The Abelian sandpile model on an infinite tree

Document status and date:
Published: 01/01/2000

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

The Abelian sandpile model on an infinite tree

C. Maes
F.H.J. Redig
E. Saada

The Abelian Sandpile Model on an Infinite Tree

Christian Maes, *K.U.Leuven* *
Frank Redig, *T.U. Eindhoven*[†]
Ellen Saada, *C.N.R.S., Rouen* [‡]

November 22, 2000

Abstract: We consider the standard Abelian sandpile process on the Bethe lattice. We show the existence of the thermodynamic limit for the finite volume stationary measures and the existence of a unique infinite volume Markov process exhibiting features of self-organized criticality.

1 Introduction

Global Markov processes for spatially extended systems have been around for about 30 years now and interacting particle systems have become a branch of probability theory with an increasing number of connections with the natural and human sciences. While standard techniques and general results have been collected in a number of books such as [Liggett (1985), Chen (1992), Toom (1990)] and are capable to treat the infinite volume construction for stochastic systems with locally interacting components, some of the most elementary questions for long range and nonlocal dynamics have remained wide open. We have in mind the class of stochastic interacting systems that during the last decade have invaded the soft condensed matter literature and are sometimes placed under the common denominator of self-organizing systems.

Since the appearance of the paper [BTW (1988)], the concept of self-organized criticality (SOC) has suscited much interest, and is applied in a great variety of domains (see e.g. [Turcotte (1999)] for an overview). From the mathematical point of view, the situation is however quite unsatisfactory. The models exhibiting SOC are in general very boundary condition dependent (especially the BTW model in dimension 2), which suggests that the definition of an infinite

---

*Instituut voor Theoretische Fysica, K.U.Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium - email: `christian.maes@fys.kuleuven.ac.be`

[†]On leave from Instituut voor Theoretische Fysica, K.U. Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium - email: `f.h.j.redig@tue.nl`

[‡]CNRS, UPRESA 6085, Université de Rouen, 76821 Mont-Saint-Aignan cedex, France. -email: `Ellen.Saada@univ-rouen.fr`

volume dynamics poses a serious problem. Even the existence of a (unique) thermodynamic limit of the finite volume stationary measure is not clear. From the point of view of interacting particle systems no standard theorems are at our disposal. The infinite volume processes we are looking for will be non-Feller and cannot be constructed by monotonicity arguments as in the case of the one-dimensional BTW model (see [MRSV (2000)]) or the long-range exclusion process (see [Liggett (1980)]). On the other hand in order to make mathematically exact statements about SOC, it is necessary to have some kind of infinite volume limit, both for statics and for dynamics.

In this paper we continue our study of the BTW-model for the case of the Bethe lattice, this is the abelian sandpile model on an infinite tree. For this system, many exact results were obtained in [DM (1990b)]. In contrast to the one-dimensional case this system has a non-trivial stationary measure. We show here that the finite volume stationary measures converge to a unique measure μ which is not Dirac and exhibits all the properties of a SOC-state. We then turn to the construction of a *stationary* Markov process starting from this measure μ. The main difficulty to overcome is the strong non-locality: Adding a grain at some lattice site x can influence the configuration far from x. In fact the cluster of sites influenced by adding at some fixed site has to be thought of as a critical percolation cluster which is almost surely finite but not of integrable size. The process we construct is intuitively described as follows: At each site x of the Bethe lattice we have an exponential clock which rings at rate $\varphi(x)$. At the ringing of the clock we add a grain at x. Depending on the addition rate $\varphi(x)$, we show existence of a stationary Markov process which corresponds to this description. We also extend this stationary dynamics to initial configurations which are typical for a measure μ' that is stochastically below μ.

The paper is organized as follows. In section 2 we introduce standard results on finite volume abelian sandpile models, and summarize some specific results of [DM (1990b)] for the Bethe lattice which we need for the infinite volume construction. In section 3 we present the results on the thermodynamic limit of the finite volume stationary measures and on the existence of infinite volume Markovian dynamics. Section 4 is devoted to proofs and contains some additional remarks.

2 Finite Volume Abelian Sandpiles

In this section we collect some results on abelian sandpiles on finite graphs which we will need later on. Most of these results are contained in the review paper [Dhar (1999)], or in [IP (1998)].

2.1 Toppling Matrix

Let V denote a finite set of sites and $\Delta^V = (\Delta^V_{x,y})_{x,y \in V}$ a matrix indexed by the elements of V satisfying the conditions :

1. For all $x, y \in V$, $x \neq y$, $\Delta^V_{x,y} = \Delta^V_{y,x} \leq 0$,

2. For all $x \in V$, $\Delta^V_{x,x} \geq 1$,

3. For all $x \in V$, $\sum_{y \in V} \Delta^V_{x,y} \geq 0$,

4. $\sum_{x,y \in V} \Delta^V_{x,y} > 0$.

Such a matrix Δ^V is called a *toppling matrix*. The fourth condition ensures that there are sites (so called *dissipative sites*) for which the inequality in the third condition is strict. This is fundamental for having a well defined toppling rule later on. In the rest of the paper we will choose Δ^V to be the lattice Laplacian with open boundary conditions on a finite simply-connected set $V \subset S$, where S is a regular graph, like the d-dimensional lattice $\mathbb{Z}^d$, or the infinite rootless tree $\mathbf{T}_d$ of degree $d + 1$. More explicitly:

$$
\begin{aligned}
\Delta^V_{x,x} &= 2d \text{ if } V \subset \mathbb{Z}^d, \\
&= d + 1 \text{ if } V \subset \mathbf{T}_d, \\
\Delta^V_{x,y} &= -1 \text{ if } x \text{ and } y \text{ are nearest neighbors.}
\end{aligned}
\tag{2.1}
$$

The dissipative sites then correspond to the boundary sites of V.

2.2 Configurations

A *height configuration* η is a mapping from V to $\mathbb{N} = \{1, 2, ...\}$ assigning to each site a natural number $\eta(x) \geq 1$ ("the number of sand grains" at site x). A configuration $\eta \in \mathbb{N}^V$ is called *stable* if, for all $x \in V$, $\eta(x) \leq \Delta^V_{x,x}$. Otherwise η is *unstable*. We denote by Ω_V the set of all stable height configurations. For $\eta \in \mathbb{N}^V$ and $V' \subset V$, $\eta|_{V'}$ denotes the restriction of η to V'.

2.3 Toppling Rule

The *toppling rule* corresponding to the toppling matrix Δ^V is the mapping

$$
T_{\Delta^V} : \mathbb{N}^V \times V \to \mathbb{N}^V
$$

defined by

$$
\begin{aligned}
T_{\Delta^V}(\eta, x)(y) &= \eta(y) - \Delta^V_{x,y} \text{ if } \eta(x) > \Delta^V_{x,x}, \\
&= \eta(y) \text{ otherwise.}
\end{aligned}
\tag{2.2}
$$

In words, site x topples if and only if its height is strictly larger than $\Delta^V_{x,x}$, by transferring $-\Delta^V_{x,y}$ grains to site $y \neq x$ and losing itself $\Delta^V_{x,x}$ grains. Toppling rules commute on unstable configurations. This means for $x, z \in V$ and η such that $\eta(x) > \Delta^V_{x,x}, \eta(z) > \Delta^V_{z,z}$,

$$
T_{\Delta^V}(T_{\Delta^V}(\eta, x), z) = T_{\Delta^V}(T_{\Delta^V}(\eta, z), x)
\tag{2.3}
$$

We write $[T_{\Delta^V}(\cdot, z) T_{\Delta^V}(\cdot, x)](\eta) = [T_{\Delta^V}(\cdot, x) T_{\Delta^V}(\cdot, z)](\eta)$.

Choose some enumeration $\{x_1, \cdots, x_n\}$ of the set V. The *toppling transformation* is the mapping

$$
\mathcal{T}_{\Delta^V} : \mathbb{N}^V \to \Omega_V
$$

defined by

$$
\mathcal{T}_{\Delta^V}(\eta) = \lim_{N \uparrow \infty} \left(\prod_{i=1}^n T_{\Delta^V}(\cdot, x_i) \right)^N (\eta).
\tag{2.4}
$$

In [IP (1998)] it is recalled that

3

1. The limit in (2.4) exists, i.e. there are no infinite cycles, due to the presence of dissipative sites.

2. The stable configuration $\mathcal{T}_{\Delta V}(\eta)$ is independent of the chosen enumeration of V. This is the *abelian property* and follows from (2.3).

2.4 Addition Operators

For $\eta \in \mathbb{N}^V$ and $x \in V$, let η^x denote the configuration obtained from η by adding one grain to site x, i.e. $\eta^x(y) = \eta(y) + \delta_{x,y}$. The *addition operator* defined by

$$a_{x,V} : \Omega_V \to \Omega_V; \eta \mapsto a_{x,V}\eta = \mathcal{T}_{\Delta V}(\eta^x) \tag{2.5}$$

represents the effect of adding a grain to the stable configuration η and letting the system topple until a new stable configuration is obtained. By (2.3), the composition of addition operators is commutative: For all $\eta \in \Omega_V$, $x, y \in V$,

$$a_{x,V}(a_{y,V}\eta) = a_{y,V}(a_{x,V}\eta).$$

2.5 Finite Volume Dynamics

Let p denote a non-degenerate probability measure on V, i.e. numbers p_x, $0 < p_x < 1$ with $\sum_{x \in V} p_x = 1$. We define a discrete time Markov chain $\{\eta_n : n \geq 0\}$ on Ω_V by picking a point $x \in V$ according to p at each discrete time step and applying the addition operator $a_{x,V}$ to the configuration. This Markov chain has the transition operator

$$P_V f(\eta) = \sum_{x \in V} p_x f(a_{x,V}\eta). \tag{2.6}$$

We can equally define a continuous time Markov process $\{\eta_t : t \geq 0\}$ with infinitesimal generator

$$L_V^\varphi f(\eta) = \sum_{x \in V} \varphi(x)[f(a_{x,V}\eta) - f(\eta)], \tag{2.7}$$

generating a pure jump process on Ω_V, with addition rate $\varphi(x) > 0$ at site x.

2.6 Recurrent Configurations, Stationary Measure

We see here that the Markov chain $\{\eta_n, n \geq 0\}$ has only one recurrent class and its stationary measure is the uniform measure on that class.

Let us call $\mathcal{R}_V$ the set of recurrent configurations for $\{\eta_n, n \geq 0\}$, i.e. those for which $P_\eta(\eta_n = \eta$ infinitely often$) = 1$, where P_η denotes the distribution of $\{\eta_n, n \geq 0\}$ starting from $\eta_0 = \eta \in \Omega_V$. In the following proposition we list some properties of $\mathcal{R}_V$. For the sake of completeness we include a proof which we could not find worked out completely in the literature.

Proposition 2.1

 1. *$\mathcal{R}_V$ contains only one recurrent class.*

 2. *The composition of the addition operators $a_{x,V}$ restricted to $\mathcal{R}_V$ defines an abelian group G.*

4

3. $|G| = |\mathcal{R}_V|$.

4. For any $x \in V$, there exists n_x such that for any $\eta \in \mathcal{R}_V$, $a_{x,V}^{n_x}\eta = \eta$.

5. $|\mathcal{R}_V| = \det \Delta^V$.

Proof: 1. We write $\eta \hookrightarrow \zeta$ if in the Markov chain ζ can be reached from η with positive probability. Since sand is added with positive probability on all sites ($p_x > 0$), the *maximal configuration* η_{max} defined by

$$\eta_{max}(x) = \Delta_{x,x}^V$$

can be reached from any other configuration. Hence, if $\eta \in \mathcal{R}_V$ then $\eta \hookrightarrow \eta_{max}$, therefore $\eta_{max} \in \mathcal{R}_V$ and $\eta_{max} \hookrightarrow \eta$ (see e.g. [Chung (1960)] p.19).

2. Fix $\eta \in \mathcal{R}_V$; then there exist $n_y \geq 1$ such that

$$\prod_{y \in V} a_{y,V}^{n_y}\eta = \eta,$$

and

$$g_x = a_{x,V}^{n_x-1} \prod_{y \in V, y \neq x} a_{y,V}^{n_y}$$

satisfies $(a_{x,V}g_x)(\eta) = (g_x a_{x,V})(\eta) = \eta$. The set

$$\mathcal{R}^x = \{\zeta \in \mathcal{R}_V : (a_{x,V}g_x)(\zeta) = \zeta\}$$

is closed under the action of $a_{x,V}$, contains η, hence also η_{max}: it is a recurrent class. By part 1, $\mathcal{R}^x = \mathcal{R}_V$, $a_{x,V}g_x$ is the neutral element e, and $g_x = a_{x,V}^{-1}$ if we restrict $a_{x,V}$ to $\mathcal{R}_V$.

3. Fix $\zeta \in \mathcal{R}_V$ and put $\Psi_\zeta : G \to \mathcal{R}_V; g \mapsto g(\zeta)$. As before $\Psi_\zeta(G)$ is a recurrent class, hence $\Psi_\zeta(G) = \mathcal{R}_V$. If for $g, h \in G$, $\Psi_\zeta(g) = \Psi_\zeta(h)$, then $gh^{-1}(\zeta) = \zeta$, and by commutativity $gh^{-1}(g'\zeta) = g'\zeta$ for any $g' \in G$. Therefore $gh^{-1}(\xi) = \xi$ for all $\xi \in \mathcal{R}_V$, thus $g = h$. This proves that Ψ_ζ is a bijection from G to $\mathcal{R}_V$.

4. Since G is a finite group, for any $x \in V$ there exists $n_x \geq 1$ such that $a_{x,V}^{n_x} = e$.

5. Adding $\Delta_{x,x}^V$ particles at a site $x \in V$ makes the site topple, and $-\Delta_{x,y}^V$ particles are transferred to y. This gives

$$a_{x,V}^{\Delta_{x,x}^V} = \prod_{y \neq x} a_{y,V}^{-\Delta_{x,y}^V}.$$

On $\mathcal{R}_V$ the $a_{x,V}$ can be inverted and we obtain the closure relation

$$\prod_{y \in V} a_{y,V}^{\Delta_{x,y}^V} = e,$$

which completely determines the one-dimensional representations of the group of addition operators, and in particular the cardinality of the latter, as obtained in [Dhar (1990a)]. $\blacksquare$

Remark. $\mathcal{R}_V$ does not depend on the p_x, and does not change by going from discrete to continuous time, i.e. from (2.6) to (2.7).

The main consequence of the group property of G is the fact that the unique stationary measure is uniform on $\mathcal{R}_V$.

Proposition 2.2

1. The measure

$$\mu_V = \sum_{\eta \in \mathcal{R}_V} \frac{1}{|\mathcal{R}_V|} \delta_\eta \qquad (2.8)$$

is invariant under the action of $a_{x,V}$, $x \in V$ (δ_η is the Dirac measure on configuration η).

2. On $L^2(\mu_V)$ the adjoint of $a_{x,V}$ is

$$a^*_{x,V} = a^{-1}_{x,V}. \qquad (2.9)$$

Proof: Since $a_{x,V} : \mathcal{R}_V \to \mathcal{R}_V$ can be inverted, we have

$$\sum_{\eta \in \mathcal{R}_V} f(a_{x,V}\eta)g(\eta) = \sum_{\eta \in \mathcal{R}_V} f(\eta)g(a^{-1}_{x,V}\eta),$$

hence (2.9). By choosing $g \equiv 1$, part 1 follows. ∎

Remark. This shows that μ_V is invariant under the Markov processes generated by (2.6) and (2.7).

2.7 Burning Algorithm

The *burning algorithm* determines whether a stable configuration $\eta \in \Omega_V$ is recurrent or not. It is described as follows: Pick $\eta \in \Omega_V$ and erase all sites $x \in V$ satisfying the inequality

$$\eta(x) > \sum_{y \in V, y \neq x} (-\Delta^V_{x,y}).$$

This means "erase the set E_1 of all sites $x \in V$ with a height strictly larger than the number of neighbors of that site in V". Iterate this procedure for the new volume $V \setminus E_1$, and the new matrix $\Delta^{V \setminus E_1}$ defined by

$$\begin{aligned}
\Delta^{V \setminus E_1}_{x,y} &= \Delta^V_{x,y} \text{ if } x,y \in V \setminus E_1 \\
&= 0 \text{ otherwise,}
\end{aligned}$$

and so on. If at the end some non-empty subset V_f is left, η satisfies, for all $x \in V_f$,

$$\eta(x) \leq \sum_{y \in V_f, y \neq x} (-\Delta^V_{x,y}).$$

The restriction $\eta|_{V_f}$ is called a *forbidden subconfiguration*. If V_f is empty, the configuration is called *allowed*, and the set $\mathcal{A}_V$ of allowed configurations satisfies

Proposition 2.3

$$\mathcal{A}_V = \mathcal{R}_V.$$

The main ingredient to prove this result (see [IP (1998)], [Speer (1993)]) is the fact that toppling or adding cannot create a forbidden subconfiguration. The set $\mathcal{A}_V$ is thus closed under the dynamics and contains the maximal configuration η_{max}.

Remark that the burning algorithm implies that for $V' \supset V$, and $\eta \in \Omega_V$, $\eta \notin \mathcal{R}_V$, then any $\zeta \in \Omega_{V'}$ such that $\zeta|_V = \eta$ satisfies $\zeta \notin \mathcal{R}_{V'}$. Indeed, the property of having a forbidden subconfiguration in V_f only depends on the heights at sites $x \in V_f$. Therefore $\eta \in \mathcal{R}_{V'}$ implies $\eta|_V \in \mathcal{R}_V$. This "consistency" property will enable us to define allowed configurations on infinite sets.

2.8 Expected Toppling Numbers

For $x, y \in V$ and $\eta \in \Omega_V$, let $n_V(x, y, \eta)$ denote the *number of topplings* at site $y \in V$ by adding a grain at $x \in V$, i.e. the number of times we have to apply the operator $T_{\Delta^V}(\cdot, y)$ to relax η^x. Define

$$G_V(x, y) = \int \mu_V(d\eta) \, n_V(x, y, \eta). \tag{2.10}$$

Writing down balance between inflow and outflow at site y, one obtains (cf. [Dhar (1990a)])

$$\sum_{z \in V} \Delta_{x,z}^V G_V(z, y) = \delta_{x,y},$$

which yields

$$G_V(x, y) = (\Delta^V)_{x,y}^{-1}.$$

In the limit $V \uparrow S$ (where S is $\mathbb{Z}^d$ or the infinite tree), G_V converges to the Green's function of the simple random walk on S.

2.9 Some specific results for the tree

When V_n is a binary tree of n generations, many explicit results have been obtained in [DM (1990b)]. We summarize here the results we need for the construction in infinite volume.

1. When adding a grain on a particular site $0 \in V_n$ of height 3, the set of toppled sites is the connected cluster $C_3(0, \eta)$ of sites including 0 having height 3. This cluster is distributed as a *random animal* (i.e. its distribution only depends on its cardinality, not on its form). Moreover

$$\lim_{n \uparrow \infty} \mu_{V_n} \left(|C_3(0, \eta)| = k \right) \simeq C k^{-3/2} \tag{2.11}$$

as k goes to infinity. The notation $\simeq$ means that if we multiply the left hand side of (2.11) by $k^{3/2}$, then the limit $k \to \infty$ is some strictly positive constant C.

2. When adding a grain on site x, the expected number of topplings at site y satisfies

$$\lim_{n \uparrow \infty} \int \mu_{V_n}(d\eta) \, n_{V_n}(x, y, \eta) = G(x, y), \tag{2.12}$$

where $G(x, y)$ is the Green's function of the simple random walk on the infinite tree, i.e.

$$G(0, x) = C2^{-|x|}, \qquad (2.13)$$

and $|x|$ is the "generation number" of x in the tree.

3. The correlations in the finite volume measures μ_{V_k} can be estimated in terms of the eigenvalues of a product of *transfer matrices*. This formalism is explained in detail in [DM (1990b)], section 5: Let f, g be two local functions whose dependence sets (see below a precise definition) are separated by n generations. To estimate the truncated correlation function

$$\mu_{V_k}(f; g) = \int fg \, d\mu_{V_k} - \int f \, d\mu_{V_k} \int g \, d\mu_{V_k}, \qquad (2.14)$$

consider the product of matrices

$$M_n^k = \prod_{i=1}^{n} \begin{pmatrix} 1 + \gamma_i^{k,n} & 1 + \gamma_i^{k,n} \\ 1 & 2 + \gamma_i^{k,n} \end{pmatrix}, \qquad (2.15)$$

where $\gamma_i^{k,n} \in [0, 1]$. Let $\lambda_m^{n,k}$ (resp. $\lambda_M^{n,k}$) denote the smallest (resp. largest) eigenvalue of M_n^k. Then

$$\mu_{V_k}(f; g) \le C(f, g) \frac{\lambda_m^{n,k}}{\lambda_M^{n,k}}. \qquad (2.16)$$

If f and g have dependence sets "deep within" V_k, then $\gamma_i^{k,n}$ is very close to one, and the correlations are governed by the maximal and minimal eigenvalues of $M_n = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}^n$.

3   Main results

3.1   Notation, definitions

From now on, S denotes the infinite rootless binary tree, $V \subset S$ a finite subset of S; Ω_V is the set of stable configurations in V, i.e. $\Omega_V = \{\eta : V \to \{1, 2, 3\}\}$, and the set of all infinite volume stable configurations is $\Omega = \{1, 2, 3\}^S$. The set Ω is endowed with the product topology, making it into a compact metric space. For $\eta \in \Omega$, $\eta|_V$ is its restriction to V, and for $\eta, \zeta \in \Omega$, $\eta_V \zeta_{V^c}$ denotes the configuration whose restriction to V (resp. V^c) coincides with $\eta|_V$ (resp. $\zeta|_{V^c}$). As in the previous section, $\mathcal{R}_V \subset \Omega_V$ is the set of all allowed (or recurrent) configurations in V, and we define

$$\mathcal{R} = \{\eta \in \Omega : \forall V \subset S \text{ finite}, \eta|_V \in \mathcal{R}_V\}. \qquad (3.17)$$

A function $f : \Omega \to \mathbb{R}$ is *local* if there is a finite $V \subset S$ such that $\eta|_V = \zeta|_V$ implies $f(\eta) = f(\zeta)$. The minimal (in the sense of set ordering) such V is called *dependence set* of f and is denoted by D_f. A local function can be seen as a function on Ω_V for all $V \supset D_f$ and every function on Ω_V can be seen as a local function on Ω. The set $\mathcal{L}$ of all local functions is uniformly dense in the set $C(\Omega)$ of all continuous functions on Ω.

All along the paper, we use the following notion of limit by inclusion for a function f on the finite subsets of the tree with values in a metric space (K, d):

Definition 3.1 *Let* $\mathcal{S} = \{V \subset S,\ V\ \text{finite}\}$, *and* $f : \mathcal{S} \to (K, d)$. *Then*

$$\lim_{V \uparrow S} f(V) = \kappa$$

if for all $\epsilon > 0$, *there exists* $V_0 \in \mathcal{S}$ *such that for all* $V \supset V_0$, $d(f(V), \kappa) < \epsilon$.

Definition 3.2 *A collection of probability measures* ν_V *on* Ω_V *is a Cauchy net if for any local* f *and for any* $\epsilon > 0$ *there exists* $V_0 \supset D_f$ *such that for any* $V, V' \supset V_0$

$$\left| \int f(\eta)\nu_V(d\eta) - \int f(\eta)\nu_{V'}(d\eta) \right| \leq \epsilon.$$

A Cauchy net converges to a probability measure ν in the following sense: The mapping

$$\Psi : \mathcal{L} \to \mathbb{R};\quad f \mapsto \Psi(f) = \lim_{V \uparrow S} \int f d\nu_V$$

defines a continuous linear functional on $\mathcal{L}$ (hence on $\mathcal{C}(\Omega)$) which satisfies $\Psi(f) \geq 0$ for $f \geq 0$ and $\Psi(1) = 1$. Thus by Riesz representation theorem there exists a unique probability measure on Ω such that $\Psi(f) = \int f d\nu$. We denote $\nu_V \to \nu$, and call this ν the *infinite volume limit* of ν_V.

We will also often consider an enumeration of the tree S, $\{x_0, x_1, \ldots, x_n, \ldots\}$, and put

$$T_n = \{x_0, \ldots, x_n\}. \tag{3.18}$$

3.2   Thermodynamic limit of stationary measures

Theorem 3.1 *The set* $\mathcal{R}$ *defined in (3.17) is an uncountable perfect set, i.e.*

1. *$\mathcal{R}$ is compact,*

2. *The interior of $\mathcal{R}$ is empty,*

3. *For all $\eta \in \mathcal{R}$ there exists a sequence $\eta_n \neq \eta$, $\eta_n \in \mathcal{R}$, converging to η.*

For $\eta \in \Omega$, we denote by $C_3(0, \eta)$ the nearest neighbor connected cluster of sites containing the origin and having height 3.

Theorem 3.2 *The finite volume stationary measures μ_V defined in (2.8) form a Cauchy net. Their infinite volume limit μ satisfies*

1. *$\mu(\mathcal{R}) = 1$,*

2. *μ is translation invariant and exponentially mixing,*

3. *$\mu\left(\eta : |C_3(0, \eta)| < \infty\right) = 1$,*

4. *$\int |C_3(0, \eta)| \mu(d\eta) = \infty$.*

Remark: Point 3. above remains true for the set $C_1(0, \eta)$, the nearest neighbor connected cluster of sites containing the origin and having height 1, and probably also for $C_2(0, \eta)$ but this we have not been able to prove.

3.3 Infinite volume dynamics

The finite volume addition operators $a_{x,V}$ (cf. (2.5)) can be extended to Ω via

$$a_{x,V} : \Omega \to \Omega; \eta \mapsto a_{x,V}\eta = (a_{x,V}\eta|_V)_V\eta_{V^c}. \tag{3.19}$$

Proposition 3.1

1. There exists a subset Ω' of $\mathcal{R}$ with $\mu(\Omega') = 1$ on which the limit

$$\lim_{V\uparrow S} a_{x,V}\eta = a_x\eta \tag{3.20}$$

   exists, and $a_x\eta \in \Omega'$.

2. The measure μ of theorem 3.2 is invariant under the action of a_x.

3. For every $\eta \in \Omega'$, $a_x(a_y\eta) = a_y(a_x\eta)$, for all $x,y \in S$.

Part 2 implies that the infinite volume addition operators a_x (cf. (3.20)) define norm 1 operators on $L^p(\mu)$, for $1 \le p \le \infty$ via

$$(a_x f)(\eta) = f(a_x\eta).$$

We now construct a Markov process on μ-typical infinite volume configurations which can be described intuitively as follows. Let $\varphi : S \to (0,\infty)$; this function will be the *addition rate function*. To each site $x \in S$ we associate a Poisson proces $N_\varphi^{t,x}$ (for different sites these Poisson processes are mutually independent) with rate $\varphi(x)$. At the event times of $N_\varphi^{t,x}$ we "add a grain" at x, i.e. we apply the addition operator a_x to the configuration. Then L_V^φ introduced in (2.7) generates a pure jump Markov process on Ω. Indeed, this operator is well-defined and bounded on any $L^p(\mu)$ space by Proposition 3.1, which implies

Proposition 3.2 L_V^φ is the $L^p(\mu)$ generator of the stationary Markov process defined by

$$\exp(tL_V^\varphi)f = \int \left(\prod_{x\in V} a_x^{N_\varphi^{t,x}} f \right) d\mathbb{P},$$

where $\mathbb{P}$ denotes the joint distribution of the independent Poisson processes $N_\varphi^{t,x}$, and $f \in L^p(\mu)$.

The following condition on the addition rate φ is crucial in our construction. Remember $|x|$ is the generation number of x:

Summability Condition: $\qquad\qquad \sum_{x\in S} \varphi(x)2^{-|x|} < \infty \qquad$ (3.21)

This condition ensures that the number of topplings at any site $x \in S$ remains finite during the addition process.

Theorem 3.3 If φ satisfies condition (3.21), then we have

1. The semigroups $S_V^\varphi(t) = \exp(tL_V^\varphi)$ converge strongly in $L^1(\mu)$ to a semigroup $S_\varphi(t)$.

2. $S_\varphi(t)$ is the $L^1(\mu)$ semigroup of a stationary Markov process $\{\eta_t : t \geq 0\}$ on Ω.

3. For any $f \in \mathcal{L}$,

$$\lim_{t \downarrow 0} \frac{S_\varphi(t)f - f}{t} = L^\varphi f = \sum_{x \in S} \varphi(x)[a_x f - f],$$

where the limit is taken in $L^1(\mu)$.

Remarks.
1. In Proposition 4.1, we show that $S_\varphi(t)$ is a strongly continuous function of φ.
2. In Proposition 4.2, we show that condition (3.21) is in some sense optimal.

Theorem 3.4 *The process $\{\eta_t : t \geq 0\}$ of Theorem 3.3 admits a cadlag version (right-continuous with left limits).*

The intuitive description of the process $\{\eta_t : t \geq 0\}$ is actually correct under condition (3.21), i.e. the process has a representation in terms of Poisson processes:

Theorem 3.5 *If φ satisfies condition (3.21), for $\mu \times \mathbb{P}$ almost every (η, ω) the limit*

$$\lim_{V \uparrow S} \prod_{x \in V} a_x^{N_\varphi^{t,x}(\omega)} \eta = \eta_t$$

exists. The process $\{\eta_t : t \geq 0\}$ is a version of the process of Theorem 3.3, i.e. its $L^1(\mu)$ semigroup coincides with $S_\varphi(t)$.

Finally, we can slightly generalize Theorem 3.5 in order to define the dynamics starting from a measure stochastically below μ. For $\eta, \zeta \in \Omega$, $\eta \leq \zeta$ if for all $x \in S$, $\eta(x) \leq \zeta(x)$. A function $f : \Omega \to \mathbb{R}$ is *monotone* if $\eta \leq \zeta$ implies $f(\eta) \leq f(\zeta)$. Two probability measures μ and ν satisfy $\mu \leq \nu$ if for all monotone functions, $\int f d\mu \leq \int f d\nu$.

Theorem 3.6 *Let $\mu' \leq \mu$. If φ satisfies condition (3.21), for $\mu' \times \mathbb{P}$ almost every (η, ω) the limit*

$$\lim_{V \uparrow S} \prod_{x \in V} a_x^{N_\varphi^{t,x}(\omega)} \eta = \eta_t$$

exists. The process $\{\eta_t : t \geq 0\}$ is Markovian with η_0 distributed according to μ'.

Remark. The last Theorem implies that $\eta \equiv 1$ can be taken as initial configuration.

4 Proofs

This section is devoted to the proofs of the results described above. Some of them will be put in a slightly more general framework so that they can be applied to other cases (where S is not a binary tree or where we have other addition

11

operators a_x) as soon as the existence of a thermodynamic limit of the finite volume stationary measures is guaranteed. The essential cause of difficulty is the non-locality of the addition operators. The essential simplification is the abelian property which enables us to think of the a_x as complex numbers of modulus one.

4.1 Thermodynamic limit of stationary measures

Proof of Theorem 3.1:

1,2. If $\eta \in \mathcal{R}$ and $\zeta \geq \eta$, then $\zeta \in \mathcal{R}$ (by the burning algorithm $\zeta|_V \geq \eta|_V$ implies that $\zeta|_V \in \mathcal{R}_V$). Since $\eta \equiv 2$ is in $\mathcal{R}$ (again by the burning algorithm), we conclude that $\mathcal{R}$ is uncountable. To see that $\mathcal{R}$ has empty interior, notice that if $\eta \in \mathcal{R}$, there does not exist $x, y \in S$ nearest neighbors such that $\eta(x) = \eta(y) = 1$ (that way, $\eta|_{\{x,y\}}$ would be a forbidden subconfiguration). Finally $\mathcal{R}$ is closed as intersection of closed sets.

3. Let η_{max} be the maximal configuration, $\eta_{max}(x) = 3$ for all $x \in S$. If $\eta|_V \in \mathcal{R}_V$, then $\eta_V(\eta_{max})_{V^c} \in \mathcal{R}$. Therefore any $\eta \in \mathcal{R}$ containing an infinite number of sites x for which $\eta(x) \neq 3$ has property 3 of Theorem 3.1. If $\eta \in \mathcal{R}$ contains only a finite number of sites having height 1 or 2, then we choose a sequence $\Sigma = \{x_n : n \in \mathbb{N}\} \subset \{x \in S : \eta(x) = 3$ and $\eta(y) = 3$ for any neighbor of $x\}$ such that two elements of Σ are never nearest neighbors, and $|x_n|$ is strictly increasing in n. We then define $\eta_n(x) = \eta(x)$ for $x \in S \setminus \{x_k \in \Sigma : 0 \leq k \leq n\}$ and $\eta_n(x_k) = 2$ for $x_k \in \Sigma$, $k \geq n + 1$. These η_n belong to $\mathcal{R}$ by the burning algorithm, and $\eta_n \to \eta$. ∎

Proof of Theorem 3.2:

We use T_n introduced in (3.18), but with the x_i such that $n \leq m$ implies that the generation numbers satisfy $|x_n| \leq |x_m|$. Then we have

$$|x_n| \simeq \log_2 n. \tag{4.22}$$

To prove that the probability measures μ_V form a Cauchy net, it is sufficient to show that for any local function $f : \Omega \to \mathbb{R}$ we have

$$\sum_n | \int f d\mu_{T_n} - f d\mu_{T_{n+1}}| < \infty. \tag{4.23}$$

We do it for $f(\eta) = \eta(x_0)$ (a general local function can be treated in the same way), by giving an upper bound of the difference $\int f d\mu_{T_n} - \int f d\mu_{T_{n+1}}$ by a truncated correlation function (cf. (2.14)). Then we estimate the latter by the transfer matrix method (cf. Section 2.9, part 3). We abbreviate in what follows $\mu_n = \mu_{T_n}$, $\mathcal{R}_n = \mathcal{R}_{T_n}$.

Lemma 4.1

$$|\mu_{n+1}[\eta(x_0)] - \mu_n[\eta(x_0)]| \leq C\mu_{n+1}[\eta(x_0); I(\eta(x_{n+1}) = 3)]$$

Proof: By the burning algorithm, every $\eta \in \mathcal{R}_n$ can be extended to an element of $\mathcal{R}_{n+1}$ by putting $\eta(x_{n+1}) = 3$. Moreover

$$\{\eta|_{T_n} : \eta \in \mathcal{R}_{n+1}, \ \eta(x_{n+1}) = 3\} = \mathcal{R}_n,$$

thus

$$\mu_{n+1}[\eta(x_{n+1}) = 3] = \frac{|\mathcal{R}_n|}{|\mathcal{R}_{n+1}|}, \tag{4.24}$$

which yields

$$
\begin{aligned}
\mu_n(\eta(x_0)) &= \sum_{\eta \in \mathcal{R}_n} \frac{1}{|\mathcal{R}_n|} \eta(x_0) \\
&= \sum_{\eta \in \mathcal{R}_{n+1}} \frac{1}{|\mathcal{R}_{n+1}|} \eta(x_0) I(\eta(x_{n+1}) = 3) \frac{|\mathcal{R}_{n+1}|}{|\mathcal{R}_n|} \\
&= \sum_{\eta \in \mathcal{R}_{n+1}} \frac{1}{|\mathcal{R}_{n+1}|} \eta(x_0) I(\eta(x_{n+1}) = 3) \frac{1}{\mu_{n+1}[\eta(x_{n+1}) = 3]}.
\end{aligned}
$$

Therefore

$$|\mu_{n+1}(\eta(x_0)) - \mu_n(\eta(x_0))| \leq \frac{\mu_{n+1}[\eta(x_0); I(\eta(x_{n+1}) = 3)]}{\mu_{n+1}[\eta(x_n + 1) = 3]}$$

The Lemma follows now from (4.24), and the fact that $|\mathcal{R}_n|$ grows like e^{cn} for some $c \geq \log 2$. ∎

Recalling Section 2.9, part 3, we have

$$\mu_n[\eta(x_0); I(\eta(x_n) = 3)] \leq C \frac{\lambda_m^{|x_n|,|x_n|}}{\lambda_M^{|x_n|,|x_n|}} \tag{4.25}$$

Lemma 4.2

$$\sum_{n=1}^{+\infty} \frac{\lambda_m^{|x_n|,|x_n|}}{\lambda_M^{|x_n|,|x_n|}} < +\infty$$

Proof: We abbreviate $\lambda_m^{(n)} = \lambda_m^{|x_n|,|x_n|}$, $\lambda_M^{(n)} = \lambda_M^{|x_n|,|x_n|}$, $M(n) = M_{|x_n|}^n$, $\gamma_i = \gamma_i^{|x_n|,|x_n|}$. Remember $0 \leq \gamma_i^{|x_n|,|x_n|} \leq 1$ is close to one for $i \ll n$ and n large. In terms of the trace and the determinant of $M(n)$ we have

$$
\begin{aligned}
\lambda_M^{(n)} &= \frac{1}{2}\left(\mathrm{Tr}(M(n)) + \sqrt{[\mathrm{Tr}(M(n))]^2 - 4\det(M(n))} \right) \\
\lambda_m^{(n)} &= \frac{1}{2}\left(\mathrm{Tr}(M(n)) - \sqrt{[\mathrm{Tr}(M(n))]^2 - 4\det(M(n))} \right).
\end{aligned}
$$

Therefore,

$$\lim_{n \uparrow \infty} \left(\frac{\lambda_m^{(n)}}{\lambda_M^{(n)}} \right) \left(\frac{[\mathrm{Tr}(M(n))]^2}{\det(M(n))} \right) = 1.$$

To prove the Lemma we show that (cf. (4.22))

$$\left(\frac{\det(M(n))}{[\mathrm{Tr}(M(n))]^2} \right) \leq \left(\frac{4}{9}\right)^{|x_n|}. \tag{4.26}$$

13

Use

$$\det(M(n)) = \prod_{i=1}^{|x_n|}(1 + \gamma_i)^2,$$

and

$$\mathrm{Tr}(M(n)) \geq \mathrm{Tr}\left(\prod_{i=1}^{|x_n|}\begin{pmatrix} 1 + \gamma_i & 0 \\ 0 & 2 + \gamma_i \end{pmatrix}\right)$$

$$= \prod_{i=1}^{|x_n|}(1 + \gamma_i) + \prod_{i=1}^{|x_n|}(2 + \gamma_i),$$

to estimate (for $1 \leq i \leq |x_n|$, $2(2 + \gamma_i) \geq 3(1 + \gamma_i)$)

$$\left(\frac{\det(M(n))}{[\mathrm{Tr}(M(n))]^2}\right) \leq \left(1 + 2\prod_{i=1}^{|x_n|}\left[\frac{2 + \gamma_i}{1 + \gamma_i}\right] + \prod_{i=1}^{|x_n|}\left[\frac{2 + \gamma_i}{1 + \gamma_i}\right]^2\right)^{-1}$$

$$\leq (1 + 2.(3/2)^{|x_n|} + (3/2)^{2|x_n|})^{-1}$$

$$\leq (4/9)^{|x_n|}.$$

For a general local function f, we have to replace $|x_n|$ by $|x_n| - N_0$, where N_0 is the number of generations involved in the dependence set of f. Since f is local, N_0 is finite, hence the convergence in (4.23) is unaffected. ■

4.2 Infinite volume toppling operators

Definition 4.1 *Given the finite volume addition operators $a_{x,V}$ (defined in (3.19)) acting on Ω, we call a configuration $\eta \in \Omega$ normal if for every $x \in S$ there exists a minimal finite set $V_x(\eta) \subset S$ such that for all $V' \supset V \supset V_x(\eta)$*

$$a_{x,V'}\eta = a_{x,V}\eta.$$

In other words, for a normal η, outside $V_x(\eta)$, no sites are affected by adding a grain at x. In our case, when a particle is added at some site $x \in S$, the cluster of toppled sites coincides with the cluster $C_3(x, \eta)$ of sites having height 3 including x, thus

$$V_x(\eta) = C_3(x, \eta) \cup \partial_e C_3(x, \eta), \tag{4.27}$$

where ∂_e denotes the exterior boundary. Notice that for a normal configuration η, by definition,

$$a_x(\eta) = \lim_{V \uparrow S} a_{x,V}(\eta) = a_{x,V_x(\eta)}(\eta) \tag{4.28}$$

Proof of Proposition 3.1:
1. We show that there is a full measure set Ω' of normal configurations. From (2.11) and Theorem 3.2,

$$\int I(|C_3(x, \eta)| = n)d\mu \simeq Cn^{-3/2}.$$

Therefore μ concentrates on the set Ω' of configurations for which all the clusters $C_3(x, \eta)$ are finite, hence for which η is normal. Moreover this set Ω' is closed under the action of the addition operators a_y, since (cf. (4.27))

$$C_3(x, a_y\eta) \subset V_x(\eta) \cup V_y(\eta). \tag{4.29}$$

2. Choose $\epsilon > 0$, pick a local function f, fix $V_n \uparrow S$ and n_0 such that $n \geq n_0$ implies

$$\mu\{\eta \in \Omega : V_x(\eta) \not\subset V_n\} \leq \frac{\epsilon}{4\|f\|_\infty + 1}. \tag{4.30}$$

This n_0 exists since μ concentrates on normal configurations. We estimate

$$
\begin{aligned}
\left| \int f(a_x\eta)d\mu - f(\eta)d\mu \right| &\leq \left| \int f(a_{x,V_n}\eta)d\mu - \int f(\eta)d\mu \right| \\
&+ 2\|f\|_\infty \mu\{\eta \in \Omega : V_x(\eta) \not\subset V_n\} \\
&\leq \lim_m \left| \int f(a_{x,V_n}\eta)d\mu_{V_m} - \int f(\eta)d\mu_{V_m} \right| + \frac{\epsilon}{2} \\
&\leq \frac{\epsilon}{2} + 2\|f\|_\infty \lim_m \mu_{V_m}\left(a_{x,V_n}(\eta) \neq a_{x,V_m}(\eta) \right) \\
&= \frac{\epsilon}{2} + 2\|f\|_\infty \left(1 - \lim_m \mu_{V_m}(V_x(\eta) \subset V_n) \right) \\
&= \frac{\epsilon}{2} + 2\|f\|_\infty (1 - \mu(V_x(\eta) \subset V_n)) \leq \epsilon.
\end{aligned}
$$

In the last step we used that the indicator $I(V_x(\eta) \subset V_n)$ is a local function.

3. Let $\eta \in \Omega'$, $x, y \in S$ be two different sites and $V \supset V_x(\eta) \cup V_x(a_x\eta) \cup V_x(a_y\eta)$. Since $a_{x,V}$ and $a_{y,V}$ commute, we have

$$
\begin{aligned}
a_x(a_y\eta) &= a_x(a_{y,V}\eta) = a_{x,V}(a_{y,V}\eta) \\
&= a_{y,V}(a_{x,V}\eta) = a_{y,V}(a_x\eta) = a_y(a_x\eta).
\end{aligned}
$$

∎

4.3 Infinite volume semigroup

We now turn to the proofs of Theorems 3.3 and 3.4.

Definition 4.2 *We define the cluster of $\eta \in \Omega$ at $x \in S$ as*

$$\mathcal{C}(x, \eta) = \{y \in S : a_y\eta(x) \neq \eta(x)\}, \tag{4.31}$$

and put

$$G_\mu(x, y) = \int I(y \in \mathcal{C}(x, \eta))d\mu(\eta). \tag{4.32}$$

Finally for $\varphi : S \to [0, \infty)$, write

$$\|f\|_\varphi = \sum_{x \in S} \varphi(x) \int \mu(d\eta)|f(a_x\eta) - f(\eta)|,$$

$$\mathcal{B}_\varphi = \{f : \Omega \to \mathbb{R} : f \text{ bounded}, \|f\|_\varphi < \infty\}.$$

15

Lemma 4.3 *If*

$$\sum_{x \in S} \varphi(x) G_\mu(y, x) < \infty \text{ for all } y \in S,$$ (4.33)

then all local functions are in $\mathcal{B}_\varphi$.

Proof: Let f be a local function with dependence set D_f. Then $f(a_x \eta) \neq f(\eta)$ if for $y \in D_f$, $a_x \eta(y) \neq \eta(y)$, i.e. $x \in C(y, \eta)$:

$$
\begin{aligned}
\|f\|_\varphi &= \sum_{x \in S} \varphi(x) \int |a_x f - f| d\mu \\
&= \int \sum_{x \in \cup_{y \in D_f} C(y,\eta)} \varphi(x) |a_x f - f| d\mu \\
&\leq 2\|f\|_\infty \sum_{x \in S} \varphi(x) \int I(x \in \cup_{y \in D_f} C(y, \eta)) d\mu \\
&\leq 2\|f\|_\infty \sum_{y \in D_f} \sum_{x \in S} G_\mu(y, x) \varphi(x) < \infty.
\end{aligned}
$$

∎

The next lemma provides a link between G_μ and the Green's function for simple random walk on S, i.e., between conditions (4.33) and (3.21).

Lemma 4.4

$$G_\mu(x, y) \leq \sum_{z \sim x} G(y, z) = \delta_{x,y} + 3G(x, y),$$

where $z \sim x$ means that z and x are neighbors.

Proof: We have to estimate the probability that $a_x \eta(y) \neq \eta(y)$. If by adding a grain at x we influence y, this can only be achieved by the toppling of one of the nearest neighbor sites of y. Since μ concentrates on normal configurations,

$$
\begin{aligned}
\mu(a_x \eta(y) \neq \eta(y)) &= \lim_{V \uparrow S} \mu(a_x \eta(y) \neq \eta(y), V_x(\eta) \cup V_y(\eta) \subset V) \\
&= \lim_{V \uparrow S} \lim_{W \uparrow S} \mu_W(a_{x,V} \eta(y) \neq \eta(y), V_x(\eta) \cup V_y(\eta) \subset V) \\
&= \lim_{V \uparrow S} \lim_{W \uparrow S} \mu_W(a_{x,W} \eta(y) \neq \eta(y), V_x(\eta) \cup V_y(\eta) \subset V) \\
&\leq \lim_{W \uparrow S} \mu_W(a_{x,W} \eta(y) \neq \eta(y)) \\
&\leq \lim_{W \uparrow S} \mu_W(\exists z \in W, z \sim y, n_W(z, y, \eta) \geq 1) \\
&\leq \lim_{W \uparrow S} \sum_{z \sim y} \int d\mu_W(\eta) n_W(z, y, \eta) \\
&= \sum_{z \sim y} G(z, y),
\end{aligned}
$$ (4.34)

where we used (2.10),(2.12), (4.27) and (4.29). ∎

The following Lemma finishes the proof of Theorem 3.3 and shows that $\mathcal{B}_\varphi$ is a natural core for the domain of the generator of the infinite volume semigroup.

16

Lemma 4.5

1. For $f \in \mathcal{B}_\varphi$ the net

$$S_V^\varphi(t)f = \exp(tL_V^\varphi)f = \exp\left(t\sum_{x \in V} \varphi(x)(a_x - I)\right) f \qquad (4.35)$$

converges in $L^1(\mu)$ (as $V \uparrow S$) to a function $S_\varphi(t)f \in L^1(\mu)$. $f \mapsto S_\varphi(t)f$ defines a semigroup on $\mathcal{B}_\varphi$ which is a contraction in both $L^1(\mu)$ and $\mathcal{B}_\varphi$ norms.

2. Under condition (4.33), the semigroup $S_\varphi(t)$ corresponds to a unique Markov process on Ω.

Proof: We denote by $\|f\|$ the $L^1(\mu)$-norm of f, and we abbreviate $S_V(t) = S_V^\varphi(t)$, $S(t) = S^\varphi(t)$, $L_V = L_V^\varphi$.

1. First note that $S_V(t)$ is well-defined on $L^1(\mu)$ by Proposition 3.2. By the abelian property (Proposition 3.1, part 3) we can write for $V \subset V' \subset S$:

$$\|S_V(t)f - S_{V'}(t)f\| = \|(S_{V'\setminus V}(t) - I)S_V(t)f\|$$

By Proposition 3.2, $S_V(t)$ is the semigroup of a stationary Markov process and hence a contraction on $L^1(\mu)$. Therefore

$$\|S_V(t)(S_{V'\setminus V}(t) - I)f\| \le \|(S_{V'\setminus V}(t) - I)f\|$$

$$= \|\int_0^t L_{V'\setminus V}S_{V'\setminus V}(s)f\,ds\|$$

$$\le \int_0^t \|L_{V'\setminus V}f\|\,ds$$

$$\le t\sum_{x \in V'\setminus V} \varphi(x)\int |(a_x - I)f|\,d\mu \to 0 \text{ as } V, V' \uparrow S, \qquad (4.36)$$

where the last step follows from $f \in \mathcal{B}_\varphi$. Hence $S_V(t)f \to S(t)f$ in $L^1(\mu)$. We show that $S(t)f \in \mathcal{B}_\varphi$:

$$\sum_{x \in S} \varphi(x)\int |S(t)f(a_x\eta) - S(t)f(\eta)|\mu(d\eta)$$

$$\le \sum_{x \in S} \varphi(x)\int S(t)|a_xf - f|\,d\mu$$

$$= \int \sum_{x \in S} \varphi(x)|a_xf - f|\,d\mu = \|f\|_\varphi.$$

Thus $S(t)$ is also a contraction for the $\|\cdot\|_\varphi$-norm. We finish with the semigroup property:

$$S(t)S(s)f = \lim_{V\uparrow S} S_V(t)[S(s)f]$$

$$= \lim_{V\uparrow S}\lim_{W\uparrow S} S_V(t)S_W(t)f,$$

and

$$S(t+s)f = \lim_{V \uparrow S} S_V(t)S_V(s)f.$$

Then, since $S_V(t)$ is a contraction in $L^1(\mu)$,

$$\|S_V(t)S_W(t)f - S_V(t)S_V(s)f\| \le \|S_W(s)f - S_V(s)f\|, \qquad (4.37)$$

By (4.36), the right hand side of (4.37) goes to zero as $V, W \uparrow S$.

2. If condition (4.33) is met, then $\mathcal{B}_\varphi$ contains all local functions by Lemma 4.3. Therefore, by contractivity the semigroup $S(t)$ on $\mathcal{B}_\varphi$ uniquely extends to a semigroup of contractions on $L^1(\mu)$. Since by Proposition 3.2, $S_V(t)$ is a Markov semigroup, so is $S(t)$, i.e. $S(t)1 = 1$, $S(t)f \ge 0$ if $f \ge 0$. Hence, by Kolmogorov's theorem there is a unique Markov process with semigroup $S(t)$. ∎

Remark. When $\varphi \equiv 1$, condition (4.33) is equivalent to

$$\sum_{x \in S} \int \mu(d\eta) I(x \in \mathcal{C}(y,\eta)) = \int |\mathcal{C}(y,\eta)| \mu(d\eta) < +\infty,$$

i.e., the clusters must be integrable under μ. For models which exhibit "self-organized criticality", $\mathcal{C}(y,\eta)$ is usually a "finite but critical percolation cluster", implying that $\int |\mathcal{C}(y,\eta)| d\mu = \infty$ (cf. Theorem 3.2 part 4, because $\mathcal{C}(y,\eta) \supset \partial_e \mathcal{C}_3(y,\eta)$). Therefore this formalism breaks down for addition rate $\varphi \equiv 1$.

The following Lemma proves Theorem 3.4.

Lemma 4.6 *Under condition (4.33), the process $\{\eta_t : t \ge 0\}$ of Theorem 3.3 is almost surely right-continuous, i.e.*

$$\mathbb{P}_\mu \left[\lim_{t \downarrow 0} d(\eta_t, \eta_0) \ge \epsilon \right] = 0, \qquad (4.38)$$

where $\mathbb{P}_\mu$ is its path-space measure, and the distance d is defined below (in (4.41)).

Proof: Pick a function $\Psi : S \to (0,1)$ such that

$$\sum_{x \in S} \Psi(x) = 1, \qquad (4.39)$$

and

$$\sum_{x,y \in S} \varphi(x) G_\mu(x,y) \Psi(y) < \infty \qquad (4.40)$$

The distance

$$d(\eta, \zeta) = \sum_{x \in S} |\eta(x) - \zeta(x)| \Psi(x) \qquad (4.41)$$

generates the product topology. Denote by $\mathbb{E}_\mu$ the expectation w.r.t. $\mathbb{P}_\mu$. For $f_y(\eta) = \eta(y)$,

$$f_y(\eta_t) - f_y(\eta_0) = \int_0^t L^\varphi f_y(\eta_s) ds + M_t^y,$$

where M_t^y is a centered martingale with quadratic variation

$$\mathbb{E}_\mu \left[(M_t^y)^2 \right] = \mathbb{E}_\mu \left[\int_0^t (L^\varphi f_y^2(\eta_s) - 2f_y(\eta_s) L^\varphi f_y(\eta_s)) ds \right]. \qquad (4.42)$$

Using stationarity of η_s and

$$\int d\mu |L^\varphi g| \leq 2\|g\|_\infty \sum_{x \in S} \sum_{y \in D_g} \varphi(x) G_\mu(y, x),$$

for a local bounded function on Ω, we obtain from (4.42)

$$\mathbb{E}_\mu \left[(M_t^y)^2 \right] \leq Ct \sum_{x \in S} \varphi(x) G_\mu(y, x).$$

Now we can estimate

$$\mathbb{P}_\mu \left[\exists s \leq t : \sum_{y \in S} |\eta_s(y) - \eta_0(y)| \Psi(y) \geq \epsilon \right]$$

$$\leq \quad \mathbb{P}_\mu \left[\int_0^t ds \sum_{y \in S} |L^\varphi f_y(\eta_s)| \Psi(y) \geq \epsilon/2 \right] + \mathbb{P}_\mu \left[\sup_{0 \leq s \leq t} \left| \sum_{y \in S} M_s^y \Psi(y) \right| \geq \epsilon/2 \right]$$

$$\leq \quad (12t/\epsilon) \sum_{x,y \in S} \varphi(x) G_\mu(y, x) \Psi(y) + (2/\epsilon)^2 \mathbb{E}_\mu \left| \sum_{y \in S} M_t^y \Psi(y) \right|^2$$

$$\leq \quad (12t/\epsilon) \sum_{x,y \in S} \varphi(x) G_\mu(y, x) \Psi(y) + (2/\epsilon)^2 \mathbb{E}_\mu \left[\sum_{y \in S} (M_t^y)^2 \Psi(y) \right]$$

$$\leq \quad tC_\epsilon \sum_{x,y \in S} \varphi(x) G_\mu(y, x) \Psi(y).$$

Here we used Markov's and Doob's inequalities in the second step and the Cauchy-Schwarz inequality combined with (4.39) in the third step. The result (4.38) follows. ∎

4.4 Poisson representation

In this section we prove Theorems 3.5 and 3.6. Intuitively it is clear from the abelian property that the process of which we showed existence in the previous subsection can be represented as $\prod_{x \in S} a_x^{N_\varphi^{t,x}} \eta$, where $N_\varphi^{t,x}$ are independent Poisson processes of intensity $\varphi(x)$.

We take T_n as in (3.18). We say that the product $\prod_{x \in S} a_x^{n_x} \eta$ exists if for every $y \in S$ there exists N_y such that for all $m, n \geq N_y$

$$\left| \left[\prod_{x \in T_n} a_x^{n_x} \eta \right] (y) - \left[\prod_{x \in T_m} a_x^{n_x} \eta \right] (y) \right| = 0.$$

This is equivalent to the convergence of the sequence $\prod_{x \in T_n} a_x^{n_x} \eta$ in the product topology.

Lemma 4.7 *Under condition (4.33), the product*

$$\prod_{x \in S} a_x^{N_\varphi^{t,x}} \eta = \bar{\eta}_t$$

19

exists for μ-almost every realization of $N_\varphi^{t,x}$ and almost every η. The process $\{\tilde{\eta}_t : t \geq 0\}$ is a version of the Markov process of Lemma 4.5.

Proof: Choose a realization of $N_\varphi^{t,x}$ such that

$$\sum_{x \in S} N_\varphi^{t,x} G_\mu(x,y) < \infty \tag{4.43}$$

for every y. This happens with probability one by condition (4.33). Define for $\eta \in \Omega'$

$$\eta_{T_n}(t) = \prod_{x \in T_n} a_x^{N_\varphi^{t,x}} \eta. \tag{4.44}$$

Under μ, $\eta_{T_n}(t)$ is stationary in n and t. We have

$$\mu \left[|(\eta_{T_n}(t))(y) - (\eta_{T_{n+1}}(t))(y)| \geq 1 \right]$$

$$\leq \int \left| \left[a_{x_{n+1}}^{N_\varphi^{t,x_{n+1}}} \eta_{T_n}(t) \right](y) - (\eta_{T_n}(t))(y) \right| \mu(d\eta)$$

$$= \int \left| \left[a_{x_{n+1}}^{N_\varphi^{t,x_{n+1}}} \eta \right](y) - \eta(y) \right| \mu(d\eta)$$

$$\leq \int \sum_{j=1}^{N_\varphi^{t,x_{n+1}}} \left| \left[a_{x_{n+1}}^j \eta \right](y) - \left[a_{x_{n+1}}^{j-1} \eta \right](y) \right| \mu(d\eta)$$

$$\leq 6 N_\varphi^{t,x_{n+1}} G_\mu(x_{n+1}, y).$$

In the second and last steps we used the invariance of μ under a_x. By the Borel Cantelli Lemma, (4.43) implies that for almost every realization of $N_\varphi^{t,x}$

$$\mu \left[\exists n_0 : \forall n \geq n_0 \ (\eta_{T_n}(t))(y) = (\eta_{T_{n_0}}(t))(y) \right] = 1.$$

This proves μ-a.s. convergence of the product. To see that $\tilde{\eta}_t$ is a version of the Markov process with semigroup $S_\varphi(t)$, combine Proposition 3.2 with Theorem 3.3, part 1 to get, for any local function f,

$$\int d\mu \left| \int d\mathbb{P} f(\tilde{\eta}_t) - S_\varphi(t) f \right| = 0.$$

In the preceding argument we used a particular enumeration of the countable set S. But changing it gives again a process with semigroup $S_\varphi(t)$. Therefore the limiting process will not depend (up to sets of measure zero) on the chosen enumeration of S. ∎

Proof of Theorem 3.6:
For $\eta \in \Omega'$ and $y \in S$ we have the relation (remember (4.44))

$$\eta_V(t)(y) = \eta(y) + I(y \in V) N_\varphi^{t,x} - \Delta n_V^t(y), \tag{4.45}$$

where $n_V^t(x)$, an integer valued random variable, is the number of topplings at site x in the time interval $[0, t]$, when sand is added in V. For T_n defined in (3.18) we will first prove that $n_{T_n}^t$ increases $\mu \times \mathbb{P}$ almost surely to an $L^1(\mu \times \mathbb{P})$ random variable n^t, interpreted as the number of topplings in $[0, t]$ when we add grains

20

according to $N_\varphi^{t,x}$. By the abelian property the sequence $n_{T_k}^t(0)$ is increasing in k. The following estimate is similar to (4.34)

$$(\mu \times \mathbb{P}) \ (\quad |n_{T_k}^t(0) - n_{T_{k+1}}^t(0)| \geq \epsilon) = (\mu \times \mathbb{P}) \left(n_{x_{k+1}}^t(0) \geq \epsilon \right)$$

$$\leq \ \frac{1}{\epsilon} \int n_{x_{k+1}}^t(0)\mu(d\eta) \times \mathbb{P}(d\omega)$$

$$\leq \ \frac{1}{\epsilon} \lim_{V \uparrow S} \int n_{x_{k+1}}^t(0) I(V_{x_{k+1}}(\eta) \cup V_0(\eta) \subset V)\mu(d\eta) \times \mathbb{P}(d\omega)$$

$$\leq \ \frac{1}{\epsilon} \lim_{W \uparrow S} \int n_W^t(x_{k+1}, 0, \eta)\mu_W(d\eta) \times \mathbb{P}(d\omega)$$

$$\leq \ \frac{1}{\epsilon} t\varphi(x_{k+1})G(0, x_{k+1}). \tag{4.46}$$

In the fourth line, $n_W^t(x_{k+1}, 0, \eta)$ denotes the number of topplings up to time t at site $0 \in W$ by adding grains at site $x_{k+1} \in W$. By the Borel Cantelli Lemma, condition (3.21) implies the a.s. convergence of $n_{T_k}^t(0)$, and analogously of every $n_{T_k}^t(x)$. Pick (η, ω) such that $n_{T_k}^t(\eta, \omega)$ converges, i.e. such that $\sup_k n_{T_k}^t(\eta, \omega) = n^t(\eta, \omega)$ is finite (indeed, $n_{T_k}^t(\eta, \omega)$ is an integer). If $\eta' \leq \eta$, then $n_{T_k}^t(\eta', \omega) \leq n_{T_k}^t(\eta, \omega)$ because we can obtain η from η' by adding sand at sites $x \in S$ for which $\eta'(x) < \eta(x)$ thereby increasing the number of topplings. We thus conclude that the convergence of $n_{T_k}^t(\eta, \omega)$ implies the convergence of $n_{T_k}^t(\eta', \omega)$ for all $\eta' \leq \eta$.

Now let $\mu' \leq \mu$ in the FKG sense. There is a coupling $\mathbb{P}_{12}$ of $\mu' \times \mathbb{P}$ and $\mu \times \mathbb{P}$ such that

$$\mathbb{P}_{12}\left(((\eta_1, \omega_1), (\eta_2, \omega_2)) : \omega_1 = \omega_2, \eta_1 \leq \eta_2\right) = 1,$$

i.e. we use the same Poisson events and couple μ' and μ according to the optimal coupling (see [Strassen (1965)]). Then

$$(\mu' \times \mathbb{P})\left(n_{T_k}^t(\eta, \omega) \to n^t(\eta, \omega)\right) = \mathbb{P}_{12}\left(n_{T_k}^t(\eta_1, \omega_1) \to n^t(\eta_1, \omega_1)\right)$$

$$= \ \mathbb{P}_{12}\left(n_{T_k}^t(\eta_1, \omega_1) \to n^t(\eta_1, \omega_1), n_{T_k}^t(\eta_2, \omega_2) \to n^t(\eta_2, \omega_2), \omega_1 = \omega_2, \eta_1 \leq \eta_2\right)$$

$$\geq \ \mathbb{P}_{12}\left(n_{T_k}^t(\eta_2, \omega_2) \to n^t(\eta_2, \omega_2)\right)$$

$$= \ (\mu \times \mathbb{P})\left(n_{T_k}^t(\eta, \omega) \to n^t(\eta, \omega)\right) = 1.$$

This shows the $\mu' \times \mathbb{P}$-almost sure convergence of $n_{T_k}^t$, hence by (4.45) the product $\prod_{x \in S} a_x^{N_\varphi^{t,x}(\omega)}\eta'$ converges $\mu' \times \mathbb{P}$ almost surely. $\blacksquare$

As a further result we show that the semigroup $S_\varphi(t)$ is continuous as a function of the addition rate φ. We define

$$\ell_1 = \{\varphi : S \to [0, \infty) : \|\varphi\| = \sum_{x \in S} \varphi(x)G(0, x) < \infty\}.$$

It is a complete metric space (as a closed subset of a Banach space) with the property: If $\varphi_n \in \ell_1$, $\varphi_n \uparrow \varphi$ (pointwise), and $\varphi \in \ell_1$ then $\varphi_n \to \varphi$ in ℓ_1.

Proposition 4.1 *The semigroup $S_\varphi(t)$ of Theorem 3.3 is a strongly continuous function of φ, i.e. if $\varphi_n \to \varphi$ in ℓ_1, then for any local function f, $S_{\varphi_n}(t)f \to S_\varphi(t)f$.*

Proof: Let $n_\varphi^t = \lim_{k \to \infty} n_{T_k}^t$ be the number of topplings in $[0, t]$ from sand addition at rate φ. In the proof of Theorem 3.5 we have shown that this random variable is $\mu \times \mathbb{P}$ almost surely well defined and, after taking limits in (4.45), satisfies

$$\eta_t = \eta_0 + N_\varphi^t - \Delta n_\varphi^t, \qquad (4.47)$$

where $N_\varphi^t = \lim_{V \uparrow S} \sum_{x \in V} N_\varphi^{t,x}$. Note that if $\psi_1 \leq \psi_2$, the Poisson processes $N_{\psi_1}^t$ and $N_{\psi_2}^t$ can be coupled in such a way that for all $x \in S$, $N_{\psi_1}^{t,x} \leq N_{\psi_2}^{t,x}$, and hence, by the abelian property, $n_{\psi_1}^t(x) \leq n_{\psi_2}^t(x)$. Consider a coupling of the four Poisson processes N_φ^t, $N_{\varphi \wedge \varphi_n}^t$, $N_{\varphi \vee \varphi_n}^t$ and $N_{\varphi_n}^t$ under which the inequalities $X_1(t) \geq X_2(t)$, $X_2(t) \leq X_3(t)$, $X_3(t) \geq X_4(t)$, are satisfied with probability one. Let $\tilde{\mathbb{P}}$ denote the law of the marginal (X_1, X_4). We have, by a reasoning similar to (4.46),

$$
\begin{aligned}
\int d\mu \left(\tilde{\mathbb{E}} |n_{\varphi_n}^t(0) - n_\varphi^t(0)| \right) &\leq \int d\mu \left(\tilde{\mathbb{E}} \left(n_{\varphi_n}^t(0) - n_{\varphi_n \wedge \varphi}^t(0) \right) \right. \\
&\quad + \tilde{\mathbb{E}} \left(n_{\varphi \vee \varphi_n}^t(0) - n_{\varphi \wedge \varphi_n}^t(0) \right) \\
&\quad + \left. \tilde{\mathbb{E}} \left(n_{\varphi \vee \varphi_n}^t(0) - n_\varphi^t(0) \right) \right) \\
&\leq t \sum_{x \in S} |\varphi_n(x) - \varphi(x)| G(0, x), \qquad (4.48)
\end{aligned}
$$

which tends to zero for $\varphi_n \to \varphi$ in ℓ_1. Take now a local function f, and denote $\tilde{D}_f = D_f \cup \partial_e D_f$,

$$
\begin{aligned}
|S_{\varphi_n}(t)(f) - S_\varphi(t)(f)| &\leq \tilde{\mathbb{P}} \left(n_{\varphi_n}^t(x) \neq n_\varphi^t(x) \text{ for some } x \in \tilde{D}_f \right) \\
&\leq \sum_{x \in \tilde{D}_f} \tilde{\mathbb{E}} |n_\varphi^t(x) - n_{\varphi_n}^t(x)|.
\end{aligned}
$$

Combining this with (4.48) concludes the proof. ∎

One might ask whether we can go beyond condition (3.21), which essentially guarantees that the expected number of topplings stays finite in the addition process. In the following proposition we show that it is impossible to keep integrable toppling numbers and "rate 1" addition. The relation (4.49) should be regarded as the infinitesimal version of (4.47), where $\alpha(x)$ replaces the rate $\varphi(x)$. We then show that φ has to depend on x.

Proposition 4.2 *Let $\alpha : S \to \{0, 1\}$ be a stationary and ergodic random field distributed according to ν. Denote by $\int \alpha(0)\nu(d\eta) = \rho$ its density. Suppose there exists a measurable transformation $T : \{0, 1\}^S \times \Omega \to \Omega$ which satisfies the conditions*

1. The measure μ of Theorem 3.2 is invariant under $T(\alpha, \cdot)$ for any α.

2.

$$T(\alpha, \eta)(x) = \eta(x) + \alpha(x) - \Delta n(\alpha, \eta, x), \qquad (4.49)$$

with $n(\alpha, \eta, .) \in L^1(\mu)$ for ν almost every α.

Then, $\rho = 0$.

Proof: Taking expectation over μ in (4.49) gives

$$\Delta\Psi(\alpha,x) = \alpha(x), \qquad (4.50)$$

where $\Psi(\alpha,x) = \int n(\alpha,\eta,x)\mu(d\eta)$. By stationarity of μ and ν, $\Psi(\alpha,x)$ is a stationary random field. Let $(x_t : t \geq 0)$ denote continuous time simple random walk on S, starting at 0. From (4.50),

$$\mathbb{E}\Psi(\alpha,x_t) = \Psi(\alpha,0) + \mathbb{E}\int_0^t \alpha(x_s)ds.$$

Divide this last line by t and let $t \uparrow +\infty$. As ν is ergodic (making the last term equal to ρ) and as the process $\Psi(\alpha,x_t)$ is stationary, we conclude that $\rho = 0$. ∎

References

[BTW (1988)] Bak, P., Tang, K. and Wiesenfeld, K., *Self-Organized Criticality*, Phys. Rev. A **38**, 364–374 (1988).

[Chung (1960)] Chung, K.L., *Markov Chains with Stationary Transition Probabilities*, Springer-Verlag, Berlin-Göttingen, Heidelberg, 1960.

[DR (1989)] Dhar, D. and Ramaswamy, R., *Exactly Solved Model of Self-Organized Critical Phenomena*, Phys. Rev. Lett. **63**, 1659–1662 (1989).

[Dhar (1990a)] Dhar, D., *Self Organised Critical State of Sandpile Automaton Models*, Phys. Rev. Lett. **64**, No.14, 1613–1616 (1990).

[DM (1990b)] Dhar, D. and Majumdar, S.N., *Abelian Sandpile Models on the Bethe Lattice*, J. Phys. A **23**, 4333–4350 (1990).

[Dhar (1999)] Dhar, D., *The Abelian Sandpiles and Related Models*, Physica A **263**, 4–25 (1999).

[IP (1998)] E.V. Ivashkevich, Priezzhev, V.B., *Introduction to the sandpile model*, Physica A **254**, 97–116 (1998).

[Liggett (1980)] Liggett, T.M., *The Long Range Exclusion Process*, Ann. Prob. **8**, 861–889 (1980).

[Liggett (1985)] Liggett, T.M., *Interacting Particle Systems*, Springer, Berlin, Heidelberg, New York, 1985.

[MRSV (2000)] Maes, C., Redig, F., Saada E. and Van Moffaert, A., *On the thermodynamic limit for a one-dimensional sandpile process*, Markov Proc. Rel. Fields, 6, 1–22 (2000).

[Chen (1992)] Mu Fa Chen, *From Markov Chains to Non-Equilibrium Particle Systems*, World Scientific, 1992.

[Priezzhev (1994)] Priezzhev, V.B., *Structure of Two Dimensional Sandpile. I.Height Probabilities*, J. Stat. Phys. **74**, 955–979 (1994).

23

[Speer (1993)] Speer, E., *Asymmetric Abelian Sandpile Models*, J. Stat. Phys. **71**, 61–74 (1993).

[Strassen (1965)] Strassen, V., *The existence of probability measures with given marginals*, Ann. Math. Statist. **36**, 423–439 (1965).

[Toom (1990)] Toom, A.L., Vasilyev, N.B., Stavskaya, O.N., Mityushin, L.G., Kurdyumov, G.L., Pirogov, S.A., *Discrete local Markov Systems*. In: Dobrushin, R.L., Kryukov V.I., Toom, A.L. (eds.), *Stochastic cellular systems: ergodicity, memory, morphogenesis*, Manchester University Press, 1–182, 1990.

[Turcotte (1999)] Turcotte, D.L., *Self-Organized Criticality*, Rep. Prog. Phys. **62**, 1377–1429 (1999).