region Z_i is called a scheduling regime.

The framework can be conceptually illustrated as in Figure 3.1. Before proceeding with the question on how to combine the models obtained for the different regimes into a global model, let us first illustrate the above terms, such as operating space and scheduling regime, by means of an example.

EXAMPLE 3.2.1 *Consider the pendulum without friction in Figure 3.2.*
Its dynamics is described by

$$J\ddot{\theta} - mgl\sin\theta = \tau \tag{3.3}$$

[1]The term '*valid*' has not been clarified at this stage and of course depends upon the purpose of modelling. In our case validity of the model will be related to a measure of closeness to the real system in such a way that they share the same qualitative structure or some system properties, at least locally.

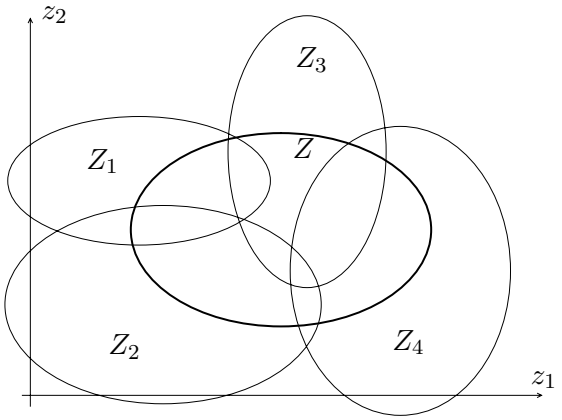


Figure 3.1: *The system's full range of operation, is completely covered by a number of possibly overlapping operating regimes Ψ_i. The models M_i are scheduled within the lower dimensional scheduling space $Z \subseteq (\cup_{i\in\{1,..,4\}} Z_i)$. In each operating regime the system is modelled by a linear state-space model. $z \in Z$ is typically a subset of the variables that constitute ψ. One could think of the remaining variables of ψ being on any axis perpendicular to z thereby not influencing validity of a specific model.*

where m is the mass of the ball, the mass of the stick is neglected, g is the gravity, l is the (variable) position of the ball, and θ is the angle of the pendulum with the vertical axis. If we write (3.3) in the form of (3.1) we obtain

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \frac{mg}{J} v \sin x_1 + \frac{1}{J} u \end{aligned} \tag{3.4}$$

with $x_1 = \theta$, $x_2 = \dot{\theta}$, $u = \tau$ and $v = l$ and $x \in X$, $u \in U$ and $v \in V$. First assume that the position of the ball is fixed, i.e. l can be viewed as a constant parameter. Then the operating space becomes $\Psi = X \times U$ with $\psi = [\theta\ \dot{\theta}\ \tau]^T$. Within a sufficient small scheduling regime Z_i, the linearized model will give an adequate description of the system. After a Taylor linearization of (3.4) in the points (x_{1i}, x_{2i}, u_i) we obtain the following non-homogeneous linear model M_i

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (\frac{mg}{J} v \cos x_{1i}) x_1 + \frac{1}{J} u + \frac{mg}{J} v (\sin x_{1i} - x_{1i} \cos x_{1i}) \end{aligned} \tag{3.5}$$

Here we see that behavior of the system qualitatively changes only as a function of x_{1i}, i.e. the linearization depends only upon x_1. The scheduling regime is therefore only a function of the state $x_1 = \theta$, i.e. s projects the operating vector

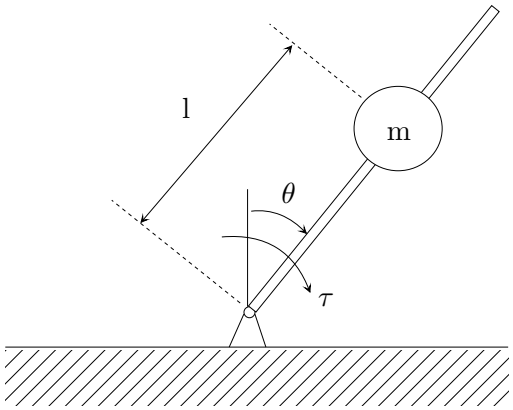


Figure 3.2: *A pendulum.*

$\psi = [\theta\ \dot{\theta}\ \tau]^T$ on the lower dimensional scheduling variable $z = \theta$. If instead, it is assumed that the position of the ball along the stick changes, through an external device, in a time varying way, that is $v = v(t)$, the operating space becomes $\Psi = X \times U \times V$ with $\psi = [\theta\ \dot{\theta}\ \tau\ l]^T$. Furthermore, since the ball position has become non-constant, the linearization in (3.5) depends also on the observed external variable $l(t) = v(t)$. In this case therefore the operating point becomes $z = [\theta\ l]$. One can interpret Figure 3.1 as follows. The operation points $z \in Z_i$ describe the region for which the linear model M_i is a valid approximation of the pendulum system. Clearly, since the pendulum system is linear in the variables $x_2 = \dot{\theta}$ and $u = \tau$ these variables do not influence the difference between the models M_i. Therefore the $\dot{\theta}$ and τ axis are perpendicular to the $z_1 = \theta$ and $z_2 = l$ axis in Figure 3.1.

Suppose we are given a set of N_m linear models that are an adequate description of the nonlinear system under different operating conditions. Next, we assume that for each local model M_i, a local model validity function $\rho_i : Z \to [0, 1]$ with local support is constructed, such that its value is close to one for operating points where the local model is an accurate description of the system, and close to zero elsewhere. Thus the relevance on a scale from zero to one is indicated by the functions ρ_i as follows:

$$\rho_i(z) \approx \begin{cases} 1 & \text{if} \quad z \in Z_i \\ 0 & \text{if} \quad z \notin Z_i \end{cases}$$

If there are N_m operating regimes with a local linear model and validity function defined for each regime, one may consider the following convex combina-

tion of locally valid linear models (3.2) to obtain a global nonlinear model

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{N_m} w_i(z) f_i(x,u,v) \\ y &= \sum_{i=1}^{N_m} w_i(z) h_i(x,u,v) \\ w_i(z) &= \frac{\rho_i(z)}{\sum_{j=1}^{N_m} \rho_j(z)} \end{aligned} \tag{3.6}$$

A model of the above form, with its specific structure, will be called a Polytopic Linear Model (PLM). To obtain a global model, it must be assumed that at any operating condition at least one model validity function is non-zero, that is $\sum_{i=1}^{N_m} \rho_i(z) > 0$ for all $z \in Z$. The scheduling function $w_i : Z \to [0,1]$ is a normalization of the model validity function ρ_i, which has the property that $\sum_{i=1}^{N_m} w_i(z) = 1$ for all $z \in Z$. Next, it will be shown that the proposed method of scheduling linear models to obtain a global nonlinear model is optimal in some sense.

3.3 Optimal Model Structure

Without loss of generality, but for notational convenience we confine ourself to the state equation of systems like (3.1). We seek a global approximate model, a substitute for the state equation of (3.1), namely a model

$$\dot{x} = g(x,u,v)$$

based on a combination of a set of N_m linear models (3.2), obtained as linearizations of (3.1) in some working points $\psi \in \Psi$. A linearized model (3.2) becomes in shorthand notation

$$\dot{x} = f_i(x,u,v)$$

with $i \in \{1,...,N_m\}$. From the knowledge of the validity of a local linear model it is natural to require that $g(\psi)$ should be close to $f_i(\psi)$ at $z = s(\psi) \in Z_i$, that is where $f_i(\psi)$ is relevant, and consequently at those points where $\rho_i(s(\psi)) > 0$. Among other possibilities, one could suggest to minimize a weighted mean square error expression that penalizes the mismatch between g and f_i the hardest whenever ρ_i is large, i.e.

$$J(g) = \sum_{i=1}^{N_m} \int_{\psi \in \Psi} \|g(\psi) - f_i(\psi)\|_2^2 \, \rho_i(s(\psi)) d\psi \tag{3.7}$$

Here $\|\cdot\|_2$ is the Euclidean norm. The following result is from [35].

THEOREM 3.3.1 *Given the functional (3.7). Suppose the functions f_i, with $i \in \{1, ..., N_m\}$, belong to $C(\Psi)$, the set of all continuous functions defined on Ψ. Assume that $\sum_{i=1}^{N_m} \rho_i(s(\psi)) > 0$ for all $\psi \in \Psi$. Then the function g defined by*

$$g(\psi) = \sum_{i=1}^{N_m} w_i(s(\psi)) f_i(\psi)$$
$$w_i(s(\psi)) = \frac{\rho_i(s(\psi))}{\sum_{j=1}^{N_m} \rho_j(s(\psi))}$$

minimizes J on $C(\Psi)$.

Proof. Notice that J is strictly convex. Hence J must have a unique global minimum. The Gateaux variation of J with respect to any perturbation $\Delta g \in C(\Psi)$ is

$$\delta J(g; \Delta g) = 2 \sum_{i=1}^{N_m} \int_{\psi \in \Psi} (g(\psi) - f_i(\psi)) \rho_i(s(\psi)) \Delta g(\psi) d\psi$$

A necessary and sufficient condition for global optimality of g is now [52]

$$\sum_{i=1}^{N_m} (g(\psi) - f_i(x, u)) \rho_i(s(\psi)) = 0$$

for all $\psi \in \Psi$. From the assumption that $\sum_{i=1}^{N_m} \rho_i(s(\psi)) > 0$ for all $\psi \in \Psi$, it follows that g is well defined and the desired result follows. ■

Of course optimality of the PLM (3.6) follows from the choice of the criterion J. It suggests that the penalty on mismatch between g and f_i should be largest whenever ρ_i is large. This is reasonable since in that region f_i is a relevant description of the true system. We can interpret w_i as a scheduling function that has its largest values in the parts of Z where the function f_i is the best approximation to the system, and close to zero elsewhere.

Next, several interpretations of the proposed PLM will be discussed.

3.4 Interpretations

Rewrite (3.6) more explicitly in case there is no auxiliary variable v

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{N_m} w_i(z)\{A_i x + B_i u + a_i\} \\ y &= \sum_{i=1}^{N_m} w_i(z)\{C_i x + D_i u + c_i\} \end{aligned} \tag{PLM}$$

where

$$\sum_{i=1}^{N_m} \omega_i(z) = 1 \text{ and } \omega_i(z) \geq 0$$

for all $z = s(x, u, v) \in Z$. The model is specified by a set of nonhomogeneous linear models

$$\{(A_i, B_i, a_i, C_i, D_i, c_i)\}$$

together with corresponding scheduling functions

$$\{w_i(x, u)\}$$

where $i \in \{1, ..., N_m\}$ finite.

The interpretation (of the representation) of the PLM will be made clear from two different viewpoints. The model based viewpoint gives insight in how to interpret the PLM, and in more detail a realization of the linear model parameters quantitatively. Secondly, the knowledge based viewpoint gives insight on how to interpret the model structure, and in more detail a realization of the scheduling functions qualitatively.

3.5 Model Based Interpretations

Performance of a model-based control strategy depends heavily on the quality of the model. For model-based control one needs either an accurate description of the system, or a 'simplified' model with an accurate description of the expected variation (uncertainty) in the system. Both objectives can be reached by the proposed model structure.

3.5.1 Linearizations

Suppose the nonlinear model (3.1) is given, in case there is no auxiliary variable v, that is

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x, u) \end{aligned} \tag{3.8}$$

Let $(x_i, u_i) \in X \times U = \Psi$ be given. Then by the Mean Value Theorem, the right-hand side of (3.8) can be rewritten if it is at least one time continuously differentiable with respect to x and u (that is $f, h \in C^1(\Psi)$) as

$$f(x, u) = f(x_{0i}, u_{0i}) + \frac{\partial f}{\partial x}(\xi)(x - x_{0i}) + \frac{\partial f}{\partial u}(\xi)(u - u_{0i}) \tag{3.9}$$

where ξ lies on the segment between $[x_{0i}^T \; u_{0i}^T]^T$ and $[x^T \; u^T]^T$ and depends on x and u. Rearranging (3.9) and applying the Mean Value Theorem to the output equation as well gives

$$\begin{aligned} \dot{x} &= \frac{\partial f}{\partial x}(\xi)x + \frac{\partial f}{\partial u}(\xi)u + f(x_{0i}, u_{0i}) - \frac{\partial f}{\partial x}(\xi)x_{0i} - \frac{\partial f}{\partial u}(\xi)u_{0i} \\ y &= \frac{\partial h}{\partial x}(\xi)x + \frac{\partial h}{\partial u}(\xi)u + h(x_{0i}, u_{0i}) - \frac{\partial h}{\partial x}(\xi)x_{0i} - \frac{\partial h}{\partial u}(\xi)u_{0i} \end{aligned} \tag{3.10}$$

This method is known as global or exact linearization, see [17], [48], as opposed to local or approximate linearization, where $\xi = [x_{0i}^T \; u_{0i}^T]^T$. A special case of the global linearization procedure occurs whenever $f(0,0) = 0$ and $h(0,0) = 0$. Then (3.10) with $(x_{0i}, u_{0i}) = (0,0)$ reduces to

$$\begin{aligned} \dot{x} &= \frac{\partial f}{\partial x}(\xi)x + \frac{\partial f}{\partial u}(\xi)u \\ y &= \frac{\partial h}{\partial x}(\xi)x + \frac{\partial h}{\partial u}(\xi)u \end{aligned} \tag{3.11}$$

suggesting a PLM consisting of homogeneous linear models as the next example indicates.

EXAMPLE 3.5.1 *Consider again the pendulum in Figure 3.2. A global linearization of the pendulum system can be obtained if (3.4) is rewritten in the form (3.9), that is*

$$\dot{x} = A(\xi_1)x + Bu + a(\xi_1)$$

with

$$\begin{aligned} A(\xi_1) &= \begin{bmatrix} 0 & 1 \\ \frac{mg}{J} v \cos\xi_1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} \\ a(\xi_1) &= \frac{mg}{J} v(\sin x_{1i} - x_{1i}\cos\xi_1) \end{aligned}$$

and with $\xi_1 \in [x_{1i}, x_1]$ and depends on x_1. This suggests, if possible, a PLM consisting of a set of N_m scheduling functions w_i, with corresponding linear models specified by (A_i, B, a_i), where

$$\sum_{i=1}^{N_m} w_i(z) \begin{bmatrix} A_i & a_i \end{bmatrix} = \begin{bmatrix} A(\xi_1) & a_i(\xi_1) \end{bmatrix}$$

This representation is not unique since x_{1i} can be given different values. With the choice $x_{1i} = 0$, the term $a(\xi_1) = 0$ vanishes and the pendulum system is rewritten as

$$\dot{x} = A(\xi_1)x + Bu$$

with $(A(\xi_1), B)$ as above. An explicit expression for ξ_1 is easily found if (3.4) is rewritten as

$$\dot{x} = A(x_1)x + Bu$$

with

$$A(x_1) = \begin{bmatrix} 0 & 1 \\ \frac{mg}{J} v \frac{\sin(x_1)}{x_1} & 0 \end{bmatrix}$$

From the equality $A(x_1) = A(\xi_1)$ it follows that $\xi_1 = \arccos(\frac{\sin(x_1)}{x_1})$. This suggests, if possible, a PLM consisting of a set of homogeneous linear models specified by (A_i, B), where

$$\sum_{i=1}^{N_m} w_i(z) A_i = A(x_1)$$

and B as above. For the above equality to hold it is necessary to choose x_1 as the scheduling variable z. Sometimes, to facilitate the analysis of PLMs, we will confine to PLMs consisting of only homogeneous linear models.

Comparing the exact linearization (3.10) to (PLM), suggests to find a PLM $\{A_i, B_i, a_i, C_i, D_i, c_i, w_i\}$ such that for all x, u

$$\begin{aligned} &\sum_{i=1}^{N_m} w_i(z) \begin{bmatrix} A_i & B_i & a_i \\ C_i & D_i & c_i \end{bmatrix} \\ &- \begin{bmatrix} \frac{\partial f}{\partial x}(\xi) & \frac{\partial f}{\partial u}(\xi) & f(x_i, u_i) - \frac{\partial f}{\partial x}(\xi)x_i - \frac{\partial f}{\partial u}(\xi)u_i \\ \frac{\partial h}{\partial x}(\xi) & \frac{\partial h}{\partial u}(\xi) & h(x_i, u_i) - \frac{\partial h}{\partial x}(\xi)x_i - \frac{\partial h}{\partial u}(\xi)u_i \end{bmatrix} \end{aligned} \tag{3.12}$$

is minimized. If (3.8) is linear in some of the variables from x or u, then the Jacobians from (3.12) do not depend on these variables and it follows that these variables do not have to appear in z. So z has to be chosen such that it captures the system nonlinearities. This was already illustrated in Example 3.2.1 and Example 3.5.1.

3.5.2 Uncertainty Model

An important issue in robust control theory is how to model or measure plant uncertainty or variation. Here an uncertainty model is proposed that is directly derived from the PLM structure. Assume that (3.12) equals zero for all (x, u), than clearly

$$\begin{bmatrix} \frac{\partial f}{\partial x}(\xi) & \frac{\partial f}{\partial u}(\xi) & f(x_i, u_i) - \frac{\partial f}{\partial x}(\xi)x_i - \frac{\partial f}{\partial u}(\xi)u_i \\ \frac{\partial h}{\partial x}(\xi) & \frac{\partial h}{\partial u}(\xi) & h(x_i, u_i) - \frac{\partial h}{\partial x}(\xi)x_i - \frac{\partial h}{\partial u}(\xi)u_i \end{bmatrix} \in \Omega \tag{3.13}$$

where Ω is a polytope[2] defined by (the vertices of) the PLM, namely

$$\Omega = \mathrm{Co}\left\{ \begin{bmatrix} A_i & B_i & a_i \\ C_i & D_i & c_i \end{bmatrix} \right\}$$

where the convex hull $\mathrm{Co}(a_1, a_2, ..., a_n) = \{a | \sum_{i=1}^{n} w_i a_i, \sum_{i=1}^{n} w_i = 1, w_i \geq 0\}$. Now it is possible to associate with the PLM an uncertainty model, a polytopic linear differential inclusion (PLDI) [17] as follows

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} \in \Omega \begin{bmatrix} x \\ u \\ 1 \end{bmatrix} \tag{3.14}$$

Of course every trajectory of the nonlinear system (3.8) is also a trajectory of the PLDI. If we can prove that every trajectory of the PLDI defined by Ω has some specific property, say it converges to zero, then also every trajectory of the nonlinear model has this property [17].

3.5.3 The Uncertainty Model Set

At this stage the uncertainty model set $\mathcal{M}$ will be introduced. The model set is defined as follows

$$\mathcal{M}(\{A_i, B_i, a_i, C_i, D_i, c_i\}) := \left\{ (\mathrm{PLM}) | \sum_{i=1}^{N_m} \omega_i(z) = 1,\ \omega_i(z) \geq 0 \right\} \tag{3.15}$$

One can think of the model set as a collection of PLMs that are all represented with the same set of nonhomogeneous linear models $\{(A_i, B_i, a_i, C_i, D_i, c_i)\}$. However every PLM from the set $\mathcal{M}(\{A_i, ..., c_i\})$ has its own unique realization of the set of scheduling functions ω_i. This means that the only difference between two PLMs from the same model set is the realization of the set of scheduling functions, which is constrained by $\sum_{i=1}^{N_m} \omega_i(z) = 1$ and $\omega_i(z) \geq 0$. By definition, the model set $\mathcal{M}$ has or satisfies a property if and only if all PLMs from $\mathcal{M}$ have or satisfy this property.

If a specific property of a PLM$\in \mathcal{M}$ is investigated then it will in general depend on the realization of the scheduling functions. If however a property has to hold for the model set then it will be independent of the scheduling functions and therefore depends solely on the parameters of the set of nonhomogeneous linear models. If we find a sufficient condition the model set $\mathcal{M}$ has to satisfy, then of course it will also be sufficient for a PLM$\in \mathcal{M}$ from the

[2] A polytope or polyhedron is a closed set whose boundary consists of (affine) linear subspaces.

model set. On the other hand, if we find a necessary condition for a PLM$\in \mathcal{M}$ from the model set, than it will also be necessary for the model set $\mathcal{M}$. We have

$$\begin{array}{rcl} (\text{necessary condition PLM} \in \mathcal{M}) & \Rightarrow & (\text{necessary condition } \mathcal{M}) \\ (\text{sufficient condition PLM} \in \mathcal{M}) & \Leftarrow & (\text{sufficient condition } \mathcal{M}) \end{array} \tag{3.16}$$

The introduction of a model set will be useful for the analysis of the polytopic linear model. The concept of a model set is closely related to the aforementioned uncertainty model, the PLDI. In fact the model set and the PLDI represent the same behavior.

3.6 Knowledge Based Interpretation

A priori knowledge about the operation of a given process in different regimes can be used in a structural way to obtain a PLM. Moreover the PLM can be explained qualitatively in terms of the operating regimes. Concepts described in a linguistic manner can be related to the validity functions ρ_i by means of fuzzy set theory [87],[84].

3.6.1 Fuzzy Model

To provide a mathematical tool for dealing with linguistic variables (i.e. concepts described in natural language) fuzzy sets have been introduced. A fuzzy set is defined as a set, the boundary of which is not sharp. Let Z_i be a fuzzy set. This means the region $Z_i \subset Z$ is assigned a linguistic label. This region is characterized by a membership function $\mu_{Z_i}(z)$ that maps the set Z into the interval [0 1]. The closer μ_{Z_i} is to 1, the more z belongs to Z_i. We may therefore also view μ_{Z_i} as the degree of compatibility of z with the concept represented by Z_i. Since in general Z defines a more than one-dimensional space we can also look at the compatibility of z_j with the concept Z_{ij}. Here z_j is the j-th coordinate of the operating vector z, and Z_{ij} the concept of the j-th dimension for regime Z_i. Also a membership function $\mu_{Z_{ij}}(z_j)$ can be assigned to the concept Z_{ij}.

The set-theoretic operations of union ($\bigcup$) and intersection ($\bigcap$) for fuzzy sets are defined through their membership functions $\mu_{Z_{ij}}$. Let Z_{i1} and Z_{i2} denote a pair of fuzzy one dimensional sets in Z with membership functions $\mu_{Z_{i1}}$ and $\mu_{Z_{i2}}$ respectively. The membership function $\mu_{Z_{i1} \bigcup Z_{i2}}$ of the union $Z_{i1} \bigcup Z_{i2}$ and the membership function $\mu_{Z_{i1} \bigcap Z_{i2}}$ of the intersection $Z_{i1} \bigcap Z_{i2}$

are defined as follows[3]:

$$\mu_{Z_{i1} \bigcup Z_{i2}}(z) = \text{softmax}(\mu_{Z_{i1}}, \mu_{Z_{i2}})$$
$$\mu_{Z_{i1} \bigcap Z_{i2}}(z) = \text{softmin}(\mu_{Z_{i1}}, \mu_{Z_{i2}})$$

The complement of the fuzzy set Z_{i1} is defined by the membership function

$$\mu_{\bar{Z}_{i1}}(z_1) = 1 - \mu_{Z_{i1}}(z_i)$$

Depending on the concept Z_i, '$z = Z_i$' could mean 'if $(z_{i1} = Z_{i1})$ and $(z_{i2} = Z_{i2})$' or 'if $(z_{i1} = Z_{i1})$ or $(z_{i2} = Z_{i2})$'. The degree of fulfillment of that statement would become $\mu_{Z_i}(z) = \text{softmin}(\mu_{Z_{i1}}(z_1), \mu_{Z_{i2}}(z_2))$ or $\mu_{Z_i}(z) = \text{softmax}(\mu_{Z_{i1}}(z_1), \mu_{Z_{i2}}(z_2))$ respectively.

EXAMPLE 3.6.1 *Consider again the pendulum from Example 3.2.1 with the position of the ball as a variable. In that case* $z = [x_1\ v]^T$ *and one could define the concepts* $z_{i1} = Z_{i1}$ *meaning 'if* x_1 *is down', and* $z_{i2} = Z_{i2}$ *meaning 'if* v *is large'. One could also have the concept* $z_i = Z_i$ *meaning*

'if $(x_1$ *is down) and* $(v$ *is large)'*

The membership functions that are associated with these concepts can be chosen as the unnormalized Gaussian functions:

$$\mu_{Z_{i1}} = e^{-\frac{(x_1 - x_{1i})^2}{2\sigma_1}} \text{ and } \mu_{Z_{i2}} = e^{-\frac{(v - v_i)^2}{2\sigma_2}}.$$

Because of the 'and' operator in the concept Z_i *its membership function becomes*

$$\mu_{Z_i}(z) = \text{softmax}(\mu_{Z_{i1}}(z_1), \mu_{Z_{i2}}(z_2)).$$

For the softmax *operator one could choose the product of the two Gaussian functions, that is*

$$\mu_{Z_i}(z) = \mu_{Z_{i1}}(z_1)\mu_{Z_{i2}}(z_2) \tag{3.17}$$
$$= e^{-\frac{(x_1 - x_{1i})^2}{2\sigma_1}} e^{-\frac{(v - v_i)^2}{2\sigma_2}} \tag{3.18}$$
$$= e^{-(z - z_i)^T \Sigma^{-1} (z - z_i)} \tag{3.19}$$

where $z_i = [x_{1i}\ v_i]^T$ *and* $\Sigma = 2\,\text{diag}(\sigma_i)$ *a diagonal matrix with* $2\sigma_i$ *on its diagonal. The result is a multivariable Gaussian membership function (3.19), which has a qualitative interpretation (3.17), since it can be factorized (3.18).*

[3]The softmax and softmin operators can be any max respectively min operator. In fact these operators function as linguistic 'or' respectively 'and' operators.

Returning to the issue at hand, the knowledge based interpretation of the PLM, the major observation is that ρ_i can be interpreted as μ_{Z_i}. The generation of the fuzzy model, see [71], that is mathematically equivalent with the PLM consists of three steps:

1. *Fuzzification*, a mapping that changes the range of values of input variables z into a degree of membership to a concept Z_i.

2. *Knowledge base*, which consists of a set of N_m linguistic rules (the knowledge base) written in the form:

$$\text{'if } (z = Z_i) \text{ then } \left(\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A_i & B_i & a_i \\ C_i & D_i & c_i \end{bmatrix} \begin{bmatrix} x \\ u \\ 1 \end{bmatrix} \right),$$

3. *Inference machine and defuzzification*, which is a decision-making logic that employs rules from the rule base to infer $[\dot{x}\ y]^T$. In the case of the PLM, Takagi-Sugeno inference and defuzzification are employed [71].

EXAMPLE 3.6.2 *Consider again the pendulum from Example 3.2.1 with the position of the ball fixed. Then one could consider the linguistic variables*

$$Z_1 = \text{'pendulum down'},\ Z_2 = \text{'pendulum horizontal'},\ Z_3 = \text{'pendulum up'}.$$

The membership functions that are associated with these concepts are the Gaussian functions:

$$\mu_{Z_i} = e^{-\frac{(x_1 - x_{1i})^2}{2\sigma_i}}$$

where $x_{11} = \pi$, $x_{12} = 0.5\pi$ *and* $x_{13} = 0$. *The rule-base becomes with* $i = \{1, 2, 3\}$*:*

$$\text{'if } (x_1 = Z_i) \text{ then } (\dot{x} = A_i x + B_i u + a_i)\text{'}$$

where a possible choice for the parameters is[4]

$$A_i = \begin{bmatrix} 0 & 1 \\ (\frac{mg}{J} v \cos x_{1i}) & 0 \end{bmatrix}\ B_i = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix}\ a_i = \begin{bmatrix} 0 \\ \frac{mg}{J} v(\sin x_{1i} - x_{1i} \cos x_{1i}) \end{bmatrix}$$

After Takagi-Sugeno fuzzy inference one obtains as a fuzzy model:

$$\dot{x} = \frac{1}{\sum_{j=1}^{3} \mu_{Z_j}(x_1)} \sum_{i=1}^{3} \mu_{Zi}(x_1)\{A_i x + B_i u + a_i\}$$

[4]Note that in the matrices A_i, B_i, a_i there are structural zeros and ones, and that furthermore $B_i = B$ for all i, which means that B_i is regime independent. These properties may make system analysis and controller synthesis easier. This is not the subject of this example however.

Example 3.6.2 shows that qualitative information about a system can be incorporated in a PLM, or extracted from a PLM via human expert's knowledge by means of the knowledge base. From this example it is also shown that equilibrium points of the real system can be preserved, i.e. in the equilibrium position 'pendulum down' and the equilibrium position 'pendulum up' the PLM is exact, at least if the scheduling functions are properly chosen.

3.7 Notes and Comments

Polytopic model structure

In this thesis, state-space polytopic models with locally valid linear models are considered. In general, one could also consider convex combinations of arbitrary complex locally valid state-space models. This possibility is discussed in [34], which also considers the case where locally valid models do not have the same (dimension of the) state-space. In the literature also the combinations of input-output models has been considered [36], [37].

Optimal model structure

Theorem 3.3.1 shows that

$$\begin{aligned} g^*(\psi) &= \arg\min_{g(\psi)} J(g(\psi)) \\ &= \arg\min_{g(\psi)} \sum_{i=1}^{N_m} \int_{\psi\in\Psi} \|g(\psi) - f_i(\psi)\|_2^2 \, \rho_i(s(\psi)) d\psi \end{aligned}$$

is a right-hand side of a PLM whenever f_i is a right-hand side of a linear model. An interesting fact is that whenever one replaces a local approximation f_i with a right-hand side of a PLM, then the function g^* that minimizes J remains a right-hand side of a PLM. Thus structural complexity does not increase as the next result shows.

COROLLARY 3.7.1 *Given the functional (3.7). Suppose the functions* $f_i = \sum_{j=1}^{N_{mi}} w_{ij}(s(\psi)) f_{ij}(\psi)$, *with* $i \in \{1, ..., N_m\}$, *belong to* $C(\Psi)$, *the set of all continuous functions defined on* Ψ. *Assume that* $\sum_{j=1}^{N_{mi}} w_{ij}(s(\psi)) = 1$, $w_{ij}(s(\psi)) \geq 0$ *for all* $\psi \in \Psi$. *Then the function* g *defined by*

$$\begin{aligned} g(\psi) &= \sum_{i=1}^{N_m} \sum_{j=1}^{N_{mi}} w_{ij}^*(s(\psi)) f_{ij}(\psi) \\ w_{ij}^*(s(\psi)) &= w_i(s(\psi)) w_{ij}(s(\psi)) \end{aligned}$$

minimizes J on $C(\Psi)$.

Proof. The result follows directly from the proof of Theorem 3.3.1. ■

Interpretations

Besides the presented interpretation, also statistical interpretations of the PLM have been discussed [24], [55], [54]. These interpretations will not be discussed here.

Chapter 4

Approximation

4.1 Introduction

In the previous chapter, the PLM structure was introduced as a candidate model structure for modelling a class of nonlinear systems. Besides the interpretation of a model also its approximation capabilities are of importance, since they restrict the potential of the model to represent the system. In this chapter we investigate the approximation properties of PLMs.

The approximation properties are important for two main reasons. Firstly, the approximation results provide us with approximation accuracy bounds, which form a starting point for the analysis of the system interconnected with controllers and/or observers based on the PLM. Secondly, the approximation results are constructive, and form the basis of some of the modelling methods that will be presented in Part II of this thesis.

For reason of clarity we will first consider autonomous systems. A useful model has to be close to the system, in the sense that it explains the behavior of the system. But then immediately the question arises of how to define and measure accuracy, that is, distance between systems. One possible choice, the one that is adopted here, is to consider the Euclidean distance between the right-hand side of two systems, that is

$$d_{fg}(E) := \sup_{x \in E} \|f(x) - g(x)\|_2 \tag{4.1}$$

where

$$\dot{x} = f(x) \tag{4.2}$$

denotes the real system with $f \in C^1(E)$, where E is an open subset of $\mathbb{R}^n$ and

$$\dot{x} = g(x) \tag{4.3}$$

denotes the approximate autonomous model. Since our main interest goes to the approximation capabilities of PLMs we will often specialize to approximate systems having the PLM structure:

$$g(x) = \sum_{i=1}^{N_m} w_i(x) g_i(x) \tag{4.4}$$

where

$$g_i(x) = A_i x + a_i$$

If the distance d_{fg} is measured on K, a compact subset of E, then the supremum becomes the maximum in the definition of d_{fg}. Other norms to measure the distance between systems could also be considered[1], but it will be shown that accuracy defined as $d_{fg} \leq \varepsilon$ will lead to the desired approximation results.

In Section 4.2, it will be demonstrated that the right-hand side $f \in C^1(E)$ of the real system (4.2) can be uniformly approximated to an arbitrary accuracy $\varepsilon > 0$ on any compact operating range $X \subset E$ with a right-hand side (4.4) of a PLM (4.3), by making the decomposition of X into a finite number of N_m operating regimes X_i sufficiently fine. A PLM turns out to be a universal approximator.

In Section 4.3, we will give an upper bound on N_m, the number of operating regimes sufficient to achieve ε-accuracy. It follows that ε-accuracy is achieved, if the locally valid nonhomogeneous linear models g_i from (4.4) are chosen as zero-th or first order Taylor series of the real system (4.2) around operating points $x_{0i} \in X_i$, uniformly distributed over the operating space. We will also observe that a PLM suffers from the 'curse of dimensionality'. This means that to achieve ε-accuracy, N_m grows exponentially with the number of scheduling variables, in this case $\dim(x)$.

In Section 4.4, it will be established that under the appropriate conditions, and for finite times, trajectories of the PLM can be made arbitrary close to trajectories of the real system, at least if ε is small enough and $d_{fg} \leq \varepsilon$. An upper bound is derived on the difference between trajectories of the real system and the ε-accurate PLM, where it is assumed that both trajectories originate from the same initial condition.

In a straightforward manner the above results generalize to systems with inputs (and outputs). This is shown in Section 4.5

In Section 4.6, it is shown that structure of a system can be exploited to reduce the complexity of the approximate PLM. Basically, the derived result states that without loosing the universal approximation property, the locally

[1]For instance the distance $\sup_{x\in E} \|f(x) - g(x)\|_2 + \sup_{x\in E} \left\|\frac{\partial f}{\partial x} - \frac{\partial g}{\partial x}\right\|_2$ as defined in [60] to analyze whether f and g are topologically equivalent.

valid models g_i from the PLM can be scheduled on a scheduling space Z consisting of only the variables z that enter f of the real system in a non-affine way. Since in this case $\dim(z) < \dim(x)$, the 'curse of dimensionality' is partially reduced. In this case, the number of operating regimes or/and the complexity of the local models can be reduced which results in a reduced number of parameters for the PLM. This is illustrated by means of an example.

Finally, in Section 4.7 some notes and comments are made regarding the presented approximation analysis of PLMs.

4.2 A Universal Approximator

In order to prove the universal approximation property for PLMs we need the following preparatory result.

LEMMA 4.2.1 *Given $f \in C^1(E)$, E an open subset of $\mathbb{R}^n$ and $\varepsilon > 0$ arbitrary. There exists a zero-th ($k = 0$) and first order ($k = 1$) Taylor series expansion f_i^k of f around x_{0i}, that is*

$$f_i^k(x) := \begin{cases} f_i^0(x) = f(x_{0i}) & \text{if } k = 0 \\ f_i^1(x) = f(x_{0i}) + \frac{\partial f}{\partial x}(x_{0i})(x - x_{0i}) & \text{if } k = 1 \end{cases} \tag{4.5}$$

such that $d_{ff_i^k}(B_{r_{ik}}(x_{0i})) \leq \varepsilon$, with $B_{r_{ik}}(x_{0i}) \subset E$ a ball of radius $r_{ik}(\varepsilon)$ centered at x_{0i}.

Proof. Define the k-th order Taylor remainder $F_i^k(x) := f(x) - f_i^k(x)$ and note that since $f \in C^1(E)$ also $F_i^k \in C^1(E)$ which implies that there exists a finite positive number L_{ik} such that $\left\|F_i^k(x) - F_i^k(x_{0i})\right\|_2 = \left\|F_i^k(x)\right\|_2 \leq L_{ik} \left\|x - x_{0i}\right\|_2$. It follows that within the ball $B_{r_{ik}}(x_{0i}) := \{x | \left\|x - x_{0i}\right\|_2 \leq \frac{\varepsilon}{L_{ik}} = r_{ik}\}$ we have $\left\|F_i^k(x)\right\|_2 = \left\|f(x) - g_i^k(x)\right\|_2 \leq \varepsilon$. ■

The universal approximation property for PLMs follows from the fact that with N_m number of locally valid models, that means models with $g_i(x) = f_i^k$ as a right-hand side, with $i \in \{1, ..., N_m\}$, a compact space $X \subset E$ can be covered. More specifically $X \subseteq \{x | x \in (\cup_i B_{r_{ik}}(x_{0i}))\}$, and it follows that the $\{x_{0i}\}$ have to be chosen sufficiently dense.

THEOREM 4.2.2 *Given the system (4.2) with $f \in C^1(E)$, E an open subset of $\mathbb{R}^n$, X any compact set such that $X \subset E$, and $\varepsilon > 0$ arbitrary. There exists a right-hand side of a PLM (4.3,4.4) with N_m finite such that $d_{fg}(X) \leq \varepsilon$.*

Proof. Construct a PLM (4.4) where $g_i(x) = f_i^k(x)$ with $k \in \{0, 1\}$, that is a zero-th or first order Taylor expansion defined as in Lemma 4.2.1. With the identity $f(x) = \sum_{i=1}^{N_m} w_i(x) f(x)$ and the use of the triangle inequality,

namely $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$, it follows immediately that $\|f(x)-g(x)\|_2 \leq \sum_{i=1}^{N_m} w_i(x) \left\|F_i^k(x)\right\|_2 \leq \sum_{i=1}^{N_m} w_i(x) L_{ik} \|x-x_{0i}\|_2$, where $F_i^k(x)$ is defined as in Lemma 4.2.1. Note that $\sum_{i=1}^{N_m} w_i(x) L_{ik} \|x-x_{0i}\|_2 \leq \varepsilon$ for all $x \in X$ implies $d_{fg}(X) \leq \varepsilon$. Rewriting the last but one inequality and choosing $w_i = \frac{\rho_i}{\sum_{j=1}^{N_m} \rho_j}$ gives $\sum_{i=1}^{N_m} \rho_i(x)\{L_{ik} \|x-x_{0i}\|_2 - \varepsilon\} \leq 0$ for all $x \in X$, which can be satisfied if for all $x \in X$ there exists at least one $i \in \{1, ..., N_m\}$ such that $L_{ik} \|x-x_{0i}\|_2 - \varepsilon \leq 0$, that means $X \subseteq (\cup_i B_{r_{ik}}(x_{0i}))$. Since X is assumed to be a compact set, it can be covered by a finite number of balls $B_{r_{ik}}(x_{0i})$, meaning that N_m is finite, as desired. Furthermore, as a result of the construction, at least one of the semi-positive definite functions $\rho_i(x)$ has to be chosen greater than zero if $x \in B_{r_{ik}}(x_{0i})$. This implies that $\sum_{i=1}^{N_m} \rho_i(x) > 0$ can be satisfied for all $x \in X$, meaning that $w_i(x)$ and therefore the PLM is well defined for all $x \in X$. ■

A graphical interpretation of Theorem 4.2.2 is given in Figure 4.1. Basically it says that as long as $x \in B_{r_{ik}}(x_{0i})$ then $\rho_i(x)$ can be chosen positive. However to get a globally well defined model the requirement $\sum_{i=1}^{N_m} \rho_i(x) > 0$ for all $x \in X$ has to be satisfied. This means that the x_{0i}'s have to be chosen sufficiently dense such that for all $x \in X$ there exists a x_{0i} such that $x \in B_{r_{ik}}(x_{0i})$. It can be seen from Figure 4.1 that with a particular choice of the model validity functions $\rho_i(x)$, and sufficient number of models N_m, Theorem 4.2.2 is satisfied. Figure 4.1 gives some insight concerning the realization of the functions $\rho_i(x)$ such that Theorem 4.2.2 holds, as will be illustrated next.

EXAMPLE 4.2.3 *A possible realization for $\rho_i(x)$ such that Theorem 4.2.2 holds could be:*

$$\rho_i(x) = \begin{cases} 1 \text{ if } L_{ik} \|x-x_{0i}\|_2 - \varepsilon \leq 0 \\ 0 \text{ if } L_{ik} \|x-x_{0i}\|_2 - \varepsilon > 0 \end{cases} \tag{4.6}$$

Another, smoother choice for $\rho_i(x)$ based on Theorem 4.2.2, which also reflects the intuition that the region of validity depends on the particular norm bound of the Taylor remainder in that specific operating region is:

$$\rho_i(x) = \frac{1}{L_{ik} \|x-x_{0i}\|_2^p} \tag{4.7}$$

for some $p > 0$. Another possible choice is the Gaussian validity function of Example 3.6.1, namely

$$\rho_i(x) = e^{-(x-x_{0i})^T \Sigma^{-1} (x-x_{0i})} \tag{4.8}$$

where $\Sigma = 2\,\mathrm{diag}(\sigma_i)$ is a diagonal matrix. Qualitatively σ_i can be chosen inversely proportional to L_{ik}, so σ_i small stands for a strong nonlinearity.

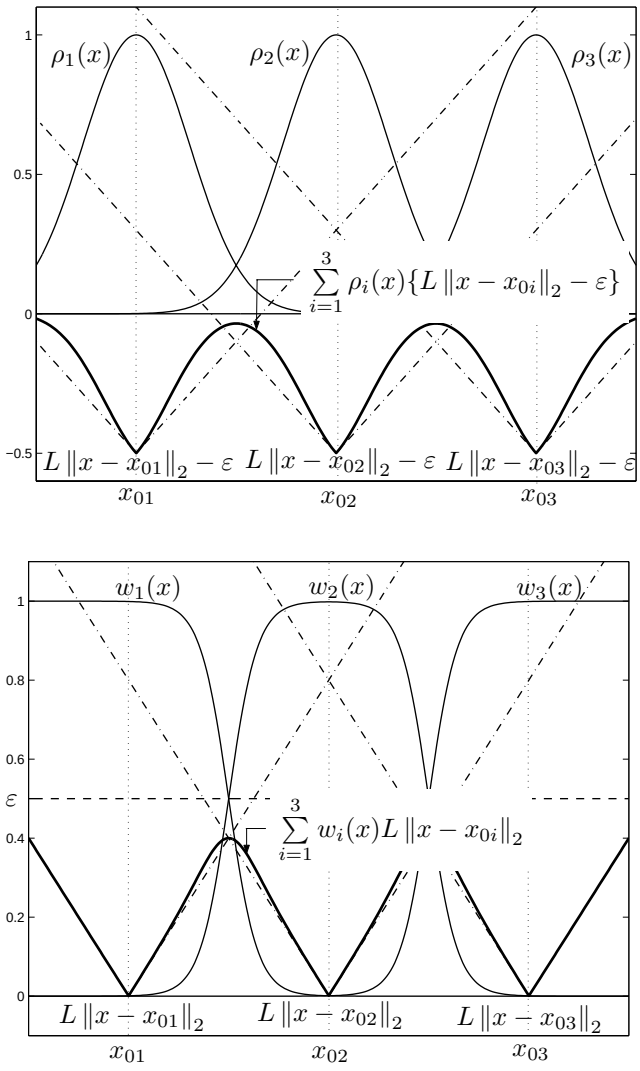


Figure 4.1: *Possible choice of the model validity functions $\rho_i(x)$ based on a worst case situation, namely for the upper bound on the Taylor remainder it is assumed that for all $i \in \{1, ..., N_m\}$ the following holds: $\left\|F_i^k(x)\right\|_2 \leq L\|x - x_{0i}\|_2$. The thick lines illustrate that ε-accuracy is achieved.*

Theorem 4.2.2 is illustrated with Gaussian validity functions in Figure 4.1. From this figure it is observed that without hard model switching as suggested in (4.6), and even with $\rho_i(x) > 0$ outside $B_{r_{ik}}(x_{0i})$ it is possible to achieve the desired accuracy.

4.3 Upper bound on the Number of Models

Next an upper bound on the number of models N_m will be derived, that is sufficient to construct a PLM with ε-accuracy. The upper bound is based on a worst case analysis, that means we use the fact that

$$\begin{aligned} \|f(x) - g(x)\|_2 &\leq \sum_{i=1}^{N_m} w_i(x) L_{ik} \|x - x_{0i}\|_2 \\ &\leq L \sum_{i=1}^{N_m} w_i(x) \|x - x_{0i}\|_2 \end{aligned} \tag{4.9}$$

where $L \geq L_{ik}$ for all $i \in \{1, ..., N_m\}$.

We are faced with the problem of covering the region X with hyperballs $B_r(x_{0i}) = \{x \mid \|x - x_{0i}\|_2 \leq r = \frac{\varepsilon}{L}\}$. Again it is assumed that X is a compact operating space, because then a finite number of models N_m suffices to cover X. It is in general easier to cover X with hypercubes $C_c(x_{0i}) = \{x \mid \|x - x_{0i}\|_\infty \leq c\}$ where $\|v\|_\infty = \max_i |v_i|$. This leads to the following idea. First inner-approximate the hyperball $B_r(x_{0i})$ with a hypercube $C_c(x_{0i})$ of maximum volume. Then cover the region X with hypercubes $C_c(x_{0i})$ to obtain the conditions the model has to satisfy to guarantee $X \subseteq (\cup_i C_c(x_{0i}))$, which, since $C_c(x_{0i}) \subseteq B_r(x_{0i})$ also implies $X \subseteq (\cup_i B_r(x_{0i}))$. A graphical interpretation of the previous idea, for a two dimensional state space is given in Figure 4.2. The radius of the circle $B_r(x_{0i})$ (hyperball) is r, the width of the square $C_c(x_{0i})$ (hypercube) is $2c$ and the centre of $B_r(x_{0i})$ and $C_c(x_{0i})$ is x_{0i}. The working points x_{0i} are uniformly distributed over the operating space as a result of the worst case analysis based on (4.9). The next result is based on the hypercube partitioning of the operating space as illustrated in Figure 4.3. Here the operating space is (in each dimension) characterized with its center d_i and width e_i as depicted in Figure 4.3.

Before stating the result we define the ceiling operator $\lceil . \rceil : \mathbb{R} \to \mathbb{N}_+$ that maps a real to the nearest integer towards infinity.

THEOREM 4.3.1 *Under the hypothesis of Theorem 4.2.2, given* $X = \{x \mid |x_i - d_i| \leq e_i, i = 1, \cdots, n\}$, *and* $\varepsilon > 0$ *arbitrary, it suffices to construct a PLM (4.3,4.4) with* $N_m = \Pi_{i=1}^n \left\lceil \frac{L\sqrt{n}}{\varepsilon} e_i \right\rceil$, *where* L *is from (4.9), to assure that* $d_{fg}(X) \leq \varepsilon$.

Proof. It holds that $\|x - x_{0i}\|_2 \leq \sqrt{n} \|x - x_{0i}\|_\infty$. Then it follows that $\|x - x_{0i}\|_2 \leq r$ is satisfied if $\|x - x_{0i}\|_\infty \leq \frac{r}{\sqrt{n}}$. This implies that to cover X with hypercubes $C_c(x_{0i})$ it is necessary that the width of the hypercubes is less or equal to $\frac{2r}{\sqrt{n}}$. This leads to the number of models N_{m_i} that is sufficient

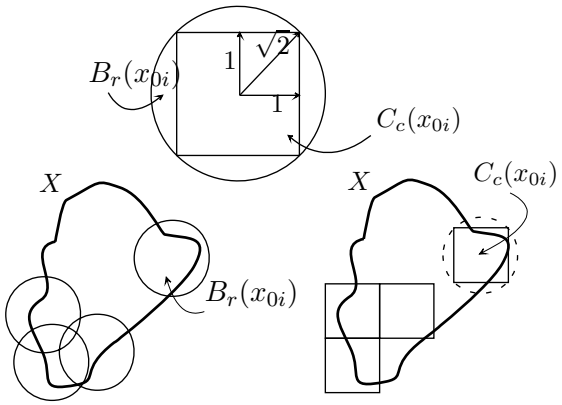


Figure 4.2: *The compact operating space X has to be covered by hyperballs $B_r(x_{0i})$. Therefore the hyperball $B_r(x_{0i})$ is inner approximated with a hypercube $C_c(x_{0i})$ of maximum volume. It holds that $\|x - x_{0i}\|_2 \leq \sqrt{n}\,\|x - x_{0i}\|_\infty$. Then it follows that $\|x - x_{0i}\|_2 \leq r$ is satisfied if $\|x - x_{0i}\|_\infty \leq \frac{r}{\sqrt{n}} = c$. The region X can now be covered constructively with hypercubes $C_c(x_{0i})$, which, since $C_c(x_{0i}) \subseteq B_r(x_{0i})$ also implies $X \subseteq (\cup_i B_r(x_{0i}))$.*

to cover X in one direction, that is $N_{m_i} = \left\lceil \frac{2e_i}{2r}\sqrt{n} \right\rceil = \left\lceil \frac{L\sqrt{n}}{\varepsilon} e_i \right\rceil$, since $r = \frac{\varepsilon}{L}$. For an n dimensional operating-space X the total number of models $N_m = \Pi_{i=1}^n N_{m_i} = \Pi_{i=1}^n \left\lceil \frac{L\sqrt{n}}{\varepsilon} e_i \right\rceil$. ■

The total number of models N_m sufficient to approximate the system (4.2) with a PLM (4.3,4.4) with ε-accuracy, increases with, $N_m \sim \Pi_{i=1}^n N_{m_i} \sim \left(\frac{L\sqrt{n}}{\varepsilon}\right)^n \Pi_{i=1}^n e_i$.

From Figure 4.4 it becomes clear that in practice, when ε is chosen in advance, $n = \dim(x)$ is a limiting factor of the modelling approach since the total number of models N_m increases exponentially with the dimension of the scheduling space. This phenomenon is known as 'the curse of dimensionality' [24]. To enlarge the applicability and to reduce the complexity of the PLM it is therefore important to reduce, if possible the dimension of z to obtain $\dim(z) < \dim(x)$. Besides the dimension of z also the nonlinearity of the system as measured with the quantity L, the size of the compact operating space $\{e_i\}$ and of course the desired accuracy ε affects the total number of models within the PLM description.

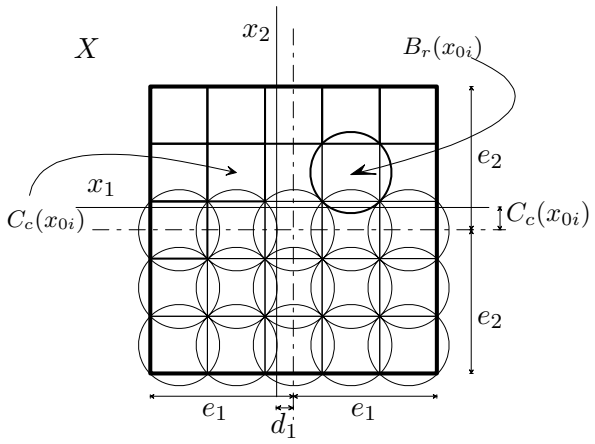


Figure 4.3: *Covering of compact region* $X = \{x \mid |x_i - d_i| \leq e_i, i = 1, \cdots, n\}$ *with hyperballs* $B_r(x_{0i})$*. First the hyperballs* $B_r(x_{0i})$ *are inner approximated with hypercubes* $C_c(x_{0i})$ *of maximum volume. Then* X *is covered easily with hypercubes* $C_c(x_{0i})$*. This implies that since* $B_r(x_{0i})$ *covers* $C_c(x_{0i})$*, also* X *is covered with hyperballs* $B_r(x_{0i})$ *as desired.*

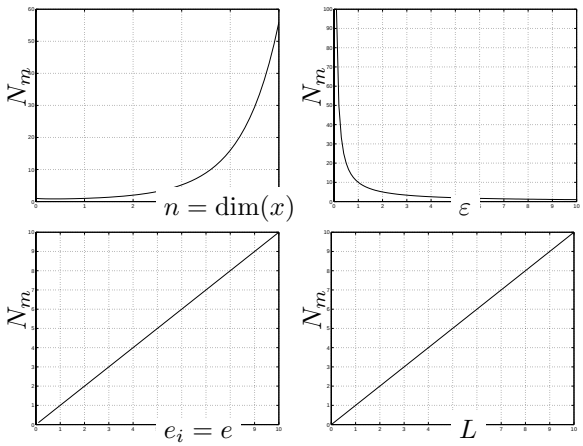


Figure 4.4: *Dependence of* N_m *on relevant parameters* $N_m \sim \left(\frac{L\sqrt{n}}{\varepsilon}\right)^n \Pi_{i=1}^n e_i$.

4.4 Upper bound on the Difference between Trajectories

We will relate the ε-accuracy between the right-hand side of two systems to the difference in solutions of the two systems.

With the bound $d_{fg} \leq \varepsilon$, that is the approximation error on the right-hand

side of two systems, an upper bound on the difference between the trajectories of the system (4.2) and the PLM (4.3) will be derived. The result is in fact based on a variation on the Gronwall Lemma, see e.g. [67] and [79].

THEOREM 4.4.1 *Given the system (4.2) with $f \in C^1$, and an approximate model (4.3), for instance a PLM, such that $d_{fg}(E) \leq \varepsilon$, then the solutions $\xi(t)$ of (4.2) and $\zeta(t)$ of (4.3) starting at $\xi(0) = \zeta(0)$ that remain in E are uniformly close on the interval $[0, T]$ in the following sense:*

$$\|\xi(t) - \zeta(t)\|_2 \leq \varepsilon \frac{e^{Lt} - 1}{L} \tag{4.10}$$

where L is an upper bound on $\left\| \frac{\partial f}{\partial x} \right\|_2$.

Proof. Note that $\xi(t) = \xi(0) + \int_0^t f(\xi(\tau))d\tau$ and $\zeta(t) = \zeta(0) + \int_0^t g(\zeta(\tau))d\tau$ from which it follows that $\xi(t) - \zeta(\tau) = \int_0^t \{f(\xi(\tau)) - g(\zeta(\tau))\}d\tau$. If we write $f(\xi(\tau)) - g(\zeta(\tau)) = f(\xi(\tau)) - f(\zeta(\tau)) + f(\zeta(\tau)) - g(\zeta(\tau))$, and substitute this in the integral equation, take norms and use the triangle inequality we have $\|\xi(t) - \zeta(t)\|_2 \leq \int_0^t \{\|f(\xi(\tau)) - f(\zeta(\tau))\|_2 + \|f(\zeta(\tau)) - g(\zeta(\tau))\|_2\}d\tau$. Since f is Lipschitz (in fact also C^1), that is $\|f(\xi(\tau)) - f(\zeta(\tau))\|_2 \leq L \|\xi(\tau) - \zeta(\tau)\|_2$ with L an upper bound on $\left\| \frac{\partial f}{\partial x} \right\|_2$, and $\sup_{\zeta(\tau) \in E} \|f(\zeta(\tau)) - g(\zeta(\tau))\|_2 \leq \varepsilon$ it is also true that

$$\|\xi(t) - \zeta(t)\|_2 \leq \int_0^t L \|\xi(\tau) - \zeta(\tau)\|_2 \, d\tau + \varepsilon t \tag{4.11}$$

Now define $F(t) := e^{-Lt}(\int_0^t L \|\xi(\tau) - \zeta(\tau)\|_2 \, d\tau - \int_0^t \varepsilon L \tau e^{-L(t-\tau)} d\tau)$ for all $t \geq 0$. Taking the time derivative of F yields $\dot{F}(t) = Le^{-Lt}(\|\xi(t) - \zeta(t)\|_2 - \int_0^t L \|\xi(\tau) - \zeta(\tau)\|_2 \, d\tau - \varepsilon t)$ from which we can conclude that in order to satisfy (4.11) for all $t \geq 0$ we have $F(t) = F(t) - F(0) = \int_0^t \dot{F}(t)d\tau \leq 0$. This implies that $\int_0^t L \|\xi(\tau) - \zeta(\tau)\|_2 \, d\tau \leq \int_0^t \varepsilon L \tau e^{-L(t-\tau)} d\tau$ has to hold. With this bound substituted in (4.11) we obtain $\|\xi(t) - \zeta(t)\|_2 \leq \int_0^t \varepsilon L \tau e^{-L(t-\tau)} d\tau + \varepsilon t = \varepsilon \frac{e^{Lt} - 1}{L}$. ■

The upper bound (4.10) will be useful if we analyze properties of trajectories of systems, such as convergence, based upon an ε-accurate approximate model. This will become clear later on.

4.5 Systems with Inputs

The derived approximation results are easily extended to handle systems with inputs and outputs. For reason of clarity we will only consider the state

equation of the system

$$\dot{x} = f(x,u) \tag{4.12}$$

$$y = h(x,u) \tag{4.13}$$

with state $x \in X \subseteq \mathbb{R}^n$, input $u \in U \subseteq \mathbb{R}^m$, and output $y \in Y \subseteq \mathbb{R}^p$. The objective is to approximate (4.12) with an ε-accurate approximate model

$$\dot{x} = g(x,u) \tag{4.14}$$

within a predefined compact region $\Psi \subseteq X \times U$. Since our main interest is in PLMs, we specialize to models with right-hand side

$$g(x,u) = \sum_{i=1}^{N_m} w_i(x,u) g_i(x,u) \tag{4.15}$$

where

$$g_i(x,u) = A_i x + B_i u + a_i$$

In that case the locally valid models, the $k-$th order Taylor series expansions of the system, are scheduled on the $n+m$ dimensional operating space Ψ to obtain an ε-accurate PLM. In particular, the following application of Theorem 4.2.2 and Theorem 4.3.1 will be a useful approximation result for systems with inputs.

COROLLARY 4.5.1 *Given $f(\psi) \in C^2(\Psi)$ the right-hand side of the state equation of (4.12) with $\Psi = \{\psi \mid |\psi_i - d_i| \leq e_i, i = 1, \cdots, n+m\}$, and $\varepsilon > 0$ arbitrary. There exists a right-hand side of a PLM, that is $g(\psi) = \sum_{i=1}^{N_m} w_i(\psi) g_i(\psi)$, with $N_m = \Pi_{i=1}^{n+m} \left\lceil \frac{e_i}{\sqrt{2\varepsilon}} \sqrt{\lambda_\xi n^{1/2}(n+m)} \right\rceil$ finite such that $d_{fg}(\Psi) \leq \varepsilon$.*

Proof. Firstly, it is assumed that $f(x,u)$ is C^2, that means at least 2 times continuously differentiable with respect to x and u. This allows us to decompose the right-hand side of the state equation of (4.12) as follows

$$f(x,u) = f_i^p(x,u) + F_i^p(x,u)$$

with $0 \leq p \leq 2$ and $f_i^p(x,u)$ a p-th order Taylor series expansion of f around $\psi_{0i} = (x_{0i}, u_{0i})$, and $F_i^p(x,u) := f(x,u) - f_i^p(x,u)$ is defined as the corresponding Taylor remainder. Since PLMs consist of (nonhomogeneous) linear models, we take $p = 1$. Application of the Mean Value Theorem allows us to rewrite the j-th component of vector $F_i^1(\psi)$ as

$$F_{i,j}^1(\psi) = \frac{1}{2}[\psi - \psi_{0i}]^T \frac{\partial^2 f_j}{\partial \psi^2}(\xi_i)[\psi - \psi_{0i}]$$

here $\psi = [x^T \ u^T]^T$, $\psi_{0i} = [x_{0i}^T \ u_{0i}^T]^T$ and the matrix $\frac{\partial^2 f_j}{\partial \psi^2}(\xi_i) = \left[\frac{\partial^2 f_j}{\partial \psi_p \partial \psi_q}(\xi_i)\right]$ with $\xi_i \in [\psi_{0i}, \psi]$. It is assumed that the triples (A_i, B_i, a_i) that specify the linear models $g_i(\psi)$ from the PLM (4.14,4.15) are chosen as Taylor linearizations of $f(\psi)$ in the points ψ_{0i}, that is $g_i(\psi) = f_i^1(\psi)$ from (4.15). In that case the triples become $(\frac{\partial f}{\partial x}(x_{0i}, u_{0i}), \frac{\partial f}{\partial u}(x_{0i}, u_{0i}), f\,(x_{0i}, u_{0i}) - \frac{\partial f}{\partial x}(x_{0i}, u_{0i})x_{0i} - \frac{\partial f}{\partial u}(x_{0i}, u_{0i})u_{0i})$ and for the difference between the system and the PLM we have

$$\begin{aligned} f(\psi) - \sum_{i=1}^{N_m} w_i(\psi) f_i^1(\psi) &= \sum_{i=1}^{N_m} w_i(\psi)(f(\psi) - f_i^1(\psi)) \\ &= \sum_{i=1}^{N_m} w_i(\psi) F_i^1(\psi) \end{aligned} \tag{4.16}$$

With $r_i = \psi - \psi_{0i}$ we obtain

$$F_i^1(\psi) = \frac{1}{2}[r_i^T \frac{\partial^2 f_1}{\partial \psi^2}(\xi_i) r_i \cdots r_i^T \frac{\partial^2 f_n}{\partial \psi^2}(\xi_i) r_i]^T \tag{4.17}$$

Then the following upper bound for the contribution of model $g_i = f_i^1$ to the approximation error can be derived

$$\begin{aligned} \left\|F_i^1(\psi)\right\|_2 &\leq \frac{1}{2}(\sum_{j=1}^{n} \lambda_{j,\xi_i}^2 (r_i^T r_i)^2)^{1/2} \\ &\leq \frac{1}{2}\sqrt{n}\lambda_{\xi_i} \left\|\psi - \psi_{0i}\right\|_2^2 \end{aligned} \tag{4.18}$$

with

$$\begin{aligned} \lambda_{j,\xi_i} &= \max_{\xi_i}\{\left|\mathrm{eig}(\frac{\partial^2 f_j}{\partial \psi^2}(\xi_i))\right|\} \\ \lambda_{\xi_i} &= \max_{j}\{\lambda_{j,\xi_i}\} \end{aligned}$$

The upper bound is conservative since by taking the maximum it is assumed that the maximum nonlinearity (as measured with the maximum absolute eigenvalue of the Hessian matrices associated with the Taylor remainder) can occur in every scalar equation of the set of equations, see (4.12). With the norm bound of the second order term as given in (4.18) it follows from (4.16) that:

$$\left\|f(\psi) - g(\psi)\right\|_2 \leq \sum_{i=1}^{N_m} w_i(\psi) \frac{1}{2}\sqrt{n}\lambda_{\xi_i} \left\|\psi - \psi_{0i}\right\|_2^2 \tag{4.19}$$

The approximation error is smaller than ε if

$$\sum_{i=1}^{N_m} w_i(\psi)\frac{1}{2}\sqrt{n}\lambda_{\xi_i}\left\|\psi-\psi_{0i}\right\|_2^2 \leq \varepsilon \quad \forall\psi\in\Psi \tag{4.20}$$

Rearranging the terms in (4.20) yields

$$\sum_{i=1}^{N_m} \rho_i(\psi)\lambda_{\xi_i}\left\|\psi-\psi_{0i}\right\|_2^2 - \frac{2\varepsilon}{\sqrt{n}}\} \leq 0 \quad \forall\psi\in\Psi \tag{4.21}$$

If we substitute $\lambda_\xi = \max\{\lambda_{\xi i}\}$ for $\lambda_{\xi i}$, which means that the maximum nonlinearity can occur everywhere within the operating space, one obtains

$$\sum_{i=1}^{N_m} \rho_i(\psi)\{\lambda_\xi\left\|\psi-\psi_{0i}\right\|_2^2 - \frac{2\varepsilon}{\sqrt{n}}\} \leq 0 \quad \forall\psi\in\Psi \tag{4.22}$$

This condition can be satisfied for finite N_m within a compact set, that is if $\Psi \subseteq (\cup_{i\in I_{N_m}} B_r(\psi_{0i}))$ with radius $r = \sqrt{\frac{2\varepsilon}{\sqrt{n}\lambda_\xi}}$. This is just an application of Theorem 4.2.2. Now by application of Theorem 4.3.1 and given $\Psi = \{\psi \mid |\psi_i - d_i| \leq e_i, i = 1, \cdots, n+m\}$ it follows that it suffices to construct a PLM with $N_m = \Pi_{i=1}^{n+m}\left\lceil \frac{e_i}{r}\sqrt{n+m}\right\rceil = \Pi_{i=1}^{n+m}\left\lceil \frac{e_i}{\sqrt{2\varepsilon}}\sqrt{\lambda_\xi\sqrt{n}(n+m)}\right\rceil$ to assure that $d_{fg}(\Psi) \leq \varepsilon$. ■

EXAMPLE 4.5.2 *Consider the following application of Corollary 4.5.1. The objective is to approximate the system*

$$\dot{x} = f(x,u) = x^2 + xu \tag{4.23}$$

with a PLM

$$\dot{x} = g(x,u) = \sum_{i=1}^{N_m} w_i(x,u)g_i(x,u)$$

where

$$g_i(x,u) = A_i x + B_i u + a_i$$

such that $d_{fg}(\Psi) \leq 1$ *for* $\Psi = \{(x,u) \mid |x| \leq 2, |u| \leq 1\}$. *The right-hand side of (4.23) can be rewritten as*

$$f(x,u) = f_i^1(x,u) + F_i^1(x,u)$$

with the first order Taylor series

$$\begin{aligned} f_i^1(x,u) &= f\left(x_{0i},u_{0i}\right) + \frac{\partial f}{\partial x}(x_{0i},u_{0i})(x-x_{0i}) + \frac{\partial f}{\partial u}(x_{0i},u_{0i})(u-u_{0i}) \\ &= -x_{0i}^2 - x_{0i}u_{0i} + (2x_{0i}+u_{0i})x + x_{0i}u \end{aligned}$$

and the corresponding Taylor remainder

$$F_i^1(x,u) := f(x,u) - f_i^1(x,u) = \frac{1}{2}\begin{bmatrix} x-x_{0i} & u-u_{0i} \end{bmatrix} \frac{\partial^2 f}{\partial \psi^2} \begin{bmatrix} x-x_{0i} \\ u-u_{0i} \end{bmatrix}$$

where $\frac{\partial^2 f}{\partial \psi^2} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$. *We obtain for* $\lambda_\xi = \max\{\left|\mathrm{eig}(\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix})\right|\} = 1+\sqrt{2}$. *Then to achieve* ε*-accuracy, it suffices to construct a PLM with* N_m *models, where*

$$N_m = \left\lceil \frac{2}{\sqrt{2}}\sqrt{2.41*2} \right\rceil * \left\lceil \frac{1}{\sqrt{2}}\sqrt{2.41*2} \right\rceil = 4*2 = 8$$

The operating points (x_{0i},u_{0j}) *with* $i \in \{1,2,3,4\}$ *and* $j \in \{1,2\}$ *are chosen equidistantly since the upper bound for the number of models is based on a worst case scenario, namely the maximum nonlinearity as measured with the bound on the Taylor remainder can occur at every place in the operating space. The corresponding triples defining the polytopic model are*

$$(A_i,B_i,a_i) = (2x_{0i}+u_{0j}, x_{0i}, -x_{0i}^2 - x_{0i}u_{0j})$$

with centers at

$$\begin{aligned} (x_{01},x_{02},x_{03},x_{04}) &= (-1.5,-0.5,0.5,1.5) \\ (u_{01},u_{02}) &= (-0.5,0.5) \end{aligned}$$

uniformly distributed over the operating space, see Figure 4.5.

The model validity functions are chosen Gaussian functions, namely

$$\rho_i(\psi) = e^{-7(\psi-\psi_{0i})^T(\psi-\psi_{0i})}$$

With the objective $d_{fg}(\Psi) \leq \varepsilon$, *model validity is made precise, meaning that the operating regimes* Ψ_i *are naturally induced by means of the local model validity regions* $B_r(x_{0i})$. *The operating regimes are constructed such that* $\Psi_i \subseteq B_r(x_{0i})$ *and* $\Psi \subseteq \cup_{i\in\{1,\ldots,N_m\}}\Psi_i$.

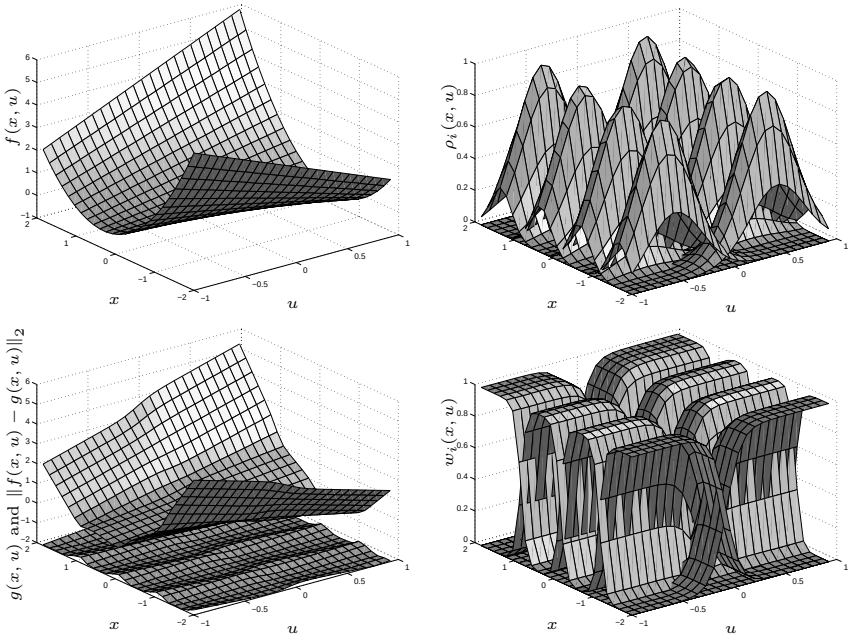


Figure 4.5: *System and ε-accurate PLM constructed as suggested in Corollary 4.5.1. Uniform distribution of the scheduling regimes Z_i over the scheduling space which equals the operating space, that is $Z = \Psi$.*

4.6 Systems with Structure

When the operating space is high dimensional, the 'curse of dimensionality' will restrict the applicability of the polytopic linear modelling approach. The core of this problem is that the number of operating regimes needed to uniformly partition the operating space increases exponentially with the dimension of the operating space, see Theorem 4.3.1 and Figure 4.4. Uniform partitioning is often not necessary (as will become clear later on), but the problem is still significant. However, in some cases the models can be scheduled on a space of lower dimension, which will reduce the curse of dimensionality considerably. On the basis of the approximation results that where derived earlier, a suitable choice of variables $z = s(\psi)$, that constitute the scheduling space, will be determined and defined. A situation in which $\dim(z)$ can be chosen smaller than $\dim(\psi)$ is illustrated in the next example.

EXAMPLE 4.6.1 *Consider the system*

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = 3x_1 + x_2^2$$

which has two important features that restrict the upper bound (4.18) in a structural way. First, one of the two scalar differential equations with right-hand side $f_i(\psi)$ is linear, and secondly only x_2 appears nonlinear in this system description. As a result we take $z = x_2$, because all other variables, in this case only x_1, appear linear in $f_i(\psi)$. In this case

$$\frac{\partial^2 f_1}{\partial \psi^2}(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$\frac{\partial^2 f_2}{\partial \psi^2}(\xi) = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

The Taylor remainder is

$$F^1(\psi) = \frac{1}{2} \begin{bmatrix} [\psi - \psi_{0i}]^T \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} [\psi - \psi_{0i}] \\ [\psi - \psi_{0i}]^T \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} [\psi - \psi_{0i}] \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ (x_2 - x_{20})^2 \end{bmatrix}$$

and depends only on x_2 since the system is linear in x_1. The Taylor remainder measures the nonlinearity of the system. In this case

$$\left\| F^1(\psi) \right\|_2 \leq \frac{1}{2} \sqrt{n} \lambda_{\xi_i} \left\| z - z_{0i} \right\|_2^2 \tag{4.24}$$

with $z = x_2$. Moreover when there are linear differential equations $f_j(\psi)$, like $f_1(\psi)$, we can further reduce conservatism of the upper bound (4.18) by replacing it with

$$\left\| F^1(\psi) \right\|_2 \leq \frac{1}{2} \sqrt{n_N} \lambda_{\xi_i} \left\| z - z_{0i} \right\|_2^2 \tag{4.25}$$

where n_N is the number of nonlinear differential equations $f_j(\psi)$.

The example shows that the upper bound on the approximation error (4.25) compared with (4.18) can be significantly reduced if the system shows some structural properties that can be exploited, e.g. linearity of the system with respect to some variables. In this section the objective is to reveal some of these structural properties. From the preceding example and the application of Theorem 4.2.2 and Theorem 4.3.1 we have the following result concerning the reduction of the operating space.

COROLLARY 4.6.2 *Given $f(\psi) = F\psi_L + f_1(\psi_N)$ with $f_1(\psi_N) \in C^2(E)$, E an open subset of $\mathbb{R}^{n_Z}$ where $\psi_L \in \Psi_L$, $\psi_N \in \Psi_N$, and $\Psi_L \times \Psi_N = \Psi$ and*

$f_1 : \Psi_N \to R^n$ is a nonlinear n-dimensional vector valued function with $n - n_N$ scalar linear components and $F \in R^{n \times \dim(\psi_L)}$, and $\varepsilon > 0$ arbitrary. Within any set Ψ, with Ψ_N compact such that $\Psi_N \subset E$ there exists a right-hand side of a PLM, with $z = \psi_N$, that is $g(\psi) = \sum_{i=1}^{N_m} w_i(z) g_i(\psi)$ with N_m finite such that $d_{fg}(\Psi) \leq \varepsilon$. Moreover given $\Psi_N = Z = \{z \mid |z_i - d_i| \leq e_i, i = 1, \cdots, n_Z\}$, it suffices to construct a PLM (4.14,4.15) with $N_m = \Pi_{i=1}^{n_z} \left\lceil \frac{e_i}{\sqrt{2\varepsilon}} \sqrt{\lambda_\xi n_N^{1/2} n_z} \right\rceil$.

Proof. Define $g_i(\psi) := F\psi_L + f_{1i}^1(\psi_N)$. Then it follows that $f(\psi) - g(\psi) = \sum_{i=1}^{N_m} w_i(z)\{f_1(\psi_N) - f_{1i}^1(\psi_N)\}$. Corollary 4.5.1 can now be applied, which shows that if Ψ_N is compact and with $z = \psi_N$ a finite number of models $N_m = \Pi_{i=1}^{n_z} \left\lceil \frac{e_i}{\sqrt{2\varepsilon}} \sqrt{\lambda_\xi n_N^{1/2} n_z} \right\rceil$ is sufficient to achieve ε-accuracy. ■

It is shown that the operating vector z must be chosen such that it captures the system's nonlinearities.

Another structure that appears often in practice, are so-called affine right-hand sides of systems:

$$f(\psi) = f(\psi_L, \psi_N) = f_1(\psi_N) + f_2(\psi_N)\psi_L \tag{4.26}$$

where $\psi_L \in \Psi_L$, $\psi_N \in \Psi_N$, and $\Psi_L \times \Psi_N = \Psi$. Furthermore $f_1 : \Psi_N \to R^n$ and $f_2 : \Psi_N \to R^{n \times \dim(\psi_L)}$ are nonlinear vector- and matrix-valued functions, respectively. As an example, models of mechanical systems are often affine in the input [57]. For the right-hand side of the state equation of such a mechanical system it holds that $\psi_N = x$, the state, and $\psi_L = u$, the input. It will be the objective to approximate (4.26) with a PLM

$$g(\psi) = \sum_{i=1}^{N_m} w_i(z) g_i(\psi) \tag{4.27}$$

where

$$g_i(\psi) = A_i^* \psi + a_i^*$$

and where $z = s(\psi)$.

THEOREM 4.6.3 *Assume $f_i \in C^1(E)$ as given by (4.26), $\Psi = \{\psi \mid |\psi_i - d_i| \leq e_i, i = 1, \cdots, n + m\}$, and $\epsilon > 0$ arbitrary. It suffices to choose $z = \psi_N$ from (4.27) such that $d_{fg}(\Psi) \leq \epsilon$. Furthermore if $f_i \in C^2(E)$ it suffices to construct a PLM with $N_m = \Pi_{i=1}^{n_Z} N_{m_i}$ models to assure that $d_{fg}(\Psi) \leq \epsilon$. Here $N_{m_i} = \left\lceil \frac{2e_i}{2r_k} \sqrt{n_Z} \right\rceil$ the number of models in direction i, n_Z equals the*

dimension of z, and

$$r_k = \begin{cases} r_1 = \frac{\varepsilon}{\lambda_{\xi,2}^{1/2}\|\psi_L\|_2}\left(-\frac{\lambda_{\xi,2}\|\psi_L\|_2^2}{\sqrt{n}\lambda_{\xi,1}\varepsilon} + \sqrt{2\frac{\lambda_{\xi,2}\|\psi_L\|_2^2}{\sqrt{n}\lambda_{\xi,1}\varepsilon} + \frac{\lambda_{\xi,2}^2\|\psi_L\|_2^4}{n\lambda_{\xi,1}^2\varepsilon^2}}\right) \\ \quad if \;\; g(\psi) = \sum_{i=1}^{N_m} w_i(\psi_N) g_i(\psi) \\ r_0 = \varepsilon\frac{\frac{1}{\lambda_{\xi,1}^{1/2}\lambda_{\xi,2}^{1/2}\|\psi_L\|_2}}{\frac{1}{\lambda_{\xi,1}^{1/2}} + \frac{1}{\lambda_{\xi,2}^{1/2}\|\psi_L\|_2}} \\ if\; g(\psi) = \sum_{i=1}^{N_m} w_i(\psi_N) g_i(\psi_L) \end{cases} \tag{4.28}$$

Proof. Fix an arbitrary $\psi = (\psi_L, \psi_N) \in \Psi = \Psi_L \times \Psi_N$. Then

$$\begin{aligned} f(\psi) - g(\psi) &= f(\psi_N, \psi_L) - \sum_{i=1}^{N_m} w_i(\psi_N)\, g_i(\psi_N, \psi_L) \\ &= \sum_{i=1}^{N_m} w_i(\psi_N)\left\{f_1(\psi_N) + f_2(\psi_N)\psi_L - g_i(\psi_N, \psi_L)\right\} \\ &= \sum_{i=1}^{N_m} w_i(\psi_N)\left\{f_1(\psi_N) - g_{i1}(\psi_N) + f_2(\psi_N)\psi_L - g_{i2}(\psi_L)\right\} \end{aligned}$$

In the last line we split the linear function $g_i : \Psi \to R^n$ into two linear functions $g_{i1} : \Psi_N \to R^n$ and $g_{i2} : \Psi_L \to R^n$, that is $g_i(\psi) = g_{i1}(\psi_N) + g_{i2}(\psi_L)$. With $g_{i2}(\psi_L) = \Gamma_i\psi_L$, where Γ_i is a not yet specified constant parameter matrix. Then we have

$$\begin{aligned} &\|f(\psi) - g(\psi)\|_2 \\ &= \left\|\sum_{i=1}^{N_m} w_i(\psi_N)\left\{f_1(\psi_N) - g_{i1}(\psi_N) + (f_2(\psi_N) - \Gamma_i)\psi_L\right\}\right\|_2 \\ &\le \left\|\sum_{i=1}^{N_m} w_i(\psi_N)\left\{f_1(\psi_N) - g_{i1}(\psi_N)\right\}\right\|_2 \\ &+ \left\|\sum_{i=1}^{N_m} w_i(\psi_N)\left\{f_2(\psi_N) - \Gamma_i\right\}\psi_L\right\|_2 \\ &\le \left\|f_1(\psi_N) - \sum_{i=1}^{N_m} w_i(\psi_N)\, g_{i1}(\psi_N)\right\|_2 \\ &+ \|\psi_L\|_2 \left\|f_2(\psi_N) - \sum_{i=1}^{N_m} w_i(\psi_N)\,\Gamma_i\right\|_2 \end{aligned}$$

By application of Theorem 4.2.2 the first and the second term can be made arbitrarily small with a finite number of models. There is some freedom left. The first term can be made arbitrarily small by choosing $g_{i1} = f_{1i}^k$ with $k \in \{0, 1\}$, that is a zero-th or first order Taylor series expansion of $f_1(\psi_N)$ around ψ_{N0i}. The case that $k = 0$, the function $g_i(\psi) = g_{i1}(\psi_{N0i}) + g_{i2}(\psi_L) = g_i(\psi_L)$ otherwise $g_i(\psi) = g_{i1}(\psi_N) + g_{i2}(\psi_L) = g_i(\psi_N, \psi_L)$.

Following the analysis as presented in Section 4 we determine the number of models N_m, and the model parameters from the PLM description needed to approximate (4.26) within the region Ψ with ε-accuracy.

First consider the case $k = 1$, where we assume $f_1(\psi_N)$ to be C^2, then

$$\begin{aligned} f_1(\psi_N) = f_1(\psi_{N0i}) + \frac{\partial f_1}{\partial \psi_N}(\psi_{N0i})(\psi_N - \psi_{N0i}) \\ + F_{1i}^1(\psi_N) \end{aligned} \tag{4.29}$$

The last term of (4.29) is the Taylor remainder. Application of the Mean Value Theorem allows us to rewrite the j-th component of $F_{1i}^1(\psi_N)$ as

$$F_{1i,j}^1(\psi) = \frac{1}{2}[\psi_N - \psi_{N0i}]^T \frac{\partial^2 f_{1j}}{\partial \psi_N^2}(\xi_i)[\psi_N - \psi_{N0i}]$$

here the matrix $\frac{\partial^2 f_{1j}}{\partial \psi^2}(\xi_i) = \left[\frac{\partial^2 f_{1j}}{\partial \psi_{Np} \partial \psi_{Nq}}(\xi_i)\right]$ with $\xi_i \in [\psi_{N0i}, \psi_N]$. If we assume that $f_2(\psi_N)$ is at least one time differentiable, then we obtain for

$$\begin{aligned} f_2(\psi_N) = f_2(\psi_{N0i}) \\ + F_{2i}^0(\psi_N) \end{aligned} \tag{4.30}$$

The last term of (4.30) is the Taylor remainder. Application of the Mean Value Theorem allows us to rewrite $F_{2i}^0(\psi_N)$ as

$$F_{2i}^0(\psi_N) = \frac{\partial f_2}{\partial \psi_N}(\xi_i)(\psi_N - \psi_{N0i})$$

with $\xi_i \in [\psi_{N0i}, \psi_N]$. We will approximate with a desired accuracy the affine system (4.26) in a predefined region Ψ with a PLM

$$\dot{x} = g(\psi) \tag{4.31}$$

$$= \sum_{i=1}^{N_m} w_i(\psi_N)\{f_1(\psi_{N0i}) + \frac{\partial f_1}{\partial \psi_N}(\psi_{N0i})(\psi_N - \psi_{N0i}) + f_2(\psi_{N0i})\psi_L\} \tag{4.32}$$

The difference between the system and the PLM becomes

$$f(\psi) - g(\psi) = \sum_{i=1}^{N_m} w_i(\psi_N) F_{1i}^1(\psi_N) + \sum_{i=1}^{N_m} w_i(\psi_N) F_{2i}^0(\psi_N)\psi_L \tag{4.33}$$

We split the Euclidean norm of (4.33) in two parts. The first and the second part are respectively the Euclidean norm of the first and the second term of (4.33). We can write

$$\begin{aligned} \|f(\psi) - g(\psi)\|_2 &\leq \varepsilon \\ &\leq \varepsilon_1 + \varepsilon_2 \end{aligned}$$

with

$$\left\| \sum_{i=1}^{N_m} w_i(\psi_N) F_{1i}^1(\psi_N) \right\|_2 \leq \varepsilon_1 \tag{4.34}$$

$$\left\| \sum_{i=1}^{N_m} w_i(\psi_N) F_{2i}^0(\psi_N)\psi_L \right\|_2 \leq \varepsilon_2 \tag{4.35}$$

We observe that (4.34) is similar to (4.16). This leads to a condition equivalent to (4.20):

$$\sum_{i=1}^{N_m} w_i(\psi_N) 1/2\sqrt{n}\lambda_{\xi,1} \|\psi_N - \psi_{N0i}\|_2^2 \leq \varepsilon_1 \tag{4.36}$$

where

$$\lambda_{\xi,1} = \max_i \left\{ \max_j \left\{ \max_{\xi_i} \left\{ \left| \text{eig}\left(\frac{\partial^2 f_{1j}}{\partial \psi^2}(\xi_i)\right) \right| \right\} \right\} \right\}$$

We obtain

$$\left\| \frac{\partial f_2}{\partial \psi_N}(\xi_i)(\psi_N - \psi_{N0i}) \right\|_2 \leq \lambda_{\xi i,2}^{1/2} \|\psi_N - \psi_{N0i}\|_2 \tag{4.37}$$

where

$$\lambda_{\xi i,2} = \max_{\xi_i} \left\{ \left| \text{eig}\left(\frac{\partial f_2}{\partial \psi_N}(\xi_i)^T \frac{\partial f_2}{\partial \psi_N}(\xi_i)\right) \right| \right\}$$

Then, with the norm given in (4.37) we obtain for (4.35)

$$\sum_{i=1}^{N_m} w_i(\psi_N)\lambda_{\xi,2}^{1/2} \left\|\psi_N - \psi_{N0i}\right\|_2 \left\|\psi_L\right\|_2 \leq \varepsilon_2 \tag{4.38}$$

where

$$\lambda_{\xi,2} = \max_i \left\{\lambda_{\xi i,2}\right\}$$

Now the question is how to choose ε_1 and ε_2 of (4.36) and (4.38) such that we can guaranty $\left\|f(\psi) - g(\psi)\right\|_2 \leq \varepsilon$. We substitute

$$\varepsilon_1 = \alpha\varepsilon \tag{4.39}$$

$$\varepsilon_2 = (1-\alpha)\varepsilon \tag{4.40}$$

where

$$\alpha \in (0,1)$$

With (4.39) and (4.40) we obtain $\varepsilon_1 + \varepsilon_2 = \varepsilon$. Rearranging (4.36) and (4.38) and substituting (4.39) and (4.40) gives

$$\sum_{i=1}^{N_m} \rho_i(\psi_N) \left(\left\|\psi_N - \psi_{N0i}\right\|_2^2 - \frac{2\varepsilon\alpha}{\sqrt{n}\lambda_{\xi,1}} \right) \leq 0 \qquad \forall \psi \in \Psi \tag{4.41}$$

$$\sum_{i=1}^{N_m} \rho_i(\psi_N) \left(\left\|\psi_N - \psi_{N0i}\right\|_2 - \frac{\varepsilon\,(1-\alpha)}{\lambda_{\xi,2}^{1/2} \left\|\psi_L\right\|_2} \right) \leq 0 \qquad \forall \psi \in \Psi \tag{4.42}$$

The upper bound on the number of models, that is N_m, is determined by the minimum of $r_1(\alpha) = \sqrt{\frac{2\varepsilon\alpha}{\sqrt{n}\lambda_{\xi,1}}}$ and $r_2(\alpha) = \frac{\varepsilon(1-\alpha)}{\lambda_{\xi,2}^{1/2}\left\|\psi_L\right\|_2}$, since both (4.41) and (4.42) have to be satisfied. To construct a PLM with N_m as small as possible, α has to be computed as the one that maximizes $\min(r_1(\alpha), r_2(\alpha))$, that is $\alpha^* = \arg\max_{\alpha\in(0,1)} \min(r_1(\alpha), r_2(\alpha))$. If we assume that there exists an $\alpha^* \in (0,1)$ such that $r_1(\alpha^*) = r_2(\alpha^*)$ then for any nonzero perturbation $\delta\alpha$ such that $\alpha^* + \delta\alpha \in [0,1]$ the following holds $\min(r_1(\alpha^*), r_2(\alpha^*)) = r_1(\alpha^*) = r_2(\alpha^*) > \max_{\{\delta\alpha\neq 0 | \alpha^*+\delta\alpha\in[0,1]\}} \min(r_1(\alpha^*+\delta\alpha), r_2(\alpha^*+\delta\alpha))$. This can be very easily seen since for any nonzero perturbation $\delta\alpha$ of α^* as suggested, either $r_1(\alpha^*+\delta\alpha) < r_1(\alpha^*)$ or $r_2(\alpha^*+\delta\alpha) < r_2(\alpha^*)$ and thus $\min(r_1(\alpha^*+\delta\alpha), r_2(\alpha^*+\delta\alpha))$ is smaller then $\min(r_1(\alpha^*), r_2(\alpha^*))$ which implies that a^* is the unique maximizer. Next it will be shown that there always exists an $\alpha^* \in (0,1)$. Therefore we set

$$r = r_1(\alpha) = r_2(\alpha) = \sqrt{\frac{2\varepsilon\alpha}{\sqrt{n_N}\lambda_{\xi,1}}} = \frac{\varepsilon\,(1-\alpha)}{\lambda_{\xi,2}^{1/2}\left\|\psi_L\right\|_2} \tag{4.43}$$

To determine α we solve the quadratic equation (4.43). So we obtain two roots

$$\alpha_1 = 1 + \frac{A}{2B^2} + \sqrt{\frac{A}{B^2} + \frac{A^2}{4B^4}}$$
$$\alpha_2 = 1 + \frac{A}{2B^2} - \sqrt{\frac{A}{B^2} + \frac{A^2}{4B^4}}$$

where

$$A = \frac{2\varepsilon}{\sqrt{n_N}\lambda_{\xi,1}}$$
$$B = \frac{\varepsilon}{\lambda_{\xi,2}^{1/2} \left\|\psi_L\right\|_2}$$

The parameters $(\varepsilon, n_N, \lambda_{\xi,1}, \lambda_{\xi,2}, \|\psi_L\|_2)$ of A and B are always positive. From this it follows that we must reject α_1 as possible solution for α^* because $\alpha_1 > 1$. Only α_2 is left as a possible solution, in Figure 4.6 we see that $0 < \alpha_2 < 1$ for every $A/B^2 > 0$. We get with $\alpha^* = \alpha_2$ for the radius, (4.43)

$$r = \frac{\varepsilon}{\lambda_{\xi,2}^{1/2} \left\|\psi_L\right\|_2} \left(-\frac{\lambda_{\xi,2} \left\|\psi_L\right\|_2^2}{\sqrt{n}\lambda_{\xi,1}\varepsilon} + \sqrt{2\frac{\lambda_{\xi,2} \left\|\psi_L\right\|_2^2}{\sqrt{n}\lambda_{\xi,1}\varepsilon} + \frac{\lambda_{\xi,2}^2 \left\|\psi_L\right\|_2^4}{n\lambda_{\xi,1}^2\varepsilon^2}} \right) \tag{4.44}$$

For the case $k = 0$, the approximation errors are based on the zero-th order

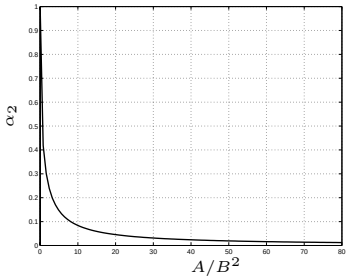


Figure 4.6: *Plot of* α_2, $0 \leq \alpha_2 \leq 1$ *is valid for any* $A/B^2 \geq 0$.

Taylor remainder both for f_1 and f_2. With an analysis similar to the one above, the radius r can be computed for the case $k = 0$, as follows

$$r = \varepsilon \frac{\frac{1}{\lambda_{\xi,1}^{1/2}\lambda_{\xi,2}^{1/2}\|\psi_L\|_2}}{\frac{1}{\lambda_{\xi,1}^{1/2}} + \frac{1}{\lambda_{\xi,2}^{1/2}\|\psi_L\|_2}} \tag{4.45}$$

The result now follows from the application of Theorem (4.3.1). ■

The result is illustrated by means of two examples. It is clear that the curse of dimensionality can not be avoided, but the reduction in dimension considerably reduces this problem for a significant number of modelling problems, as we shall illustrate with these examples. The first example shows the case that $r_k = r_1$. The second example illustrates the case that $r_k = r_0$.

EXAMPLE 4.6.4 *Consider the following application of Theorem 4.6.3 for the case $k = 1$. The objective is to approximate the system*

$$\dot{x} = f(x,u) = x^2 + xu \tag{4.46}$$

with a PLM

$$\dot{x} = g(x,u) = \sum_{i=1}^{N_m} w_i(x) g_i(x,u)$$

where

$$g_i(x,u) = A_i x + B_i u + a_i$$

such that $d_{fg}(\Psi) \leq 1$ for $\Psi = \{(x,u) \mid |x| \leq 2, |u| \leq 1\}$. The right-hand side of (4.46) can be rewritten as follows

$$f(x,u) = f_1(x) + f_2(x)u$$

with

$$f_1(x) = x^2 \tag{4.47}$$
$$f_2(x) = x \tag{4.48}$$

Rewrite (4.47) as follows

$$f_1(x) = f_{1i}^1(x) + F_{1i}^1(x)$$

with the first order Taylor series

$$\begin{aligned} f_{1i}^1(x) &= f_1(x_{0i}) + \frac{\partial f_1}{\partial x}(x_{0i})(x - x_{0i}) \\ &= -x^2 + 2x_{0i}x \end{aligned}$$

and the corresponding Taylor remainder

$$F_{1i}^1(x) := f_1(x) - f_{1i}^1(x) = \frac{1}{2}\frac{\partial^2 f_1}{\partial x^2}(x - x_{0i})^2$$

where $\frac{\partial^2 f_1}{\partial x^2} = 2$. *We obtain* $\lambda_{\xi,1} = 2$. *Next rewrite (4.48) as follows*

$$f_2(x) = f_{2i}^0(x) + F_{2i}^0(x)$$

with the zero-th order Taylor series

$$\begin{aligned} f_{2i}^0(x) &= f_2(x_{0i}) \\ &= x_{0i} \end{aligned}$$

and the corresponding Taylor remainder

$$F_{2i}^1(x) := f_2(x) - f_{2i}^0(x) = \frac{\partial f_2}{\partial x}(x - x_{0i})$$

where $\frac{\partial f_2}{\partial x} = 1$. *We obtain* $\lambda_{\xi,2} = 1$. *We find for the radius (4.28)* $r_1 = 0.618$. *Then it suffices to construct a PLM with* N_m *models, where*

$$N_m = \left\lceil \frac{2}{0.618}\sqrt{1} \right\rceil = 4$$

Given that the operating regimes are uniformly distributed over the operating space we find for the centres of the operating regimes

$$(x_{01}, x_{02}, x_{03}, x_{04}) = (-1.5, -0.5, 0.5, 1.5)$$

The corresponding triples defining the polytopic model are

$$(A_i, B_i, a_i) = (2x_{0i}, x_{0i}, -x_{0i}^2)$$

see Figure 4.7. The model validity functions are chosen

$$\rho_i(x) = \frac{1}{2\left\|x - x_{0i}\right\|_2^2}$$

Compared to Example 4.5.2 the scheduling space is reduced. The result is a reduction of local model parameters by a factor two, that is 12 *model parameters now against* 24 *for Example 4.5.2.*

An important practical application of the approximation result, Theorem 4.2.2, for the case $k = 0$, is that it is always possible to eliminate dependence of the locally valid models on the variables z, at least if Ψ is compact. This was also shown in Theorem 4.2.2 and Theorem 4.6.3, and will be illustrated in the next example.

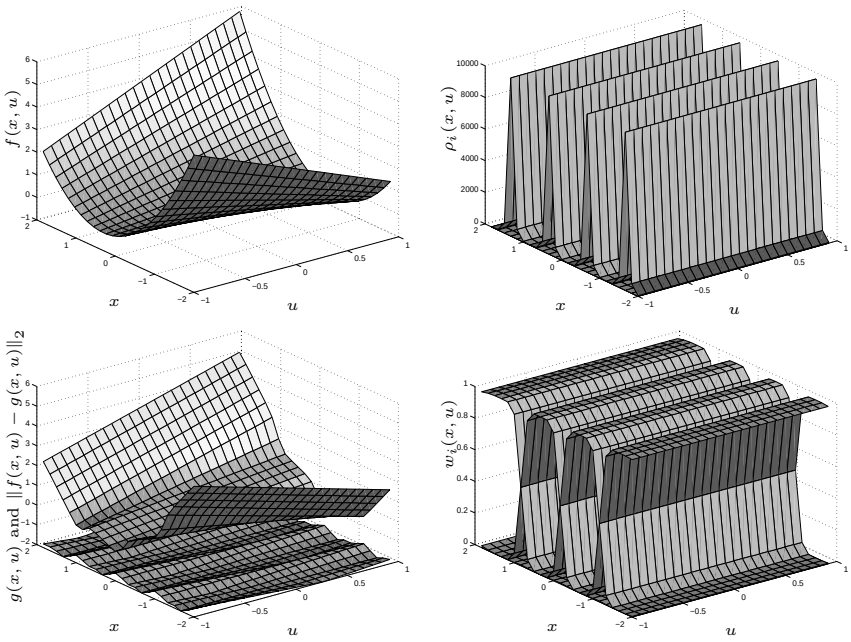


Figure 4.7: *System and ε-accurate PLM constructed as suggested in Theorem 4.6.3 for the case $k = 1$. Uniform distribution of the scheduling regimes Z_i over the scheduling space which in this case equals $Z = \Psi_N$.*

EXAMPLE 4.6.5 *Consider the following application of Theorem 4.6.3 for the case $k = 0$. The objective is to approximate the system*

$$\dot{x} = f(x,u) = x^2 + xu \tag{4.49}$$

with a PLM

$$\dot{x} = g(x,u) = \sum_{i=1}^{N_m} w_i(x) g_i(u)$$

where

$$g_i(u) = B_i u + a_i$$

such that $d_{fg}(\Psi) \leq 1$ for $\Psi = \{(x,u) \mid |x| \leq 2, |u| \leq 1\}$. The right-hand side of (4.49) can be rewritten as follows

$$f(x,u) = f_1(x) + f_2(x)u$$

with

$$f_1(x) = x^2 \tag{4.50}$$

$$f_2(x) = x \tag{4.51}$$

Rewrite (4.50) as follows

$$f_1(x) = f_{1i}^0(x) + F_{1i}^0(x)$$

with the zeroth order Taylor series

$$\begin{aligned} f_{1i}^0(x) &= f_1(x_{0i}) \\ &= x_{0i}^2 \end{aligned}$$

and the corresponding Taylor remainder

$$F_{1i}^0(x) := f_1(x) - f_{1i}^0(x) = \frac{\partial f_1}{\partial x}(\xi)(x - x_{0i})$$

where $\left|\frac{\partial f_1}{\partial x}\right| \leq |x + x_{0i}| = 4$. *We obtain* $\lambda_{\xi,1} = 4$. *Next rewrite (4.51) as follows*

$$f_2(x) = f_{2i}^0(x) + F_{2i}^0(x)$$

with the zero-th order Taylor series

$$\begin{aligned} f_{2i}^0(x) &= f_2(x_{0i}) \\ &= x_{0i} \end{aligned}$$

and the corresponding Taylor remainder

$$F_{2i}^1(x) := f_2(x) - f_{2i}^0(x) = \frac{\partial f_2}{\partial x}(x - x_{0i})$$

where $\frac{\partial f_2}{\partial x} = 1$. *We obtain* $\lambda_{\xi,2} = 1$. *We find for the radius (4.28)* $r_1 = 0.2$. *Then it suffices to construct a PLM with* N_m *models, where*

$$N_m = \left\lceil \frac{2}{0.2}\sqrt{1} \right\rceil = 10$$

Given that the operating regimes are uniformly distributed over the operating space we find for the centres of the operating regimes

$$(x_{01}, x_{02}, ..., x_{09}, x_{010}) = (-1.8, -1.4, ..., 1.4, 1.8)$$

The corresponding triples defining the polytopic model are

$$(A_i, B_i, a_i) = (0, x_{0i}, x_{0i}^2)$$

see Figure 4.8. Note that in this case the PLM description consists of $10 * 2 = 20$ *model parameters (of course not taken into account the* 10 *structural zeros). In the case of Example 4.6.4 there were* 12 *model parameters. The model validity functions are chosen*

$$\rho_i(x) = \begin{cases} 1 \text{ if } 4\left\|x - x_{0i}\right\|_2 - \varepsilon \leq 0 \\ 0 \text{ if } 4\left\|x - x_{0i}\right\|_2 - \varepsilon > 0 \end{cases}$$

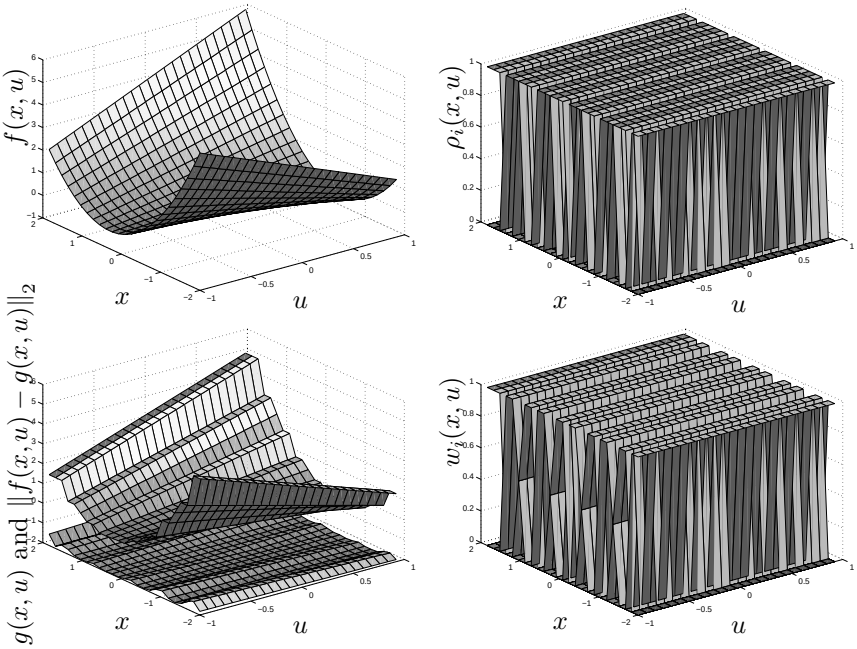


Figure 4.8: *System and ε-accurate PLM constructed as suggested in Theorem 4.6.3 for the case $k = 0$. Uniform distribution of the scheduling regimes Z_i over the scheduling space which in this case equals $Z = \Psi_N$.*

4.7 Notes and Comments

A universal approximator

We have shown that the right-hand side of a C^k ($k \in \{1, 2\}$) ordinary differential equation that describes the real system can be uniformly approximated to an arbitrary accuracy ε on any compact operating space with a PLM, by making the decomposition into operating regimes sufficiently fine. Sometimes also systems defined on an unbounded operating space can be approximated to an arbitrary accuracy ε with a finite number of operating regimes. The idea is to use a one to one coordinate transformation that maps the unbounded operating space into a compact operating space. In the new coordinates the derived results can be applied. More on how the one to one mapping may be constructed can be found in [83].

Systems with structure

We have seen that system structure, as presented by Corollary 4.6.2 and Theorem 4.6.3, can reduce the dimension of the operating space and thereby complexity of the model. Of course, the structures that we have analyzed and for

which model reduction, in the sense as described before was possible, depends on the choice of coordinates for the system. Stated otherwise, the distance d_{fg} that we have used and system structure as we have analyzed are not invariant under coordinate transformations. Therefore it could be useful to search for coordinate transformations, in such a way, that within the new coordinates the system can be approximated with a PLM consisting of fewer parameters, without affecting the approximation accuracy. One can think of transformations that transform the system to one of the proposed structures in this section.

More formally the question is under which conditions do there exist, and how can we find, a nonsingular coordinate transformation

$$\tilde{x} = T^{-1}(x)$$

such that within the new coordinates $\tilde{x}$ the system (4.2)

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x})$$

can be approximated with a PLM (4.3)

$$\dot{\tilde{x}} = \tilde{g}(\tilde{x})$$

in such a way that $d_{fg}(X) \leq \varepsilon$ is achieved, and in such a way that the PLM $\tilde{g}(\tilde{x})$ consists of less models and/or parameters then $g(x)$, the PLM obtained for the untransformed system. Since $f(x) - g(x) = \frac{\partial T(\tilde{x})}{\partial \tilde{x}} \left(\tilde{f}(\tilde{x}) - \tilde{g}(\tilde{x}) \right)$, it follows that $d_{fg} \leq \varepsilon$ holds if $\left\| \frac{\partial T(\tilde{x})}{\partial \tilde{x}} \right\|_2 \left\| \tilde{f}(\tilde{x}) - \tilde{g}(\tilde{x}) \right\|_2 \leq \varepsilon$ or equivalently

$$(d_{\tilde{f}\tilde{g}}(\tilde{X}) \leq \tilde{\varepsilon}) \Rightarrow (d_{fg}(X) \leq \varepsilon)$$

where $\tilde{\varepsilon} = \frac{\varepsilon}{\left\| \frac{\partial T(\tilde{x})}{\partial \tilde{x}} \right\|_2}$ and $\tilde{X} = T(X)$.

It is imaginable that within the new coordinates $\tilde{x}$, the transformed system $\tilde{f}(\tilde{x})$ exhibits some desired system structure in contrast to the untransformed system $f(x)$, that could reduce 'the curse of dimensionality' and therefore reduce the upper bound N_m. It is conceivable however, that the state transformation comes at the expense of a reduced accuracy margin, $\varepsilon^* < \varepsilon$ and an increased operating region $\tilde{X} > X$, that has to be covered by N_m models. It seems natural to search for transformations that reduce $\dim(z)$, since N_m increases inverse proportional with at most $\sqrt{\varepsilon}$, while N_m increases exponentially with $\dim(z)$. Also state transformations that transform a set of n first-order nonlinear differential equations into $n - n_N$ first-order linear differential equations together with n_N nonlinear first order differential equations

could considerably reduce the number of PLM parameters, at least if the contribution of $\left\| \frac{\partial T(\tilde{x})}{\partial \tilde{x}} \right\|_2$, $\lambda_{\xi,\tilde{x}}$, and $\tilde{X}$ is of less importance[2]. A more detailed treatment of simplifying state transformations is not the subject of this thesis, however.

[2]Note that under observability conditions it is always possible to find locally such state transformations [57].

Chapter 5

Stability, Controllability and Observability by Duality

5.1 Introduction

In the previous chapters, the PLM structure was introduced as a candidate model structure for modeling a class of nonlinear systems, and was shown to be flexible. From an approximation point of view, every other universal approximator could equally well be chosen, e.g. a polynomial or radial basis function expansion or a neural network [24], [30]. The choice for a PLM is however motivated by the intended application of the model, that is model based control. Model based control imposes serious restrictions on the model's complexity and structure. For control it is important that the model is accurate, and interpretable to some extent, since this may facilitate the design of a successful control strategy. Furthermore the model by itself should give the designer information about relevant properties of the system, such as equilibria, stability etc.. In addition these system properties have to be revealed from the PLM, or incorporated in the PLM constructively, without too much effort.

In this chapter the system analytic properties of PLMs are investigated. Besides the practical relevance of PLMs, these models are of course a subject of interest and study on their own. Hence, sometimes only properties of the PLM are revealed, while in addition sometimes the relation between properties of the real system and the PLM are given. Due to the resemblance between PLMs and linear state-space models, the intention is to derive easy verifiable (algebraic) conditions of practical use for control synthesis, by taking advantage of linear (and nonlinear) control theory.

In Section 5.3 stability is analyzed, which is a state property of the system. From a control point of view, stability of the closed loop system, is

one of the most important properties to analyze. Stability will play a central role in this thesis. A definition of stability is given, and two commonly used methods, namely the so-called first and second method of Lyapunov, are introduced to establish stability properties of the system. In Section 5.3 we specialize to PLMs. The presented Lyapunov methods are used to establish stability properties of the system based upon its approximation, the PLM. In Section 5.4, input-to-state properties of systems are analyzed. More specifically, controllability conditions are derived for PLMs. Controllability will play a role in checking feasibility of a control strategy. Moreover, these properties give insight on how a PLM structure can be chosen, such that after modelling a controllable model results. In Section 5.5, observability conditions are presented based on similarities between controllability and observability for PLMs. These conditions are dual to the ones derived for controllability. Finally, in Section 5.6, some notes and comments are made regarding the presented analysis of PLMs.

5.2 Lyapunov Stability

In this section stability of a continuous-time autonomous system

$$\dot{x} = f(x) \tag{5.1}$$

and more specifically also stability of its approximation, the PLM

$$\dot{x} = \sum_{i=1}^{N_m} w_i(x, v)\{A_i x + a_i\} \tag{5.2}$$

will be discussed. In particular it will be discussed how the PLM description can be used to establish local or global stability properties of the real system. To guarantee stability of (5.1) based on (5.2) we need to know how the PLM is related to the real system. At that stage the foregoing analysis concerning approximation with PLMs comes into play. Especially, one can think of the approximation error bounds derived earlier for the PLM, or the interpretation of the PLM as an uncertainty description for the real system. The ideas and concepts that will be introduced, will also play a central role if one is interested in control. A main objective then will be the stability of the closed loop system.

To analyze the qualitative behavior of a nonlinear system (5.1) around an equilibrium point x_{0i}, that is

$$0 = f(x_{0i}) \tag{5.3}$$

one often uses Lyapunov methods. More specifically, to decide about stability properties of (4.2) there are two Lyapunov methods available. These are the so-called first and second (or direct) method of Lyapunov. Before stating these two methods it will be made precise what is meant by the notion of stability and the type of Lyapunov stability that we want to discuss [64].

Definition 5.2.1 (Stability) *Consider the differential equation (4.2) with unique solution $\xi(t, x_0)$ passing through x_0 at time $t = 0$, i.e. $\xi(0, x_0) = x_0$.*

1. *An equilibrium point $x_{0i} \in X$ is called stable (in the sense of Lyapunov) if for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|x_{0i} - x_0\|_2 \leq \delta \Longrightarrow \|\xi(t, x_0) - x_{0i}\|_2 \leq \varepsilon \text{ for all } t \geq 0$$

2. *The equilibrium point $x_{0i} \in X$ is called an attractor if there exists $\delta > 0$ with the property that*

$$\|x_{0i} - x_0\|_2 \leq \delta \Longrightarrow \lim_{t\to\infty} \xi(t, x_0) = x_{0i}$$

3. *The equilibrium point is called locally asymptotically stable (in the sense of Lyapunov) if x_{0i} is both stable and an attractor.*

4. *The equilibrium point is called (locally) exponentially stable (in the sense of Lyapunov) if x_{0i} is (locally) asymptotically stable (in the sense of Lyapunov) with exponentially decay rate $\alpha > 0$, i.e.*

$$\lim_{t\to\infty} e^{\alpha t} \|\xi(t, x_0) - x_{0i}\|_2 = 0$$

The region of attraction associated with an equilibrium point x_{0i} is defined to be the set of all initial states $x_0 \in X$ for which $\xi(t, x_0) \to x_{0i}$ as $t \to \infty$. If this region coincides with X then x_{0i} is said to be a global attractor. We will say that an equilibrium x_{0i} is globally asymptotically stable if it is stable and globally attractive. We restrict ourself to (global) asymptotic stability, and sometimes more specifically to (global) exponential stability of an equilibrium point x_{0i} of (5.1).

In the first method of Lyapunov the local stability of the equilibrium x_{0i} for the system (5.1) is related to the stability of the linearization of (5.1) around the equilibrium point (5.3). Thus the linear dynamics in the local coordinates $x_i = x - x_{0i}$

$$\dot{x}_i = A_i x_i \tag{5.4}$$

with

$$A_i = \frac{\partial f}{\partial x}(x_{0i}) \tag{5.5}$$

is considered.

THEOREM 5.2.2 (FIRST METHOD OF LYAPUNOV) *The equilibrium point x_{0i} of (5.1) is locally asymptotically stable if the matrix A_i given in (5.5) is asymptotically stable, i.e. the matrix A_i has all its eigenvalues in the open left half plane. The equilibrium point is not stable if at least one of the eigenvalues of the matrix A has a positive real part.*

Essentially local asymptotic stability and instability of (5.1) can be decided via Theorem 5.2.2 provided x_{0i} is a hyperbolic equilibrium point, that means the linearized dynamics (5.4) has no eigenvalues with zero real part.

The second or direct method of Lyapunov for deciding about the (local) asymptotic stability of an equilibrium point x_{0i} involves the introduction of positive/negative definite functions and invariant sets. A function V is called positive definite on some neighborhood $N(x_{0i})$ of x_{0i} if $V(x_{0i}) = 0$, and $V(x) > 0$ for each $x \in N(x_{0i})$, $x \neq x_{0i}$. A set $W \subset X$ is called an invariant set for (4.2) if for all $x^1 \in W$ the solution $\xi(t, x^1)$ of (4.2) remains in W for all $t \in \mathbb{R}^+$. The idea of an invariant set is therefore that a solution remains in the set once it started there. A point x^1 is called an equilibrium point of (4.2) if the singleton $W = \{x^1\}$ is an invariant set. Equivalently, an equilibrium point defines a constant solution $\xi(t, x_0) = x_0$ of (4.2).

THEOREM 5.2.3 (SECOND METHOD OF LYAPUNOV) *Consider the dynamics of (5.1) around the equilibrium point x_{0i}. Let V be a C^1 positive definite function on some neighborhood $N(x_{0i})$ of x_{0i}. Then we have*

i *x_{0i} is locally stable if*

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \leq 0 \text{ for each } x \in N(x_{0i}) \tag{5.6}$$

ii *x_{0i} is locally asymptotically stable if (5.6) holds and the largest invariant set under the dynamics (5.1) contained in the set*

$$W = \{x \in N(x_{0i}) \mid \frac{\partial V}{\partial x} f(x) = 0\}$$

equals x_{0i}, that is

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) < 0 \text{ for each } x \in N(x_{0i}),\ x \neq x_{0i} \tag{5.7}$$

The equilibrium x_{0i} is globally stable respectively globally asymptotically stable if Theorem 5.2.3 holds with $N(x_{0i}) = X$, and if in addition $V(x)$ is (globally) proper [67]. A function V satisfying Theorem 5.2.3 is called a Lyapunov function for (5.1). One can interpret a Lyapunov function V as a sort of 'energy' measure of states, and in the case of asymptotic stability energy decreases along every nonconstant solution of (5.1) and approaches zero. Since $V(x) \to 0$ implies $x \to x_{0i}$, then this means that $\xi(t) \to x_{0i}$. The closed (energy) level sets $\{x \in X \mid V(x) = C\}$ are always approached from their exterior to their interior with increasing t and therefore define invariant sets, that is $x^1 \in W_C := \{x \in X \mid V(x) \leq C\}$ implies $\xi(t, x^1) \in W_C \ \forall t \in \mathbb{R}^+$. The second method of Lyapunov is useful in establishing stability properties of systems that have nonhyperbolic equilibrium points, that means systems of which the linearized dynamics (5.4) has some eigenvalues with real part identically zero. Moreover, in contrast to the first method of Lyapunov, the direct method of Lyapunov may be used to determine the region of attraction of an asymptotically stable equilibrium point. This is a desirable property we want to exploit, since one of the main ideas of the PLM framework is to extend model validity with respect to a single linearized model. In the case of stability this means that we want a more precise characterization of the domain of validity of the stability analysis with respect to an analysis based on the linearization. A drawback of the second method of Lyapunov for the study of the stability of an equilibrium point x_{0i}, is that in general there is no systematic way to come up with Lyapunov functions. For linear systems however, the existence of a quadratic Lyapunov function is a necessary and sufficient condition for asymptotic stability of a linear system. Also for nonlinear systems it is often useful to try a quadratic candidate Lyapunov function to establish some stability property. The stability results that follow are all based upon the existence of a quadratic Lyapunov function. Without loss of generality stability of the origin is considered. We start with a well known stability result.

PROPOSITION 5.2.4 *The following statements are equivalent.*

1. *The origin is an asymptotic stable equilibrium point of (5.4).*
2. *All eigenvalues $\lambda(A_i)$ of A_i have a strictly negative real part.*
3. *The linear matrix inequality*

$$A_i^T P + P A_i < 0 \tag{5.8}$$

admits a positive definite solution $P = P^T > 0$.

Moreover, if one of these statements hold, then the equilibrium x_{0i} of (4.2) is asymptotically stable.

Proof. We will proof only that 3 implies 1 because of its importance for the results to come. Proofs can be found in standard systems and control literature, for instance [67].

From Theorem 5.2.3 we know that the origin ($x_i = 0$) is an asymptotically stable equilibrium point of (5.4) if there exists a quadratic Lyapunov function $V(x_i) = x_i^T P x_i$ with $P = P^T > 0$ such that $\frac{dV(\xi_i(t))}{dt} < 0$ along every nonzero trajectory of (5.4). $\frac{dV(x_i)}{dt} = x_i^T(A_i^T P + P A_i)x_i$ and we see that $V(x_i) = x_i^T P x_i$ is indeed a Lyapunov function for the system (5.4) if $A_i^T P + P A_i < 0$. ■

Determining whether or not there exist solutions P to the Linear Matrix Inequalities (LMIs) $P = P^T > 0$ and $A^T P + PA < 0$ is called a feasibility problem. These LMIs define a convex optimization problem and can be solved efficiently and in a reliable way using convex optimization routines that are readily available [1]. Many problems originating from linear systems and control theory can be reformulated using LMIs, see [17], [64]. A lot of work is done to extend some of these LMI results for different types of nonlinear and uncertain systems, confer [17], [64], [39], [61]. Since the PLM shows similarities with a linear model, we only consider quadratic candidate Lyapunov functions and try to arrive at sufficient conditions for stability.

5.3 Stability of a PLM and Implications for the System

The objective is to derive easy verifiable stability conditions for both the system and its approximation, the PLM. The stability conditions are based on the PLM representation, together with a description of the mismatch between the system and the approximate PLM. Three PLM based stability conditions are derived that, under appropriate conditions, imply stability of the system. For each of these three conditions, a different viewpoint for the PLM will be adopted. Consecutively, the PLM will be viewed as

An uncertainty model That is, with the PLM an uncertainty model set will be associated, and under appropriate conditions, the system will be part of that model set. Thus, stability of the system is implied by stability of the model set.

An ε-accurate model with ε small enough That is, under appropriate conditions, stability of an ε-accurate PLM implies stability of the system with the same region of attraction, at least if ε is small enough and if the system possesses a local stability property. Since the PLM is a universal approximator, ε can be made as small as desired with a finite parametrization.

An ε-accurate model with bounds on Taylor remainders That is, the mismatch between an ε-accurate PLM based upon local linearizations (first order Taylor series) is bounded by Taylor remainders associated with the first order Taylor series. Under appropriate conditions, stability of the PLM implies stability of the system.

5.3.1 An Uncertainty Model

Firstly, an uncertainty model, a model set $\mathcal{M}$ will be associated with the PLM. The model set was already defined by (3.15), but since stability issues are addressed, which is a property of the state, it is restricted to

$$\mathcal{M}(\{A_i, a_i\}) := \left\{ (5.2) | \sum_{i=1}^{N_m} \omega_i(x,v) = 1,\ \omega_i(x,v) \geq 0 \right\} \tag{5.9}$$

Here $v \in V \subseteq \mathbb{R}^l$ is an external scheduling variable. $\mathcal{M}$ is defined as a collection of PLMs (5.2), where every PLM from the model set is represented by the same set of autonomous nonhomogeneous linear models (A_i, a_i). This means that the only difference between two PLMs from the same model set is the realization of the set of scheduling functions, which is constrained by $\sum_{i=1}^{N_m} \omega_i(x,v) = 1$ and $\omega_i(x,v) \geq 0$.

THEOREM 5.3.1 *$\mathcal{M}(\{A_i, 0\})$ is globally asymptotically stable if there exists $P = P^T > 0$ such that*

$$A_i^T P + P A_i < 0 \text{ for all } i \in \{1, ..., N_m\} \tag{5.10}$$

Proof. From Theorem 5.2.3 we know that the origin is an asymptotic stable equilibrium point of a PLM$\in \mathcal{M}(\{A_i, 0\})$ if there exists a quadratic Lyapunov function $V(x) = x^T P x$ with $P = P^T > 0$ that decreases along every nonzero trajectory of the PLM. That is, $\frac{dV(x)}{dt} = \sum_{i=1}^{N_m} w_i(x,v)\{x^T(A_i^T P + P A_i)x\} < 0$ for all $x \neq 0$. Note that by definition for all x, $\sum_{i=1}^{N_m} \omega_i(x,v) = 1$ and $\omega_i(x,v) \geq 0$. If (5.10) is satisfied then $\frac{dV(x)}{dt} < 0$ for all $x \neq 0$ independent of the particular realization of the scheduling functions. ■

COROLLARY 5.3.2 *If for the system (5.1) it holds that $f(0) = 0$ and $\frac{\partial f}{\partial x} \in \text{Co}\{A_i\}$ for all x with $i \in \{1, ..., N_m\}$, then global asymptotic stability of the origin of $\mathcal{M}(\{A_i, 0\})$ implies global asymptotic stability of (5.1).*

Proof. By the mean-value theorem we have $f(x) = f(x_{0i}) + \frac{\partial f}{\partial x}(\xi)(x - x_{0i})$ for some ξ that lies on the line segment between x and x_{0i}. Take $x_{0i} = 0$ and $f(x) = \frac{\partial f}{\partial x}(\xi)x$. Since by assumption $\frac{\partial f}{\partial x} \in \text{Co}\{A_i\}$, and by definition

$\text{Co}\{A_i\} = \{\sum_{i=1}^{N_m} w_i A_i \mid \sum_{i=1}^{N_m} w_i = 1, w_i \geq 0\}$, it follows that all trajectories of the system are also trajectories of the model set $\mathcal{M}(\{A_i, 0\})$. ■

Corollary 5.3.2 states that stability of (5.1) can be verified by studying stability of the model set $\mathcal{M}$, i.e. Theorem 5.3.1 can be applied. In this case, the model set $\mathcal{M}$ gives a description of the expected variation (uncertainty) in the system, and can be seen as an uncertainty model of the system.

5.3.2 An ε-Accurate Model with ε Small Enough

The next stability result is based upon [79] and the approximation results that were derived earlier. Informally, it states that stability of a PLM implies stability of the real system with at least the same region of attraction, if the approximation of the PLM to the real system is close enough and if the real system possesses a local stability property. First we make clear that conditions like Theorem 5.3.1 imply also exponential stability of the origin of a corresponding PLM. We therefore need the following preliminary result.

PROPOSITION 5.3.3 ([17]) *Suppose that* $\frac{dV(x)}{dt} \leq -2\alpha V(x)$ *for all trajectories, then* $V(x(t)) \leq V(x(0))e^{-2\alpha t}$, *so that* $\|x(t)\| \leq e^{-\alpha t}\kappa(P)^{1/2}\|x(0)\|$. *Here* $\kappa(P) = \|P\|\left\|P^{-1}\right\|$ *the condition number of the matrix* P.

The condition that $\frac{dV(x)}{dt} \leq -2\alpha V(x)$ for all x, implies exponential stability of the origin of the corresponding PLM and is equivalent to the LMI

$$A_i^T P + P A_i + 2\alpha P \leq 0 \text{ for all } i \in \{1, ..., N_m\} \tag{5.11}$$

The fact that Corollary 5.3.1 implies exponential stability can be seen as follows. Suppose that indeed $A_i^T P + P A_i < 0$ is satisfied. This means that there exists a positive definite matrix M such that $A_i^T P + P A_i + M \leq 0$. Since there always exists an $\alpha > 0$ such that $2\alpha P \leq M$, i.e. $\alpha \leq \frac{1}{2}\lambda_{\min}(D^{-T} M D^{-1})$ with $P = D^T D$, exponential stability is implied.

THEOREM 5.3.4 *Assume that the origin of the system (5.1) is locally asymptotically stable. Let the origin of an* ε*-accurate PLM (5.2) be exponentially stable with a region of attraction* X *and such that (5.11) holds with* $P = P^T > 0$. *Then this implies asymptotic stability of the origin of the system with region of attraction* X.

Proof. By application of the triangle inequality on norms it holds that

$$\|\xi(t)\| \leq \|\zeta(t)\| + \|\xi(t) - \zeta(t)\|$$

where $\xi(t)$ is the solution of (5.1) and $\zeta(t)$ of (5.2). By using the aforementioned norm bound together with Proposition 5.3.3 and (4.10) we obtain

$$\|\xi(t)\| \leq \delta(t,\varepsilon) = e^{-\alpha t}\kappa(P)^{1/2}\|\zeta(0)\| + \varepsilon\frac{e^{Lt}-1}{L}$$

From this expression it is clear that with ε small enough, $\delta(t,\varepsilon)$ and as a consequence $\|\xi(t)\|$ will decrease for small times. If $\delta(t,\varepsilon)$ decreases as a function of time, then its minimum is attained at

$$t = t_{\min} = \frac{1}{-\alpha - L}\ln\left(\frac{\varepsilon}{\alpha\kappa(P)^{1/2}\|\zeta(0)\|}\right)$$

Assume that the origin of the system (5.1) is locally asymptotically stable with region of attraction $B_{r_1}(0) \subseteq X$. If now ε is small enough such that $\delta(t_{\min},\varepsilon) \subset B_{r_1}(0)$, global stability follows. Since $\delta(t_{\min},\varepsilon) = \delta(\varepsilon)$ is a monotonically increasing function of ε on the domain $(0,\infty)$ and its limit, i.e. $\lim_{\varepsilon\to 0}\delta(\varepsilon) = 0$ it follows that there always exists an $\varepsilon > 0$ such that $\delta(t,\varepsilon) \subset B_{r_1}(0)$ for some positive time t. ■

Thus under appropriate conditions the PLM preserves global stability properties of the real system, at least if the approximation is close enough, that is ε small enough. Since the PLM is a universal approximator, ε can be made as small as desired with a finite parametrization.

5.3.3 An ε-Accurate Model with Bounds on Taylor Remainders

There is yet another natural way, and from a computational point of view attractive way to find sufficient conditions for stability of the origin of the system based upon an ε-accurate PLM. Consider again the system (5.1) and its ε-accurate approximation, the PLM (5.2), constructed as in Theorem 4.2.2 i.e. with $A_i = \frac{\partial f}{\partial x}(x_{0i})$ and $a_i = f(x_{0i}) - \frac{\partial f}{\partial x}(x_{0i})x_{0i}$. Note that if x_{0i} is an equilibrium point of (5.1), then also x_{0i} is an equilibrium of the PLM. So, equilibria of interest are easily preserved by a PLM. Let us consider asymptotic stability of the origin of the system.

The system (5.1) can be rewritten as

$$\dot{x} = \sum_{i=1}^{N_m} w_i(x)\{A_i x + a_i + F_i(x)\} \tag{5.12}$$

where $F_i(x) = f(x) - f(x_{0i}) - \frac{\partial f}{\partial x}(x_{0i})(x - x_{0i})$. From the proof of Theorem 4.2.2 we know that the mismatch between the system and the PLM is bounded

by

$$F_i(x)^T F_i(x) \leq L_i(x - x_{0i})^T (x - x_{0i}) \tag{5.13}$$

$$L_i(x - x_{0i})^T (x - x_{0i}) \leq \varepsilon^2 \tag{5.14}$$

for some finite positive number L_i. Remember that ε-accuracy was implied by letting $w_i(x) > 0$ only if (5.14) holds. The following condition, again based upon an ε-accurate PLM, and a norm-bound on the mismatch between the system and the PLM, is sufficient for stability of the origin of the system (5.1).

THEOREM 5.3.5 *Given a PLM (5.2) constructed as suggested in Theorem 4.2.2. If there exist a matrix $P = P^T > 0$, and scalars $\tau_{ij} \geq 0$ with $j \in \{1, 2\}$ satisfying for $i \in \{1, ..., N_m\}$ such that $L_i {x_{0i}}^T x_{0i} \leq \varepsilon^2$*

$$\begin{bmatrix} A_i^T P + P A_i + \tau_{i1} L_i I & P \\ P & -\tau_{i1} I \end{bmatrix} < 0$$

and for all remaining $i \in \{1, ..., N_m\}$ such that $L_i {x_{0i}}^T x_{0i} > \varepsilon^2$

$$\begin{bmatrix} A_i^T P + P A_i + (\tau_{i1} - \tau_{i2}) L_i I & P & P a_i + x_{0i} L_i (\tau_{i2} - \tau_{i1}) \\ P & -\tau_{i1} I & 0 \\ a_i^T P + x_{0i}^T L_i (\tau_{i2} - \tau_{i1}) & 0 & L_i x_{0i}^T x_{0i} (\tau_{i1} - \tau_{i2}) + \varepsilon^2 \tau_{i2} \end{bmatrix} < 0$$

then the system (5.1) is asymptotically stable. Furthermore $V(x) = x^T P x$ serves as a Lyapunov function for the system (5.1) and the corresponding PLM (5.2).

Proof. From Theorem 5.2.3 we know that the origin is an asymptotic stable equilibrium point of the system (5.12) if there exists a quadratic Lyapunov function $V(x) = x^T P x$ with $P = P^T > 0$ that decreases along every nonzero trajectory of the system, that is

$$\frac{dV(x)}{dt} = \sum_{i=1}^{N_m} w_i(x) \{ \begin{bmatrix} x \\ F_i(x) \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P + P A_i & P & P a_i \\ P & 0 & 0 \\ a_i^T P & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ F_i(x) \\ 1 \end{bmatrix} \}$$

Clearly $\frac{dV(x)}{dt} < 0$ is implied by

$$\begin{bmatrix} x \\ F_i(x) \\ 1 \end{bmatrix}^T \begin{bmatrix} A_i^T P + P A_i & P & P a_i \\ P & 0 & 0 \\ a_i^T P & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ F_i(x) \\ 1 \end{bmatrix} < 0$$

for all $x \neq 0$, $w_i(x) > 0$. Note that by construction of the ε-accurate PLM $w_i(x) > 0$ only if (5.14) holds, i.e.

$$\begin{bmatrix} x \\ F_i(x) \\ 1 \end{bmatrix}^T \begin{bmatrix} L_i I & 0 & -x_{0i} L_i \\ 0 & 0 & 0 \\ -x_{0i}^T L & 0 & L_i x_{0i}^T x_{0i} - \varepsilon^2 \end{bmatrix} \begin{bmatrix} x \\ F_i(x) \\ 1 \end{bmatrix} \leq 0$$

Furthermore $F_i(x)$ satisfies (5.13), i.e. the norm bound

$$\begin{bmatrix} x \\ F_i(x) \\ 1 \end{bmatrix}^T \begin{bmatrix} -L_i I & 0 & x_{0i} L_i \\ 0 & I & 0 \\ x_{0i}^T L_i & 0 & -L_i x_{0i}^T x_{0i} \end{bmatrix} \begin{bmatrix} x \\ F_i(x) \\ 1 \end{bmatrix} \leq 0$$

Using the S-method [17] a sufficient condition for stability is that there exist a matrix $P = P^T > 0$, and scalars $\tau_{ij} \geq 0$ satisfying for all $i \in \{1, ..., N_m\}$ such that $L_i {x_{0i}}^T x_{0i} > \varepsilon^2$ (that is for regimes that do not contain the origin)

$$\begin{bmatrix} A_i^T P + P A_i + (\tau_{i1} - \tau_{i2}) L_i I & P & P a_i + x_{0i} L_i (\tau_{i2} - \tau_{i1}) \\ P & -\tau_{i1} I & 0 \\ a_i^T P + x_{0i}^T L_i (\tau_{i2} - \tau_{i1}) & 0 & L_i x_{0i}^T x_{0i} (\tau_{i1} - \tau_{i2}) + \varepsilon^2 \tau_{i2} \end{bmatrix} < 0$$

and for $i \in \{1, ..., N_m\}$ such that $L_i {x_{0i}}^T x_{0i} \leq \varepsilon^2$ (that is for the remaining regime that does not contain the origin)

$$\begin{bmatrix} A_i^T P + P A_i + \tau_{i1} L_i I & P \\ P & -\tau_{i1} I \end{bmatrix} < 0$$

Note that by construction of the PLM it holds that for the regime that contains the origin, $a_i = 0$. ■

5.4 Controllability

Before any controller design is attempted it is important to know if one can steer any state to any other state. In other words we want to know if a system is controllable or not.

A general definition of controllability is given by

Definition 5.4.1 (Controllability) *A PLM is (complete or globally) controllable, if for each pair of states $x^1, x^2 \in X$ it holds that x^1 can be steered to x^2 with a suitable chosen input $u(t)$.*

Just as in the linear case controllability is a central property for nonlinear systems. From the Kalman controllability rank condition [41], we know that a n-dimensional continuous-time linear system

$$\dot{x} = Ax + Bu$$

represented by the pair (A, B), is controllable if and only if $\operatorname{rank} R(A, B) = n$ with

$$R(A, B) := [B, AB, A^2 B, ..., A^{n-1} B]$$

The column space (the space generated by the column vectors) of the controllability matrix $R(A, B)$, is the controllability space of (A, B) and will be denoted by $\mathcal{R}(A, B)$.

There is a big resemblance between PLMs and linear systems. In fact, the PLM is based upon N_m representations of linear models (A_i, B_i, a_i) together with the scheduling functions w_i, i.e.

$$\dot{x} = \sum_{i=1}^{N_m} \omega_i(x, v)\{A_i x + B_i u + a_i\} \tag{5.15}$$

In this section, the objective is to find out if one can decide about controllability of the PLM (5.15) by means of easily verifiable conditions, i.e. through simple manipulations on its representation like in the linear case.

Firstly, an uncertainty model, a model set $\mathcal{M}$ will be associated with the PLM. The model set was already defined by (3.15), but since controllability issues are addressed, which is an input-to-state property, it is restricted to

$$\mathcal{M}(\{A_i, B_i, a_i\}) := \left\{ (5.15) | \sum_{i=1}^{N_m} \omega_i(x, v) = 1,\ \omega_i(x, v) \geq 0 \right\} \tag{5.16}$$

Here $v \in V \subseteq \mathbb{R}^l$ is an external scheduling variable. $\mathcal{M}$ is defined as a collection of PLMs (5.15), where every PLM from the model set is represented by the same set of nonhomogeneous linear models (A_i, B_i, a_i). This means that the only difference between two PLMs from the same model set is the realization of the set of scheduling functions, which is constraint by $\sum_{i=1}^{N_m} \omega_i(x, v) = 1$ and $\omega_i(x, v) \geq 0$.

The general problem of finding necessary and sufficient conditions for deciding when a nonlinear system is controllable is still open, even if one restricts to classes of systems with much extra structure, such as bilinear systems. For PLMs this is also the case. Hence, we will sometimes specialize to PLMs of which the model set $\mathcal{M}$ is restricted. This is done by imposing extra structure on the representation. Sometimes the model set will be restricted. As an example, consider the model set $\mathcal{M}(\{A_i, B, 0\} \mid w_i = w_i(v))$, meaning the restricted model set, consisting of PLMs represented by homogeneous linear models (A_i, B), and with scheduling functions w_i that depend only on the external variable v.

In the sequel of this section we mention a few easy verifiable controllability conditions derived for PLMs.

5.4.1 Kalman Controllability Decomposition

We start with a controllability result which follows from Kalman's controllability decomposition of linear systems. The column space of the controllability

matrix $R(A,B)$, is the controllable space of (A,B) and will be denoted by $\mathcal{R}(A,B)$. We have the following necessary condition for controllability of the PLM.

THEOREM 5.4.2 *The n-dimensional PLM*$\in \mathcal{M}(\{A_i,B_i,a_i\} \mid w_i = w_i(v))$ *is controllable only if* $\cup_{i\in\{1,\ldots,N_m\}}\mathcal{R}(A_i,B_i) = \mathbb{R}^n$.

Proof. Assume that $\mathbf{R} = \cup_{i\in\{1,\ldots,N_m\}}\mathcal{R}(A_i,B_i) = \mathbb{R}^r$, $r < n$. Then there exists a subspace $\mathbf{S}$ such that $\mathbf{R}\oplus\mathbf{S} = \mathbb{R}^n$. Let $\{v_1,\ldots,v_r\}$ be a basis of $\mathbf{R}$ and $\{w_1,\ldots,w_{n-r}\}$ be a basis of $\mathbf{S}$. With the change of variables $x = Tx^N$, where $T := [v_1,\ldots,v_r,w_1,\ldots,w_{n-r}]$, the system is transformed into (Kalman controllability decomposition)

$$\dot{x}_1^N = \sum_{i=1}^{N_m} w_i(v)\left\{A_{i1}^N x_1^N + A_{i2}^N x_2^N + B_{i1}^N u + a_{i1}^N\right\}$$
$$\dot{x}_2^N = \sum_{i=1}^{N_m} w_i(v)\left\{A_{i3}^N x_2^N + a_{i2}^N\right\}$$

where x_1^N, x_2^N are r-dimensional and $(n-r)$-dimensional respectively. It is clear that the x_2^N component of the state cannot be controlled in any way. ■

COROLLARY 5.4.3 *The n-dimensional PLM*$\in \mathcal{M}(\{A_i,B_i,a_i\} \mid w_i = w_i(x_2^N,v))$ *is controllable only if* $\cup_{i\in\{1,\ldots,N_m\}}\mathcal{R}(A_i,B_i) = \mathbb{R}^n$.

Theorem 5.4.2 is exactly the same for discrete-time PLMs [42].

5.4.2 Feedback Equivalence

Sufficient conditions for controllability are derived by explicitly constructing a controller for a PLM in such a way that the PLM is linearized. If the linearized PLM is controllable then this implies that also the PLM is controllable.

We first restrict to PLMs from the model set $\mathcal{M}(\{A_i,B,a_i\})$, i.e. consisting of nonhomogeneous linear models sharing the same input matrix.

THEOREM 5.4.4 $\mathcal{M}(\{A_i,B,a_i\})$ *is feedback linearizable (into a nonhomogeneous linear model) if for all* $i,j \in \{1,\ldots,N_m\}$ *it holds that* $(A_i - A_j, a_i - a_j)$ *is in the range of* B, *i.e.* (A_i,B,a_i) *and* (A_j,B,a_j) *are feedback equivalent.* $\mathcal{M}(\{A_i,B,a_i\})$ *is controllable if also* (A_i,B) *is controllable.*

Proof. If we apply the feedback control law

$$u = \sum_{i=1}^{N_m} w_i(x,v)\{F_i x + f_i\} + u^*$$

to the PLM

$$\dot{x} = \sum_{i=1}^{N_m} w_i(x,v)\,\{A_i x + Bu + a_i\}$$

the closed-loop system becomes

$$\dot{x} = \sum_{i=1}^{N_m} w_i(x,v)\,\{(A_i + BF_i)x + Bf_i + a_i\}$$

Now, since $A_i + BF_i = A_{cl}$, and $Bf_i + a_i = a_{cl}$, $\forall i \in \{1, \ldots, N_m\}$, then

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{N_m} w_i(x,v)\{A_{cl}x + Bu^* + a_{cl}\} \\ &= A_{cl}x + Bu^* + a_{cl}\end{aligned}$$

is linearized. This linearizing feedback law is possible if $a_i - a_j = B(f_j - f_i)$ and $A_i - A_j = B(F_j - F_i)$, $\forall (i,j) \in \{1, ..., N_m\}$. This condition can be satisfied if $(A_i - A_j, a_i - a_j)$ is in the range of B, $\forall (i,j) \in \{1, ..., N_m\}$.

Next we prove that the closed-loop

$$\dot{x} = A_{cl}x + Bu^* + a_i \tag{5.17}$$

is controllable if and only if (A_{cl}, B) is controllable. From the Kalman controllability decomposition, see Theorem 5.4.2, we know that (5.17) is controllable only if (A_{cl}, B) is controllable. If (5.17) is controllable then $\text{rank}[A_{cl}, B] = n$, and thus a_{cl} is in the range of $[A, B]$. This implies that there are x_0,u_0 such that

$$\dot{x} = A_{cl}x + Bu^* + a_i = \dot{x} - \dot{x}_0 = A_{cl}(x - x_0) + B(u^* - u_0)$$

From which it follows that (5.17) is controllable if and only if (A_{cl}, B) is controllable. If $(A_{cl}, B) = (A_i + BF_i, B)$ is controllable then also the PLM is controllable. But $\text{rank}\, R(A_i + BF_i, B) = n$ is equivalent with $\text{rank}\, R(A_i, B) = n$, see [67] and the result follows. ■

We restrict to single input PLMs from the model set $\mathcal{M}(\{A_i, b_i B, 0\})$ with (A_i, B) in controller form and b_i a scalar.

$$A_i := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ a_1^i & a_2^i & \cdots & \cdots & a_n^i \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

The idea is to use a preliminary transformation of the control value space, in such a way that Theorem 5.4.4 can be applied.

THEOREM 5.4.5 *The PLM* $\in \mathcal{M}(\{A_i, b_iB, 0\})$ *with* (A_i, B) *in controller form and* b_i *a scalar, is controllable if the scheduling functions* $w_i(x, v)$ *are such that* $\sum_{i=1}^{N_m} w_i(x, v)b_n^i \neq 0$ *for all* x, v.

Proof. If we use the following input transformation

$$u = \left(\sum_{i=1}^{N_m} w_i(x, v)b_i \right)^{-1} u^*$$

for the PLM

$$\dot{x} = \sum_{i=1}^{N_m} w_i(x, v) \{A_i x + b_i B u\} \tag{5.18}$$

which is well defined if the representation is such that

$$\sum_{i=1}^{N_m} w_i(x, v)b_i \neq 0 \qquad \forall\ x, v, \tag{5.19}$$

we obtain the transformed system

$$\dot{x} = \sum_{i=1}^{N_m} w_i(x, v) \{A_i x + B u^*\} \tag{5.20}$$

with (A_i, B) in controller form. From Theorem 5.4.4 it follows that (5.20) is controllable. This implies that the PLM (5.18) is also controllable, at least if condition (5.19) is satisfied. ■

COROLLARY 5.4.6 $\mathcal{M}(\{A_i, b_iB, 0\})$ *with* (A_i, B) *in controller form and* b_i *a scalar, is controllable if and only if* $0 \notin \text{Co}\{b_n^i\}$.

Proof. The sufficiency part follows from Theorem 5.4.5. To prove that this condition is also necessary we show that $0 \in \text{Co}\{b_n^i\}$ implies the existence of at least one PLM $\in \mathcal{M}$ that is uncontrollable. Since $\mathcal{M}$ is controllable only if each PLM $\in \mathcal{M}$ is controllable, we must conclude that the model set is uncontrollable, implying necessity of the condition.

Assume $0 \in \text{Co}\{b_n^i\}$, then with a constant realization of the scheduling functions $\{w_i \mid \sum_{i=1}^{N_m} w_i b_n^i = 0\}$, the PLM becomes the autonomous linear model $\dot{x} = \sum_{i=1}^{N_m} w_i A_i x$ that is clearly uncontrollable. ■

The derived controllability results are illustrated with the next example.

EXAMPLE 5.4.7 *Consider a system described by*

$$\begin{aligned} \dot{x}_1 &= 3x_1 + 2\sin(x_2) + 5\cos(x_2)u \\ \dot{x}_2 &= 1.5x_1 + 1.25x_2 \\ y &= x_2 \end{aligned} \tag{5.21}$$

The approximate PLM

$$\dot{x} = \sum_{i=1}^{2} w_i(x_2)\left\{A_i x + b_i B u\right\}$$

is obtained from [18].

$$\begin{aligned} A_1 &= \begin{bmatrix} 3 & 2 \\ 1.5 & 1.25 \end{bmatrix} & A_2 &= \begin{bmatrix} 3 & 4/\pi \\ 1.5 & 1.25 \end{bmatrix} \\ b_1 B &= 5\begin{bmatrix} 1 \\ 0 \end{bmatrix} & b_2 B &= 5\xi\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C_1 &= \begin{bmatrix} 0 & 1 \end{bmatrix} & C_2 &= \begin{bmatrix} 0 & 1 \end{bmatrix} \end{aligned}$$

and $\xi = \cos(88^0)$. *This model describes the dynamics of the system (5.21) in the range* $x_1 \in (-\pi/2, \pi/2)$. *The scheduling functions, which depend upon* x_2, *are depicted in Figure 5.1.*

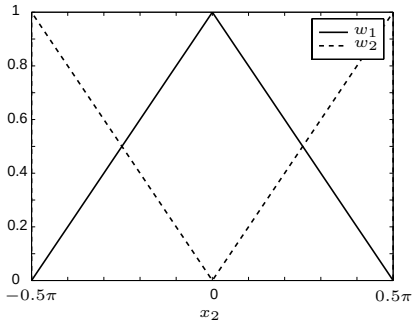


Figure 5.1: *Scheduling functions* $w_i(x_2)$ *with* $i = \{1, 2\}$ *of a PLM.*

Using the control law

$$u = (\sum_{i=1}^{2} w_i(x_2) b_i)^{-1} \tilde{u}$$

$$\tilde{u} = \sum_{i=1}^{2} w_i(x_2) F_i x + u^*$$

we obtain the linear system

$$\dot{x} = A_{cl}x + Bu^* \tag{5.22}$$

where

$$A_{cl} = \begin{bmatrix} * & * \\ 1.5 & 1.25 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the $*$ can be assigned arbitrary values. Since $\operatorname{rank} R(A_{cl}, B) = 2$ *we conclude that the PLM is controllable.*

5.4.3 Stabilizability

A sufficient condition for stabilizability is derived. This is done by constructing a controller for the PLM in such a way that the PLM is controlled to the origin.

We restrict to PLMs from the model set $\mathcal{M}(\{A_i, B_i, 0\})$.

THEOREM 5.4.8 *$\mathcal{M}(\{A_i, B_i, 0\})$ is stabilizable if there exists a (n, n) matrix $Q = Q^T > 0$ and a scalar σ such that*

$$A_iQ + QA_i^T - \sigma B_iB_i^T < 0 \text{ for all } i \in \{1, ..., N_m\} \tag{5.23}$$

Proof. Consider an arbitrary feedback $u = Fx$. The closed-loop model set becomes $\mathcal{M}(\{A_i + B_iF, 0, 0\})$. By application of Corollary 5.10 a sufficient condition for stabilizability is the existence $P = P^T > 0$, F such that

$$(A_i + B_iF)^TP + P(A_i + B_iF) < 0 \text{ for all } i \in \{1, ..., N_m\}$$

or equivalently, after a congruence transformation with $Q = P^{-1}$, the existence $Q = Q^T > 0$, F such that

$$Q(A_i + B_iF)^T + (A_i + B_iF)Q < 0 \text{ for all } i \in \{1, ..., N_m\}$$

After a change of variables, $Y = FQ$, this yields

$$QA_i^T + A_iQ + Y^TB_i^T + B_iY < 0 \text{ for all } i \in \{1, ..., N_m\}$$

By application of Finler's lemma (see [17]) this is equivalent to (5.23). ■

5.5 Observability by Duality

Before any controller design is attempted based on output information, it is important to know to which extend state information can be recovered by means of input and output information. In other words we want to know if a system is observable or not. The output of the PLM at time t for the initial state x_0 and admissible input function $\omega \in \mathcal{U}$ is denoted by $\eta(t, x_0, \omega)$

A definition of observability is given by

DEFINITION 5.5.1 (OBSERVABILITY) *A PLM is (complete or globally) observable if for some $T > 0$ and for every admissible input function ω, from the output $\eta(t, x^1, \omega) = \eta(t, x^2, \omega)$ for $0 \leq t \leq T$ it follows that $x^1 = x^2$.*

Just as in the linear case observability is a central property to be checked for nonlinear systems. From the Kalman observability rank condition [41], we know that a n-dimensional continuous-time linear system

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

represented by the pair (A, C), is observable if and only if $\operatorname{rank} O(A, C) = n$ with

$$O(A, C) := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

or, equivalently, by duality if and only if the pair (A^T, C^T) is controllable, i.e. $\operatorname{rank} R(A^T, C^T) = n$. The row space (the space generated by the row vectors) of the observability matrix $O(A, B)$, is the observable space of (A, C) and will be denoted by $\mathcal{O}(A, C)$.

There is a big resemblance between PLMs and linear systems. In fact, the PLM is based upon N_m representations of linear models (A_i, a_i, C_i, c_i) together with the scheduling functions w_i, that is

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{N_m} \omega_i(x, v)\{A_i x + a_i\} \\ y &= \sum_{i=1}^{N_m} \omega_i(x, v)\{C_i x + c_i\} \end{aligned} \tag{5.24}$$

In this section, the objective is to find out if one can decide about observability of the PLM (5.24) by means of easily verifiable conditions, i.e. through simple manipulations on its representation like in the linear case.

Firstly, an uncertainty model, a model set $\mathcal{M}$ will be associated with the PLM. The model set was already defined by (3.15), but since observability issues are addressed, it is restricted to

$$\mathcal{M}(\{A_i, a_i, C_i, c_i\}) := \left\{ (5.24) | \sum_{i=1}^{N_m} \omega_i(x, v) = 1,\ \omega_i(x, v) \geq 0 \right\} \tag{5.25}$$

Here $v \in V \subseteq \mathbb{R}^l$ is an external scheduling variable. $\mathcal{M}$ is defined as a collection of PLMs (5.24), where every PLM from the model set is represented by the same set of nonhomogeneous linear models (A_i, a_i, C_i, c_i). This means that the only difference between two PLMs from the same model set is the realization of the set of scheduling functions, which is constraint by $\sum_{i=1}^{N_m} \omega_i(x,v) = 1$ and $\omega_i(x,v) \geq 0$.

The general problem of finding necessary and sufficient conditions for deciding when a nonlinear system is observable are still open, even if one restricts to classes of systems with much extra structure, such as bilinear systems. For PLMs this is also the case. Hence, we will sometimes specialize to PLMs of which the model set $\mathcal{M}$ is restricted. This is done by imposing extra structure on the representation. Sometimes the model set will be restricted. As an example, consider the model set $\mathcal{M}(\{A_i, 0, C, 0\} \mid w_i = w_i(v))$, meaning the restricted model set, consisting of PLMs specified by homogeneous linear models (A_i, C), and with scheduling functions w_i that depend only on the external variable v.

In the sequel we mention a few easy verifiable observability conditions derived for PLMs. These conditions are dual to the controllability conditions and a derivation can be found in [42].

COROLLARY 5.5.2 *The following duality relations hold:*

1. $\dot{x} = Ax + Bu$ controllable if and only if $R(A,B) = n$ or equivalently $\mathcal{R}(A,B) = \mathbb{R}^n \iff \dot{x} = A^T x,\ y = B^T x$ observable if and only if $O(A^T, B^T) = n$ or equivalently $\mathcal{O}(A^T, B^T) = \mathbb{R}^n$

2. $\dot{x} = \sum_{i=1}^{N_m} w_i(v)\{A_i x + B_i u\}$ controllable only if $\cup_{i=1}^{N_m} \mathcal{R}(A_i, B_i) = \mathbb{R}^n \iff \dot{x} = \sum_{i=1}^{N_m} w_i(v) A_i^T x,\ y = \sum_{i=1}^{N_m} w_i(v) B_i^T x$ observable only if $\cup_{i=1}^{N_m} \mathcal{O}(A_i^T, B_i^T) = \mathbb{R}^n$

3. $\dot{x} = \sum_{i=1}^{N_m} w_i(x,v)\{A_i x + Bu\}$ controllable if $\operatorname{rank} R(A_i, B) = n$ and $A_i - A_j$ is in the range of $B \iff \dot{x} = \sum_{i=1}^{N_m} w_i(v,y) A_i^T x,\ y = B^T x$ observable if $\operatorname{rank} O(A_i^T, B^T) = n$ and $A_i - A_j$ is in the range of B

4. $\mathcal{M}(\{A_i, b_i B, 0\})$ with (A_i, B) in controller form is controllable if and only if $0 \notin \operatorname{Co}\{b^i\} \iff \mathcal{M}(A_i^T, 0, b_i B^T, 0)$ with (A_i^T, B^T) in observer form is observable if and only if $0 \notin \operatorname{Co}\{b^i\}$

5. $\mathcal{M}(\{A_i, B_i, 0\})$ is stabilizable if there exists $Q = Q^T > 0$ and a scalar σ such that $A_i Q + Q A_i^T - \sigma B_i B_i^T < 0$ for all $i \in \{1, ..., N_m\} \iff$

$\mathcal{M}(A_i^T, 0, B^T, 0)$ is detectable if there exists $Q = Q^T > 0$ and a scalar σ such that $A_iQ + QA_i^T - \sigma B_i B_i^T < 0$ for all $i \in \{1, ..., N_m\}$

5.6 Notes and Comments

Lyapunov stability

Stability results for PLMs and related models are also reported in [17], [33], [53], [61] and cited references. The stability conditions formulated as LMI problems can be constructed with ease from an ε-accurate PLM and are easy verifiable. However since the PLM is based on linear models (often linearizations) only stability of nonhyperbolic equilibria can be proven in this way. This is the price to pay for these easy verifiable conditions.

First method of Lyapunov

A PLM can be constructed on the basis of N_m linearizations of (5.1) around operating points x_{0i}. Note that in that case, equilibria x_{0i} and local stability properties of the nonlinear system (5.1) are easily preserved by a PLM (5.2). For instance by constructing a PLM (5.2) with $A_i = \frac{\partial f}{\partial x}(x_{0i})$, $a_i = -\frac{\partial f}{\partial x}(x_{0i})x_{0i}$ and the corresponding scheduling function $w_i(x)$ such that $w_i(x_{0i}) = 1$ and also $\frac{\partial w_j}{\partial x}(x_{0i}) = 0$ for $j \neq i$. In this case the linearization of the PLM (5.2) equals the linearization of the system (5.1) around equilibria x_{0i}, and the linearization is given by (5.4,5.5). Local stability properties are now preserved as can be verified by application of the first method of Lyapunov, i.e. Theorem 5.2.2.

Controllability and observability

Most of the controllability and observability results are weak in the sense that necessary conditions are far from sufficient and vice-versa. Moreover the PLM class is often restricted, by imposing structure on the representation (A_i, B_i, a_i), of the locally valid models. However, the controllability and observability results are constructive, since they reveal how to design a controller or an observer for the PLM. Moreover, a nice property of a PLM is that, in the extreme case, where all scheduling functions are constants, the PLM degenerates to a LTI system. As a consequence all the results that were derived are consistent with the well understood theory of linear systems.

Structural controllability

Reducing the structural complexity of a PLM doesn't have to limit the significance of the model. For instance the system matrices of second order

mechanical systems, with state variables, position and velocity, have a structural zero and one since the derivative of the position equals the velocity. This means that controllability results for PLMs represented by pairs in controller canonical form can readily be applied to nonlinear mechanical systems.

Studying controllability for a restricted model set is related to the field of structural controllability, see i.e. [67], [33]. Furthermore since with the model set that we consider a PLDI is associated also the work from [45] could be of interest for PLMs. Studying structural controllability within the PLM framework, implies that the PLM is interpreted as an uncertainty model.

An interesting controllability result for differential inclusions is given in [62]. There the same measure of closeness between systems as in this thesis is adopted. Since the PLM can be interpreted as an uncertainty model, i.e. a polytopic linear differential inclusion, this result is of relevance to this work. Informally, it states that under appropriate conditions reachable sets of the real system are preserved with an ε accurate differential inclusion.

Also controllability results for time variant linear systems can be applied, if one views the PLM as an uncertain time varying linear system of which the parameters vary within a polytope and depend explicitly on time [67].

Nonlinear controllability

The proposed model fits within the class of control affine nonlinear systems. For these systems a lot of work is done regarding controllability and observability, see for instance [57] and cited references. However it seems that the PLM structure does not improve the practical applicability of Lie algebraic controllability an observability conditions from nonlinear geometrical control theory [57], with regard to general control affine systems, except for some special cases [42].

First-order local controllability

For nonlinear systems tests based upon the linearization, sufficient for local controllability and observability are reported [67]. Since a PLM is based upon multiple linearizations of the system, it is clear that also local controllability and observability properties can be preserved.

It is easy to preserve local controllability in a neighborhood of a finite set of equilibrium points (x_{0i}, u_{0i}) of the nonlinear system

$$\dot{x} = f(x, u) \tag{5.26}$$

with a PLM (5.15), for instance by choosing $A_i = \frac{\partial f}{\partial x}(x_{0i}, u_{0i})$, $B_i = \frac{\partial f}{\partial u}(x_{0i}, u_{0i})$ and $a_i = -(\frac{\partial f}{\partial x}(x_{0i}, u_{0i})x_{0i} + \frac{\partial f}{\partial u}(x_{0i}, u_{0i})u_{0i}$ and the corresponding scheduling function $w_i(x, u)$ such that $w_i(x_{0i}, u_{0i}) = 1$ and also $\frac{\partial w_j}{\partial x}(x_{0i}, u_{0i}) = 0$ for

$j \neq i$. In this case the linearization of the PLM (5.15) equals the linearization of the nonlinear system (5.1) around equilibria (x_{0i}, u_{0i}). The obtained linear dynamics in the local coordinates $x_i = x - x_{0i}$ and $u_i = u - u_{0i}$ is given by

$$\dot{x}_i = A_i x_i + B_i u_i$$

with

$$A_i = \frac{\partial f}{\partial x}(x_{0i}, u_{0i}),\ B_i = \frac{\partial f}{\partial u}(x_{0i}, u_{0i})$$

Local controllability properties of the system are now preserved by the PLM as can be verified by application of the following result.

THEOREM 5.6.1 ([67]) *If the Taylor linearization (A_i, B_i) of a nonlinear system in an equilibrium point (x_{0i}, u_{0i}) is controllable, then the nonlinear system is locally controllable around x_{0i}.*

Part II

MODELLING

Chapter 6

Introduction to Modelling

In the first part of this thesis PLMs were introduced, and the system analytic properties of PLMs were investigated. We presented and dealt with the basis concepts underlying the analysis of systems for the purpose of control. This resulted in the development of mathematical control and system theory for PLMs.

This part is devoted to modelling methods specifically developed for PLMs, and describes in detail, how to construct a system model of the desired PLM structure. The main objective is to construct PLMs from a priori available physical knowledge and data obtained from the system. A distinction has been made between model based and data based modelling methods.

In Chapter 7, a model based modelling method is considered; so it is assumed that a nonlinear model of the system is available. The method is based upon the (worst-case) approximation results that were derived in Chapter 4. It automatically decomposes the operating space in qualitatively different operating regimes, and subsequently specifies a scheduling function and a linear model for each regime. In contrast to Chapter 4, it constructs the simplest PLM with the pre-imposed desired accuracy. The method is illustrated by means of two examples.

In Chapter 8, two data based modelling methods are considered; so it is assumed that measurements have been obtained from the system. The first method is based upon the model based modelling method presented in Chapter 7. This method is partially generalized, in the sense that the condition of a model of the system being available is relaxed. It will be shown that it suffices that data generated from the system are available. The method will be illustrated by an example. This method is followed by a standard parameter identification method, referred to in the literature as least squares filtering [22]. The applicability of this method is illustrated by means of an example. It is shown how prior knowledge can be incorporated to enhance the applicability

of the data based modelling methods.

The remainder of this chapter addresses general modelling issues that are a requisite for a successful modelling methodology.

6.1 Model Building

In general, building a model from observed data and prior physical knowledge of the system has to be approached with great care. It consists of a number of basic steps, that force the modeler to make a lot of choices and assumptions that can strongly influence the success of the finally obtained model for the problem at hand. The route to a successful model for the intended application is therefore in general of an iterative nature. This means that the choices and assumptions made have to be reconsidered by the modeler if the model performance is not satisfactory. The situation is sketched in Figure 6.1, followed by a brief description of the basic ingredients of model building and the main aspects that should be considered. More detailed treatments can be found in [19], [78], [77], [51].

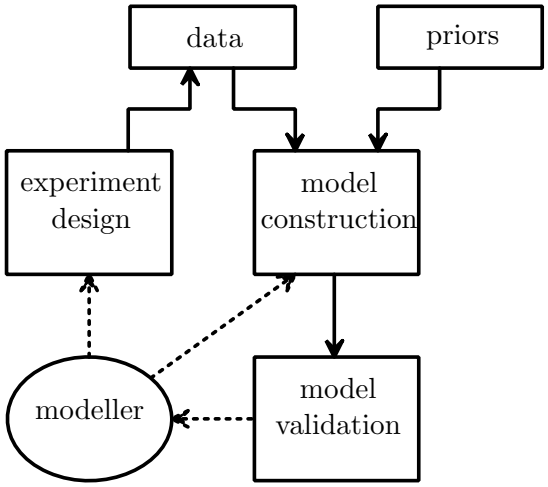


Figure 6.1: *Model bulding and its basis steps. If the model is invalidated by the modeller, then choices and assumptions that are made regarding experiment design and model construction have to be reconsidered. Hopefully this will lead to an improved and valid model.*

6.2 Experiment Design

To design an experiment means to design an input signal for the data generating system in order to extract information from the generated data, sufficient to identify[1] a PLM from an a priori proposed model set, that is successful for the modelling task at hand. Design of a suitable input signal is not an easy task. The shape of the input signal together with the required number of data points have to be determined. Since nonlinear systems are considered, not only the frequency range of the input signal but also its amplitude is of great importance, in order to excite the system everywhere in the operating space. Satisfactory results were obtained for the PLM by applying a low frequent signal with a large amplitude, with on top of it a small amplitude signal containing a broad spectrum of higher frequencies [68], [32], [72]. Similar input shaping strategies are reported in literature, e.g. [47]. Intuitively, it is clear that the number of data points that are needed depends on the smoothness of the system, and the dimensionality of the operating space. If one views approximation as an hypersurface reconstruction problem then because of limited data it is not known how regular this hypersurface is in between the data points. On the other hand, many more data points are needed to fill a three dimensional cube than a two dimensional cube. This is known as the curse of dimensionality, because complexity grows exponentially with the dimension. Furthermore, the number of data points needed depends upon the number of unknown parameters that have to be identified. One could say that the total amount of information that is present in a data sequence is bounded; when this information has to be divided over a larger set of unknown parameters the information-per-parameter is reduced leading to a lower confidence level about the estimates. Of course prior knowledge can reveal some of these problems. One can use regime information to create more dense data sets for some regions, thereby reflecting complexity of the underlying system. If the number of scheduling variables can be reduced with respect to the number of operating variables then also the number of parameters that have to be identified is reduced. Also, constraining the model set by restricting to locally valid models with predefined structure, results in less parameters that have to be identified. This also positively affects the amount of data needed for the identification.

[1] Identification means here building a representative model from the collected data of the system.

6.3 Model Construction

This aspect, the core of model building, asks to choose for a specific model structure, from which a set of candidate models, the model set is derived. Finally the 'best' approximate model in the model set has to be selected. Therefore a criterion has to be defined that reflects and measures model validity of the candidate models for the modelling task at hand. This last part of model construction, the parameter estimation phase, is highly algorithmic. The different stages of model construction are depicted in Figure 6.2.

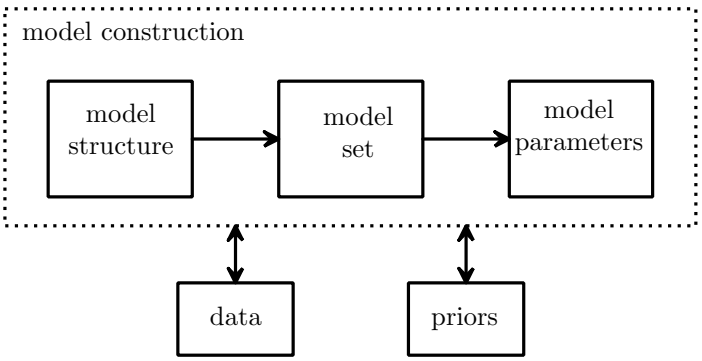


Figure 6.2: *Model construction or model selection consists of three phases. Firstly, a promising model structure is selected. Secondly, a more specific collection of models with the selected model structure is selected. This model set constrains the number of candidate models. Finally, a model from this model set is selected. In all three phases prior knowledge can be utilized to enhance the selection capacity. The needed data properties depend on the model set that is selected. The estimated parameters are often those that best explain the data.*

In the case of PLMs, model construction boils down to approximate modelling rather than exact modelling. This is the more realistic situation since in this case it is not assumed that the data generating system can be modelled exactly. This assumption is consistent with the interpretation of a PLM as an ε-accurate approximate model of the real system. Since constructing a model means finding an approximate rather than an exact representation, it is not trivial which type of approximation to choose. The approximate model should be close to the system in the sense that it is a useful substitute for the specific modelling task at hand. The model construction phase is in contrast to the other model building phases very much dependent on the model structure. In view of this research, we will specialize to model construction for PLMs. The choice for this model structure was already motivated at the beginning of this chapter.

Despite the fact that validity of a given model will depend on its ultimate use, there are also application independent criteria that reflect the quality of an approximate model. That is the trade-off between the conflicting requirements, a small approximation error and a small estimation error, which is similar to the bias-variance trade-off in statistics and in system identification [58]. Firstly, if the system is assumed to be outside the representation capacity of the selected model set then always an error between the system and its approximation will exist, even if an infinite data set was available. In this context, a class of models that has large representational power is referred to as a class of high model complexity or flexibility. The error that the best approximate model in the model set makes is referred to as the approximation error (bias in statistics). From the preceding chapter on approximation we know that for PLMs it holds that the approximation error d_{fg} is proportional to $N_m^{-1/n}$, where N_m is the number of locally valid models and n equals the dimension of the scheduling space. From this it follows that for high dimensional modelling problems $n \gg 1$, the curse of dimensionality complicates model construction. Secondly, we do not have an infinite amount of data available but only a finite set of data points from which we construct the model. Therefore, different data sets convey different information on the system, and the model constructed from the finite data set might not be the best approximate model. This results in an additional error, the estimation error (variance in statistics). With respect to the data side of the problem, the amount and type of data needed to ensure a small estimation error is referred to as the sample complexity. The approximation error and the estimation error are two components of the generalization error. The model complexity, the sample complexity and the generalization error are therefore related. The relation is illustrated in Figure 6.3.

Simply, the approximation error results from a finite parameter set and the estimation error is due to finite data (and/or noise). Simple models would have high approximation errors but low estimation errors while for complex models the opposite would be true. Schematized this is shown in Figure 6.4. Two different sets of eight data points generated by an underlying mapping are given and two models different in complexity are fitted to the data. Both have a large generalization error. Whereas in Figure 6.4a this is the result of a large estimation error, in Figure 6.4b this is the result of a large approximation error. With the more complex model the two different data sets lead to two models that differ substantially; the model is sensitive to the finite data set. The model of low complexity is much less sensitive to the finite data set; the two different data sets lead to two models that do not differ substantially. The trade-off between model complexity and sample complexity is manifest in that complex models can fit many functions but need large amount of data, while simple

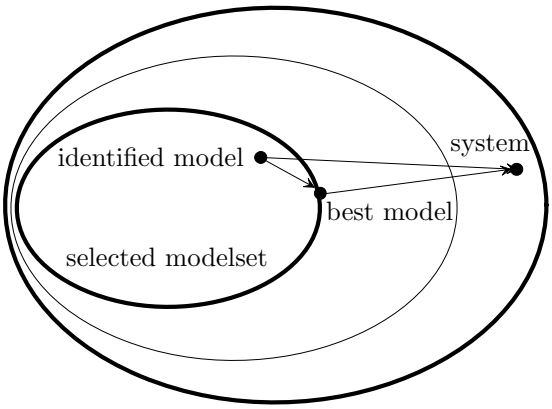
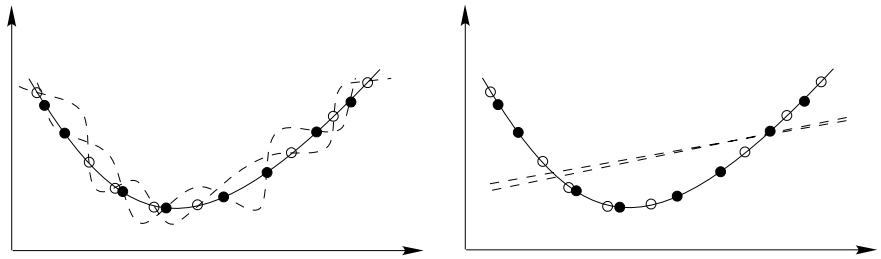


Figure 6.3: *Illustration of the relation between the approximation, estimation and generalization error. The distance between the best model from the selected model set and the system is the approximation error. The distance between the identified model and the best model from the model set is the estimation error. The generalization error is a sum of the estimation error and the approximation error.*

models need only small amounts of data but have a small representational capacity. The trade-offs to be made between a small approximation error and a small estimation error, and more generally between model simplicity and model accuracy have to be reflected as much as possible by the choice of a model set. A model should not be more flexible then necessary. To overcome the problem of a high variance or estimation error it is important to restrict the model set as much as possible by incorporating prior knowledge. For the PLM one can distinguish between regime information, i.e. information on the scheduling variables or functions, and information on (the structure of) the models that are scheduled. This knowledge reduces the number of parameters to be estimated and thus, also reduces the computational cost associated with this model construction phase.

In the parameter estimation phase the objective is to select the best model from the model set. The model is selected that is in some way the closest to the system, e.g. that explains best the data generated by the system. The criterion of closeness has to be defined by the modeler. This criterion has to convey the important properties of the system to be preserved. In this way the parameter estimation phase is reduced to an optimization problem, i.e. the minimization of a criterion that measures closeness over all candidate models from the model set. Since the model structure is fixed, this boils down to finding a set of optimal model parameters. In the case of PLMs a suitable criterion to identify a set of PLM parameters for the 'modelling for control' task could be the



(a) Higher complexity model (b) Lower complexity model

Figure 6.4: *Bias-variance trade-off. Two time-series are fitted to respectively a higher and lower complexity model.*

error measure d_{fg} as defined in Chapter 4. From a computational point of view, a convex optimization problem is to be preferred. This is often achieved by choosing quadratic criteria and model sets that consist of models that are linear in the free parameters. This can also be done for PLMs. Once scheduling functions have been chosen, the PLMs becomes linear in the local model parameters. So, a priori available knowledge about these scheduling functions could simplify the model construction problem.

6.4 Model Validation

Finally, we give some short comments on another aspect of modelling, namely validation. Consider the case where an experiment was designed to acquire a suitable data set, a model structure was chosen to approximate the underlying system, and its parameters were given values by means of a not yet specified estimation method. Do we thus have a model that we can exploit for our purposes? No, not yet. Different considerations bring us to postpone such a conclusion. How do we know that our model captures the desired aspects of the system sufficiently well?

By construction, the estimation process will always yield a set of parameter values. When the model structure is not able to describe the system this will at least result in an approximation error. Furthermore, if our model structure is nonlinear in the parameters, often a nonconvex optimization problem results, and no guarantee can be given that the parameter values found are the best possible, in the sense that only a local minimum of the estimation criterium is found. This is often referred to as the problem of local minima.

But even if our estimated model 'explains' the data used for estimation to a sufficient degree of accuracy, we should not forget that only a finite noisy

data set was used. It is quite possible that when confronted with a 'fresh' data set from the system, the model will perform not as well due to the estimation error, which, is the result of a model structure of high complexity combined with a small data set. On the other hand, it is also possible that the data set used for estimation of the model parameter values did not cover all the system's peculiarities or regions to a sufficient degree; this is a problem of persistence of excitation.

The last problem can be solved by arranging an experiment which gives us a data set that does fulfill the condition of persistence of excitation. But as already mentioned, in general this is not an easy task.

To reduce the probability of not reaching the minimal possible value of the estimation criterium, the estimation of the parameter values is repeated several times for the same model structure under slightly varying (starting) conditions. This gives us some assurance that the 'best' model within a particular model structure, given a particular data set, has been found. However, it is not clear if this is what we want, since this 'best' model that minimizes the estimation criterium is based on a particular finite data set, and a large estimation error may be the result. We actually want a small generalization error. It is possible that the 'best' model as the term is used in this paragraph has a larger generalization error than a slightly 'worse' model.

Several validation methods have been devised which all try to alarm us in case we come across some of the mentioned pitfalls. The most obvious and pragmatic way of validation is to investigate how well the model is capable of reproducing the system behavior on a new set of data (the cross-validation data) that was not used to fit the model. That is, we supply both the model and the system with an input different from the one used to obtain the estimation data set, and compare the model and system output afterwards. We may visually inspect the result or use a more quantitative measure of fit, e.g., the root-mean-square value of the difference between the outputs.

A second basic method for model validation is to examine the residuals ('leftovers') from the estimation process, i.e. the differences between the measured and the estimated output. The advantage hereby is that separate validation data sets are not necessary. A number of statistical residual analysis tests can be performed to see if what the model could not 'explain' still contains traces of a deterministic nature, e.g., unmodelled dynamics. For a detailed treatment the reader is referred to [21], [50], [15]. These tests can also be applied to a validation data set.

If possible, it is wise to use all the available a priori information to validate the model, for instance, does the model make physically sense? Since the PLM has locally a clear physical interpretation, its parameter values can be qualitatively validated by inspection.

Besides these application independent criteria for assessing the quality of

a model, validity of a proposed model should in the first place be judged by its success within the intended model application.

Chapter 7

Building PLMs from known Nonlinear Models

7.1 Introduction

In this chapter a model based modelling method is considered. It is assumed that a nonlinear model of the system is available. The method is based upon the approximation results that were derived in Chapter 4. It automatically decomposes the operating space in qualitatively different operating regimes, and subsequently specifies a scheduling function and a linear model for each regime. For notational convenience the modelling method is presented for systems without an output equation, i.e. (7.1), nonetheless (by augmenting the right-hand side of the system with the right-hand side of the output equation) it naturally extends to systems with outputs [81].

A novel method is developed that, given an a priori sufficiently smooth nonlinear continuous time state space description of the system

$$\dot{x} = f(x, u) \tag{7.1}$$

with the state $x \in X \subseteq \mathbb{R}^n$, the manipulated input $u \in U \subseteq \mathbb{R}^m$, automatically constructs a PLM

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{N_m} w_i(x, u)\{A_i x + B_i u + a_i\} \\ &= g(x, u) \end{aligned} \tag{7.2}$$

with N_m as small as possible, and such that within the region of interest, a compact operating space $\Psi \subseteq X \times U$,

$$d_{fg}(\Psi) \leq \varepsilon \tag{7.3}$$

with $\varepsilon > 0$ pre-specified by the modeler.

The distance d_{fg} between the right-hand side of the system and the PLM was already defined in (4.1). It is known from Theorem 4.6.3 that with this particular choice of distance between the system and the PLM, an upper bound for the number of scheduling regimes N_m can be computed, sufficient to ensure the existence of an ε-accurate PLM. A PLM that achieves this accuracy is built from a set of N_m Taylor series expansions of the system, equidistantly distributed over the scheduling space Z.

The above-mentioned upper bound, will be denoted by N_m^{upper} and is based on a worst-case situation. It is assumed that the maximum nonlinearity as measured with an upper bound on the Taylor remainders corresponding to the aforementioned Taylor series expansions, is attained everywhere in the scheduling space, and as a consequence the scheduling space is uniformly partitioned into scheduling regimes.

Thus the objective is to select an ε-accurate PLM from the model set

$$\mathcal{M}(N_m \leq N_m^{upper}) := \{(7.2) \mid N_m \leq N_m^{upper}\} \tag{7.4}$$

with N_m as small as possible. Determination of a PLM$\in \mathcal{M}(N_m \leq N_m^{upper})$ implies that a PLM from the model set has to be selection. The modelling problem can be divided into three subproblems that are solved sequentially.

problem 1: operating space → scheduling space The scheduling space is determined. The dimension of the scheduling space is reduced as much as possible compared to the operating space, since N_m grows exponentially with the dimension of the scheduling space Z.

problem 2: scheduling space → scheduling regimes The N_m scheduling regimes Z_i are determined. The number of scheduling regimes N_m is reduced as much as possible. The result is a nonuniform partitioning of the scheduling space.

problem 3: scheduling regimes → PLM parameters The parameters are determined either by linearization of the system in the centers of the obtained operating regimes or by a least squares fitting procedure.

In each of the three steps the model set is restricted and the number of candidate PLMs is eventually reduced to one. In Section 7.2 through Section 7.4 a more in depth description of the three subproblems is given as well as a solution to each of the three subproblems is suggested. The situation is depicted in Figure 7.1.

In Section 7.5 the modelling method is illustrated by two examples, the modelling of two nonlinear mechanical systems. Finally, in Section 7.6, some notes and comments are made regarding the proposed method.

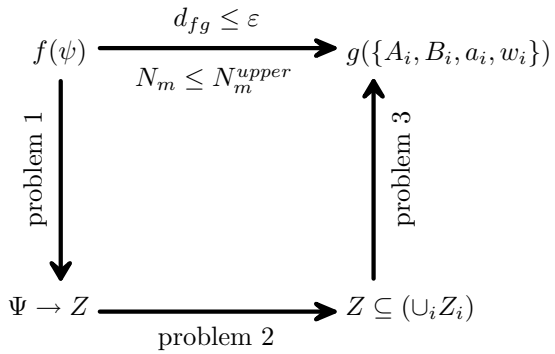


Figure 7.1: *Sketch of the modelling problem. The modelling problem is divided into three subproblems. These subproblems are solved sequentially and in such a way that ε-accuracy of the finally obtained PLM is assured.*

7.2 From Operating Space to Scheduling Space

Since the system (7.1) operates on the operating space, it seems natural to schedule the models (A_i, B_i, a_i) from the PLM (7.2) on the operating space. However, if the scheduling space is of high dimension, the curse of dimensionality will restrict the applicability of the polytopic linear modelling approach. The core of the problem is that the number of scheduling regimes, needed to uniformly partition the scheduling space, increases exponentially with the dimension of the scheduling space, see Theorem 4.3.1. In some cases the models can be scheduled on a space of lower dimension then the operating space, which will reduce the modelling problem considerably. In these cases the structure of the system is exploited to reduce dimensionality.

Thus, first of all a scheduling space Z is defined of which the dimension is reduced as much as possible compared to the dimension of the operating space Ψ, the space on which the system (7.1) operates. In this step the structure of the right-hand side $f(\psi)$, of the system (7.1) is exploited. This means components of the operating vector ψ that enter $f(\psi)$ in an affine way are excluded from the scheduling variables z, since this does not negatively effect the approximation capabilities of PLMs. This completely agrees with Theorem 4.6.3, where it was already shown that an exponential reduction of the number of models N_m involved in the PLM description can be achieved in this way.

In summary, the set of candidate models is reduced, since the model set (7.4) is restricted, i.e. $\mathcal{M}(\{w_i(z)\} \mid Z \text{ determined}) \subset \mathcal{M}(N_m \leq N_m^{upper})$

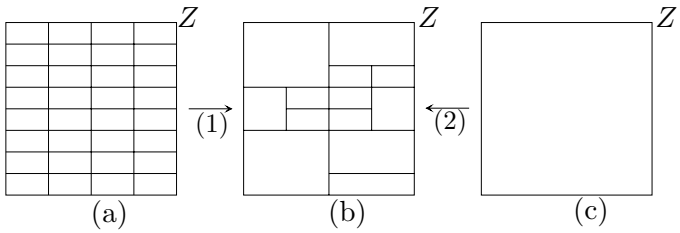


Figure 7.2: *The procedures aggregation* (1) *and segregation* (2) *for a two dimensional scheduling space Z. The aggregation procedure reduces complexity of the PLM by means of regime aggregation. The segregation procedure increases complexity of the PLM by means of regime segregation.*

7.3 From Scheduling Space to Scheduling Regimes

This subproblem is the core of the modelling problem. In this step the number of models N_m is reduced compared to N_m^{upper} by dropping the idea of uniform partitioning. The key assumption is that in general, nonlinearity, as measured with a norm-bound on the first (or zeroth) order Taylor remainder in an operating point of the system, varies as a function of the operating point and therefore depends on the scheduling variable z. In the process of reduction the computation of N_m^{upper} will play an important role.

In this step a minimum number of scheduling regimes Z_i is constructed that covers the compact scheduling space Z. These regimes have to be constructed in such a way that for this scheduling regime configuration an ε-accurate PLM exists. Theorem 4.6.3, and in particular the formula to compute the number of regimes N_m sufficient to achieve ε-accuracy, act as a guide to arrive at a non-uniformly partitioned scheduling space consisting of a minimum number of partitions. As opposed to Theorem 4.6.3 however, the formulas will be applied to evaluate N_m within local operating regions instead of over the whole operating space of interest.

Two procedures will be introduced; segregation and aggregation, that automatically decompose Z in qualitatively different scheduling regimes Z_i such that $Z \subseteq (\cup_i Z_i)$ and the modelling objective, i.e. ε-accuracy can be satisfied. The idea of both procedures is illustrated by Figure 7.2.

Aggregation is a procedure that describes how to aggregate regimes when starting from a PLM consisting of $N_m = N_m^{upper}$ regimes, such that iteratively a simpler realization is obtained. Hence, complexity is reduced by aggregation. Segregation is a procedure that describes how to segregate regimes when starting from a simple PLM consisting of $N_m = 1$ regime, such that iteratively a more complex realization is obtained. Hence, complexity is increased by seg-

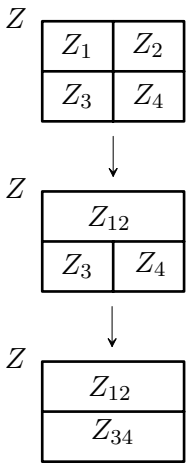


Figure 7.3: *Regime aggregation.*

regation. It is also possible to alternate between the procedures to enlarge the number of possible scheduling regime configurations.

7.3.1 Regime Aggregation

The aggregation procedure starts from a uniformly decomposed scheduling space consisting of N_m^{upper} regimes, as defined in Chapter 4. The situation is depicted in Figure 7.2a. The preliminary analysis in Chapter 4, and in particular the computation of N_m, allows to state a very useful and almost trivial definition of a scheduling regime. Namely, a scheduling regime is a region Z_i within the scheduling space Z for which it holds that $N_m = 1$, or equivalently in shorthand notation $N_m(Z_i) = 1$. We start with N_m^{upper} regimes and it is tried to unite scheduling regimes conceptually as follows: if $N_m(Z_i \cup Z_j) = 1$ for two adjoining scheduling regimes Z_i and Z_j then these regimes are aggregated to obtain a new scheduling regime Z_{ij}. The model is simplified since N_m is reduced. This action is repeated until no further model simplification occurs. Conceptually the procedure is illustrated in Figure 7.3.

In practice, to automate this procedure of finding and aggregating two adjoining operating regimes, several choices and restrictions have to be made that may influence the final result. There may be several candidate regimes that equally well qualify for aggregation. Hence, priority has to be given to certain combinations for avoiding possible deadlock situations in the implementation of the procedure.

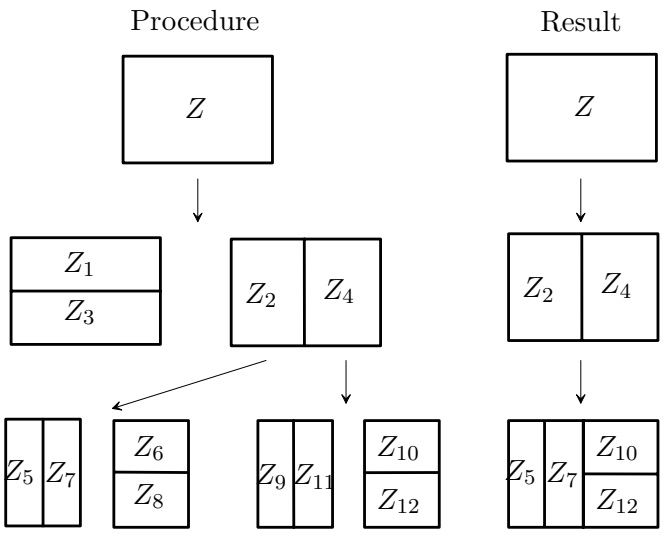


Figure 7.4: *Regime segregation.*

7.3.2 Regime Segregation

Contrary to the aggregation procedure, the segregation procedure starts with the most simple PLM, consisting of a single linear model, which covers the entire scheduling space as depicted in Figure 7.2c. Step by step the model complexity is increased conceptually as follows: split the n_Z dimensional scheduling space 'in the middle' in two scheduling regimes Z_i and Z_{i+n_Z}. This can be done in n_Z different ways, i.e. $i = \{1, ..n_Z\}$. Compute the number of models $N_m = N_m(Z_i) + N_m(Z_{i+n_Z})$ for all obtained configurations that is sufficient to achieve ε-accuracy. Select the configuration with the lowest number of models. One segregation step is then completed. Segregation is repeated until $N_m(Z_j) = 1$. Conceptually the procedure is illustrated in Figure 7.4.

Notice that the segregation procedure is restricted to decompose the scheduling space in the middle into two regions. This restricts the possible regime configurations that can be obtained after executing the segregation procedure. Nevertheless the procedure can be further improved by adding the possibility to split the scheduling space not only into two, but in every larger prime number $i = 3, 5, 7, 11, 13, 17, ...$ This allows for more different regime configurations. The improvement is a reduction of the number of models N_m. However we observed that adding these additional configurations, for $i > 3$, will not necessarily yield a significant reduction of the number of models N_m [81]. Moreover, considering more candidate configurations is more computational intensive, which may be undesirable. Therefore, only splitting in two and three regions is considered.

7.4 From Scheduling Regimes to PLM Parameters

Two procedures were presented, aggregation and segregation, that decompose the scheduling space Z in N_m disjoint regions Z_i, such that ε-accuracy can be achieved. A scheduling regime Z_i is characterized by its center z_{0i} and width r_i. For each scheduling regime Z_i a model M_i, the triple (A_i, B_i, a_i) will be defined together with a scheduling function $w_i(\theta_i)$, ergo the PLM will be specified. If the scheduling function parameters θ_i are to be computed simultaneously with the model parameters M_i a complex nonlinear programming problem results. We like to point out that our experience obtained with the procedures aggregation and segregation, indicates that the qualitative information, concerning the validity of the scheduling regimes, provides enough knowledge to construct the scheduling functions.

The scheduling functions are chosen as normalized basis functions, centered in the middle of operating regime i and with support in accordance with the size of scheduling regime Z_i. The parameters of the linear models are either obtained by linearization of the system in the centers of the scheduling space, or by a least squares fitting procedure.

7.4.1 Constructing Scheduling Functions

The scheduling functions are the result of a normalization of a set of basis functions, the model validity functions ρ_i. For instance as in (3.6)

$$w_i(z, \theta_i) = \frac{\rho_i(z, \theta_i)}{\sum_{j=1}^{N_m} \rho_j(z, \theta_i)}$$

A few possible choices for the ρ_i functions are given in Example 4.2.3. The basis functions ρ_i have support in the region Z_i and tend to zero outside this region. Furthermore $\sum_{i=1}^{N_m} \rho_i(z) \geq 0$ for all $z \in Z$. A common choice is the multivariate Gaussian function (4.8):

$$\rho_i(z, \theta_i) = e^{-(z-\theta_{i1})^T \Theta_i^{-1} (z-\theta_{i1})}$$

where $\Theta_i = 2\,\mathrm{diag}(\sigma_{ij})$ with $j = 1, ..., \dim(z)$, and $\theta_i = [\theta_{i1}, \Theta_i]$ are the unknowns to be determined. The unknowns denoted by θ_i are expressed as a function of the centre z_{0i} and width r_i of regime Z_i. Typically the maximum value of $\rho_i(z)$ is attained in z_{0i}, which is 'the centre' of operating regime Z_i, that was determined with the procedure aggregation or segregation. Therefore $\theta_{i1} = z_{0i}$. The variance matrix Θ_i is expressed as a function of the width r_i of the scheduling regime, i.e. $\Theta_i = \gamma I r_i$. Here γ is a scaling parameter that considers the overlap between the validity functions. A typical value of γ is between 0.25 and 2. There will almost be no overlap when $\gamma = 0.25$ and a

large overlap when $\gamma = 2$. Sometimes prior knowledge about the system is a good guide for the choice of γ. It may appear that the choice of the type of validity function has significant impact on the model. However it is our experience that both the efficacy of the algorithm and the model predictions are quite insensitive with respect to this choice, hence the specification of the type of validity function does not seem to require much prior knowledge of the system. On the other hand, the choice of the user specified parameter γ appears to be of major influence.

7.4.2 Constructing Linear Model Parameters by Linearizations

It is known from Chapter 4 that with the adopted choice of distance between systems, an ε-accurate PLM is ensured if the model parameters are chosen as linearizations of the system around ψ_{0i}, the centers of the operating regimes. This statement still being valid, a useful choice for the model parameters is:

$$A_i = \frac{\partial f}{\partial x}(\psi_{0i}) \quad B_i = \frac{\partial f}{\partial u}(\psi_{0i}) \quad a_i = f(\psi_{0i}) - \frac{\partial f}{\partial \psi}(\psi_{0i})\psi_{0i} \tag{7.5}$$

7.4.3 Constructing Linear Model Parameters by Least Squares

Another method which yields good results, is to determine the model parameters with a least squares method. Therefore the system is linearized in N_g points ψ_{0i}, uniformly scattered over the scheduling space. Evidently it is not possible to linearize the system in every point of the operating space. It is important that $N_g \gg N_m$, to assure enough information is available to compute reliable model parameters. At the points of linearization, i.e. ψ_{0i} with $i = 1, ..., N_g$ the obtained linearized model is an exact representation of the system and becomes

$$\dot{x} = A(\psi_{0i})x + B(\psi_{0i})u + a(\psi_{0i})$$

Here the matrices are defined as in (7.5). For the N_g points we can write for the error between the system and the PLM

$$A(\psi_{0i})x + B(\psi_{0i})u + a(\psi_{0i}) - \sum_{j=1}^{N_m} w_j(\psi_{0i})\{A_i x + B_i u + a_i\} = E_{ABa,i}$$

where $i = \{1, ..., N_g\}$. This error can be split in different parts to reflect the contribution of the different components of the PLM description, i.e. $E_{ABa,i} = E_{A,i} + E_{B,i} + E_{a,i}$ with

$$E_{A,i} = A(\psi_{0i}) - \sum_{j=1}^{N_m} w_j(\psi_{0i})A_j$$

Similar expressions hold for the matrices $E_{B,i}$ and $E_{a,i}$. Note that the number N_m and the scheduling functions w_j are already determined as a result of the application of the aggregation or segregation procedure.

For the first part of $E_{ABa,i}$, i.e. $E_{A,i}$, the objective is to minimize the cost function

$$J(\{A_j\}) = \sum_{i=1}^{N_g} \|E_{A,i}\|_F^2 \tag{7.6}$$

with respect to the unknown parameters $\{A_j\}$ with $j = \{1, ..., N_m\}$. Here

$$\|E\|_F = \sqrt{\sum_{\substack{1\le i\le n\\ 1\le j\le m}} |E(i,j)|^2}$$

is the Frobenius norm of the matrix $E \in \mathbb{R}^{n*m}$, i.e. the square root of the sums of the squares of the entries of the matrix E. Similar expressions hold for $J(\{B_j\})$ and $J(\{a_j\})$. In the sequel the least squares estimates for $\{A_j\}$ are derived. The remaining unknown parameters $\{B_j, a_j\}$ can be obtained similarly. Thus

$$\begin{aligned}
\|E_{A,i}\|_F^2 &= \sum_{j,k=1}^{n} |E_{A,i}(j,k)|^2 \\
&= \sum_{j,k=1}^{n} \left| A(\psi_{0i})(j,k) - \sum_{l=1}^{N_m} w_l(\psi_{0i}) A_l(j,k) \right|^2 \\
&= \sum_{j,k=1}^{n} \left| A(\psi_{0i})(j,k) - w(\psi_{0i})^T A_{PLM}(j,k) \right|^2
\end{aligned}$$

where

$$w(\psi_{0i})^T = [w_1(\psi_{0i}), \cdots, w_{N_m}(\psi_{0i})]$$

and

$$A_{PLM}(j,k) = [A_1(j,k), \cdots, A_{N_m}(j,k)]^T.$$

With

$$W^T = [w(\psi_{01}), \cdots, w(\psi_{0N_g})]$$

and

$$A(j,k) = [A(\psi_{01})(j,k), \cdots, A(\psi_{0N_g})(j,k)]^T$$

the cost function (7.6) becomes

$$J(\{A_i\}) = \sum_{j,k=1}^{n} \|A(j,k) - W A_{PLM}(j,k)\|^2$$

The least square estimate is

$$A_{PLM}(j,k) = (W^T W)^{-1} W^T A(j,k)$$

The other free parameters of the PLM can be determined in the same way, i.e.,

$$B_{PLM}(j,l) = (W^T W)^{-1} W^T B(j,l)$$

and

$$a_{PLM}(j) = (W^T W)^{-1} W^T a(j)$$

7.5 Examples

The modelling method, as it was described in this section will be illustrated by two examples. In each of the two examples the objective is to derive the simplest ε accurate PLM. It is assumed that a suitable white-box model for the system is available. The two systems under consideration belong to the class of nonlinear mechanical systems that are often encountered in applications [57]. The state space equations describing systems of this particular class are

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -M(q)^{-1}(C(q,\dot{q}) + k(q)) \end{bmatrix} + \begin{bmatrix} 0 \\ M(q)^{-1}B \end{bmatrix} u \tag{7.7}$$

with input u and state $x = [q, \dot{q}]^T$, where q are the displacements and $\dot{q}$ the velocities. Here $M(q)$ is the positive definite mass matrix, B is denoted the distribution matrix, $k(q)$ represents the conservative generalized forces, and $C(q,\dot{q})$ comprehends the remaining generalized forces such as the Coriolis, centrifugal and friction components. The particular structure of the model (7.7) will be exploited in the two examples. More specifically, since the input enters the system in an affine way, this variable can be excluded from the set of scheduling variables without affecting the approximation capability of the PLM, thereby reducing the complexity of the modelling problem. In this way the set of candidate PLMs is reduced. This is in fact the result of the application of Theorem 4.6.3, and as already mentioned, a possible solution to the first of the three subproblems in which the modelling problem was divided (see Figure 7.1).

The first example, i.e. modelling of a rotating robot arm subjected to significant friction in the drive line, shows a local nonlinearity. This implies that on any non-compact operating region an ε-accurate PLM exists, consisting of a finite number of models, in contrast to a system that features a global nonlinearity. Because of the strongly local character of the nonlinearity, i.e. the friction torque characteristic, we expect that the number of models N_m will be reduced significantly as compared to N_m^{upper}, see Theorem 4.6.3, by dropping the idea of uniform partitioning and applying the aggregation or segregation procedure.

The second example, i.e. modelling of an inverted pendulum on a cart, shows a global nonlinearity. This system is more complex in the sense that the scheduling space is two dimensional instead of one dimensional. Because of the global nonlinearity, the reduction of N_m is expected to be less significant, as compared to the previous example.

The examples will illustrate the modelling method and its features. The choice for the two white-box models used in the examples is motivated by the fact that they show great similarity with those located in our laboratory. The modelling method was automated as a Matlab implementation. A more detailed description of the modelling method for these specific examples can be found in [81] and [11].

7.5.1 A Rotational Robotic Manipulator System

Consider the rotating robot arm from Figure 7.5, which is subjected to significant friction in the driveline. The rotating robot arm is a nonlinear mechanical

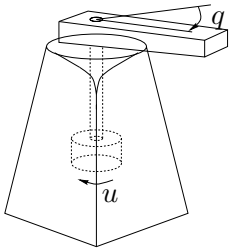


Figure 7.5: *Rotational robotic manipulator.*

system with one degree of freedom q, where q represents the angular displacement of the arm, and u is the motor input current. The state space equations (7.7) reduce to

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -M^{-1}C(\dot{q}) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}B \end{bmatrix} u \tag{7.8}$$

with $M = 0.0292$ $[Nms^2]$ the effective inertia of the motor-transmission-rotating arm combination, $B = 16$ $[NmV^{-1}]$ is the motor gain, and $C(\dot{q})$ is the friction torque. Friction models that are often used for modelling purposes are depicted in Figure 7.6. A survey on friction models is given in [32]. The

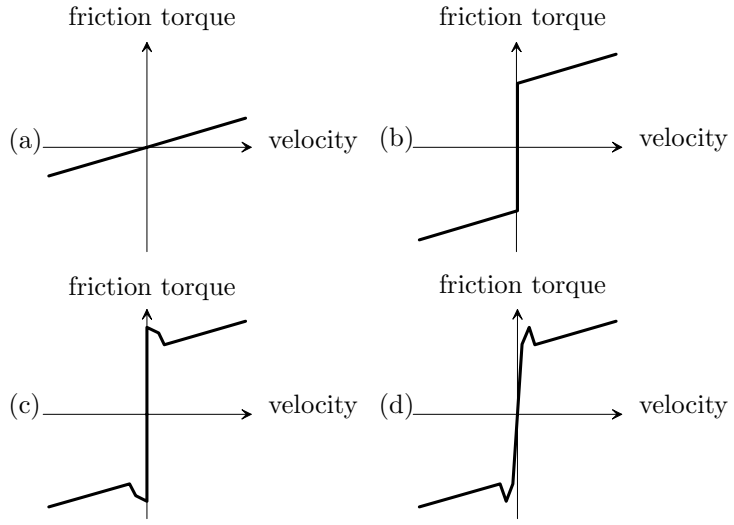


Figure 7.6: *Friction models. (a) viscous friction model, (b) Coulomb+viscous friction model, (c) Coulomb+viscous+Stribeck friction model, (d) continuous Coulomb+viscous+Stribeck friction model.*

friction torque is assumed to be a continuous friction model consisting of a Coulomb part (C_c), a viscous friction part (C_v), and the Stribeck friction effect C_s, i.e.

$$C(\dot{q}) = C_c(\dot{q}) + C_v(\dot{q}) + C_s(\dot{q})$$

where

$$C_c(\dot{q}) = T_c \frac{2}{\pi} \arctan(\kappa \dot{q})$$
$$C_s(\dot{q}) = (T_s - T_c) e^{-\left(\frac{\dot{q}}{v_s}\right)^2} \frac{2}{\pi} \arctan(\kappa \dot{q})$$
$$C_v(\dot{q}) = \sigma \dot{q}$$

See Figure 7.6d for the qualitative relation between friction torque and angular velocity. The friction parameters values are $T_c = 0.416$ $[Nm]$, $T_s = 0.4657$ $[Nm]$, $v_s = 0.2$ $[rad/s]$, $\sigma = 0.0135$ $[Nms]$, $\kappa = 1000[-]$. All the parameter values are estimates obtained from measurements on a real rotating robot arm in the mechanical engineering laboratory. The unknown parameters were estimated with standard parameter estimation methods [22].

In this example the objective is to derive the simplest PLM that achieves ε-accuracy, this means that it is tried to minimize the number of scheduling regimes N_m. The modelling problem is divided into 3 subproblems as depicted in Figure 7.1, and solved sequentially following the modelling approach presented earlier in this section.

Subproblem 1: From Operating Space to Scheduling Space

The operating space is defined as the product space of the state space with the input space, i.e. $\Psi = X \times U$ where $x = [q, \dot{q}]^T \in X$, $q \in X_1$, $\dot{q} \in X_2$, and $u \in U$. From Corollary 4.6.2 we know that it is sufficient to schedule the models over the angular velocity $\dot{q}$ only, instead of over all the operating variables q,$\dot{q}$ and u. This is because the variables q and u enter the model in an affine way. Hence the scheduling space $Z = X_2$. Corollary 4.6.2 tells us that there will always exist a PLM with N_m finite, at least if Z is compact. Therefore it is assumed that $Z = \{\dot{q} \mid |\dot{q}| \leq 25[rad/s]\}$[1]. The result is a dimensionality reduction of the next modelling subproblem.

Subproblem 2: From Scheduling Space to Scheduling Regimes

In this step the number of models N_m is reduced as much as possible compared to the upper bound from Corollary 4.6.2. This upper bound is based on a worst case scenario, and the result is a uniform partitioning of the scheduling space in N_m^{upper} scheduling regimes, where

$$N_m^{upper} = \Pi_{i=1}^{n_z} \left\lceil \frac{e_i}{\sqrt{2\varepsilon}} \sqrt{\lambda_\xi n_N^{1/2} n_z} \right\rceil$$

Here $n_z = 1$ the dimension of the scheduling space, $n_N = 1$ the number of scalar nonlinear differential equations involved in the state-space description, $e_1 = 25$ half the width of the scheduling space, ε the user specified demanded accuracy of the PLM, and $\lambda_\xi = 53.59$ the maximum eigenvalue of the Hessian matrices evaluated over the scheduling space and associated with the Taylor remainder, i.e. a measure of nonlinearity of the system. Since λ_ξ varies as a function of $\dot{q}$, we can use the procedure aggregation or segregation to reduce the number of models. In Table 7.1 the number of models N_m, sufficient to assure ε-accuracy for, respectively, uniform decomposition, aggregation and segregation of the operating space are given.

The result of the segregation method was improved by an extra aggregation step and summarized as the 'extended' method in the last column of Table 7.1. Choosing for $\varepsilon = 2$ we obtain for the extended procedure 6 models.

[1] In the case of a local nonlinearity however, Corollory 4.6.2 also holds when Z is not compact.

Table 7.1: *Number of models, N_m*

ε	uniform	aggregation	segregation	extended
0.1	410	13	18	13
0.2	290	10	14	10
0.5	184	7	13	9
1.0	130	6	10	8
2.0	92	6	10	6
4.0	65	4	9	5
8.0	46	4	8	4

The advantage of segregation over aggregation, is the smaller computational effort. The obtained center $d(Z_i)$ and width $2e(Z_i)$ of each scheduling regime $Z_i : d(Z_i) - e(Z_i) \leq \dot{q} \leq d(Z_i) + e(Z_i)$ are summarized in Table 7.2.

Table 7.2: *Operating regimes and model parameters*

i	$d(Z_i)$	$e(Z_i)$	$A_i(2,2)$	$B_i(2)$	$a_i(2)$
1	−13.8889	11.1111	−0.4624	547.9452	14.2453
2	−1.6204	1.1574	−0.4658	547.9452	14.2354
3	−0.2315	0.2315	4.5093	547.9452	15.8029
4	0.2315	0.2315	4.5093	547.9452	−15.8029
5	1.6204	1.1574	−0.4658	547.9452	−14.2354
6	13.8889	11.1111	−0.4624	547.9452	−14.2453

Subproblem 3: From scheduling regimes to PLM parameters

The triples

$$\left(A_i = \begin{pmatrix} 0 & 1 \\ 0 & A_i(2,2) \end{pmatrix}, B_i = \begin{pmatrix} 0 \\ B_i(2) \end{pmatrix}, a_i = \begin{pmatrix} 0 \\ a_i(2) \end{pmatrix} \right)$$

are determined by linearizing $f(q, \dot{q}, u)$ in the points $(*, d(Z_i), *)$, where because of linearity, $*$ can be chosen arbitrary. The obtained model parameters for each scheduling regime are summarized in Table 7.2. As we could expect $B_i(2)$ is the same for the six operating regimes, because the system is affine in u. From $B_i(2) = M^{-1}B$ it follows that the estimated inertia $M = B/B_i(2) = 0.0292$ which is identical to the inertia of the system. The scheduling functions are chosen as normalized radial basis functions that are placed in the center $d(Z_i)$ of each operating regime Z_i, i.e. $w_i(\dot{q}) = \frac{\rho_i(\dot{q})}{\sum_{i=1}^{N_m} \rho_j(\dot{q})}$ with $\rho_i(\dot{q}) = e^{-\frac{(\dot{q}-d(Z_i))^2}{2\gamma e(Z_i)}}$. The user-specified parameter γ is chosen 0.25, indi-

cating almost no overlap between the models. The scheduling functions are shown in Figure 7.7.

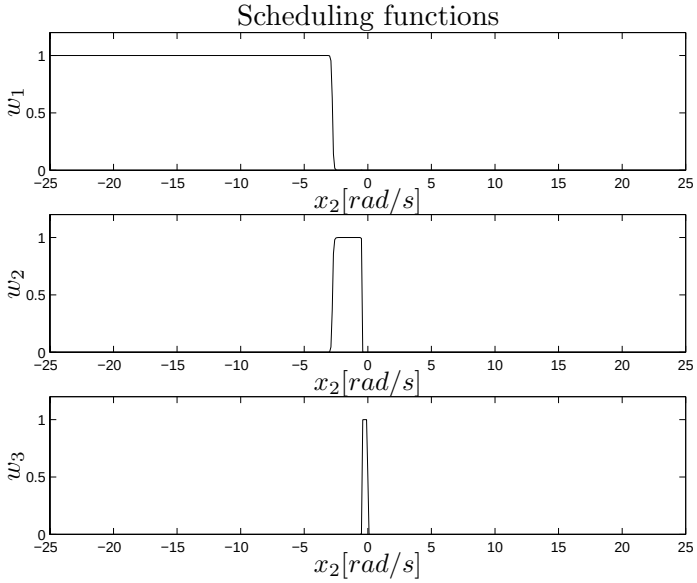


Figure 7.7: *The scheduling functions w_1, w_2 and w_3. The remaining scheduling functions are omitted since they follow from the symmetry around zero velocity.*

Model Validation

The friction torque of the PLM is validated and depicted together with the friction torque of the real system in Figure 7.8. A visually identical friction curve compared to the system is observed. In this case

$$\begin{aligned} d_{fg} &= M^{-1} * \max_{\dot{q} \in X_2} \|C(\dot{q}) - C_{PLM}(\dot{q})\|_2 \\ &= 1/0.0292 * 0.0165 \\ &= 0,5638 \end{aligned}$$

and thus the demanded accuracy $d_{fg} \leq 2$ is achieved.

The PLM can be unraveled as follows. Regime Z_1, Z_2 and Z_5, Z_6 indicate the viscous friction, while the regimes for low velocities, i.e. regimes Z_3 and Z_4 introduce a Coulomb friction and Stribeck friction component. The symmetry of the friction torque follows from the PLM parameters in Table 7.2. The regimes are centered symmetric around zero velocity, and these symmetric regimes share the same absolute model parameter values. The symmetry of the system is revealed by the modelling method.

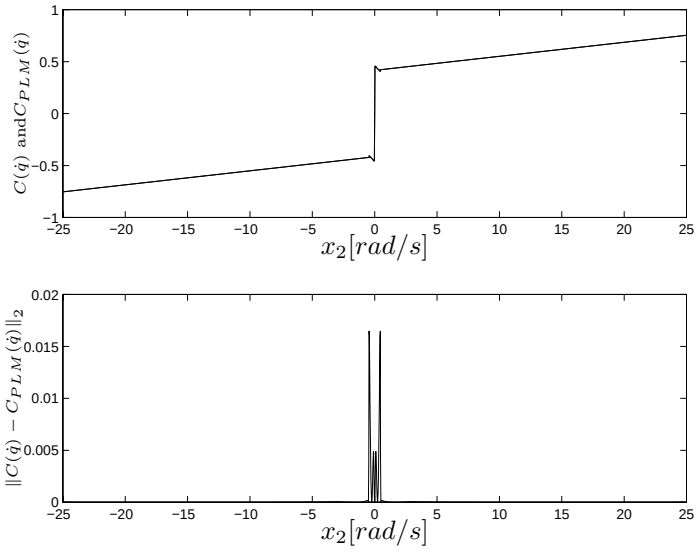


Figure 7.8: *Validation of the modelled friction torque.*

7.5.2 An Inverted Pendulum on a Cart

In this example the inverted pendulum as depicted in Figure 7.9 is considered as a subject for modelling. This is a benchmark example in the literature, confer [82], [44], [85]. The inverted pendulum is again a nonlinear mechanical

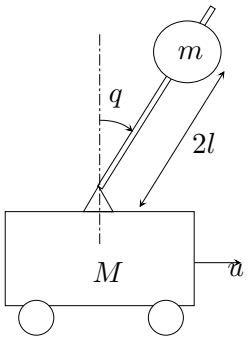


Figure 7.9: *Inverted pendulum on a cart.*

system with one degree of freedom q, where q denotes the angular displacement of the arm from the vertical axis, and u is the motor input. The state space equations (7.7) reduce to

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \frac{g\sin(q) - 0.5aml\dot{q}^2\sin(2q)}{4l/3 - aml\cos^2(q)} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-a\cos(q)}{4l/3 - aml\cos^2(q)} \end{bmatrix} u \tag{7.9}$$

and are taken from [13]. Here $g = 9.8m/s^2$ is the gravity constant, $m = 2.0kg$ is the mass of the pendulum, $M = 8.0kg$ is the mass of the cart, $2l = 1.0m$ is the length of the pendulum, and $a = 1/(m + M)$.

Once again the objective is to derive the simplest PLM that achieves ε-accuracy, that is the PLM that minimizes the number of scheduling regimes N_m. The modelling problem is divided into 3 subproblems as depicted in Figure 7.1, and solved sequentially following the modelling approach presented earlier in this section.

Subproblem 1: From Operating Space to Scheduling Space

The operating space is defined as the product space of the state space with the input space, i.e. $\Psi = X \times U$ where $x = [q, \dot{q}]^T \in X$, $q \in X_1$, $\dot{q} \in X_2$, and $u \in U$. From Theorem 4.6.3 we know that it is sufficient to schedule the models over the angle q and angular velocity $\dot{q}$, instead of over all the operating variables q,$\dot{q}$ and u, because the variable u enters the model in an affine way. Hence the scheduling space $Z = X$. Theorem 4.6.3 tells us that there will always exist a PLM with N_m finite that achieves ε-accuracy, at least if Z is compact, and Theorem 4.6.3 gives an upper bound for N_m. It is assumed that $Z = \{q, \dot{q} \mid |q| \leq \frac{\pi}{2}[rad], |\dot{q}| \leq 6.5[rad/s]\}$ covers the working range of the inverted pendulum.

Subproblem 2: From Scheduling Space to Scheduling Regimes

In this step the number of models N_m is reduced as much as possible compared to the upper bound from Theorem 4.6.3. This upper bound is based on a worst case scenario, and the result is a uniform partitioning of the scheduling space. Since the nonlinearity varies as a function of $z \in Z$, we can use the procedure aggregation or segregation to reduce the number of models. In Table 7.3 the number of models N_m, sufficient to assure ε-accuracy for, respectively, uniform decomposition, aggregation and segregation of the operating space are given.

Table 7.3: *Number of models, N_m*

ε	uniform	aggregation	segregation	extended
0.5	324	306	281	275
1.0	150	144	137	137
2.0	90	88	85	85
4.0	39	39	43	41
8.0	27	27	21	21
16.0	14	14	15	13
32.0	10	8	11	9

The result of the segregation method was improved by an extra aggregation step and summarized as the 'extended' method in the last column of Table 7.3. It can be noticed that, because of the global geometric nonlinearities in the system, the upper bound is much less conservative than in the previous example with the local nonlinearity. Choosing $\varepsilon = 8$, we obtain for the segregation method 21 models. It is difficult to determine a suitable value of ε in advance. Typically, one starts with a value, validates the result, and increases or decreases the value if one is not satisfied with the validation result.

The centres $d(Z_i)$ and the widths $2e(Z_i)$ of each operating regime Z_i : $d(Z_i) - e(Z_i) \leq (z = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}) \leq d(Z_i) + e(Z_i)$, where $d(Z_i) = \begin{bmatrix} d_1(Z_i) \\ d_2(Z_i) \end{bmatrix}$ and $e(Z_i) = \begin{bmatrix} e_1(Z_i) \\ e_2(Z_i) \end{bmatrix}$ are summarized in Table 7.4.

Table 7.4: *Operating regimes and model parameters*

i	$d_1(Z_i)$	$d_2(Z_i)$	$e_1(Z_i)$	$e_2(Z_i)$	$A_i(2,1)$	$A_i(2,2)$	$B_i(2)$	$a_i(2)$
1	-1.0472	-1.4444	0.5236	0.7222	6.0328	-0.1949	-0.0779	-7.0498
2	-1.0472	0	0.5236	0.7222	5.8512	0	-0.0779	-7.0992
3	-1.0472	1.4444	0.5236	0.7222	6.0328	0.1949	-0.0779	-7.0498
4	-0.7854	2.8889	0.7854	0.7222	9.5247	0.4685	-0.1147	-4.4332
5	-0.7854	4.3333	0.7854	0.7222	9.6619	0.7027	-0.1147	-5.1713
6	-0.7854	5.7778	0.7854	0.7222	9.8539	0.9369	-0.1147	-6.2047
7	-0.7854	-5.7778	0.7854	0.7222	9.8539	-0.9369	-0.1147	-6.2047
8	-0.7854	-4.3333	0.7854	0.7222	9.6619	-0.7027	-0.1147	-5.1713
9	-0.7854	-2.8889	0.7854	0.7222	9.5247	-0.4685	-0.1147	-4.4332
10	0	0	0.5236	0.7222	17.2941	0	-0.1765	0
11	0	-1.4444	0.5236	0.7222	16.9259	0	-0.1765	0
12	0	1.4444	0.5236	0.7222	16.9259	0	-0.1765	0
13	0.7854	2.8889	0.7854	0.7222	9.5247	-0.4685	-0.1147	4.4332
14	0.7854	4.3333	0.7854	0.7222	9.6619	-0.7027	-0.1147	5.1713
15	0.7854	5.7778	0.7854	0.7222	9.8539	-0.9369	-0.1147	6.2047
16	0.7854	-5.7778	0.7854	0.7222	9.8539	0.9369	-0.1147	6.2047
17	0.7854	-4.3333	0.7854	0.7222	9.6619	0.7027	-0.1147	5.1713
18	0.7854	-2.8889	0.7854	0.7222	9.5247	0.4685	-0.1147	4.4332
19	1.0472	-1.4444	0.5236	0.7222	6.0328	0.1949	-0.0779	7.0498
20	1.0472	0	0.5236	0.7222	5.8512	0	-0.0779	7.0992
21	1.0472	1.4444	0.5236	0.7222	6.0328	-0.1949	-0.0779	7.0498

The distribution of the centres $d(Z_i)$ over the scheduling space and for the obtained configuration is depicted in Figure 7.10.

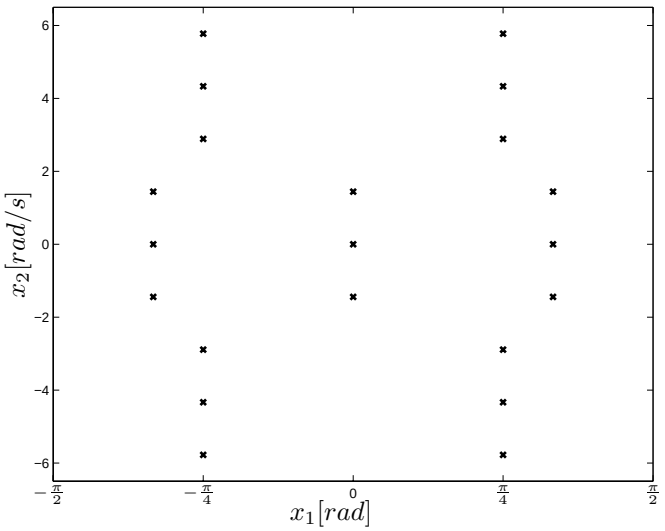


Figure 7.10: *Distribution of the centres $d(Z_i)$ over the scheduling space.*

Subproblem 3: From scheduling regimes to PLM parameters

The triples

$$\left(A_i = \begin{pmatrix} 0 & 1 \\ A_i(2,1) & A_i(2,2) \end{pmatrix}, B_i = \begin{pmatrix} 0 \\ B_i(2) \end{pmatrix}, a_i = \begin{pmatrix} 0 \\ a_i(2) \end{pmatrix}\right)$$

are determined by linearization, as described in [81]. The obtained model parameters for each scheduling regime are summarized in Table 7.4. The symmetry of the system is revealed by the PLM, i.e. regimes that are point symmetric compared to the origin in Figure 7.10 have the same quantitative and qualitative contribution to the overall behavior of the system. The scheduling functions are chosen as normalized radial basis functions that are placed in the center $d(Z_i)$ of each operating regime Z_i, i.e. $w_i(z) = \frac{\rho_i(z)}{\sum_{i=1}^{N_m} \rho_j(z)}$ with $\rho_i(z) = e^{-\frac{1}{2}(z-d(Z_i))^T \Sigma (z-d(Z_i))}$ where $\Sigma = \begin{bmatrix} \frac{1}{(\gamma e_1(Z_i))^2} & 0 \\ 0 & \frac{1}{(\gamma e_2(Z_i))^2} \end{bmatrix}$. The user-specified parameter γ is chosen 0.25, indicating almost no overlap between the models.

Model Validation

In Figure 7.11 the phase portrait of the system and the identified PLM, both with $u = 0$, are plotted for the compact operating space Z. The initial conditions of the plotted trajectories are indicated with a star (*). The boundaries

of the ellipsoidal validity regions of the operating regimes Z_i are also shown by the dotted lines. It is interesting to see how the trajectories are going through the different operating regimes Z_i. Almost all of the system trajectories diverge to infinity. The phase portrait shows a saddle point, as indicated with the asymptotes along the arrows. As a result in the centre of the plot an unstable equilibrium point can be observed. The difference between the trajectories of the system and the PLM can be made smaller by repeating the procedure for a smaller ε. Qualitatively, the phase portrait of the system and the PLM are very much alike; the system and the PLM are said to be topological equivalent [60].

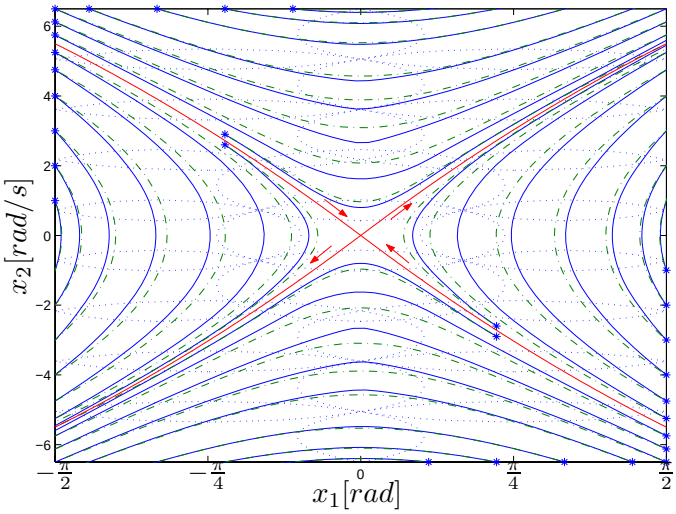


Figure 7.11: *Phase portrait of both the system (dashed line) and the PLM (solid line) within the compact scheduling space Z. The dotted ellipsoidal regions centered at $d(Z_i)$ are validity regions.*

7.6 Notes and Comments

Nonlinearity

Frequently, the term nonlinearity is used in a loose sense when system behavior is discussed. Think about phrases as: This system is very or strongly nonlinear, or, this system is locally nonlinear. These assertions are often not made precise, i.e. not supported by mathematical definitions. Within the presented modelling framework, terms as strongly nonlinear, and locally nonlinear, can be given a precise meaning. The decomposition of the system in operating regimes, allows for an intuitively clear and natural interpretation

of these terms. That is, these terms can be qualified and quantified. Next, two definitions are proposed that can contribute to the classification of nonlinearities, at least for systems that are defined on a non-compact operating space.

DEFINITION 7.6.1 (LOCALLY NONLINEAR) *A system is locally nonlinear if for any $\varepsilon > 0$, and for any non-compact operating region, an ε-accurate PLM exists, consisting of a finite number of models.*

DEFINITION 7.6.2 (RELATIVELY STRONGLY NONLINEAR) *A system is relatively strongly nonlinear if for any $\varepsilon > 0$ an ε-accurate PLM exists for the region Ψ such that for some operating region $\Psi_1 \subset \Psi$ it holds that $N_m(\Psi_1) \simeq N_m(\Psi)$ and $\Psi_1 \ll \Psi$.*

Segregation

The segregation procedure relies on a tree construction algorithm for structure optimization. It orthogonally partitions the scheduling space into hyperrectangles. Similar procedures but for discrete time systems are reported in literature, see [56] and [34]. A main difference between all these procedures, is the criterion that is used to select the 'best' partitioning.

Chapter 8

Building PLMs from Measured Data

8.1 Introduction

As depicted in Figure 6.2, model construction consists of three phases, namely the selection of a model structure, followed by a model set and finally a model from the model set. Since the model structure is chosen to be a PLM, what remains to be determined is a model set and the best model has to be selected from the model set. The determination of the model set and a suitable model from the model set depends besides prior knowledge and the type and amount of available data, of course on the objective of modelling.

Two data based modelling methods will be described that automatically generate a PLM from data and prior knowledge. These two methods are output error methods, based upon a well known augmented state least squares filtering method [22]. A distinction has been made between local and global parameter estimation methods.

Before presenting the modelling methods, in Section 8.2, the basic aspects of least squares filtering will be discussed. After that, in Section 8.3, the first method, a local parameter estimation method is presented. It is founded upon the regime segregation method described earlier. This means that the scheduling parameters are identified independently from the local model parameters. In this case the decision to segregate is specified by the output error criterion that an augmented state least squares filter tries to minimize. For each segregated region, (that is a scheduling regime,) a linear model is estimated. This method is illustrated by an example in Section 8.4. In Section 8.5, the second method, a global parameter estimation method is presented. With the second method it is shown that it is feasible, at least for small scale problems, to utilize a standard augmented state least squares filter, to estimate for the

entire operating space of interest, both the unknown scheduling parameters and the unknown local model parameters simultaneously. This method is illustrated by an example in Section 8.6. Finally, in Section 8.7, some notes and comments are made regarding the proposed methods.

To make the data based modelling methods more clear, first the relevant aspects of least squares filtering will be explained. We will closely follow the formulation presented in [74].

8.2 Least Squares Filtering

In fact, the least squares filter is used as a state reconstruction and parameter estimation method for the determination of the unknown state x and the unknown parameters θ of a continuous time model

$$\begin{aligned} \dot{x} &= f(x,u,\theta), \quad x(t_0) = x_{t_0} \\ y &= h(x,u,\theta) \end{aligned} \tag{8.1}$$

with state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$, parameters $\theta \in \mathbb{R}^k$ and output $y \in \mathbb{R}^p$. It is the task of the filtering procedure to estimate the unknowns in some optimal sense given measurements $\{u, y_r\}$ of the input and the output of the system. The way to formulate and to solve this problem will be outlined below.

By adding a set of k trivial differential equations, $\dot{\theta} = 0$, to the description (8.1), the combined parameter and state estimation problem is converted into a pure state reconstruction problem.

$$\dot{x} = f(x,u,\theta), \quad x(t_0) = x_{t_0} \tag{8.2}$$

$$\dot{\theta} = 0, \quad \theta(t_0) = \theta_{t_0} \tag{8.3}$$

$$y = h(x,u,\theta) \tag{8.4}$$

Mathematically, the resemblance that exists between the state equation (8.2) and the parameter equation (8.3) clearly suggests that there is no need to distinguish the states x from the parameters θ. Therefore the unknown state x is augmented with the unknown parameters θ, and the new augmented state $x_a = [x^T \ \theta^T]^T$ is introduced.

$$\begin{aligned} \dot{x}_a &= f(x_a,u), \quad x_a(t_0) = x_{at_0} \\ y &= h(x_a,u) \end{aligned} \tag{8.5}$$

It is assumed that measurements $\{u(t), y_r(t)\}$ for times $t \in [t_0, t_e]\}$ are available. A model like (8.5) however is only a limited description of reality and there will always be some discrepancy between the model behavior and the

measurements. Thus, the measurements will not satisfy (8.5). This discrepancy can be taken into account by assuming that the system can be described by

$$\begin{aligned} \dot{x}_a &= f(x_a, u) + \xi, \quad x_a(t_0) = x_{at_0} + \zeta \\ y_r &= h(x_a, u) + \eta \end{aligned} \tag{8.6}$$

where $\xi(t)$ and $\eta(t)$ are called the residuals on the state equation and the output equation respectively, and ζ represents the difference between the estimated initial state $x_a(t_0)$ and the real initial state x_{t_0} of the system. If the estimate x_a is such that the residuals are small in some sense, then this will give confidence in the chosen model structure, the measured data and the obtained estimate. More specifically, optimal filtering refers to the 'best' estimate $x_a(t)$ at the current time t, based upon all past measurements, and is specified as a least squares estimate $x_a(t)$ which minimizes a weighted squares of the residuals, over the whole interval $[t_0, t]$. For this choice the estimate $x_a(t)$ has to minimize the measure

$$J := \frac{1}{2} \int_{t=t_0}^{t} \{\xi(t)^T W \xi(t) + \eta(t)^T V \eta(t)\} dt + \frac{1}{2} \zeta^T R_0 \zeta \tag{8.7}$$

The matrices W, V and R_0 are user specified weighting matrices that weight the residuals on the state and output equation respectively, they express confidence in these equations. As an example, suppose one restricts to model structures with constant parameters θ, as is the case in (8.3). By choosing very large values for those elements in W that weight the residuals on the parameter equations, it is made sure that the parameter estimates in the augmented state x_a will not change significantly with time. In a similar way other prior model assumptions and knowledge can be incorporated. For instance, the elements of $\xi(t)$ and $\zeta(t)$, besides having different dimensions, often take values that differ substantially from each other in magnitude. Then, such values for the elements in W and V are chosen that the weighted residuals are all of the same order of magnitude. After that, to express the confidence we have in certain model equations, we can choose larger values for the corresponding elements in W and V. The matrix W has to be positive definite and V should be a semi-positive definite matrix; often they are chosen diagonal. The column x_{at_0} from $\zeta = x_a(t_0) - x_{at_0}$ represents initial knowledge about the real initial state, and R_0 is a positive definite matrix expressing confidence in knowledge of the initial state.

The optimal estimation or filtering problem is treated extensively in a survey given in [32] and references therein. An approximate solution x_a to the

optimal filtering problem is given by [22]

$$\dot{x}_a = f(x_a, u) + K(y_r - h(x_a, u)), \qquad x_a(t_0) = x_{at_0} \tag{8.8}$$

$$\dot{P}(t) = F(t)P(t) + P(t)F(t)^T + W - P(t)H(t)^T V^{-1} H(t)P(t), \qquad P(t_0) = R_0 \tag{8.9}$$

$$K(t) = P(t)H(t)^T V^{-1} \tag{8.10}$$

where $F = \frac{\partial f}{\partial x_a}(x_a, u)$ and $H = \frac{\partial h}{\partial x_a}(x_a, u)$, both evaluated at the current estimated state and current input.

The above filter can also be derived in a statistical setting and interpreted in statistical terms, then called the extended Kalman filter. The state estimate equation (8.8) of the above filter, is nothing more then a copy of the state equation of the model (8.5), the state of which is improved by taking into account measurements y_r of the real system. The measurements are compared to the output of the model (8.5). The difference between the estimated outputs and the measurements is propagated by the gain matrix K to produce an innovation signal and to improve the state estimate. The gain K, see (8.9) is computed in such a way that it optimally balances the confidence in different parts of the model, as specified by (8.7).

There is however no guarantee that the approximate filter (8.8,8.9,8.10) will converge when applied to a system [49]. Nevertheless, in practical application the filter is commonly used, which often leads to useful estimates.

8.3 Local Parameter Estimation

As explained earlier the segregation procedure has as a point of departure the most simple PLM, consisting of only one linear model, which covers the entire scheduling space as depicted in Figure 7.4, and step by step model complexity is increased conceptually as follows: split the n_Z dimensional scheduling space 'in the middle' in two scheduling regimes Z_i and Z_{i+n_Z}. This can be done in n_Z different ways, i.e. $i = \{1, ..n_Z\}$. After that a cost criterion is evaluated for each partitioning, from which the most promising partitioning follows. One segregation step is then completed, and segregation is repeated until some prescribed accuracy is achieved, meaning that the criterion that is evaluated is below some prescribed value.

If there is no model of the system available, the number of models N_m sufficient to achieve ε-accuracy within some region, cannot be computed and therefore the suggested criterion for splitting the operating space fails. A useful criterion however has to measure the distribution of the nonlinearity of the system over a grid in the scheduling space.

It is assumed that the only information of the system available are measurements $\{u, y_r\}$ of the input u and the output y_r, on some time interval $[t_0, t_e]$, and some prior knowledge on the system, such as the order of the system and structure of the matrices $(A_i, B_i, a_i, C_i, D_i, c_i)$, such as structural zeros and ones, involved in the representation of the PLM. The decision to segregate is specified by the output error criterion that the least squares filter minimizes. For each segregated regime Z_i, a linear model $(A_i, B_i, a_i, C_i, D_i, c_i)$, that is part of the PLM, will be estimated.

8.3.1 Data Based Regime Segregation

The least squares filter (8.8,8.9,8.10) is used as a state reconstruction and parameter estimation method for the determination of the unknown state and the unknown parameters of the locally valid continuous time models that are part of the PLM. A locally valid model, i.e. a linear model valid within some region of the operating space, is described as follows:

$$\begin{aligned} \dot{x} &= A_i x + B_i u + a_i, \quad x(t_0) = x_{t_0} \\ y &= C_i x + D_i u + c_i \end{aligned} \tag{8.11}$$

with state $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. However, in general the state and in particular the initial state x_{t_0} is unknown, as are some elements $\theta \in \mathbb{R}^k$ of the matrices $\{A_i, B_i, ..., c_i\}$. It is the task of the filter (8.8,8.9,8.10) to estimate these unknowns.

By adding a set of k trivial differential equations, $\dot{\theta} = 0$ to the description (8.11), the combined parameter and state estimation problem is converted into a state reconstruction problem.

$$\begin{aligned} \dot{x} &= A_i(\theta)x + B_i(\theta)u + a_i(\theta), & x(t_0) &= x_{t_0} \\ \dot{\theta} &= 0, & \theta(t_0) &= \theta_{t_0} \\ y &= C_i(\theta)x + D_i(\theta)u + c_i(\theta) \end{aligned}$$

The unknown state x is augmented with the unknown parameters θ, and the augmented state $x_a = [x^T \ \theta^T]^T$ is introduced to obtain (8.5).

First the (optimal) filter (8.8,8.9,8.10) is applied to reconstruct the augmented state x_a, i.e. the parameters θ and the state x, of the linear model (8.11) based on all the measurements available. Next, the obtained constant parameters θ are substituted in (8.11) and the state x of the model (8.11) is again reconstructed by a filter. If the underlying nonlinear system is sufficiently smooth, then it is expected that within a region small enough, a linear model like (8.11) is an acceptable model for the system. Acceptability of the

model (8.11) will be evaluated by a measure similar to (8.7), i.e.

$$J := \frac{1}{2}\int_{t=t_0}^{t} \{\xi(t)^T W \xi(t) + \eta(t)^T V \eta(t)\} dt + \frac{1}{2}\zeta^T R_0 \zeta \tag{8.12}$$

where $\xi(t) = \dot{x}(t) - A_i x(t) - B_i u(t) - a_i = K(y_r(t) - C_i x(t) - D_i u(t) - c_i)$ is the filter innovation, $\eta(t) = y_r(t) - C_i x(t) - D_i u(t) - c_i$ the difference between the output of the filter and the measurements, and $\zeta = x(t_0) - x_{t_0}$ represents uncertainty about the real initial condition. Thus ξ, η and ζ represent errors on the state and output equation due to unmodelled phenomena. A model (8.11) is said to be acceptable if $J \leq \varepsilon$, with $\varepsilon > 0$ to be specified by the user. The decision to segregate is specified by a normalized version of (8.12), to account for the time that is spent within some region Z_i.

More specifically, we evaluate $J(Z_i) + J(Z_{i+n_Z})$ for all n_Z obtained configurations, and select the configuration of which the evaluated measure is the smallest. Here

$$J(Z_i) = \frac{1}{\tau(Z_i)} \sum_{j=1}^{k_{Z_i}} J(Z_i, T_{j,Z_i}) \tag{8.13}$$

$$J(Z_i, T_{j,Z_i}) := \int_{t \in T_{j,Z_i}} \{\xi(t)^T W \xi(t) + \eta(t)^T V \eta(t)\} dt + \frac{1}{2}\zeta^T R_0 \zeta \tag{8.14}$$

were $\tau(Z_i)$ stands for the time that is spent in region Z_i, and $T_{j,Z_i} = [t_{0,j,Z_i}$ $t_{e,j,Z_i}]$ stands for one of the k_{Z_i} time intervals spent in Z_i. Thus $\sum_{j=1}^{k_{Z_i}} (t_{e,j,Z_i} - t_{0,j,Z_i}) = \tau(Z_i)$. The situation is depicted in Figure 8.1. Basically, segregation is repeated until for each obtained partitioning $J(Z_i) \leq \varepsilon$. There are however some limitations. If segregation gives no further improvement regarding the error measure J, in comparison to the error measure of the undivided space, segregation for that region is stopped. Furthermore, when there are not enough measurements left to divide, the procedure is stopped. This means that either the chosen accuracy, as specified by ε, is too high or the number of measurements is too small.

For each scheduling regime Z_i obtained by segregation, a model (8.1), the set $(A_i, B_i, a_i, C_i, D_i, c_i)$, was already determined by the filter. These models will be scheduled by the scheduling functions w_i, that are defined as described in Subsection 7.4, hence the PLM

$$\dot{x} = \sum_{i=1}^{N_m} w_i(z)\{A_i x + B_i u + a_i\}$$

$$y = \sum_{i=1}^{N_m} w_i(z)\{C_i x + D_i u + c_i\}$$

is specified.

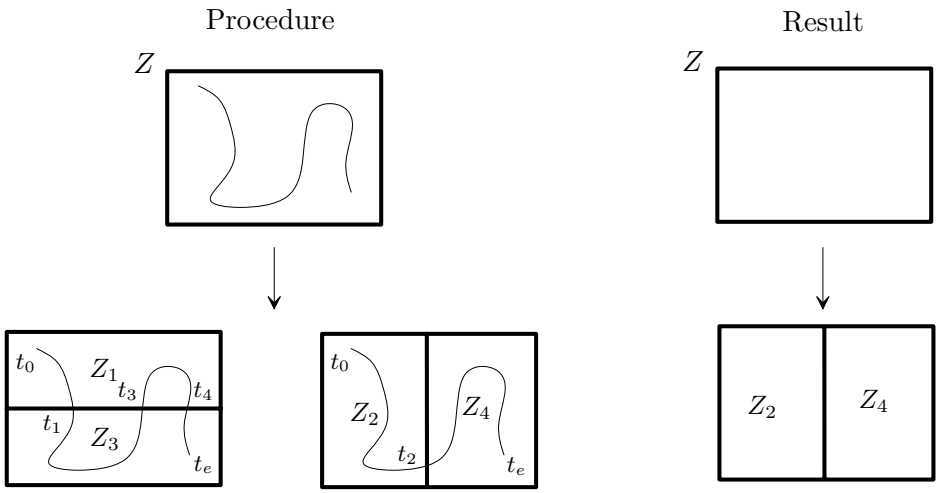


Figure 8.1: *Data based regime segregation. For each candidate regime Z_i a linear model is identified. The parametrization of each linear model, i.e. $(A_i, B_i, ..., c_i)$, is based on the data $\{u(t), y_r(t) \mid z(t) \in Z_i\}$ associated with regime Z_i. That means, the measurements at times t for which it holds that the estimated scheduling variable $z(t)$, obtained from a previous segregation step, is in the region Z_i. The configuration for which the evaluated measure $J(Z_i) + J(Z_{i+n_Z})$ is the smallest is selected. In this example* $\dim(Z) = n_Z = 2$ *and we evaluate* $J(Z_1) = \frac{1}{(t_1-t_0)+(t_4-t_3)}\{J(Z_1, [t_0\ t_1]) + J(Z_1, [t_3\ t_4])\}$ *and* $J(Z_3) = \frac{1}{(t_3-t_1)+(t_e-t_4)}\{J(Z_3, [t_1\ t_3]) + J(Z_3, [t_4\ t_e])\}$. *And also* $J(Z_2) = \frac{1}{(t_2-t_0)}J(Z_2, [t_0\ t_2])$ *and* $J(Z_4) = \frac{1}{(t_e-t_2)}J(Z_4, [t_2\ t_e])$. *In this case* $J(Z_1) + J(Z_3) > J(Z_2) + J(Z_4)$ *and the configuration with* $Z = Z_2 + Z_4$ *is selected.*

8.4 Example

8.4.1 A Rotational Robotic Manipulator System

In this example a real rotational robotic manipulator system, as described earlier and depicted in Figure 7.5, is considered as a subject to be modelled. However, in contrast to the model based segregation method, the objective is to derive a PLM based upon measured data on the system. More specifically, the data based segregation method is illustrated as a possible solution to the modelling problem.

The resolution of the encoder used to measure the angular displacement is 1.9175×10^{-4} $[rad]$. The induction motor is supplied by a 'Pulse Width Modulation' source inverter which translates the input signal, i.e., the desired torque expressed in a voltage, into three phase signals with a fundamental frequency. This source inverter actually controls the torque produced by the

motor to the desired torque. The input signal of the source inverter and the TTL encoder signals are respectively sent and read by a dSPACE system [20]. During the experiments the sample frequency is set to 1 $[kHz]$.

Prior Knowledge

It is assumed that the robot arm is a nonlinear control affine mechanical system (7.7) with one degree of freedom q, where q represents the angular displacement of the arm and u is the input motor voltage. The output y of the system is the angular displacement q of the arm.

The mechanical model structure makes it possible to restrict the representation of the PLM by means of structural zeros and ones, i.e.,

$$A_i = \begin{pmatrix} 0 & 1 \\ A_i(2,1) & A_i(2,2) \end{pmatrix} \quad B_i = \begin{pmatrix} 0 \\ B_i(2) \end{pmatrix} \quad a_i = \begin{pmatrix} 0 \\ a_i(2) \end{pmatrix}$$
$$C_i = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad D_i = 0 \qquad c_i = 0$$

and only the nontrivial parameters, the ones different from 0 and 1 have to be estimated.

From Operating Space to Scheduling Space

The operating space is defined as the product space of the state space with the input space, i.e. $\Psi = X \times U$ where $x = [q\ \dot{q}]^T \in X$, $q \in X_1$, $\dot{q} \in X_2$, and $u \in U$. From Theorem 4.6.3 and the prior knowledge available on the system, we know that it is sufficient to schedule the models over the angle q and angular velocity $\dot{q}$, instead of over all the operating variables q,$\dot{q}$ and u. This is because the variable u enters the model in an affine way. Hence the scheduling space $Z = X$. Theorem 4.6.3 tells us that there will always exist a PLM that achieves arbitrary accuracy, at least if Z is compact and if the system is assumed to be smooth enough. It is expected that the system exhibits nonlinear behavior for low velocities due to nonlinear friction characteristics. Therefore, it is assumed that $Z = \{q, \dot{q} \mid |q| \leq \frac{\pi}{10}[rad], |\dot{q}| \leq 4[rad/s]\}$ covers the interesting working range of the inverted pendulum.

Experiment Design

First, an experiment has to be designed, i.e. an input signal u has to be applied to the robot arm such that the measurements $\{u, y_r\}$ contain enough information to determine together with a priori knowledge on the system a PLM that suffices for the modelling task at hand. It is not an easy task to excite the system in the working range of interest. As mentioned earlier, several case studies indicated that a relatively high frequency input signal of small

amplitude, superimposed on a low frequency input signal with a relatively large amplitude, leads to a data set that contains sufficient information for the modelling task to be successful. Figure 8.2 shows the data set obtained from the system, that is the input signal together with the output response of the system. The measurement noise is not visible on the scale of the plot.

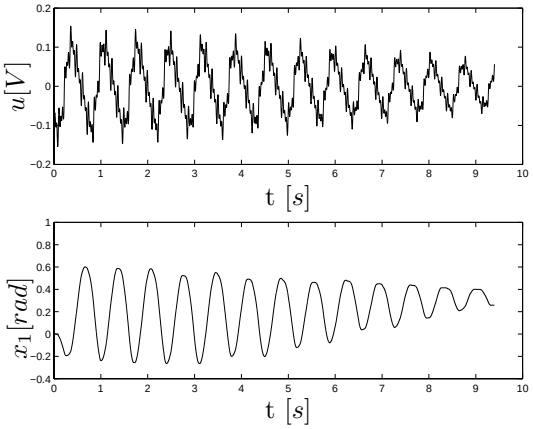


Figure 8.2: *Experimental data.*

The measured angle is differentiated numerically by a zero-phase high pass filter with a cut-off frequency of 200 Hz to reconstruct the angular velocity $\dot{q}$. This reconstruction makes it possible to depict the system response in the scheduling space. From Figure 8.3 it can be seen that this trajectory covers the estimated scheduling space fairly well, which is required for the segregation procedure to work. The reconstructed angular velocity will not be used

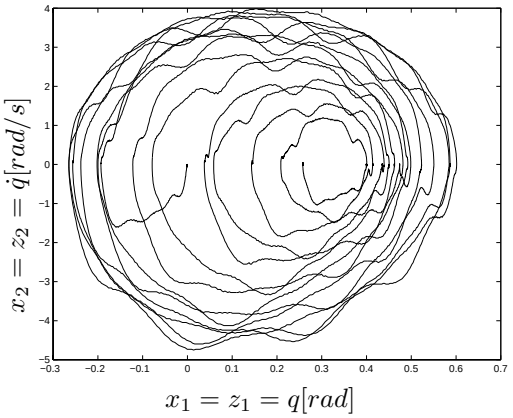


Figure 8.3: *System trajectory in the scheduling space Z.*

as an additional measurement for the segregation procedure.

Segregation

The data set $\{u, y_r\}$ is offered to the segregation procedure. The user specified accuracy ε is set as follows, $J(Z_i) \leq \varepsilon$, and the measure $J(Z_i)$ is defined in (8.13). Choosing $\varepsilon = 0.01$ we obtain for the segregation method 21 models.

The centres $d(Z_i)$ and the widths $2e(Z_i)$ of each operating regime Z_i : $d(Z_i) - e(Z_i) \leq (z = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}) \leq d(Z_i) + e(Z_i)$, where $d(Z_i) = \begin{bmatrix} d_1(Z_i) \\ d_2(Z_i) \end{bmatrix}$ and $e(Z_i) = \begin{bmatrix} e_1(Z_i) \\ e_2(Z_i) \end{bmatrix}$ are summarized in Table 8.1. The unknown param-

Table 8.1: *Operating regimes and model parameters*

i	$d_1(Z_i)$	$d_2(Z_i)$	$e_1(Z_i)$	$e_2(Z_i)$	$A_i(2,1)$	$A_i(2,2)$	$B_i(2)$	$a_i(2)$
1	0.1683	-2.5673	0.4331	2.1844	-19.6306	-1.3388	474.5737	23.2652
2	-0.0483	2.8937	0.2165	1.0922	-24.3043	-3.3836	406.8478	-7.0186
3	-0.0483	1.2554	0.2165	0.5461	-15.0759	-2.9403	369.6105	-5.9302
4	0.4931	2.8937	0.1083	1.0922	-24.5307	-6.5162	374.8093	1.5409
5	0.1683	0.2997	0.4331	0.1365	-14.6128	-10.8237	354.7136	-7.4879
6	0.2224	2.8937	0.0541	1.0922	-20.1223	-4.6237	373.2493	0.2875
7	0.3307	2.8937	0.0541	1.0922	-25.8769	-3.6231	366.1476	-2.5554
8	-0.0483	-0.2464	0.2165	0.1365	-21.7033	-9.8994	337.5465	19.8181
9	0.1683	0.0267	0.4331	0.1365	-12.3130	-41.1949	230.6125	5.4659
10	0.1683	0.5045	0.4331	0.0683	-22.4484	-10.9273	344.9230	-2.8270
11	0.3307	1.2554	0.0541	0.5461	-24.7331	-6.5735	351.7486	1.6675
12	0.4931	1.5285	0.1083	0.2731	-23.5115	-9.2669	359.3913	3.1342
13	0.2765	-0.2464	0.1083	0.1365	-20.1245	-8.6843	338.2395	14.3558
14	0.1683	0.6069	0.4331	0.0341	-16.7735	-11.4670	332.0291	-2.4134
15	0.1683	0.6752	0.4331	0.0341	-20.3161	-10.9481	335.3594	-1.2280
16	0.2224	0.9824	0.0541	0.2731	-21.7525	-8.0048	337.8189	5.0920
17	0.2224	1.5285	0.0541	0.2731	-22.8421	-9.6383	337.5241	4.3743
18	0.4931	0.8458	0.1083	0.1365	-25.8317	-11.3728	345.0546	-1.3407
19	0.4931	1.1189	0.1083	0.1365	-23.5545	-10.6700	343.2975	1.1010
20	0.4931	-0.3146	0.1083	0.0683	-17.1133	-9.0823	337.7909	13.8845
21	0.4931	-0.1781	0.1083	0.0683	-15.7904	-8.4212	339.0424	16.1129

eters from $A_i = \begin{pmatrix} 0 & 1 \\ A_i(2,1) & A_i(2,2) \end{pmatrix}, B_i = \begin{pmatrix} 0 \\ B_i(2) \end{pmatrix}, a_i = \begin{pmatrix} 0 \\ a_i(2) \end{pmatrix}$ are determined by the augmented state least squares filter, as a part of the segregation procedure. The scheduling functions are chosen as normalized radial basis functions that are placed in the center $d(Z_i)$ of each operating

regime Z_i, i.e. $w_i(z) = \frac{\rho_i(z)}{\sum_{i=1}^{N_m} \rho_j(z)}$ with $\rho_i(z) = e^{-\frac{1}{2}(z-d(Z_i))^T \Sigma (z-d(Z_i))}$ where $\Sigma = \begin{bmatrix} \frac{1}{(\gamma e_1(Z_i))^2} & 0 \\ 0 & \frac{1}{(\gamma e_2(Z_i))^2} \end{bmatrix}$. The user-specified parameter γ is chosen 0.25, indicating almost no overlap between the models.

Model Validation

The identified model is now used for validation. Figure 8.4 shows both the system response, and the response of the identified PLM in the compact scheduling space Z. The model validity regions as computed by the segregation pro-

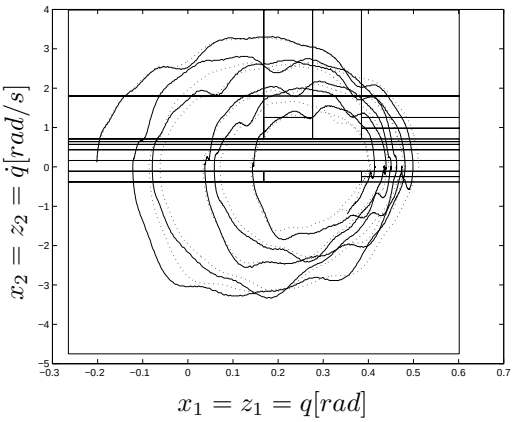


Figure 8.4: *Response of the system (solid) and the identified PLM (dotted) on the input signal used for identification. Also the segregated scheduling space (in rectangular regions) is shown.*

cedure are also depicted in the same figure. One can clearly see a preference to segregate in the $z_2 = \dot{q}$ direction. Furthermore, the regimes are concentrated around $\dot{q} = 0$. This suggests that the system is nonlinear, as a function of $\dot{q}$, and highly nonlinear in the region around $\dot{q} = 0$. These are typically properties of the nonlinear friction models that were proposed earlier. The time response of the system and the identified PLM on the same input signal is shown in Figure 8.5. The small difference between the two responses, also on other input signals, confirms the good long-term prediction capabilities of the PLM.

8.5 Global Parameter Estimation

The least squares filter (8.8,8.9,8.10) is used as a state reconstruction and parameter estimation method for the determination of the unknown state and

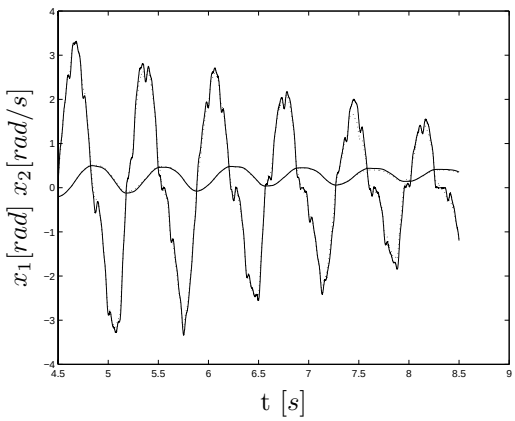


Figure 8.5: *Time response of the system (solid) and the identified PLM (dotted) on the input signal used for identification.*

the unknown parameters of a PLM

$$\dot{x} = \sum_{i=1}^{N_m} w_i(x,u)\{A_i x + B_i u + a_i\}, \quad x(t_0) = x_{t_0} \tag{8.15}$$
$$y = \sum_{i=1}^{N_m} w_i(x,u)\{C_i x + D_i u + c_i\}$$

with state $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. In general the state and in particular the initial state x_{t_0} is unknown, as are some elements $\theta \in \mathbb{R}^k$ of the matrices $\{A_i, B_i, ..., c_i\}$ and the scheduling functions w_i. The scheduling functions are defined as $w_i(z,\theta) = \frac{\rho_i(z,\theta)}{\sum_{j=1}^{N_m} \rho_j(z,\theta)}$ where $z = s(x,u)$ is a set of scheduling variables. The unknown parameters of the scheduling functions are as before the centres d_i and the widths e_i of the basis of the a priori chosen basis functions $\rho_i(z,\theta)$. It is the task of the filter (8.8,8.9,8.10) to estimate all these unknowns simultaneously.

By adding a set of k trivial differential equations, $\dot{\theta} = 0$ to the description (8.15), the combined parameter and state estimation problem is converted into a state reconstruction problem.

$$\dot{x} = \sum_{i=1}^{N_m} w_i(z,\theta)\{A_i(\theta)x + B_i(\theta)u + a_i(\theta)\}, \qquad x(t_0) = x_{t_0} \tag{8.16}$$
$$\dot{\theta} = 0, \qquad \theta(t_0) = \theta_{t_0}$$
$$y = \sum_{i=1}^{N_m} w_i(z,\theta)\{C_i(\theta)x + D_i(\theta)u + c_i(\theta)\}$$

The unknown state x is augmented with the unknown parameters θ, and the augmented state $x_a = [x^T \ \theta^T]^T$ is introduced to obtain (8.5).

The optimal filter (8.8,8.9,8.10) is applied to reconstruct the augmented state x_a, i.e. the parameters θ and the state x, of the PLM (8.15) based on all the measurements available. Prior system knowledge can be incorporated into the PLM by restricting the structure, thereby facilitating the applicability of the parameter estimation method. Next an experimental case study is reported to illustrate the modelling method, and in particular it is shown how prior knowledge can be incorporated.

8.6 Example

8.6.1 A Rotational Robotic Manipulator System

In this example a real rotational robotic manipulator system, as described earlier and depicted in Figure 7.5, is considered as a subject for modelling. In conformance with the data based segregation method, the objective is to derive a PLM based upon measured data on the system. More specifically, the global parameter estimation method is illustrated as a possible solution to the modelling problem.

Prior Knowledge

The state space equations (7.7) describing the robotic manipulator system are assumed to simplify to

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -M^{-1}C(\dot{q}) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}B_m \end{bmatrix} u$$

with M the unknown effective inertia of the motor-transmission-rotating arm combination, $B_m = 16\ [NmV^{-1}]$ is the motor gain, and $C(\dot{q})$ is the friction torque. The system is depicted in Figure 7.5. The friction torque is a simplified model of reality, and assumed to be a function of the angular velocity only. The validity of this assumption is partly confirmed by the outcome of the previous example on the subject of data based segregation. There it could be seen that segregation dominantly occurred in the $\dot{q}$ direction. Furthermore, it is assumed that the friction can be modelled by an odd continuous function, i.e.

$$C(\dot{q}) = -C(-\dot{q})$$

This is consistent with well known friction models, see Figure 7.6. For angular velocity equal to zero the model friction torque is zero, which results

in the preservation of equilibrium points of the system. The friction model does not describe stiction, since the continuous friction model will always slide (i.e. move) for an applied input u unequal to zero. However if stiction is present in the system, the stiction regime will be approximated, if the friction function near $\dot{q} = 0$ is very steep. Then the model can still give acceptable simulation results, that is angular displacement during stiction will be negligible on our scale of interest.

The PLM is composed of N_m locally valid models. The structure of each model is chosen equal to the topology of a mechanical system. This makes it possible to restrict the representation of the PLM, i.e.,

$$A_i = \begin{pmatrix} 0 & 1 \\ 0 & A_i(2,2) \end{pmatrix} \quad B_i = \begin{pmatrix} 0 \\ B_i(2) \end{pmatrix} \quad a_i = \begin{pmatrix} 0 \\ a_i(2) \end{pmatrix}$$
$$C_i = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad D_i = 0 \qquad c_i = 0$$

and only the nontrivial parameters have to be estimated. With each local model (A_i, B_i, a_i) a model validity function ρ_i is associated which, by definition is close to one for those regions were the corresponding linear model is valid, and zero elsewhere. Here, the validity depends only on the angular velocity $\dot{q}$, due to the choice of the nonlinear friction as a function of only $\dot{q}$. Hence the scheduling variable $z = \dot{q}$. The scheduling functions are defined as normalized model validity functions, i.e. $w_i(z) = \frac{\rho_i(z)}{\sum_{i=1}^{N_m} \rho_j(z)}$. A typical choice for ρ_i is the Gaussian function

$$\rho_i(z) = e^{-\frac{(z-d(Z_i))^2}{(2e(Z_i))^2}}$$

where $d(Z_i)$ marks the center and $2e(Z_i)$ the width of operating region Z_i. The PLM becomes

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \sum_{i=1}^{N_m} w_i(z,\theta)\{A_i(\theta) + B_i(\theta)u + a_i(\theta)\} \tag{8.17}$$

For the PLM the unknown model parameters that have to be estimated are

$$\theta = [A_i(2,2), B_i(2), a_i(2), d(Z_i), e(Z_i)]$$
$$i = 1, ..., N_m$$

Note that $B_i(2) = B(2) = M^{-1}B_m$ with M unknown. The unknowns θ are further reduced by restricting the PLM to construct only odd friction functions in the following way

- Choose an odd number of local models, i.e. $N_m = 2k + 1$ with $k \in \{0, 1, 2, ...\}$, where one local model has no offset, i.e. $a_i(2) = 0$ and the corresponding center $d(Z_i) = 0$.

- The other $N_m - 1$ models are divided in pairs of two symmetrical around $z = 0$, i.e., the centers are opposite $d(Z_i) = -d(Z_{i+1})$, the widths are equal $e(Z_i) = e(Z_{i+1})$ as well as the slopes $A_i(2,2) = A_{i+1}(2,2)$ and the offsets are again opposite $a_i = -a_{i+1}$ with $i = 2k$ with $k \in \{1, 2, ..., 1/2N_m\}$

An advantage of this construction is the reduction of parameters by about a factor two.

Experiment Design

This same data set was used as for the data based segregation method (see Figure 8.2).

Filtering

The augmented state least squares filter (8.8,8.9,8.10) was implemented off-line. For the PLM the number N_m is set to three, in order to facilitate the estimation procedure. If this PLM does not lead to a satisfactory model, then complexity can be increased gradually by increasing N_m.

In this case $\theta = [A_1(2,2), e(Z_1), A_2(2,2), a_2(2), d(Z_2), e(Z_2), M]$, i.e. 7 parameters have to be determined; 6 for the friction model and the inertia M. The filter parameters R_0, W, V from (8.8,8.9,8.10), which express consecutively confidence or uncertainty in initial states, model equations and measurements have to be tuned in order to obtain reliable augmented state estimates. The initial state x of the system is known, but the model parameters θ are not known. The initial model parameters are chosen in such a manner that physical known properties, e.g., positive inertia value or positive viscous damper value, are met. (Though we have no guarantee that that will stay that way during the estimation phase.) Hence, the uncertainty for the initial state estimates is small while we are not sure of the initial estimates for the model parameters. These considerations lead to the initial augmented state uncertainty matrix $R_0 = \text{diag}(0, 0, 1, \dots, 1)$ where the non-zero elements express the uncertainty of initially uncertain parameters. The matrix W can be seen as expressing uncertainty on the augmented state model equations. The model equations describing constant model parameters $\dot{\theta} = 0$ and $\frac{d}{dt}q = \dot{q}$ are regarded as true. The confidence in these equations being large, is expressed by a zero at the corresponding places at the diagonal matrix $W = \text{diag}(0, W_{22}, 0, \dots, 0)$. Here $Q_{22} = 0.001$ expresses uncertainty in the proposed friction model. Due to the finite encoder resolution of 1.9175 10^{-4} $[rad]$ and the differentiation scheme an uncertainty on the angular velocity reconstruction is introduced. To take this into account the matrix V is constructed as a diagonal matrix $V = \text{diag}(0.001, 0.01)$ where 0.001 corresponds to the uncertainty in the angle

measurement and 0.01 to the uncertainty in the arm velocity reconstruction. The filter tuning is mainly based on experience and trial and error. It is important that the parameters converge to constant values. Different filter tuning will result in different convergence speeds and even parameter divergence can occur. The estimated data are passed through the filter several times until the parameter estimates converge. After 10 filter passes the parameter estimates become constant and the sum of eigenvalues of the covariance matrix $P(t)$ has become minimal. The identified inertia value is $M = 0.0323\ [kgm^2rad^{-1}]$. The obtained model parameters for each scheduling regime Z_i are summarized in Table 8.2. The scheduling functions are shown in Figure 8.6.

Table 8.2: *Operating regimes and model parameters*

i	$d(Z_i)$	$e(Z_i)$	$A_i(2,2)$	$B_i(2)$	$a_i(2)$
1	0	0.0290	−2040.3	495.7782	0
2	1.6162	1.4033	−2.2466	495.7782	18.7012
3	1.6162	1.4033	−2.2466	495.7782	−18.7012

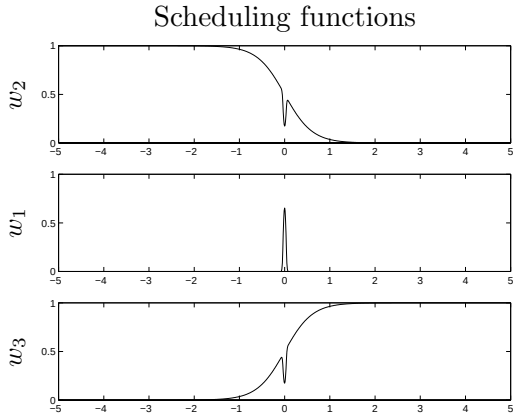


Figure 8.6: *The scheduling functions w_1, w_2, and w_3.*

Model Validation

The identified model is now used for validation. Figure 8.7 shows the time response of the system and of the identified PLM. An input signal similar to the one used for identification purposes is applied.

The identified friction torque of the PLM with superimposed on it the scaled Kalman filter innovation signal (dotted line) is depicted in Figure 8.8.

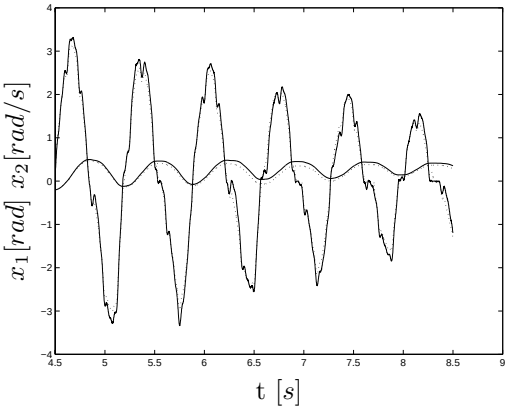


Figure 8.7: *Time response of the system (solid) and the identified PLM (dotted) on the input signal used for identification.*

The innovation is small and seems random and therefore gives no further information on how to improve the friction characteristic of the PLM. From this observation together with the fact that the filtered output and even the simulated output is very close to the system, it is concluded that the model captures the main nonlinearity of the system.

Coulomb, viscous and Stribeck friction phenomena can be observed and distinguished from the estimated friction model. The PLM can be interpreted in terms of regimes as follows. Regimes Z_2, Z_3 and the associated models are responsible for the viscous friction part. Regime Z_1 and the associated model is responsible for the Coulomb part. The non-empty intersection of the regions Z_1, Z_2 and Z_3 and the associated convex combination of the associated models is responsible for the Stribeck part of the friction.

8.7 Notes and Comments

Least squares filtering

The least squares filtering toolbox, which includes simulation and validation facilities, that was used throughout this chapter for the estimation of unknown parameters, is available as a non-commercial Matlab toolbox with a user friendly graphical interface [26].

In principle, the presented second modelling method could be applied online in real time applications.

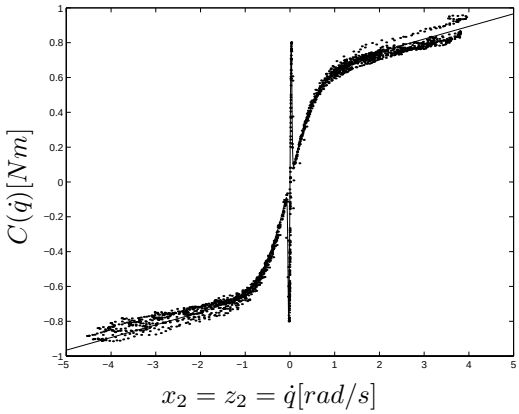


Figure 8.8: *The identified friction characteristic of the PLM (solid line) and superimposed on it the least squares filter innovation signal (dots).*

Optimal estimation

Also another well-known parameter estimation method, referred to in the literature as optimal estimation due to its analogy to optimal control, see [76] and [80], is employed to estimate the unknown parameters of a PLM. A solution to this estimation problem is found as a solution to a two point boundary value problem. Algorithms are available that can solve two point boundary value problems. We have reported on these algorithms in [32]. Within our group, software packages were developed especially for solving the optimal estimation problem [76], [73]. It was shown in several simulation studies, i.e. the modelling of nonlinear mechanical systems, that it is feasible to estimate the unknown parameters of a PLM [73], [32]. These examples include: modelling of a spring with nonlinear hardening characteristic, a second order system with a Coulomb-like friction component, and a fourth order system with two Coulomb-like friction components. For these simulation studies it was also shown how prior knowledge can be incorporated to enhance the applicability of this method.

Part III

CONTROL

Chapter 9

Introduction to Control

In the first two parts of this thesis, consecutively the analysis and modelling possibilities of PLMs were explored. We introduced and dealt with the basic concepts underlying the analysis and modelling of systems for the purpose of control. This resulted in the development of mathematical control and system theory for PLMs. Also novel modelling methods for PLMs were proposed. In this part the acquired knowledge from the previous parts will be utilized to design controllers for nonlinear systems that meet pre-imposed control objectives.

The main control objective will be the stabilization of the system. Besides stabilization, sometimes also more demanding performance specifications, such as optimality and robustness of the closed-loop will be addressed. To investigate applicability of the proposed synthesis algorithms, the controllers have been evaluated on several bench-mark examples. These examples include experimental as well as simulation case studies such as:

- Friction compensation for a rotational robotic manipulator system [28], [29].
- Optimal control of an inverted pendulum system[42], [9], [3].
- Adaptive optimal friction control for a rotational robotic manipulator system [27].
- Robust controller design with optimality bounds for a fictitious system [10].
- Stable controller design for a translational inverted pendulum on a cart [81].
- Vibration amplitude reduction with small control effort in a harmonically excited system with one sided spring [25], [5].

- Stable observer design for a harmonically excited system with one sided spring [75].
- Piecewise constant controls for systems with discrete measurements [4].
- Improving performance of single input stable linear time invariant systems with positive controls[6], [2].

Only on the first four of these case-studies will be reported since they illustrate the most relevant aspects and features of the PLM based control approach. They show how PLM based system analysis and modelling can be utilized to systematically design controllers for nonlinear systems. The main observation is that representing a nonlinear system by a PLM, that is to say 'a linear model scheduler' leads to a controller structure that can be characterized as 'a linear controller scheduler'. This controller is subsequently applied to the system, and it will be shown that it outperforms linear controllers.

In Chapter 10, a friction compensator design for a real rotational robotic manipulator system is reported. Analysis of the identified PLM reveals that a nonlinear state feedback can be designed, that stabilizes and linearizes the system. An experimental case study illustrates the applicability of the controller.

In Chapter 11, an optimal regulator design is reported. That is, besides stability of the closed-loop system also a cost functional associated with the control is minimized. In fact, the proposed design method generalizes the well-known Linear Quadratic Regulator (LQR) optimal control problem to a class of nonlinear systems, the PLMs. Analysis of the PLM reveals that under stabilizability conditions a solution exists to the optimal control problem. A simulation study, the control of an inverted pendulum on a cart illustrates the method.

In Chapter 12, a model based controller design will be proposed with the objective to perform servo tasks on mechanical systems that exhibit friction. Again the control of a real rotational robotic manipulator system is considered. The controller consists of three parts, namely, a feedforward part on the basis of estimated PLM parameters, an adaptive part to compensate for parameter estimation errors, and finally an optimal regulator for the remaining error dynamics. The controller is evaluated for performing servo tasks, by means of experiments on the robotic manipulator system.

In Chapter 13, a stabilizing controller is designed for a family of (nonlinear) systems. Thus the controller is robust against parametric uncertainty of the system. In contrast to the previous chapters the PLM is interpreted as an uncertainty model of the system. The applicability of the synthesis algorithms is illustrated with a simulation example.

Chapter 10

Friction Compensation for a Rotational Robotic Manipulator System

10.1 Introduction

We have seen in previous chapters that a PLM is capable of identifying nonlinear friction characteristics of a rotational robotic manipulator, that are left unexplained by first principles modelling. In an experimental case-study, a PLM was applied to identify a rotating arm subjected to friction. The applicability of the identified PLM is illustrated in a PLM based control scheme. A nonlinear state feedback will be designed with the objective to linearize the system, that is to compensate for the nonlinear friction force and in addition to stabilize the system. Other friction compensation designs are described by [40] and [46] who consider also nonlinear state feedback, in contrast [59] considers dynamic friction models, [31] applies a dithering technique, and an acceleration feedback control approach is reported by [70].

The purpose of this chapter is twofold. First we illustrate how modelling and analysis based on a PLM, very naturally leads to nonlinear control designs for the problem at hand. The suggested controller is a gain scheduling controller, and the linear gains are scheduled by the scheduling variables of the identified PLM. Secondly, the aim is to validate the identified PLM within a control scheme. The controller performance is compared to a linear design based upon a linear approximate model of the system and experiments reveal that the PLM approach outperforms the linear approach.

In the next section the robotic manipulator system and the approximate PLM will be introduced. After that in Section 10.3, analysis of the PLM reveals that the PLM can be linearized and that the PLM is also control-

lable. Based upon this information a linearizing and stabilizing controller is designed. Next, in Section 10.4 this controller is implemented on the real system, analyzed for positioning tasks and compared to a linear design. Finally, the conclusions of this chapter are drawn.

10.2 A PLM Representing a Rotational Robotic Manipulator System

In this experimental case study it is the objective to feedback linearize the rotating robotic manipulator as described in Section 8.4.1. For the friction compensator feedback law to work, an accurate PLM, which includes a description of the nonlinear friction characteristic of the rotational robotic manipulator system, is needed. The controller will be based on the PLM from Section 8.6.1. The structure of the PLM was based upon prior knowledge on the system, and the unknown parameters were identified from experimental data obtained from the robotic manipulator. The obtained PLM is

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \sum_{i=1}^{3} w_i(\dot{q}) \left\{ \begin{bmatrix} 0 & 1 \\ 0 & A_i(2,2) \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ a_i(2) \end{bmatrix} + \begin{bmatrix} 0 \\ B_i(2) \end{bmatrix} u \right\} \tag{10.1}$$

and the identified parameters are given in Table 8.2. Note that $B_i(2) = B(2)$.

10.3 Nonlinear State Feedback Friction Compensation

From Theorem 5.4.4 it follows that the PLM (10.1) is indeed feedback linearizable, since $(A_i - A_j, a_i - a_j)$ is in the range of B. Furthermore the system is controllable since (A_i, B) is controllable. The nonlinear controller

$$u = \sum_{i=1}^{3} w_i(\dot{q}) \left\{ K_i x + k_i + u^* \right\} \tag{10.2}$$

with $x = [q\ \dot{q}]^T$ suggested by Theorem 5.4.4 linearizes the system and therefore compensates the friction. The controller gains $\{K_i, k_i\}$ are chosen in such a way that $A_i + BK_i = A_{cl}$ and $a_i = Bk_i$. In the ideal case, i.e. if it is assumed that the PLM is an exact representation of the system, the closed loop system is described by the linear system

$$\dot{x} = A_{cl}x + Bu^*$$

The PLM based controller design will be compared to a linear controller design

$$u = Kx + u^* \tag{10.3}$$

based upon an identified linear model of the system, namely

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & A(2,2) \end{bmatrix}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ B_i(2) \end{bmatrix} u$$

that completely neglects the nonlinear friction of the system. The controller gain K is chosen in such a way that $A + BK = A_{cl}$.

10.4 Experiments

The closed loop parameters $A_{cl}(2,1)$ and $A(2,2)$ are set to -10. The friction compensator (10.2) and the linear controller (10.3) are tested on the real system, and analyzed for the following two positioning tasks

1. Step response performed with $u^* = \frac{100}{B(2)}\varepsilon(t-2)$, where $\varepsilon(\tau)$ is the Heaviside function:

$$\varepsilon(\tau) = \begin{cases} 0, & \forall\, \tau < 0 \\ 1, & \forall\, \tau \geq 0 \end{cases}$$

2. Velocity reversal response performed with $u^* = \frac{-100}{B(2)}\sin(1.5t)$.

For these tasks the closed-loop response, with and without friction compensation, is compared to the theoretical linear response. It is well-known that if friction phenomena such as coulomb friction, static friction and the Stribeck effect, are neglected by the model, it can limit the performance of industrial model-based control systems due to increasing tracking errors and limit cycles [12]. In Figure 10.1 the responses on the step input are shown.The solid line is the desired linear response, and the dotted line the experimentally observed response with and without friction compensation. As expected, with friction compensation a more accurate step response can be observed than without friction compensation. The responses for the velocity reversal task are shown in Figure 10.2. Also in this case the solid line is the desired linear response, and the dotted line the experimentally observed response with and without friction compensation. During the velocity reversal task no stiction is recorded if the friction compensator is applied, but the stiction is considerable without friction compensation. Hence, the performance of the two tasks can be improved by using a friction compensator based on a PLM. Moreover, it can be concluded that the PLM is a useful model for the control task at hand.

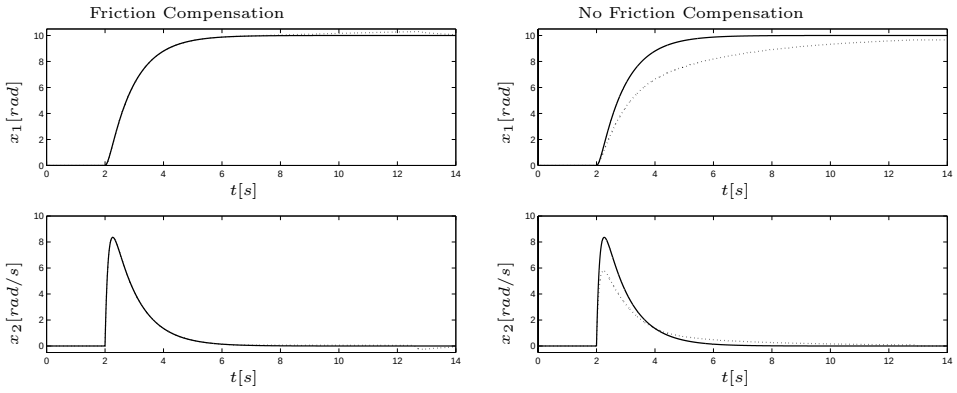


Figure 10.1: *Step input responses.*

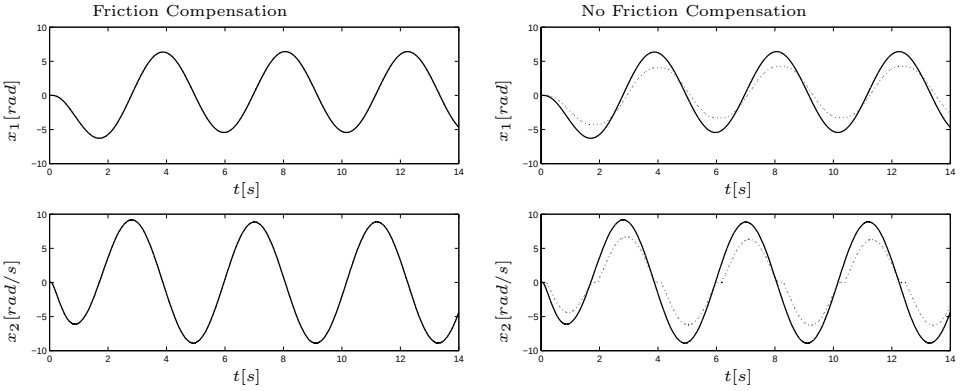


Figure 10.2: *Velocity reversal responses.*

10.5 Conclusion

A simple analysis based upon the identified PLM shows that the PLM can be linearized by compensating for the nonlinear friction phenomena. Furthermore, it can be concluded that the PLM is controllable. Hence, the PLM can be stabilized. The example shows that an earlier identified PLM is applicable within a model based control scheme with the objective to compensate for the nonlinear friction phenomena. The controller design based upon a PLM outperforms a linear controller design.

Chapter 11

Optimal Control

11.1 Introduction

In this chapter a novel regulator design method for PLMs is suggested and formalized. Controller synthesis is based upon optimal control theory. It is shown that under controllability assumptions a solution exists to the Hamilton Jacobi Bellman (HJB) equation, which is known to be a sufficient condition for optimality of the closed loop system. More specifically, the HJB equation admits a solution, if there exists a common positive definite solution to a set of algebraic Riccati equations (AREs), involving the different linear models by which the PLM is represented. An optimal static state feedback controller can then be computed as a solution of a convex optimization program. It turns out that not only the PLM but also the resulting controller and cost functional that is minimized can be interpreted in terms of regimes. Moreover, the optimal control system has an infinite gain margin, a requisite for robustness of the control system.

This chapter shows that the proposed design method generalizes the well-known Linear Quadratic Regulator (LQR) optimal control problem to a class of nonlinear systems, a class of PLMs. Analysis of the PLM reveals that under stabilizability conditions, just as in the linear case, a solution exists to the optimal control problem. This chapter clearly also shows how the similarities between the PLM and a linear model can be exploited for controller design. Well-known optimality and stabilizability conditions from linear systems theory have also meaning if one considers nonlinear systems, namely PLMs.

In the next section the optimal control problem is stated and sufficient conditions for optimality (of a PLM interconnected with a state feedback controller) are derived. After that, in Section 11.3 a control algorithm is suggested that solves a version of the optimal control problem. The control algorithm is given as a convex optimization problem that can be solved efficiently. In

Section 11.4 the controller design method is illustrated by an example, the control of a translational inverted pendulum on a cart. Finally, in Section 11.5 this chapter is concluded with some notes and comments.

Next the PLM will be introduced and the optimal stabilization problem for PLMs will be addressed.

11.2 Stabilizing Optimal Controls

We consider PLMs from the model set $\mathcal{M}$, defined as

$$\mathcal{M} : \left\{ \text{PLM} : \dot{x} = \sum_{i=1}^{N_m} w_i(x)\{A_i x + B_i u\} \right\} \tag{11.1}$$

with state $x \in X \subseteq \mathbb{R}^n$ and input $u \in U \subseteq \mathbb{R}^m$, and where the realization of the scheduling functions w_i is constrained by $\sum_{i=1}^{N_m} w_i(x) = 1$ and $w_i(x) \geq 0$. The model set $\mathcal{M}$ is defined as a collection of PLMs, where every PLM from the model set is represented by the same set of pairs $\{(A_i, B_i)\}$. The only difference between two PLMs from the same model set is the realization of the set of scheduling functions, which is constraint by the convexity property of the scheduling functions.

We have to specify a class of admissable inputs $\mathcal{U} = \{\omega \mid \omega : I = [0, \infty) \rightarrow U\}$, sufficiently rich to have practical meaning for the model, i.e. we restrict to the class of piecewise continuous functions on U. We assume that unique solutions $\xi(t)$ of the PLM for initial conditions $\xi(0) = x_0 \in X$ and $\omega \in \mathcal{U}$ exist.

The objective is to derive a sufficient condition that assures the existence of a state feedback $u = k(x)$ that achieves asymptotic stability of the equilibrium $x = 0$ of the PLM and furthermore minimizes the cost functional

$$J_\infty(x_0, \omega) := \int_0^\infty \{Q(\xi(t)) + \omega(t)^T R(\xi(t))\omega(t)\} dt \tag{11.2}$$

over all stabilizing controls ω for which $\lim_{t\rightarrow\infty} \xi(t) = 0$, and where it is assumed that $Q(x) > 0$ for all $x \neq 0$ with $Q(0) = 0$ and $R(x) > 0$ for all x. Here $\xi(t)$ denotes the solution of the PLM from (11.1) with initial condition $\xi(0) = x_0$ and input $\omega(.)$. It is well known that any solution $u = k(x)$ that satisfies the HJB equation guarantees optimality, i.e. minimizes J_∞, confer [41][14]:

Proposition 11.2.1 ([67]) *Assume $V(x)$ is a radially unbounded positive definite function on $\mathbb{R}^n$ with $V(0) = 0$. Let $k : X \rightarrow U$ be so that*

$$\begin{aligned} &\forall x, u \quad \dot{V}(x, k(x)) + Q(x) + k(x)^T R(x) k(x) \\ &= \min_{u\in\mathcal{U}} \{\dot{V}(x, u) + Q(x) + u^T R(x) u\} = 0 \end{aligned} \tag{HJB}$$

holds. Then for each state x_0 the control $\omega(t) = k(\xi(t))$ is optimal, and $V(x^0) = J_\infty(x_0, \omega)$ is the corresponding minimal cost. Moreover, V is a global Lyapunov function for the closed-loop system.

For convenience we define $A(x) := \sum_{i=1}^{N_m} w_i(x)A_i$ and $B(x) := \sum_{i=1}^{N_m} w_i(x)B_i$. Note that $\dot{V}(x,u) = \nabla V(x)\{A(x)x + B(x)u\}$. Then the following optimality condition holds.

LEMMA 11.2.2 *Suppose there exists a C^2 positive definite radially unbounded function $V(x)$ which satisfies*

$$\forall x \quad Q(x) + \nabla V(x)A(x)x - \frac{1}{4}\nabla V(x)B(x)R(x)^{-1}B(x)^T\nabla V(x)^T = 0 \tag{11.3}$$

and define

$$k(x) := -\frac{1}{2}R(x)^{-1}B(x)^T\nabla V(x)^T \tag{11.4}$$

then the conclusions of Proposition 11.2.1 hold.

Proof. By 'completing the squares' of

$$\begin{aligned} &u^T R(x)u + \nabla V(x)B(x)u \\ &\quad = (u + \frac{1}{2}R(x)^{-1}B(x)^T\nabla V(x)^T)^T R(x)(u + \frac{1}{2}R(x)^{-1}B(x)^T\nabla V(x)^T) \\ &\qquad\qquad - \frac{1}{4}\nabla V(x)B(x)R(x)^{-1}B(x)^T\nabla V(x)^T \end{aligned}$$

it follows that (11.4) minimizes (HJB), i.e.

$$k(x) = \arg\min_u\{\nabla V(x)A(x)x + Q(x) + \nabla V(x)B(x)u + u^T R(x)u\}$$

By substituting (11.4) in (HJB) one obtains the left hand side of (11.3) which is assumed to be equal to zero, so (HJB) holds. ■

The idea is to use Lemma 11.2.2 together with a quadratic candidate Lyapunov function to prove stability and optimality of the closed-loop polytopic system. We are then able to state the following

THEOREM 11.2.3 *If there exists a (n,n) matrix $P = P^T > 0$ satisfying $\forall i,j,k \in \{1,\ldots,N_m\}$ with $w_i(x)w_j(x)w_k(x) > 0$ for some $x \in X$, the following set of AREs*

$$Q_{ijk} + PA_i + A_i^T P - PB_iR_k^{-1}B_j^T P = 0 \tag{ARE}$$

then the input

$$u_{opt} = k(x) = -\sum_{i,j=1}^{N_m} w_i(x) w_j(x) R_i^{-1} B_j^T P x$$

minimizes the cost functional

$$\mathcal{J}_\infty(x_0, u) := \int_0^\infty \{u(t)^T (\sum_{l=1}^{N_m} w_l(x) R_l^{-1})^{-1} u(t) + \sum_{i,j,k=1}^{N_m} w_i(x(t)) w_j(x(t)) w_k(x(t)) x(t)^T Q_{ijk} x(t)\} dt$$

over all other controls, for the PLM $\in \mathcal{M}$*. Furthermore* $V(x) = x^T P x$ *serves as a Lyapunov function for the PLM.*

Proof. Apply Lemma 11.2.2 together with $V(x) = x^T P x$, $P = P^T > 0$ (which serves as a Bellman or Value function) as a candidate Lyapunov function to prove stability and optimality of the closed-loop polytopic system. The algebraic condition (11.3) can then be written as

$$\sum_{i,j=1}^{N_m} w_i(x) w_j(x) \{Q(x) + x^T (P A_i + A_i^T P - P B_i R(x)^{-1} B_j^T P) x\} = 0$$

Now, by choosing a suitable structure for the cost function, i.e.

$$Q(x) = \sum_{i,j,k=1}^{N_m} w_i(x) w_j(x) w_k(x) x^T Q_{ijk} x$$

$$Q_{ijk} = Q_{ijk}^T > 0$$

$$R(x) = \left(\sum_{l=1}^{N_m} w_l(x) R_l^{-1} \right)^{-1}$$

$$R_l = R_l^T > 0$$

one obtains the following condition for optimality

$$\sum_{i,j,k=1}^{N_m} w_i(x) w_j(x) w_k(x) \{x^T (Q_{ijk} + P A_i + A_i^T P - P B_i R_k^{-1} B_j^T P) x\} = 0 \tag{11.5}$$

This condition is satisfied if (ARE) holds. ■

From Theorem 11.2.3 a few interesting observations can be made. Firstly, if P is a solution of (ARE) for all $i, j, k \in \{1, \ldots, N_m\}$ then Theorem 11.2.3 holds for the model set $\mathcal{M}$, this means for all PLMs in $\mathcal{M}$. Secondly, the optimal controller is a gain scheduling (regime scheduled) controller. Also the cost associated with a control input is regime dependent. This makes it possible to interpret not only the PLM in terms of regimes, but also the control inputs as well as the control objectives. Thirdly, if the functions w_i depend on $v(t)$, an external C^1 scheduling signal, then Theorem 11.2.3 remains valid. Finally, the case the functions w_i are constant for all $x \in X$, i.e. the PLM becomes a LTI model, Theorem 11.2.3 gives the well-known solution of the infinite time linear-quadratic regulator (LQR) optimal control problem [41],[67]. So, in fact Theorem 11.2.3 generalizes the LQR problem to a class of PLMs.

11.3 Inverse Optimal Controls

Optimal stabilization guarantees several desirable properties for the control system, such as stability margins. These stability margins are largely independent of the particular choice of functions $Q(x) \geq 0$ and $R(x) > 0$. This assertion, together with the fact that solving (HJB) with a pre-specified cost function $Q(x) + u^T R(x) u$ is often not feasible, has motivated people to solve inverse optimal control problems, confer [65],[10]. This means, given a candidate Lyapunov function or controller, compute a cost functional such that the controller becomes optimal with respect to this cost functional. In the sequel a quadratic Lyapunov function $V(x) = x^T P x$, $P > 0$ (which serves as a Value function) is shown to exist if a certain system property holds. The system property the PLM has to obey is the analogue of the controllability condition that a linear system has to satisfy in order that the ARE corresponding to the optimal control problem has a solution. Furthermore, solutions to the inverse optimality problem can be obtained efficiently since they can be computed as a solution of a convex optimization program.

THEOREM 11.3.1 *If there exist matrices $Q = Q^T > 0$, $R_k > 0$ that satisfy $\forall i, j, k \in \{1, \ldots, N_m\}$ with $w_i(x) w_j(x) w_k(x) > 0$ for some $x \in X$*

$$A_i Q + Q A_i^T - B_i R_k^{-1} B_j^T < 0 \tag{11.6}$$

then the input

$$u_{opt} = - \sum_{i,j=1}^{N_m} w_i(x) w_j(x) R_i^{-1} B_j^T P x$$

with $P = Q^{-1}$ minimizes the cost functional

$$\mathcal{J}_\infty(x_0, u) := \int_0^\infty \{u(t)^T (\sum_{l=1}^{N_m} w_l(x) R_l^{-1})^{-1} u(t) + \sum_{i,j,k=1}^{N_m} w_i(x(t)) w_j(x(t)) w_k(x(t)) x(t)^T Q_{ijk} x(t)\} dt$$

where $Q_{ijk} := -(PA_i + A_i^T P - PB_i R_k^{-1} B_j^T P)$ over all other controls, for the PLM $\in \mathcal{M}$. Furthermore $V(x) = x^T P x$ serves as a Lyapunov function for the PLM.

Proof. The idea is to apply Theorem 11.2.3 and to construct solutions P, R_k, Q_{ijk} that satisfy (ARE). We proceed as follows. Rewrite (ARE) as

$$PA_i + A_i^T P - PB_i R_k^{-1} B_j^T P = -Q_{ijk}$$

This implies that if a solution $P = P^T > 0$, $R_k > 0$ exists that satisfies $\forall i, j, k \in \{1, \ldots, N_m\}$ with $w_i(x) w_j(x) w_k(x) > 0$ for some $x \in X$

$$PA_i + A_i^T P - PB_i R_k^{-1} B_j^T P < 0 \tag{11.7}$$

then the optimal control problem is solved, simply by defining

$$Q_{ijk} := -(PA_i + A_i^T P - PB_i R_k^{-1} B_j^T P) \tag{11.8}$$

After a congruence transformation with $Q = P^{-1}$, i.e. multiplying (11.7) on the left and right by $Q = P^{-1}$, (11.7) can be rewritten as the dual and equivalent condition (11.6). By the same congruence transformation the inequality $P = P^T > 0$ transforms to $Q = Q^T > 0$. ■

Theorem 11.3.1 makes it easy to automate inverse optimal controller design, since it can be written as a convex optimization routine in the variables Q and R_k^{-1}. Furthermore, it is possible to add extra LMI constraints on the variables Q, R_k^{-1} and Q_{ijk}. An example of such a control strategy is given in [27].

11.4 Optimal Control of a Translational Inverted Pendulum on a Cart

To illustrate the controller design method, consider the problem of swing-up and balancing of an inverted pendulum on a cart as given by Figure 7.9. See, (7.9) for the equations of motion of the pendulum.

For the derived control strategy to work we need a PLM to represent the plant. The following representation for the PLM was used in [85] to represent the inverted pendulum on a cart:

$$A_1 = \begin{bmatrix} 0 & 1 \\ \frac{g}{4l/3-aml} & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ -\frac{\alpha}{4l/3-aml} \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi(4l/3-aml\beta^2)} & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ -\frac{\alpha\beta}{4l/3-aml\beta^2} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi(4l/3-aml\beta^2)} & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 \\ \frac{\alpha\beta}{4l/3-aml\beta^2} \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} 0 \\ \frac{\alpha}{4l/3-aml} \end{bmatrix}$$

with $\beta = \cos(88^0)$. The corresponding scheduling functions w_i, which are supposed to depend on x_1 only, are shown in Figure 11.1.

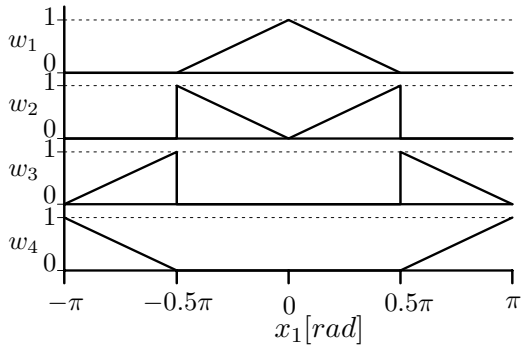


Figure 11.1: *The scheduling functions $w_i(x_1)$, for $i \in \{1, 2, 3, 4\}$.*

In the sequel this model is used for controller design and will represent the plant during evaluation of the control strategy.

We apply Theorem 11.2.3 and solve the LMIs (11.6) for P and R_i^{-1} Notice that since $w_1 w_3 \equiv 0$, $w_1 w_4 \equiv 0$, $w_2 w_3 \equiv 0$, $w_2 w_4 \equiv 0$ the corresponding LMIs need not to be solved. We obtain

$$P = \begin{bmatrix} 2.9064 \cdot 10^3 & 0.9470 \cdot 10^3 \\ 0.9470 \cdot 10^3 & 0.3092 \cdot 10^3 \end{bmatrix} \tag{11.9}$$

and

$$R_1 = 3.7883 \cdot 10^3, \quad R_2 = 3.7883 \cdot 10^3 \qquad (11.10)$$
$$R_3 = 5.0802 \cdot 10^3, \quad R_4 = 5.0802 \cdot 10^3$$

as a feasible solution. The control law

$$u = -\sum_{i,j \in I_{N_m}} w_i(x) w_j(x) R_i^{-1} B_j^T P x \qquad (11.11)$$

guarantees global asymptotic stability of the PLM and optimality with respect to the cost functional proposed in Theorem 11.2.3.

Figure 11.2 illustrates the response of the closed-loop systems for the initial condition $(x_{10}, 0)$ with $x_{10} = \frac{\pi}{4}, \frac{85}{180}\pi, \frac{3\pi}{4}, \pi$ respectively.

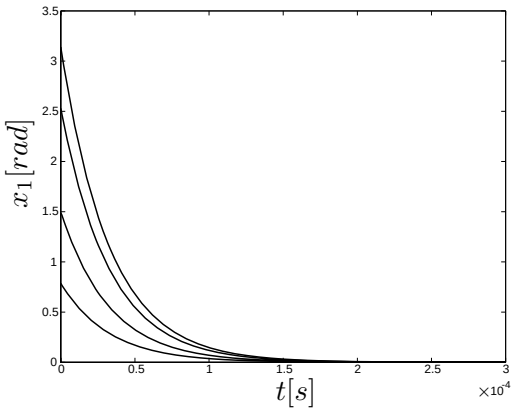


Figure 11.2: *Angle response x_1 of optimal PLM based control system.*

We can also confirm that (HJB) holds, i.e. $\dot{V}(x, u_{opt}) + Q(x) + u_{opt}^T R(x) u_{opt} = 0$, see Figure 11.3, and the minimum cost to go $\mathcal{J}_\infty(x_0, u_{opt})$ equals $V(x_0)$.

11.5 Notes and Comments

Optimal controls

Analogously to the LQR problem there are several variations on the optimal control problem for PLMs, worthwhile to investigate. With the output equation $y = \sum_{i=1}^{N_m} w_i(x) C_i x$ added to the PLM description (11.1), it is natural to consider

$$\mathcal{J}_\infty(x_0, u) := \int_0^\infty \{u(t)^T R(x(t)) u(t) + y^T Q y\} dt$$

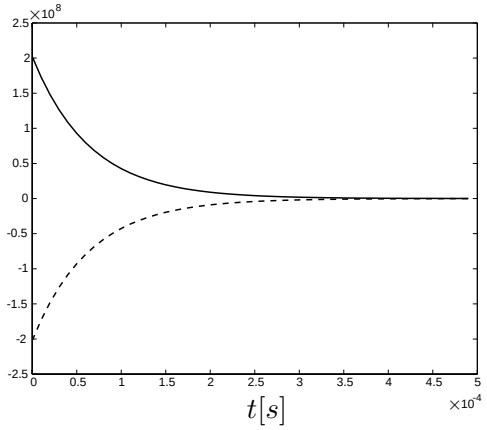


Figure 11.3: *Time derivative of the Lyapunov energy $\dot{V}$ (dashed line) is depicted together with the instantaneous cost $Q(x) + u_{opt}^T R(x) u_{opt}$ (solid line) as defined in Theorem 11.2.3. Simulation started at initial condition $x_0 = (0.7\pi, 0)$. (HJB) holds, i.e. $\dot{V}(x, u_{opt}) = -\{Q(x) + u_{opt}^T R(x) u_{opt}\}$.*

as the cost associated with control, i.e. a measure of the input u and output y of the PLM. This cost functional can be rewritten as

$$\mathcal{J}_\infty(x_0, u) := \int_0^\infty \{u(t)^T R(x(t)) u(t) + Q(x)\} dt \tag{11.12}$$

where $Q(x) = \sum_{i,j=1}^{N_m} w_i(x) w_j(x) x^T C_i^T Q C_j x$ is assumed to be positive semi definite. Just like for the LQR case, one will need besides a stabilizability condition also an detectability condition to be satisfied to solve this problem.

The cost functional (11.12) with $Q(x)$ positive semi definite is of major importance, since it occurs naturally if one formulates the optimal tracking problem and the optimal deterministic filtering problem, which in the linear case is dual to the optimal tracking problem [67].

Furthermore, it is worthwhile to investigate to what extend the choice for a common quadratic Lyapunov function is conservative with respect to the applicability and generality of optimal controller design.

Another interesting feature of the control strategy is the fact that the feedback system has infinite gain margin. If $u = -\sum_{i,j=1}^{N_m} w_i(x) w_j(x) R_i^{-1} B_j^T P x$ solves the optimal control problem, i.e. Theorem 11.2.3 is satisfied, then $u = -\sum_{i,j=1}^{N_m} w_i(x) w_j(x) a_{ij} R_i^{-1} B_j^T P x$ with $a_{ij} > 1$ also stabilizes the PLM. Furthermore optimal stabilization of the PLM is connected to passivity of the PLM with output equation $y = \sum_{i,j=1}^{N_m} w_i(x) w_j(x) R_i^{-1} B_j^T P x$, this observation leads to well defined stability margins as shown in [65].

Inverse optimal controls

There exist solutions to the inverse optimal control problem if (11.7) is satisfied. Inequality (11.7) can be interpreted as a sufficient condition for V to be a control Lyapunov function, i.e.

$$\nabla V(x)B(x) = 0 \text{ and } x \neq 0 \Rightarrow \nabla V(x)A(x)x < 0$$

from [67] is satisfied.

The inverse optimal controller synthesis algorithm admits a solution, only if the pairs (A_i, B_j) are stabilizable $\forall i, j \in \{1, \ldots, N_m\}$ with $w_i(x)w_j(x) > 0$ for some $x \in X$. This stabilizability condition can be relaxed. Clearly, $\forall i \in \{1, \ldots, N_m\}$, $w_i(x) > 0$ for some $x \in X$, and the pairs (A_i, B_i) have to be stabilizable. However, Theorem 11.2.1 is also satisfied if (11.6) for $i \neq j$ holds with non-strict inequality instead of strict inequality.

Conservatism can even be further reduced if one confines to the case that $R_i = R$ for all $i \in \{1, \ldots, N_m\}$ in Theorem 11.2.3. In that case the condition to be satisfied for optimality, that is (11.5), reduces to

$$\sum_{i,j=1}^{N_m} w_i(x)w_j(x)\{x^T(Q_{ij} + PA_i + A_i^T P - PB_i R^{-1} B_j^T P)x\} = 0$$

which can be rewritten as

$$\begin{aligned}
&\sum_{i=1}^{N_m} w_i^2(x) x^T L_{ii} x + \frac{1}{2} \sum_{i,j=1, j\neq i}^{N_m} w_i(x)w_j(x)\{x^T(L_{ij} + L_{ji})x\} \\
&= -\sum_{i=1}^{N_m} w_i^2(x)\{x^T Q_{ii} x\} - \frac{1}{2} \sum_{i,j=1, j\neq i}^{N_m} w_i(x)w_j(x)\{x^T(Q_{ij} + Q_{ji})x\}
\end{aligned}$$

where $L_{ij} = PA_i + A_i^T P - PB_i R^{-1} B_j^T P$.

If one introduces $\tilde{x} = [w_1 x^T, w_2 x^T, \cdots, w_{N_m} x^T]^T$ this yields

$$\begin{aligned}
&\tilde{x}^T \begin{bmatrix} L_{11} & * & \cdots & * \\ \frac{1}{2}(L_{12} + L_{21}) & \ddots & & \vdots \\ \vdots & & \ddots & * \\ \frac{1}{2}(L_{1N_m} + L_{N_m 1}) & \cdots & \cdots & L_{N_m N_m} \end{bmatrix} \tilde{x} \\
&= \tilde{x}^T \begin{bmatrix} Q_{11} & * & \cdots & * \\ \frac{1}{2}(Q_{12} + Q_{21}) & \ddots & & \vdots \\ \vdots & & \ddots & * \\ \frac{1}{2}(Q_{1N_m} + Q_{2N_m}) & \cdots & \cdots & Q_{N_m N_m} \end{bmatrix} \tilde{x}
\end{aligned}$$

The $*$ elements are induced by the symmetry of the matrices. From Theorem 11.3.1 it follows that there exists a solution to the optimal control problem if

$$\begin{bmatrix} L_{11} & * & \cdots & * \\ \frac{1}{2}(L_{12}+L_{21}) & \ddots & & \vdots \\ \vdots & & \ddots & * \\ \frac{1}{2}(L_{1N_m}+L_{N_m1}) & \cdots & \cdots & L_{N_mN_m} \end{bmatrix} < 0$$

After a congruence transformation with the block diagonal N_m-blocks by N_m-blocks matrix

$$\tilde{Q} = \begin{bmatrix} P^{-1} & 0 & \cdots & 0 \\ 0 & P^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & P^{-1} \end{bmatrix}$$

one obtains the equivalent and convex condition in $Q = P^{-1}$

$$\begin{bmatrix} M_{11} & * & \cdots & * \\ \frac{1}{2}(M_{12}+M_{21}) & \ddots & & \vdots \\ \vdots & & \ddots & * \\ \frac{1}{2}(M_{1N_m}+M_{N_m1}) & \cdots & \cdots & M_{N_mN_m} \end{bmatrix} < 0 \tag{11.13}$$

where $M_{ij} = A_iQ + QA_i^T - B_iR^{-1}B_j^T$. It follows that in general, for (11.13) to admit a solution, the condition $M_{ij} \geq 0$ for $i \neq j$ not necessarily has to be satisfied. Hence, conservatism is reduced with respect to condition (11.6).

Chapter 12

Adaptive Optimal Friction Control for a Robotic Manipulator System

12.1 Introduction

Friction is to some extent present in all mechanical systems. In servo systems this phenomenon can limit the performance considerably due to increasing tracking errors and the appearance of limit cycles. Especially limit cycling is an undesirable friction induced phenomenon in controlled servo systems because of its oscillatory and persistent behavior. Limit cycling is mainly caused by the combination of the difference in static and Coulomb friction, and integral action in the control loop. One way of avoiding limit cycles is to use solely a PD-controller, which consequently implies steady state errors. In literature as discussed in [12], stiff PD-controllers have been proposed to suppress these steady state errors.

Here, an adaptive optimal PLM based state feedback controller will be derived with the objective to perform servo tasks on mechanical systems that are subject to friction. Two desirable properties of the proposed controller design are: (i) small positioning errors through high gain feedback for low velocities, that means for the region where friction is important, and (ii) the stability of the closed-loop system is guaranteed and limit cycling is avoided. The controller design is based upon Lyapunov stability theory and consists of three parts:

1. Feedforward control on the basis of estimated PLM parameters.

2. Adaptive control to compensate for parameter estimation errors.

3. A nonlinear (regime dependent) optimal controller for the remaining error dynamics.

The resulting controller design for performing servo tasks, is tested and evaluated, by means of experiments on a rotational robotic manipulator system.

To illustrate the benefits of a nonlinear regime dependent controller over a linear controller for the remaining error dynamics, these two are compared. The derived controller is compared to a classical PID controller with a feedforward part based on the inertia of the mechanical system.

In the next section the robotic manipulator system and the approximate PLM will be introduced. After that in Section 12.3, an adaptive optimal control design based upon a PLM will be derived. Next in Section 12.4, the derived controllers are implemented on the real system, analyzed for positioning tasks and compared to a linear design. Finally, the conclusions of this chapter are drawn.

12.2 A PLM Representing a Rotational Robotic Manipulator System

For the model based control scheme to work, an accurate PLM, which includes a description of the nonlinear friction characteristic of the real rotational robotic manipulator system, as described in Section 8.4.1, is needed. The controller will be based on a PLM that is identified with the filtering method presented in Section 8.6.1. The structure of the PLM is based upon prior knowledge on the system. It is chosen according to the topology of the mechanical system, that means control affine and with the angular displacement and its time derivative as state variables. Furthermore, it is assumed that the friction can be modelled as an odd continuous function of angular velocity.

The unknown parameters were identified from experimental data obtained from the robotic manipulator. The obtained PLM is

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \sum_{i=1}^{4} w_i(\dot{q}) \left\{ \begin{bmatrix} 0 & 1 \\ 0 & A_i(2,2) \end{bmatrix} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} + \begin{bmatrix} 0 \\ a_i(2) \end{bmatrix} + \begin{bmatrix} 0 \\ B_i(2) \end{bmatrix} u \right\} \tag{12.1}$$

and the identified parameters are given in Table 12.1.

To construct an odd friction function, the PLM was a priori composed of an even (in this case $N_m = 4$) number of local models, divided in pairs of two. Note that the input matrix $B_i = \begin{bmatrix} 0 & 533.33 \end{bmatrix}^T$ is considered to be a priori

Table 12.1: *Operating regimes and model parameters*

i	$d(Z_i)$	$e(Z_i)$	$A_i(2,2)$	$B_i(2)$	$a_i(2)$
1	-1.50	0.21	-0.90	533.33	-13.33
2	-0.167	0.026	7	533.33	-16.43
3	1.50	0.21	-0.90	533.33	13.33
4	0.167	0.026	7	533.33	16.43

known exactly, and therefore not estimated. In fact B_i was identified in [28]. The scheduling functions

$$w_i = \frac{\rho_i}{\sum_{j=1}^{N_m} \rho_j}$$

are defined as normalized Gaussian model validity functions with

$$\rho_i(z) = e^{-\frac{1}{(2e(Z_i))^2}(z-d(Z_i))^2}$$

centered around $d(Z_i)$ and with a width (variance) $e(Z_i)$. In the upper plot of Figure 12.1 the identified friction model is shown. In the lower plot of the same

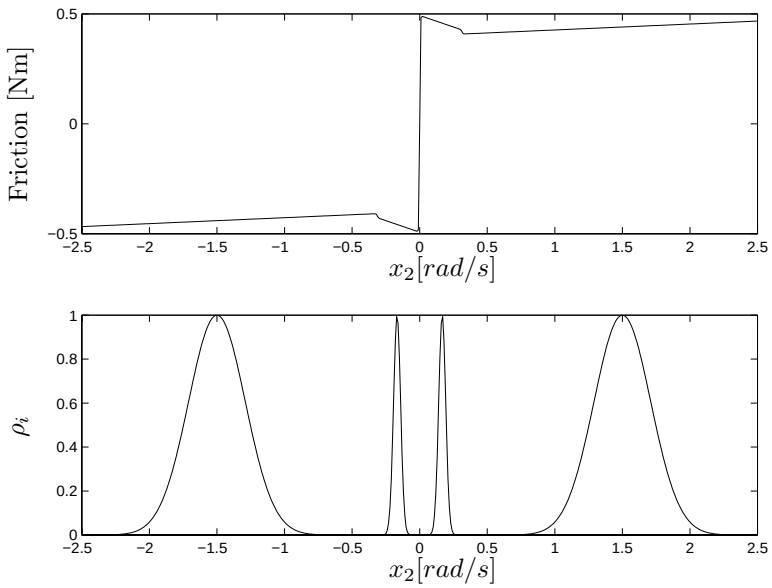


Figure 12.1: *Identified friction model.*

figure, the Gaussian validity functions of the different locally valid models are given.

12.3 Adaptive Optimal Control

The objective of the closed-loop system is to perform a high precision servo or positioning task. It is assumed that the state x is available for control. As mentioned already earlier the controller consists of (i) a feedforward part u_{ff}, (ii) an adaptive part u_{ad} which compensates for parametric uncertainty and (iii) a nonlinear optimal regulator u_{fb} for the remaining error dynamics.

The identified PLM (12.1) can be represented as

$$\dot{x} = \sum_{i=1}^{N_m} w_i(\dot{q}) \left\{ \hat{A}_i x + \hat{a}_i \right\} + Bu$$

where $x = [q\ \dot{q}]^T$ and the hat symbol ($\hat{}$) is used to emphasize that we deal with estimates, and that the PLM is only an approximation of the real system. It is assumed that the real system belongs to the PLM class and can be written as

$$\dot{x} = \sum_{i=1}^{N_m} w_i(\dot{q}) \left\{ A_i x + a_i \right\} + Bu$$

The feedforward control part[1] which is based on the approximate model can be written as

$$u_{ff} = \left(B^T B\right)^{-1} B^T \sum_{i=1}^{N_m} w_i(\dot{q}) \left\{ \dot{x}_d - \hat{A}_i x_d + \hat{a}_i \right\}$$

where x_d is a desired state trajectory and $\left(B^T B\right)^{-1} B^T$ is the pseudo-inverse of B. With $u = u_{ff} + u^*$ the remaining error dynamics becomes

$$\begin{aligned} \dot{e} &= \sum_{i=1}^{N_m} w_i(\dot{q}) \left\{ A_i x - \hat{A}_i x_d + a_i - \hat{a}_i \right\} + Bu^* \\ &= \sum_{i=1}^{N_m} w_i(\dot{q}) \left\{ \hat{A}_i e + \varepsilon_i \right\} + Bu^* \end{aligned}$$

where the tracking error $e = x - x_d$ and the model error $\varepsilon_i = (A_i - \hat{A}_i)x + a_i - \hat{a}_i$.

Now, define the following Lyapunov function candidate:

$$V = e^T P e + \sum_{i=1}^{N_m} \frac{1}{\gamma} \tilde{\varepsilon}_i^T \tilde{\varepsilon}_i, \tag{12.2}$$

[1]Note that this control part is not a feedforward in the strict sense, since it depends on (the measurement) $\dot{q}$.

with $\tilde{\varepsilon}_i = \varepsilon_i - \hat{\varepsilon}_i$ and $P = P^T > 0$, where $\hat{\varepsilon}_i$ is an estimate for ε_i and γ a positive constant. The time derivative of (12.2) becomes

$$\begin{aligned}\dot{V} &= \dot{e}^T P e + e^T P \dot{e} + \sum_{i=1}^{N_m} \frac{1}{\gamma}(\dot{\tilde{\varepsilon}}_i^T \tilde{\varepsilon}_i + \tilde{\varepsilon}_i^T \dot{\tilde{\varepsilon}}_i) \\ &= \sum_{i=1}^{N_m} w_i(\dot{q}) e^T \left\{ \hat{A}_i^T P + P \hat{A}_i \right\} e + \sum_{i=1}^{N} w_i(\dot{q}) \left\{ \varepsilon_i^T P e + e^T P \varepsilon_i \right\} \\ &+ \left\{ u^{*T} B^T P e + e^T P B u^* \right\} + \sum_{i=1}^{N} \frac{1}{\gamma}(\dot{\tilde{\varepsilon}}_i^T \tilde{\varepsilon}_i + \tilde{\varepsilon}_i^T \dot{\tilde{\varepsilon}}_i) \end{aligned} \tag{12.3}$$

Now we determine the remaining control part as discussed above by $u^* = u_{ad} + u_{fb}$. An adaptive compensation of model errors $\tilde{\varepsilon}_i$ is only possible if $\tilde{\varepsilon}_i$ is in the range of B for all i. To design an adaptive mechanism the following condition is imposed as demonstrated by [43].

$$\begin{aligned} u_{ad}^T B^T P e + e^T P B u_{ad} + \sum_{i=1}^{N_m} w_i(\dot{q}) \left\{ \varepsilon_i^T P e + e^T P \varepsilon_i \right\} \\ + \sum_{i=1}^{N_m} \frac{1}{\gamma}(\dot{\tilde{\varepsilon}}_i^T \tilde{\varepsilon}_i + \tilde{\varepsilon}_i^T \dot{\tilde{\varepsilon}}_i) = 0 \end{aligned} \tag{12.4}$$

By choosing

$$u_{ad} = -\sum_{i=1}^{N_m} w_i(\dot{q}) \left(B^T B \right)^{-1} B^T \hat{\varepsilon}_i \tag{12.5}$$

and substituting (12.5) in (12.4) the following condition is obtained

$$\begin{aligned} \sum_{i=1}^{N_m} w_i(\dot{q}) \tilde{\varepsilon}_i^T B (B^T B)^{-1} B^T P e + e^T P B (B^T B)^{-1} B^T \tilde{\varepsilon}_i \\ + \frac{1}{\gamma}(\dot{\tilde{\varepsilon}}_i^T \tilde{\varepsilon}_i + \tilde{\varepsilon}_i^T \dot{\tilde{\varepsilon}}_i) = 0 \end{aligned} \tag{12.6}$$

By defining the update rule of the adaptive controller as

$$\dot{\tilde{\varepsilon}}_i = -\gamma w_i(\dot{q}) B (B^T B)^{-1} B^T P e, \quad \forall i \in \{1, ..., N_m\}$$

we assure that (12.6) is satisfied, implying boundedness of $\tilde{\varepsilon}_i$. Here, it is assumed that at least $\ddot{q}$ or $(A_i - \hat{A}_i)$ is small, that means $\dot{\varepsilon}_i \approx 0$ for all $i \in \{1, ..., N_m\}$, which results for the update rule in

$$\dot{\hat{\varepsilon}}_i = \gamma w_i(\dot{q}) B (B^T B)^{-1} B^T P e, \quad \forall i \in \{1, ..., N_m\}$$

The adaptive controller becomes

$$u_{ad} = -\sum_{i=1}^{N_m} w_i(\dot{q}) \left(B^T B\right)^{-1} B^T \hat{\varepsilon}_i$$
$$\dot{\hat{\varepsilon}}_i = \gamma w_i(\dot{q}) B (B^T B)^{-1} B^T P e, \quad \forall i \in \{1, ..., N_m\} \tag{12.7}$$

Applying this adaptive controller, (12.3) reduces to

$$\dot{V} = \sum_{i=1}^{N_m} w_i(\dot{q}) \left\{ e^T (\hat{A}_i^T P + P \hat{A}_i) e \right\} + u_{fb}^T B^T P e + e^T P B u_{fb}$$

This is in fact the expression that appears when taking the derivative of a quadratic Lyapunov function, $V(e) = e^T P e$, $P = P^T > 0$, along a trajectory of the PLM

$$\dot{e} = \sum_{i=1}^{N_m} w_i(\dot{q}) \hat{A}_i e + B u_{fb} \tag{12.8}$$

To obtain an optimal regulator for (12.8), Theorem 11.2.3 is applied.

Theorem 11.2.3 states that if there exists a $P = P^T > 0$ satisfying $\forall i, j \in \{1, \ldots, N_m\}$ with $w_i(x) w_j(x) > 0$ for some $x \in X$, the following set of AREs

$$Q_{ij} + P \hat{A}_i + \hat{A}_i^T P - P B R_j^{-1} B^T P = 0 \tag{ARE}$$

then the input

$$u_{fb} = -\sum_{i,j=1}^{N_m} w_i(\dot{q}) R_i^{-1} B^T P e \tag{12.9}$$

minimizes the cost functional

$$\mathcal{J}_\infty(e_0, u) := \int_0^\infty \{u(t)^T (\sum_{i=1}^{N_m} w_i(\dot{q}(t)) R_i^{-1})^{-1} u(t) + \sum_{i,j=1}^{N_m} w_i(\dot{q}(t)) w_j(\dot{q}(t)) e(t)^T Q_{ij} e(t)\} dt$$

over all other controls, for the error dynamics described by (12.8).

Two different optimal controllers will be derived, a static gain controller and a regime (or gain) scheduled controller. The static gain controller is derived by setting $R_i = R$ in (12.9) such that the regulator becomes

$$u_{fb} = -R^{-1} B^T P e \tag{12.10}$$

At this stage P, R have to be determined. For the robotic manipulator system this implies that two AREs have to be solved for $P = P^T > 0$, $R > 0$, $Q_1 > 0$, $Q_2 > 0$, that is the above matrices are sought such that

$$\hat{A}_1^T P + P\hat{A}_1 - P^T B R^{-1} B^T P + Q_1 = 0 \tag{12.11}$$

$$\hat{A}_2^T P + P\hat{A}_2 - P^T B R^{-1} B^T P + Q_2 = 0 \tag{12.12}$$

Solutions exist to the AREs (12.11,12.12) if the stabilizability condition (11.6) is satisfied. This condition defines a convex optimization problem and is checked easily. It confirms that the AREs admit a solution. The (inverse) optimal control design suggested by Theorem 11.3.1 can now be applied to obtain an optimal regulator.

For this example however, an alternative solution to the optimal regulator design problem is proposed, to assure acceptable input torques. First, (12.11) is solved, yielding a P for given R and Q_1. Next, given R and P, Q_2 is computed from (12.12). If $Q_2 > 0$, a solution has been found for the optimal control problem. For the rotating arm performance is initially specified on the first ARE (by means of Q_1, R), to assure acceptable input torques in the high velocity region, that is the region where the maximum input torques are expected. Q_1 and R are chosen as follows

$$Q_1 = \begin{bmatrix} 2500 & 0 \\ 0 & 2 \end{bmatrix}, \quad R = 10$$

after an iterative process, where the matrices are initialized as suggested in [16]. We find for P and Q_2

$$P = \begin{bmatrix} 80.5129 & 0.2965 \\ 0.2965 & 0.0095 \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} 2500 & -2.4 \\ -2.4 & 1.8 \end{bmatrix} > 0$$

and the optimal state feedback becomes

$$u_{fb} = -R^{-1} B^T P e = -Fe,$$

where $F = \begin{bmatrix} 15.8114 & 0.5085 \end{bmatrix}$.

For the situation where different input weights R_i are used the optimal control law becomes

$$u_{fb} = -\sum_{i=1}^{4} w_i(\dot{q}) R_i^{-1} B^T P e$$

and can be obtained by solving four AREs, namely

$$Q_{ij} + P\hat{A}_i + \hat{A}_i^T P - PBR_j^{-1}B^T P = 0 \quad \forall i, j \in \{1, 2\}$$

If we denote the input weight R from the previous controller design as R_1, Q_1 as Q_{11} and Q_2 as Q_{21} we have to find R_2, Q_{22} and Q_{12} for the given P. Due to the fact that by using the weights R_1, Q_{11} and Q_{21} and the optimal P the third and fourth ARE are satisfied, we propose for R_2

$$R_2 = \alpha R_1 \qquad 0 < \alpha \leq 1,$$

With α small we obtain a high gain controller for the low velocity regime. The design variable α is limited due to the fact that high gain control will amplify measurement noise, and due to the fact that the control torque u is bounded in practice. For the low velocity regime a stiff PD-controller is required in order to reduce steady state errors due to the static friction.

12.4 Experiments

The proposed gain scheduled controller is tested on the rotational robotic manipulator system, and analyzed for performing the following two servo tasks

1. Tracking a smooth setup function

$$x_d = \begin{cases} 0 & \text{if} \quad t \leq 8.1 \\ \frac{-5}{2\pi}\sin(2\pi(t-8.1)) + 5(t-8.1) & \text{if} \quad 8.1 \leq t \leq 9.1 \\ 5 & \text{if} \quad t \geq 9.1 \end{cases}$$

   as depicted in the upper plot of Figure 12.2.

2. Tracking a (velocity reversal) reference trajectory

$$x_d = \begin{cases} 0.1\sin(t-4.2) & \text{if} \quad 4.2 \leq t \leq 4.2 + 4\pi \\ 0 & \text{if} \quad 4.2 \geq t \geq 4.2 + 4\pi \end{cases}$$

   as depicted in the upper plot of Figure 12.3.

Since only the angular displacement of the arm is measured in the experiment, we use a weak differentiator scheme to reconstruct the angular velocity. The adaptive optimal controller is compared to a PID controller with a feedforward action $u_{ff} = (B^T B)^{-1} B^T \dot{x}_d$ that is tuned for the robotic manipulator system. The closed-loop bandwidth of the PID controller connected to the system is set to 25 [Hz]. Two notch filters are introduced to suppress higher order dynamics. Furthermore, a second order low pass filter is introduced to

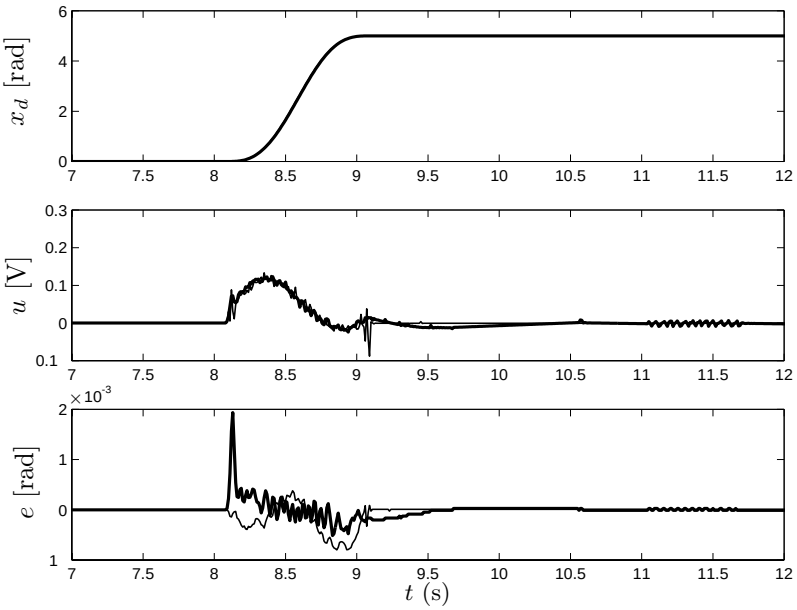


Figure 12.2: *Setpoint, controller effort and servo errors for both the PID controller (thick line) and the PLM based gain scheduled controller (thin line).*

suppress the influence of measurement noise. The adaptive gain for the PLM based controller is set to $\gamma = 1000$, and for the gain scheduled feedback a gain $\alpha = 0.5$ is used.

The control effort for the different controllers can for the first servo task be found in Figure 12.2, where the PID controller is shown by a thick line, and the gain scheduled controller by a thin line. In the same figure the tracking errors are shown (the same line definitions are used). During the setpoint change the maximum absolute tracking error for the gain scheduled controllers is smaller than the error achieved by the PID controller. At the end of the setpoint the PID controller needs an additional 0.5 second to reach the steady state. This settling behavior is not desirable, since the additional control effort might result, for certain controller settings and friction characteristics, in a limit cycle. The gain scheduled controller clearly outperforms the PID controller.

To investigate the controllers for velocity reversals, a sinusoidal reference signal, as shown in the upper plot of Figure 12.3, is used. The corresponding controller effort u and the servo errors are depicted in the same figure for both the PID controller and the PLM based controller. During motion the difference in performance between the two proposed controllers is small. However, the PID controller needs an additional 0.3 second to reach the steady state.

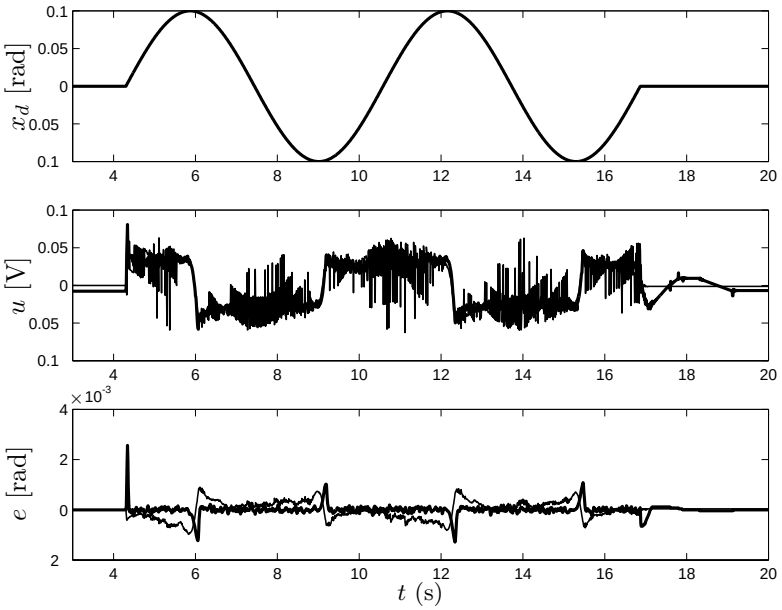


Figure 12.3: *Setpoint, controller effort and servo errors for both the PID controller (thick line) and the PLM based gain scheduled controller (thin line).*

Additional modifications of the proposed controller design are possible, such as scheduled adaptive gains, $\gamma = \gamma_i$. Furthermore, an integral action can be used in the high velocity region, and as a result, performance during motion might increase considerably for the described controllers.

Correct velocity reconstruction is very important for the applicability of the adaptive optimal controller. We experienced that due to loss of phase or amplification of measurement noise the performance of the proposed controllers can be considerably limited.

12.5 Conclusion

We have presented an adaptive optimal controller which leads to a stable closed-loop system under the assumption that the system is within the model class. The proposed controller is investigated for two tracking tasks and compared to a PID controller. The gain scheduled outperforms the PID controller with respect to settling times. However, the servo error for the gain scheduled controller and the PID controller are almost equal for position tasks. Due to the absence of integral action for the proposed gain scheduled controller, limit cycling is always avoided.

Chapter 13

Robust Control with Bounds on Performance

13.1 Introduction

The purpose of this chapter is to design a stabilizing controller for a family of (nonlinear) systems. This means that the controller is robust against parametric uncertainty of the system. In contrast to the previous chapters the PLM is interpreted as an uncertainty model of the system.

The objective is to compute a stabilizing gain scheduling controller, which is robust against parametric uncertainty. A similar discrete time robust stabilization problem is discussed in [66], formulated as a bilinear matrix inequality feasibility problem and locally solved using LMI algorithms. However, contrary to [66] we also demand that some performance level of the closed loop system is achieved in certain regions of the state-space. Related work can be found in [23], [63] though restricted to piecewise-affine systems. We also formulate the synthesis problem as a matrix inequality feasibility problem. But as mentioned in [23] and in conformance with [66] for the case of affine state feedback this leads to nonlinear matrix inequalities. To avoid the nonlinear matrix inequalities that are hard to solve, an iterative algorithm involving LMIs will be proposed to solve the synthesis problem.

In Section 13.2 an algorithm will be proposed that partitions the state-space into a number of disjoint clusters. With this partitioning a gain scheduling controller will be associated. After that in Section 13.3 relevant performance issues will be addressed. An analysis tool is presented that associates with certain regions of the state-space, (a measure of) performance of the closed-loop. The analysis tool builds naturally on the outcome of the iterative robust controller synthesis algorithm that will be presented in Section 13.4. This algorithm is proposed to solve the robust stabilization problem. In Sec-

tion 13.5, the synthesis algorithm and the analysis tool are illustrated with an example. Finally, the conclusions of this chapter are drawn.

13.2 State-space Partitioning and Feedback Law

It is assumed that the system to be stabilized can be described locally, that is within the region $X \times U \times V$ where $x \in X \subseteq \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, $v \in V \subseteq \mathbb{R}^l$, sufficiently accurate by a PLM

$$\dot{x} = \sum_{i=1}^{N_m} w_i(x,u,v)\left\{A_i x + B_i u + a_i\right\} \tag{13.1}$$

A partitioning of the state-space, similar to [66], and the associated feedback law will be formalized. The only knowledge that will be exploited to stabilize the PLM is the region of support in the state space of the w_i's. First these regions are identified and described by the sets X_j^s, where s stands for support

$$X_i^s := \bigcup_{(u,v)\in U\times V} \text{supp } w_i(\cdot,u,v) \qquad \forall i \in \{1,...,N_m\} \tag{13.2}$$

where for w_i the support $\text{supp}\, w_i(\cdot,u,v)$ is defined as

$$\text{supp } w_i(\cdot,u,v) := \{x \in X \mid w_i(x,u,v) > 0\} \tag{13.3}$$

Now, a partitioning of the state space in clusters X_J^c where c stands for cluster can be recursively computed as follows:

$$\begin{aligned}
&\text{for } k = 0 \text{ to } k = N_m - 1 \text{ do}\\
&\forall J \in V := \{L \mid L \subseteq \{1,...,N_m\}, \#L = N_m - k\}\\
&X_J^c = \left(\bigcap_{i\in J} X_i^s\right) \setminus \left(\bigcup_{i\in I_{Nm}\setminus J} X_{J\cup\{i\}}^c\right)
\end{aligned} \tag{13.4}$$

Some of the clusters are possibly empty, and $\bar{X}_{01}^c = \{X_J^c \mid X_J^c \neq \emptyset\}$ is the set containing all non-empty clusters. $\bar{X}_{01}^c$ can be associated with $\bar{J}_{01} = \{J \mid X_J^c \in \bar{X}_{01}^c\}$, the set pointing to all polytopes that have non-empty clusters. Furthermore it will be helpful to have another non-overlapping partitioning of the state-space, i.e. the cluster X_J^c that contains the origin will be called $\bar{X}_0^c$ and the set of clusters $\bar{X}_{01}^c \setminus \bar{X}_0^c$ will be called $\bar{X}_1^c$. In resemblance with this partitioning, $\bar{J}_0$ is the set pointing to models associated with region $\bar{X}_0^c$. In

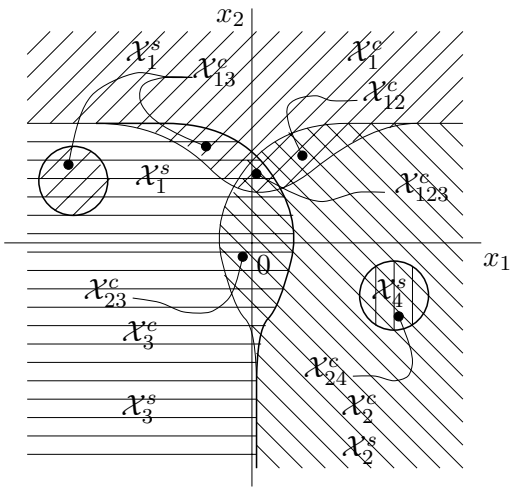


Figure 13.1: *The PLM with $x \in \mathbb{R}^2$ consists of the triples (A_i, B_i, c_i) with $i \in \{1, 2, 3, 4\}$. With the knowledge of the w_i's the support sets $\mathcal{X}_i^s$ are identified and are given a different hatching. The algorithm, (13.4) is initialized, $k = 0$, by taking the intersection of all the support sets $\mathcal{X}_i^s$ which in this case is empty, $\mathcal{X}_{1,2,3,4}^c = \emptyset$. For $k = 1$ the algorithm returns $N_m!(k!(N_m - k)!)^{-1} = 4$ polytopes of size $N_m - k = 3$. Only cluster $\mathcal{X}_{1,2,3}^c$ is non-empty. For $k = 2$ there are 6 polytopes of size 2. Again, some of them have non-empty clusters e.g. $\mathcal{X}_{2,3}^c$. Along the same lines all the clusters are defined resulting in a non-overlapping partitioning of the state-space.*

the same way, $\bar{J}_1$ can be associated with $\bar{X}_1^c$, $\bar{J}_1 = \bar{J}_{01} \backslash \bar{J}_0$. An example of the suggested state-space partitioning (13.4), is given in Figure 13.1. Figure 13.1 shows that the state-space X is divided in a number of disjoint regions X_J^c, in such a way that with every region an uncertainty model (a PLDI) is associated as follows:

$$\dot{x} \in \Omega_J \begin{bmatrix} x \\ u \\ 1 \end{bmatrix} \quad \text{if} \quad x \in X_J^c \in \bar{X}_{01}^c \tag{13.5}$$

where $\Omega_J = \text{Co}\{\begin{bmatrix} A_i & B_i & a_i \end{bmatrix} \mid i \in J\}$. With the robust stabilization of (13.1) the stabilization of the associated uncertainty model (13.5) is meant. It is assumed that the models indexed with $i \in \bar{J}_0$ reduce to $(A_i, B_i, 0)$, from which it follows that $0 \in \bar{X}_0^c$ is an equilibrium point of (13.1) and (13.5). The objective is to stabilize the origin.

With the uncertainty model (13.5) it seems natural to associate the fol-

lowing piecewise affine state feedback:

$$u = \begin{cases} K_J x & \text{if} \quad x \in X_J^c \in \bar{X}_0^c \\ K_J x + k_J & \text{if} \quad x \in X_J^c \in \bar{X}_1^c \end{cases} \tag{13.6}$$

This controller can be interpreted as a gain scheduling controller, where the scheduling of the controller parameters K_J, k_J is defined by the clusters X_J^c.

13.3 Performance and Bounds on the Associated Cost

From earlier expositions on stability it is clear that a sufficient condition for asymptotic stability of the closed-loop system, i.e. (13.5) interconnected with (13.6), is the existence of a quadratic function

$$V(x) = x^T P x \tag{13.7}$$

with $P = P^T > 0$ that decreases along every nonzero trajectory of the closed-loop system, i.e. $\dot{V}(x) < 0$ for all $x \neq 0$. If there exists such a P then $V(x)$ is a Lyapunov function for the closed-loop system.

Next, the quality of control will be considered. This means that the energy function $V(x)$ will not only be demanded to decrease, but also to decrease in a prescribed way along every nonzero trajectory of the closed-loop system to achieve some desired performance. With respect to performance, the cost functional

$$J_\infty(x_0, u) = \int_0^\infty (q(x(t), u(t))dt \tag{13.8}$$

with $x_0 = x(0)$, $q(x, u) = x^T Q^x x + u^T R^u u$ the instantaneous cost function, and $Q^x > 0$, $R^u > 0$ is introduced. Two problems will be considered.

PROBLEM 13.3.1 (PERFORMANCE) *Given (13.1) with $x_0 = x(0)$ and the instantaneous cost function $q(x, u)$, the problem is to determine $P > 0$ from (13.7) and $\{K_J, k_J\}$ from (13.6) that minimize $J_\infty(x_0, u)$.*

PROBLEM 13.3.2 (INVERSE PERFORMANCE) *Given (13.1) with $x_0 = x(0)$ and a Lyapunov function (13.7) with $P > 0$ proving stability of the closed-loop system, the problem is to determine the pair (Q, R) from the instantaneous cost function $q(x, u)$ and $\{K_J, k_J\}$ from (13.6), that minimize $V(x(0))$.*

It is well known that under appropriate conditions[1], any solution $u = k(x)$ that satisfies (HJB), which in the context of the considered performance

[1]In contrast to the linear case, where under the appropriate controllability conditions there always exists a solution to (HJB), here it has to be assumed that V exists.

problems can be rewritten as

$$\min_{u\in\mathcal{U}} \left\{ \dot{V}(x,u) + q(x,u) \right\} = 0 \tag{13.9}$$

minimizes J_∞, see Proposition 11.2.1. Suppose that (13.9) holds with $u^* = k(x)$, that is

$$u^* = k(x) = \arg\min_{u\in\mathcal{U}} \left\{ \dot{V}(x,u) + q(x,u) \right\} = 0$$

then

$$\dot{V}(x,u^*) = -q(x,u^*) \tag{13.10}$$

and

$$\begin{aligned} V(x(T)) - V(x(0)) &= \int_0^T \frac{d}{dt} V(x(t))dt \\ &= \int_0^T \dot{V}(x(t),u^*(t))dt = -\int_0^T (q(x(t),u^*(t))dt \end{aligned}$$

for all $T \geq 0$, from which can be concluded that

$$V(x(0)) = V(x(T)) + \int_0^T (q(x(t),u^*(t))dt$$

Using the fact that $\lim_{T\to\infty} V(x(T)) = 0$ since by assumption $\lim_{T\to\infty} x(T) = 0$, it follows that

$$V(x(0)) = \min_{u\in\mathcal{U}} J_\infty(x_0,u)$$

That is $V(x(0))$ is the optimal cost.

In this case the structure of V see (13.7), u see (13.6), and $q(x,u)$ see (13.8) are a priori specified and therefore it is unlikely that (13.9) will hold. However, lower bounds and upper bounds of the cost associated with (inverse) performance analogous to [63], [17], as well as regions of performance level in the state-space X will be derived. Therefore it is assumed that a stabilizing feedback $u = k(x)$ is designed such that

$$\dot{V}(x,k(x)) \geq -q(x,k(x)) \text{ for all } x \tag{13.11}$$

or

$$\dot{V}(x,k(x)) \leq -q(x,k(x)) \text{ for all } x \tag{13.12}$$

This means equality of (13.10) is replaced by an inequality. First the performance problem will be addressed, and after that the same ideas are applied to the inverse performance problem.

Performance: lower bound for $J_\infty(x_0, u)$

Assume $q(x, u)$ given, and suppose that (13.11) holds. Then with an exposition similar to the one following (13.10) it follows that

$$V(x(0)) \leq J_\infty(x_0, k(x)) \text{ for all } x \tag{13.13}$$

That is $V(x(0))$ is a lower bound for the cost that is achieved. The best lower bound can be computed as the one that maximizes

$$V(x(0)) = x(0)^T P x(0).$$

subject to (13.13) and the closed-loop system, that is the system interconnected with the feedback $u = k(x)$. The P that achieves this bound will be denoted P_{lo}.

Performance: upper bound for $J_\infty(x_0, u)$

Assume $q(x, u)$ given, and suppose that (13.12) holds. Then with an exposition similar to the one following (13.10) it follows that

$$V(x(0)) \geq J_\infty(x_0, k(x)) \text{ for all } x \tag{13.14}$$

That is $V(x(0))$ is an upper bound for the cost that is achieved. The best upper bound can be computed as the one that minimizes

$$V(x(0)) = x(0)^T P x(0).$$

subject to (13.14) and the closed-loop system. The P that achieves this bound will be denoted P_{up}.

Following a similar analysis as above, bounds on the cost associated with inverse performance are derived.

Inverse performance: lower bound for $V(x(0))$

Assume the function $V(x)$ proving stability of the system interconnected with the feedback $u = k(x)$ given, and suppose that (13.12) holds. Then with an exposition similar to the one following (13.10) it follows that

$$J_\infty(x_0, k(x)) \leq V(x(0)) \tag{13.15}$$

That is $J_\infty(x_0, k(x))$ is a lower bound for the energy $V(x(0))$. The best lower bound can be computed as the one that maximizes

$$J_\infty(x_0, k(x)) = \int_0^\infty \left\{ x^T Q^x x + k(x)^T R^u k(x) \right\} dt.$$

subject to (13.15) and the closed-loop system. The pair (Q^x, R^u) that achieves this bound will be denoted (Q^x_{lo}, R^u_{lo}).

Inverse performance: upper bound for $V(x(0))$

Assume $V(x)$ proving stability of the closed-loop system given, and suppose that (13.11) holds. Then with an exposition similar to the one following (13.10) it follows that

$$J_\infty(x_0, k(x)) \geq V(x(0)) \tag{13.16}$$

That is $J_\infty(x_0, u)$ is an upper bound for the energy $V(x(0))$. The best upper bound can be computed as the one that minimizes

$$J_\infty(x_0, k(x)) = \int_0^\infty \left\{x^T Q^x x + k(x)^T R^u k(x)\right\} dt.$$

subject to (13.16) and the closed-loop system. The pair (Q^x, R^u) that achieves this bound will be denoted (Q^x_{up}, R^u_{up}).

Regions of performance (accuracy)

It follows that if $P_{lo} = P_{up}$ then necessarily (13.10) holds, since $V(x)$ is a common solution to (13.12) and (13.11). This implies that (13.9) holds, and $V(x(0)) = J_\infty(x_0, k(x))$ is minimized for every initial condition $x(0)$, i.e. optimal performance is achieved everywhere in the state-space. As a measure of accuracy of the upper bound and lower bound it is therefore natural to look at the following ratio of quadratic terms:

$$R(x(0)) = \frac{x(0)^T P_{lo} x(0)}{x(0)^T P_{up} x(0)} \tag{13.17}$$

Here it is assumed that $x(0) \neq 0$. Of course $0 \leq R(x(0) \leq 1$. If $R(x(0) \approx 1$ then the bounds are very accurate. $R(x(0))$, and therefore performance of the controller depends on the initial condition $x(0)$. Hence there will be regions in the state-space with maximum and minimum performance (accuracy). With the factorization $P_{up} = F_{up}^T F_{up}$ and the nonsingular transformation $z(0) = F_{up} x(0)$ (13.17) becomes

$$R(z(0)) = \frac{z(0)^T F_{up}^{-T} P_{lo} P F_{up} z(0)}{z(0)^T z(0)} \tag{13.18}$$

The minimum and maximum of the Rayleigh quotient (13.18), can be obtained from the eigenvector-eigenvalue decomposition $F_{up}^{-T} P_{lo} F_{up} = Q_z \Lambda Q_z^T$ with Q_z a matrix containing the orthonormal eigenvectors, and Λ a diagonal matrix containing the eigenvalues λ_i arranged in order of magnitude. Now the $\max_{z \neq 0} R(z(0)) = \lambda_{\max}$ and occurs in the direction of the corresponding eigenvector $z_{\max}$ which is the corresponding column of Q_z. Equally,

$\min_{z\neq 0} R(z(0)) = \lambda_{\min}$ with eigenvector $z_{\min}$. With the transformation $Q_x = F_{up}^{-1} Q_z$ the direction of minimum and maximum performance (accuracy) in the state-space can be recovered. The regions for constant performance accuracy are given by $R(x(0)) = c$ with $\lambda_{\min} \leq c \leq \lambda_{\max}$. This analysis makes it possible to associate with certain regions of the state-space a measure of performance.

13.4 Controller Synthesis

An iterative synthesis algorithm will be proposed that guarantees robust stability for the PLM (13.1) via piecewise affine state feedback. The conditions involved can be written as LMIs. Feasibility of these conditions can therefore be checked efficiently by means of convex optimization routines. If these conditions are feasible then also the corresponding feedback can be computed. Furthermore, the synthesis algorithm provides upper and lower bounds, that are required to investigate performance of the closed loop.

Robust performance algorithm

The algorithm follows from standard Lyapunov arguments and LMI results using the $\mathcal{S}$-method. The $\mathcal{S}$-method is introduced to reduce conservatism of the synthesis inequalities [17]. Therefore a quadratic function

$$s_J(x) = [x1]^T S_J [x1]$$

has to be identified that outer approximates X_J^c, that is it satisfies

$$\{x \mid s_J(x) \geq 0\} \supseteq X_J^c.$$

More about how the outer approximation can be obtained can be found in [23]. A complete derivation of the synthesis inequalities can be found in [8] and [7].

step 1: iteration 1, compute the best lower bound P_{lo} as a function of the desired performance given by the fixed pair (Q^x, R^u). Compute this best lower bound for the cost without assuming a specific controller structure within a cluster. This can be done by solving the appropriate LMIs based on (13.11) together with the maximization of trace(P).

step 2: iteration 1, solve the inverse performance problem, this means with $Q_{lo} = P_{lo}^{-1}$ fixed solve the appropriate LMIs based on (13.12) together with the minimization of $\text{trace}((Q^x)^{-1}) + \text{trace}((R^u)^{-1})$. The computed performance matrices will be denoted Q_{lo}^x respectively R_{lo}^u. By doing so P_{lo} becomes an upper bound P_{up} for the cost, given by the obtained performance pair (Q_{lo}^x, R_{lo}^u).

step 1: iteration 2, compute the best lower bound P_{lo} as a function of the pair (Q^x_{lo}, R^u_{lo}) obtained in **step 2: iteration 1** and iterate further to see if some demanded agree of performance accuracy is achieved. At any moment one could stop the iteration and validate performance by comparing the computed upper- and lower bounds for performance. This means compare P_{up} obtained at **step 2: iteration k** with P_{lo} obtained at **step 1: iteration k+1** following the analysis presented after (13.18). If one is satisfied about the (accuracy of) performance then the controller can be computed.

This leads to the following iterative procedure:

step 1: (find P_{lo})

check if $\exists P$, $\tau_J \geq 0$ such that for $J \in \bar{J}_{01}$, $j \in J$

$$\begin{bmatrix} A_j^T P + P A_j + Q^x - \tau_J S_J(x,x) & * & * \\ c_j^T P - \tau_J S_j(x,1)^T & -\tau_J S_J(1,1) & * \\ B_j^T P & 0 & R^u \end{bmatrix} \geq 0 \qquad (13.19)$$

and also maximize $\gamma > 0$

$$\text{trace}(P) > \gamma$$

step 2: (find P_{up})

and if for $Q_{lo} = P_{lo}^{-1}$ $\exists$ $\tau_J \geq 0$, $(Q^x)^{-1} > 0$, $(R^u)^{-1} > 0$, $\{Y_J\}$, $\{y_J\}$ such that for $J \in \bar{J}_{01}$, $j \in J$

$$\begin{bmatrix} -L_J - \tau_J Q_{lo} S_J(x,x) Q_{lo} & * & * & * \\ -M_J^T - \tau_J S_j(x,1)^T Q_{lo} & -\tau_J S_J(1,1) & * & * \\ Q_{lo} & 0 & (Q^x)^{-1} & * \\ Y_J & y_J & 0 & (R^u)^{-1} \end{bmatrix} \geq 0 \qquad (13.20)$$

with $L_J = Q_l A_j^T + Y_J^T B_j{}^T + A_j Q_l + B_j Y_J$, $M_J^T = c_j^T + y_J^T B_j{}^T$ and also minimize $\gamma > 0$

$$\text{trace}((Q^x)^{-1}) + \text{trace}((R^u)^{-1}) < \gamma$$

The $*$ elements are induced by symmetry of the matrices. $S_J(.,.)$ is partitioned according to x and 1.

If the LMIs are feasible, i.e. there exists a solution to the two step synthesis algorithm, then the controller (13.6) with $K_J = Y_J P_l$ and $k_J = y_J$ robustly stabilizes (13.1).

13.5 Example

Assume that the PLM (13.1) consists of three models, parametrized by (A_i, B_i, c_i), $i \in \{1, 2, 3\}$ where

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 \\ -4 & 2 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad c_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

In this example it is assumed that the clusters are identified such that $\bar{X}^c_{01} = \{X^c_1, X^c_{12}, X^c_3\}$ and $\bar{X}^c_0 = X^c_1$, see Figure 13.2. In order to be able to introduce the $\mathcal{S}$-method the different regions $X^c_J \in \bar{X}^c_{01}$ have to be outer approximated by quadratic functions $s_J(x)$. In this typical example

$$\begin{aligned} X^c_1 &= \mathbb{R}^2 \backslash \bar{X}^c_1 \\ X^c_{12} &= \{x \mid -((x_1 - 3)^2 + (x_2 - 3)^2)) + 1 > 0\} \\ X^c_3 &= \{x \mid -((x_1 + 3)^2 + (x_2 + 3)^2) + 1 > 0\} \end{aligned}$$

and the associated quadratic outer approximations $s_J(x)$ are equal except $s_1(x) = 0$. Some conservatism is introduced since $s_1(x) = 0$ for all $x \in \mathbb{R}^2$. Without the $\mathcal{S}$-method, for every polytope defined by $J \in \bar{J}_{01}$, the triples (A_i, B_i, c_i) have to be stabilizable via the feedback (13.6). Therefore the synthesis LMIs without the $\mathcal{S}$-method would fail. In this case however (13.19,13.20) are feasible and the following results are obtained.

step 1: iteration 1(find P_{lo})

Solve (13.19) with the suggested maximization of P for fixed

$$Q^x = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} \text{ and } R^u = 1.$$

The following result is obtained:

$$P_{lo} = \begin{bmatrix} 11.8148 & 1.7044 \\ 1.7044 & 2.7034 \end{bmatrix}.$$

step 2: iteration 1 (find P_{up})

Solve (13.20) with the suggested minimization of $(Q^x)^{-1}$ and $(R^u)^{-1}$, the off-diagonal elements of Q^x are chosen to be structural zeros. The following solution is obtained:

$$Q^x_{lo} = \begin{bmatrix} 7.9147 & 0 \\ 0 & 14.1451 \end{bmatrix} \text{ and } R^u_{lo} = 0.2547.$$

For the performance pair (Q_{lo}, R_{lo}) the fixed P_{lo} becomes P_{up}. The computed controller parameters are

$$\begin{array}{ll} K_1 = \begin{bmatrix} -6.6911 & -10.6129 \end{bmatrix} & k_1 = 0 \\ K_{12} = \begin{bmatrix} -6.6917 & -10.6084 \end{bmatrix} & k_{12} = 0.2714 \\ K_3 = \begin{bmatrix} -6.7007 & -10.5092 \end{bmatrix} & k_3 = -0.0117 \end{array}$$

step 1: iteration 2(find P_{lo})

Solve (13.19) with the suggested maximization of P however with the pair (Q^x_{lo}, R^u_{lo}). The outcome of this step is

$$P_{lo} = \begin{bmatrix} 11.3612 & 1.0562 \\ 1.0562 & 1.6998 \end{bmatrix}.$$

The quality of the two step synthesis algorithm can be quantized by looking at the distance between the performance achieved by the controller and the desired performance. The achieved performance of the controller (13.6)

$$u_c = \begin{cases} K_J x & \text{if} \quad x \in X^c_J \in \bar{X}^c_0 \\ K_J x + k_J & \text{if} \quad x \in X^c_J \in \bar{X}^c_1 \end{cases}$$

is measured by the cost functional

$$J_\infty(x_0, u_c) = \int_0^T \left\{ x(t)^T Q^x_{lo} x(t) + u_c(t)^T R^u_{lo} u_c(t) \right\} dt.$$

$V_{up}(x(0)) = x^T(0) P_{up} x(0)$ is an upper bound for the desired cost function $J_\infty(x_0, u)$, see (13.12,13.20). The corresponding lower bound for the cost function reads $V_{lo}(x(0)) = x^T(0) P_{lo} x(0)$, see (13.11,13.19). So clearly

$$V_{lo}(x(0)) \leq J_\infty(x_0, u_c) \leq V_{up}(x(0))$$

.

The upper bound V_{up}, lower bound V_{lo}, achieved performance J_∞ and performance accuracy R for the associated regions of minimal performance accuracy $\beta x_{\min}$, and regions of maximal performance accuracy $\beta x_{\max}$ in the state-space, is given in Table 13.1.

In Figure 13.2 simulation results are shown for the polytopic system with $w_1 = w_2 = 0.5$ for $x \in X^c_{12}$. Trajectories are initialized from: $x(0) = [3\ 4.5]^T$,

Table 13.1: *Upper bound, lower bound and achieved performance.*

	minimal accuracy	maximal accuracy
$x(0)$	$x_{\min} = [-0.019\ -4.853]^T$	$x_{\max} = [3.051\ -1.971]^T$
$R(x(0))$	$\lambda_{\min} = 0.6287$	$\lambda_{\max} = 0.9967$
$V_{up}(x(0))$	64.0072	99.9826
$J_\infty(x_0, u_c)$	61.9262	99.9071
$V_{lo}(x(0))$	40.2440	99.6574

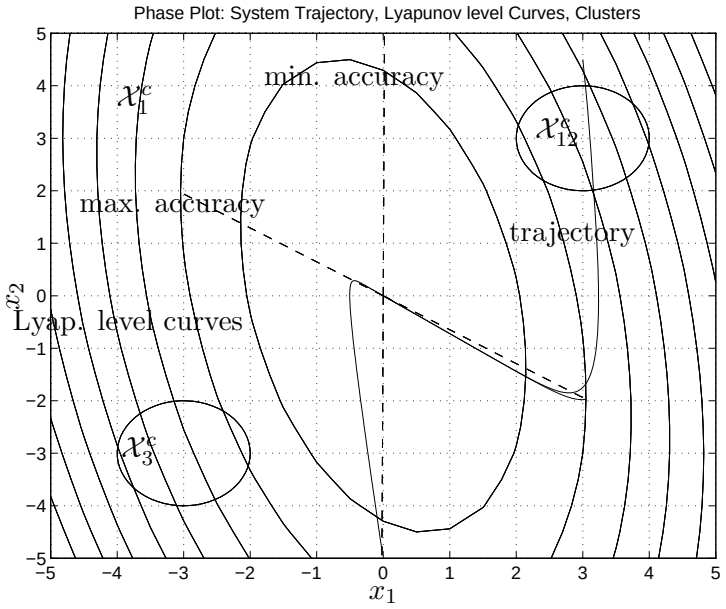


Figure 13.2: *Trajectories in the state-space for different initial conditions of the closed loop PLM.*

$x(0) = x_{\min}$ and $x(0) = x_{\max}$. Also the (dotted) lines along which maximal and minimal performance accuracy is obtained are drawn in this figure. From the elliptical Lyapunov level curves it can be seen that, as expected, along the trajectories energy decreases. Since trajectories of the closed loop system with initial values $x(0) = x_{\max}$ and $x(0) = x_{\min}$ stay within cluster X_1^c one could compare the performance of the controller for these initial conditions with a LQR design for the triple (A_1, B_1, c_1) with performance pair (Q_{lo}^x, R_{lo}^u). This means solving the algebraic Riccati equation for the unknown P,

$$PA_1 + A_1^T P - PB_1(R_{lo}^u)^{-1}B_1^T P + Q_{lo}^x = 0$$

with optimal controller $K = -(R_{lo}^u)^{-1}B_1^T P$. The following solution is ob-

tained:

$$P = \begin{bmatrix} 11.8036 & 1.6972 \\ 1.6972 & 2.6835 \end{bmatrix} \text{ and } K = [-6.6635 - 10.5359].$$

This leads to the costs $E_{x_{\max}} = 99.8224$ and $E_{x_{\min}} = 56.6821$ evaluated for initial conditions of maximal and minimal performance accuracy as given in Table 13.1. Comparing these costs with the costs from Table 13.1 shows that indeed performance is achieved for initial conditions $x(0)$ close to $ax_{\max}$.

Yet another interesting observation can be made about performance of the closed loop system. For initial conditions $x(0)$ such that trajectories go through an uncertain region (in this case cluster X_{12}^c), one may not expect the upper and lower bound for the cost function to be the same. This is because for such an initial condition different trajectories of the closed-loop are expected to be possible. Different trajectories will most likely result in different costs. If different costs are possible for the same initial condition then this will lead to upper bounds and lower bounds that are not the same. This means that in general the uncertainty reduces the performance.

13.6 Conclusion

In this chapter a stabilizing controller for a family of (nonlinear) systems is designed. This means that the controller is robust against parametric uncertainty of the system. In contrary to the previous chapters the PLM is interpreted as an uncertainty model of the system.

An algorithm is proposed that partitions the state-space into a number of disjoint clusters on which a gain scheduling controller is defined. The partitioning relies on the support of the scheduling functions of the PLM. The objective is to parametrize this controller such that robust stabilization and performance of the closed loop PLM is achieved.

An iterative algorithm involving LMIs is proposed to solve the robust stabilization problem. If the LMIs are feasible then the synthesis problem is solved. Furthermore performance of the closed loop in the state-space can be analyzed with the presented analysis tool that builds naturally on the outcome of the iterative synthesis algorithm presented in this chapter.

Chapter 14

Conclusions

14.1 Main Conclusions

The hypothesis that the choice for a PLM structure enlarges the possibility to develop control theory and modelling methodologies, that are in agreement with and partly generalize the linear theory, is justified by this thesis. It is demonstrated that a substantial variety of analysis, modelling and control problems for nonlinear systems can be approached constructively with the presented framework. The approach is attractive since it shows resemblance with the linear approach and, therefore, already well-known theoretical concepts and ideas can be formalized and generalized in the PLM setting. The presented framework is far from finished. The contents of this thesis, the problems considered, are meant only to be representative and certainly not exhaustive. The reader however will easy catch on and elaborate on the ideas presented in this thesis. Next a few conclusions by part will be given.

Analysis

The proposed model structure, and the different interpretations of the PLM are easily understood, which facilitates the transfer of information between the model and the users of the model. Furthermore, a PLM is a universal approximator, thus the model class is rich. However, the 'curse of dimensionality' restricts the practical applicability of PLMs to low dimensional scheduling spaces. Stability of the system can be checked efficiently with the proposed methods. The controllability and observability results are restricted to PLMs which have additional structure.

Modelling

The presented modelling methods demonstrate that it is feasible to model a system adequately with a PLM. The modelling problems that we have considered, however, are of limited complexity. The practical applicability of the modelling methods to more complex systems is not within the scope of this thesis. It is our experience that, for the small-scale problems considered, the transparency of the PLM, contributes to the understanding of system properties, in particular nonlinearities.

Control

The presented control approaches show that it is viable to design controllers for a nonlinear system on the basis of an approximate PLM representation of the system. The presented control approaches are constructive and can be automated to a large extent. They show how PLM analysis and modelling can be utilized and combined to design controllers for the system. From the presented examples it can be concluded that the same objective or performance could never be achieved with a linear control design. Furthermore, the resulting controllers are easy to understand and intuitively appealing.

One can conclude that model scheduling leads to controller scheduling. Within industry controller scheduling has already been done for years, however on an ad-hoc basis. Gain scheduling controller designs are often a first step beyond linear designs. The presented research work is a step towards theoretically sound and automated model-scheduling and gain-scheduling and is therefore of practical relevance.

14.2 Main Contributions

The main contributions of this thesis are classified by part and ordered by appearance.

Analysis

- The approximation results Theorem 4.3.1 and Theorem 4.6.3.
- The stability results Theorem 5.3.1, Corollary 5.3.2, Theorem 5.3.4 and Theorem 5.3.5.
- The controllability results Theorem 5.4.2, Theorem 5.4.4, Theorem 5.4.5, Corollary 5.4.6 and the observability conditions that follow from duality.

Modelling

- The model based modelling method presented in Chapter 7.
- The data based modelling method presented in Section 8.3.

Control

- The optimal control results Theorem 11.2.3 and Theorem 11.3.1.
- The robust controller design algorithm presented in Chapter 13.

14.3 Suggestions for further Research

The PLM structure is of a form lending itself to computer aided modelling and controller design. Preliminary work indicates that it is feasible and paying to implement the presented algorithms for system analysis, modelling and control in a software environment with an interface especially developed for PLMs.

I also suggest to elaborate on the ideas presented in this thesis and to contribute to the framework. Next a few suggestions by part will be given.

Analysis

The polytopic linear model structure as suggested, is constrained by the fact that each regime consists of the same set of state variables. However, if a system is identified in several characteristic operating regimes by a linear state space model, it is imaginable that these linear models will have different state-spaces. Often on the basis of these locally valid models, locally valid controllers are designed. How to schedule these controllers over the operating space is far from trivial, and often approached in an ad-hoc way. If it is possible to combine the locally valid models into a polytopic linear model, then controller scheduling could be handled in the way presented in this thesis. A next step is thus to extend the PLM approach such that linear models with different state spaces can be handled. A point of departure can be [34], who addresses this problem.

Modelling

The practical applicability of the modelling methods has to be evaluated on more complex systems. A user friendly software implementation of the derived identification algorithms, will facilitate the applicability of the modelling method.

Control

The stabilizing controllers that are presented for PLMs are all based upon the existence of a quadratic Lyapunov function for the closed-loop system. Quadratic candidate Lyapunov functions are only sufficient for establishing stability of the nonlinear systems under consideration and many systems do not admit a quadratic Lyapunov function. Therefore, more powerful candidate Lyapunov functions can reduce the conservatism of quadratic candidate Lyapunov functions. A next step, that naturally generalizes the presented approach, can be to search for piecewise quadratic Lyapunov functions. These functions are more powerful then quadratic candidate Lyapunov functions [39], and because of the structure of these functions, we expect that in some cases it is possible to arrive at more powerful controller synthesis algorithms that are nearly as tractable as the ones presented here.

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List of Acronyms

Acronym[1]	Meaning
ARE	Algebraic Riccati Equation
HJB	Hamilton Jacobi Bellman
LMI	Linear Matrix Inequality
LQR	Linear Quadratic Regulator
LTI	Linear Time Invariant
PD	Proportional Derivative
PID	Propotional Integral Derivative
PLDI	Polytopic Linear Differential Inclusion
PLM	Polytopic Linear Model

[1] See the index for the page numbers.

Some Symbols

Symbol	Description
A	system matrix
A_i	system matrix of model i
$A(i,j)$	system matrix entry located at the i-th row and j-th column of A
$(A_i, B_i, a_i, C_i, D_i, c_i)$	parametrization of nonhomogeneous model i
a_i	system vector of nonhomogeneous linear model i
B	control input matrix
$B_r(x_{0i})$	ball with radius r centered at x_{0i}
$C_c(x_{0i})$	cube with width c centered at x_{0i}
Co	convex hull
C	output matrix
$C^k(E)$	functions k-times continuously differentiable on the domain E
c_i	output vector of nonhomogeneous linear model i
D	direct feedthrough matrix
$d(Z_i)$	center of scheduling regime i
$d_{fg}(E)$	distance between system f and model g on the domain E
$e(Z_i)$	width of scheduling regime i
F_i^k	k-th order Taylor remainder of f around x_{0i}
$f(\cdot)$	right-hand side for system
f_i^k	k-th order Taylor series expansion of f around x_{0i}
$g(\cdot)$	right-hand side for (approximate) model
h	output map
$J(\cdot)$	cost function(al)
λ_ξ	maximum absolute eigenvalue of the Jacobian or Hessian matrices associated with the Taylor remainder
$\mathcal{M}$	model set
N_m	number of (local) models or operating regimes

$O(A, B)$	observability matrix
$\mathcal{O}(A, B)$	observable space
Ψ	operating space
Ψ_i	operating regime i
$\rho_i(z)$	validity function i
$R(A, B)$	controllability matrix
$\mathcal{R}(A, B)$	controllable space
θ	parameter vector
$w_i(z)$	scheduling function i
x	state
u	input
v	external (observed) variable
y	output
Z	scheduling space
Z_i	scheduling regime i
$\lceil \cdot \rceil$	ceiling operator
$\|\cdot\|_2$	Euclidean norm
$\|\cdot\|_F$	Frobenius norm
$\|\cdot\|$	absolute value

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Samenvatting

Dit onderzoek beschrijft de analyse, het modeleren, en het regelen van een klasse van niet-lineaire dynamische systemen op basis van polytope lineaire modellen. De polytope lineaire modelstructuur wordt geïntroduceerd als een alternatieve benaderende beschrijvingswijze van niet-lineaire dynamische systemen ten behoeve van systeem analyse en regelaarontwerp. De modelstructuur heeft een drietal eigenschappen die we willen benutten.

Ten eerste, een PLM is opgebouwd uit een aantal lineaire modellen die ieder afzonderlijk binnen een werkgebied, een zogenaamd regime, het systeem kwalitatief beschrijven. Door deze modellen op geschikte wijze aan elkaar te koppelen, door middel van werkpuntsafhankelijke convexe combinaties van de parameters behorend bij de lineaire modellen, ontstaat een PLM. De parameterwaarden van het PLM variëren dan ook binnen een polytoop die gedefiniëerd wordt door alle mogelijke convexe combinaties van parameterwaarden te nemen behorende bij de afzonderlijke lineaire modellen. Aan deze eigenschap dankt een PLM zijn naam. Een PLM is dus te interpreteren op basis van een regime decompositie. Ten tweede, daar een PLM gebaseerd is op meerdere lineaire modellen, is het mogelijk om het niet-lineaire systeem meer globaal te beschrijven dan mogelijk zou zijn geweest met één enkel lineair model. Ten derde wordt er aangetoond dat onder geschikte condities, niet lineaire systemen tot iedere gewenste nauwkeurigheid benaderd kunnen worden door een PLM, geparametriseerd met een eindig aantal parameters. Er wordt een bovengrens gegeven voor het aantal benodigde parameters, voldoende om een vooraf gegeven gewenste nauwkeurigheid van de benadering te behalen.

Een belangrijke motivatie voor het beschouwen van een PLM berust op de overeenkomsten in structuur met een linear model. Voor lineaire systemen is de systeem- en regeltheorie goed ontwikkeld en begrepen. Het al dan niet voldoen aan de voor dit vakgebied belangrijke systeem eigenschappen, zoals stabiliteit, regelbaarheid enzovoort, kan aangetoond worden middels (vaak relatief eenvoudige) mathematische manipulaties van de parametrisatie. Het regelaarontwerp kan vaak geautomatiseerd worden op basis van de parametrisatie en de regeldoelstelling. Denk hierbij aan regelwetten op basis van stabiliteit, optimaliteit enz.. Voor niet-lineaire systemen is dit maar ten dele het

geval en is dus het ontwikkelen van systeem- en regeltheorie van groot belang. Gezien de overeenkomsten tussen een lineair model en een PLM bestaat de verwachting dat (van resultaten en concepten) van de goed ontwikkelde lineaire systeem- en regeltheorie gebruik gemaakt kan worden. Deze hypothese wordt ten dele bevestigd door dit onderzoek.

Onder geschikte voorwaarden kunnen middels een eenvoudige analyse van de parametrisatie van het PLM systeem theoretische en regeltechnisch gezien relevante systeemeigenschappen van het PLM vastgesteld worden. Eén van die eigenschappen is stabiliteit. Onder geschikte voorwaarden impliceert stabiliteit van het PLM stabiliteit van het werkelijke systeem. Verder worden er enkele eenvoudig te verifiëren condities afgeleid met betrekking tot de begrippen regelbaarheid en waarneembaarheid. Hierbij dient opgemerkt te worden dat daartoe de modelstructuur van het PLM in verdere mate beperkt is.

Het afleiden van systeemeigenschappen van het PLM heeft tot doel om tot een geschikt model, en in het bijzonder een model gebaseerd regelaar ontwerp te komen. Dit onderzoek beschrijft dan ook enkele constructieve methoden om tot een PLM representatie van het werkelijke systeem te komen.

Op basis van het PLM zijn enkele regelwetten geformuleerd. Deze regelwetten hebben als hoofddoel het systeem in een gewenst werkpunt te stabiliseren. Enkele geautomatiseerde stabiliserende regelaarontwerpen die als neven regeldoelstelling optimaliteit of robuustheid beogen, behoren tot het resultaat van dit onderzoek.

Het complete traject van het representeren van een systeem met een benaderend PLM, vervolgens het analyseren van het PLM, tot en met het uiteindelijk regelen van het systeem op basis van een op het PLM gebaseerd regelaarontwerp is geïllustreerd aan de hand van enkele voorbeelden. Het betreft hier zowel experimentele alsook simulatie studies waarbij niet-lineaire dynamische (mechanische) systemen het onderwerp van onderzoek zijn.