Recent results on Coxeter groups

Recent Results on Coxeter Groups

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Abstract

The last few years have been good for the knowledge of Coxeter groups: the conjugacy problem has been solved, Coxeter groups have been shown to be automatic, and the structure of subgroups has been further exploited. In these notes, we survey some of these results, thus providing a sequel to three earlier ASI lectures on Coxeter groups.

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1. Basic definitions

Although this paper updates (Cohen [1991]), we facilitate its reading by repeating some of the basic definitions. The "classical" reference for Coxeter groups is (Bourbaki [1968]). Besides that, the recent introductions in (Humphreys [1990]; Scharlau [1993]) are recommended.

1.1. *Coxeter groups*

A *Coxeter matrix* of rank n is an $n \times n$ matrix $M = (m_{i,j})_{1\leq i,j\leq n}$ with $m_{i,i} = 1$ and $m_{i,j} = m_{j,i} > 1$ (possibly ∞) for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. The *Coxeter group* associated with the Coxeter matrix M is the group generated by a set R of elements ρ_i ($i = 1, \ldots, n$) subject only to the relations

$$(\rho_i\rho_j)^{m_{i,j}} = 1.$$

It is denoted by $W(M)$ or just W. Since $\rho_i \neq \rho_j$ for $i \neq j$, the two sets I and R will often be identified by means of the map $i \mapsto \rho_i$. The pair (W, R) is called a *Coxeter system* of type M. The number n is called the *rank* of the system. The rank will be assumed finite throughout these notes.

It is common practice to provide a pictorial presentation of M by means of the labeled graph (I, M) with vertex set I; the pair $\{i, j\}$ is an edge whenever $m_{i,j} > 2$ and this edge is labeled $m_{i,j}$. If $m_{i,j} = 3$, the label is often omitted.

1.2. *Chamber systems*

Let (W, R) be a Coxeter system of type M. The free monoid on the alphabet R with unit is denoted by R^*, and $\rho : R^* \to W(R)$ stands for the monoid morphism determined by $\rho(r) = r$ $(r \in R)$. A typical element $\mathbf{r} \in R^*$ of length q will be written $[r_1, \ldots, r_q]$ to distinguish it from its image $\rho(\mathbf{r}) = r_1 \cdots r_q$ in W. We shall write $\ell(\mathbf{r})$ to indicate the length q of $\mathbf{r}$. In particular, $[\,]$ denotes the unit of R^*.

The *length* of an element $w \in W$, denoted by $\ell_R(w)$ or just $\ell(w)$ if no confusion is imminent, is $\min\{\ell(\mathbf{r}) \mid \rho(\mathbf{r}) = w\}$. Each element $\mathbf{r} \in R^*$ with $\rho(\mathbf{r}) = w$, is called an *expression* for w; if $\ell(\mathbf{r}) = \ell(w)$, the expression is called *reduced*.

The *chamber system associated with* (W, R), denoted by $\mathcal{C}$, is the labeled graph whose vertex set is W and in which the edges labeled r (for $r \in R$) are all $\{w, wr\}$ for $w \in W$. Note that the edges are undirected as $r^2 = 1$ for all $r \in R$. The (label-preserving) automorphism group of $\mathcal{C}$ contains W via left multiplication, and is actually equal to W. The graph-theoretic distance between the 'chambers' $w, w' \in W$ of $\mathcal{C}$, denoted by $d_{\mathcal{C}}(w, w')$, equals $\ell_R(w^{-1}w')$.

Given $\mathbf{r} = [r_1, \ldots, r_q] \in R^*$ and $c \in \mathcal{C}$, there is a unique path in the graph starting at c along edges with subsequent labels $r_1, \ldots, r_q$. It runs through cr_1, cr_1r_2, $\ldots$ and ends at $c\rho(\mathbf{r})$; it is denoted by $c \cdot \mathbf{r}$. Thus, if $w = \rho(\mathbf{r})$, the chamber wc is the image of $c \in \mathcal{C}$ under $w \in W$ in the isometric regular action of W on $\mathcal{C}$ from the left, and the image cw of c under w (in the regular action on W from the right) is the end point of the path $c \cdot \mathbf{r}$ in $\mathcal{C}$.

The chamber system can be given the structure of a combinatorial cell complex by letting its cells be the empty set and all subsets of the form $c\langle J\rangle := \{cw \mid w \in \langle J\rangle\}$, where J runs through all subsets of R and $c \in \mathcal{C}$. Instead of $\langle J\rangle$, we also write W_J. Observe that cW_J is a coset of W_J in W.

The cell complex construction works because of some beautiful properties of Coxeter groups, the most remarkable of which are

- for each subset J of R, the pair $(\langle J\rangle, J)$ is again a Coxeter system; its type is $M\,|_{J\times J}$, and $\ell_J(w) = \ell_R(w)$ for each $w \in \langle J\rangle$;
- for every pair J, K of subsets of R and every pair x, y of elements of W, the intersection $xW_J \cap yW_K$ is either empty or of the form $zW_{J\cap K}$, for some $z \in W$.

By leaving out all cells with infinitely many vertices, one arrives at a topological description of the metric cell complex that will be described in §2. Obviously, the cell cW_J has finitely many vertices if and only if $\langle J\rangle$ is a finite group.

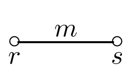
1.3. *Examples*

For the time being, let M be a Coxeter matrix of rank n, and let (W, R) be a Coxeter system of type M, so that $n = |R|$.

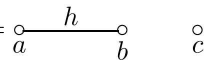
(o) If $n = 0$, then W is generated by the empty set, and so W is the trivial group.

(i) If $n = 1$ then $M = (1)$ and $W = \{1\} \cup R \cong \mathbf{Z}/(2)$, the group of order 2.

(ii) If $n = 2$, then

$$M = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix} =$$ 

for some $m \in \mathbf{N} \cup \{\infty\}$, and $W(M) = \langle r, s \mid r^2 = s^2 = (r\,s)^m = 1\rangle$ is the dihedral group of order $2m$, where, if $m = \infty$, the relation $(r\,s)^m = 1$ is void. If $m = 2h$ with $h \in \mathbf{N}$ odd, then

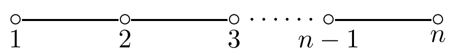
$M' =$ 

satisfies $W(M') \cong W(M)$, so the Coxeter diagram of a Coxeter group is not uniquely determined by the group.

(iii) If $M = A_n$, where

$$A_n = \begin{pmatrix} 1 & 3 & 2 & \cdots & \cdots & 2 \\ 3 & 1 & 3 & 2 & \cdots & 2 \\ 2 & 3 & 1 & 3 & \cdots & 2 \\ \vdots & 2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1 & 3 \\ 2 & \cdots & \cdots & 2 & 3 & 1 \end{pmatrix},$$

or, in diagram form

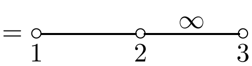
$A_n =$ 

then $W(M)$ is the symmetric group Sym_{n+1} on $n+1$ letters. The evident morphism $W(A_n) \to \mathrm{Sym}_{n+1}$ sending ρ_i to $(i, i+1)$ for each $i \in I$ is in fact an isomorphism.

(iv) Take

$$\rho_1 = \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \pm\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho_3 = \pm\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{in } PGL(2, \mathbf{Z}).$$

Then, putting $R = \{\rho_1, \rho_2, \rho_3\}$ and $W = \langle R\rangle$, we obtain a Coxeter system (W, R) of type

$$M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & \infty \\ 2 & \infty & 1 \end{pmatrix} =$$ 

so that $W(M) \cong PGL(2, \mathbf{Z})$.

(v) If $R = R_1 \cup R_2$ is a partition of (R, M) into disjoint graphs (here disjoint means: $m_{i,j} = 2$ whenever $i \in R_1$ and $j \in R_2$), then $W(M) = W(M_1) \times W(M_2)$, where M_k is the restriction of M to $R_k \times R_k$ $(k = 1, 2)$. This explains why, in addressing many questions concerning Coxeter groups, we can restrict to the case where M is connected.

1.4. *The Reflection Representation*

Consider a Coxeter system (W, R) of type $M = (m_{rs})_{r,s\in R}$. Let V be a real vector space with basis $(e_r)_{r\in R}$. There is a real linear representation $\sigma : W \to GL(V)$ such that, for $r \in R$, the transformation $\sigma(r)$ is a reflection. We shall give a more specific description of this important representation.

Denote by B the symmetric bilinear form on V defined by

$$B(e_r, e_s) = -2\cos(\pi/m_{rs}) \quad (r, s \in R)$$

with the understanding that $B(e_r, e_s) = -2$ if $m_{rs} = \infty$. We call B *the symmetric bilinear form associated with* M. For each $r \in R$, consider the linear transformation σ_r of V defined by

$$\sigma_r(x) = x - B(x, e_r)e_r \qquad (x \in V)$$

This defines a reflection of $GL(V)$ in the hyperplane $e_r^\perp$ with *root* (i.e., eigenvector with eigenvalue -1) e_r.

A representation is called faithful if it is an injective morphism of groups. Faithfulness of the reflection representation is due to Tits, cf. (Bourbaki [1968]).

1.5. Proposition. *Let B be the symmetric bilinear form associated with the Coxeter matrix M, and let σ_r $(r \in R)$ be as above. Then the mapping σ on R given by $\sigma(r) = \sigma_r$ extends to a faithful orthogonal representation $\sigma : W \to O(V, B)$.*

1.6. *The Tits cone*

Tits' proof that the above representation σ is faithful makes use of the representation of W on V^*, the dual of V. The action σ^* of W on V^* is induced from its action on V. Explicitly, for $w \in W$, the image $\sigma^*(w)f$ of $f \in V^*$ is given by

$$(\sigma^*(w)f)x = f(\sigma(w^{-1})x) \qquad \text{for } x \in V.$$

Let $f_1, \ldots, f_n$ be the basis of V^* that is dual to $e_1, \ldots, e_n$. Then

$$A = \mathbf{R}_{\geq 0} f_1 + \cdots + \mathbf{R}_{\geq 0} f_n$$

is called the *fundamental chamber* of (W, R). The stabilizer in W of a vector $a = \sum_{i=1}^n a_i f_i$ in A is W_J, where $J = \{1 \leq i \leq n \mid a_i = 0\}$. This shows there is some control over the part $T = \bigcup_{w \in W} wA$ of V that is covered by images of A under W. This set is a cone and is called the *Tits cone*.

1.7. *Finite Coxeter groups*

A Coxeter group W is finite if and only if the quadratic form $x \mapsto B(x, x)$ on V for V and B as in Proposition 1.4 is positive definite. This, combined with Example 1.3(v), gives a straightforward method of classifying the finite Coxeter groups.

For instance, a Coxeter system (W, R) of type

$$\begin{pmatrix} 1 & p & q \\ p & 1 & r \\ q & r & 1 \end{pmatrix}$$

is finite if and only if the quadratic form with matrix

$$\begin{pmatrix} 2 & -2\cos(\pi/p) & -2\cos(\pi/q) \\ -2\cos(\pi/p) & 2 & -2\cos(\pi/r) \\ -2\cos(\pi/q) & -2\cos(\pi/r) & 2 \end{pmatrix}$$

is positive definite, which in turn is straightforwardly checked to be equivalent to

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

Let T be the Tits cone as above. In the finite case, $T = V^*$ and V^* can be identified with V as B (being positive definite) is non-degenerate. Conversely, if $-v$ lies in T for some v in the interior of A, then W is finite.

1.8. *Connection with diagram geometry*

The chamber system $\mathcal{C}$ of a Coxeter system is *residually connected*. That means that, for any collection pairwise non-disjoint cells $c_1\langle J_1\rangle, \ldots, c_t\langle J_t\rangle$ of $\mathcal{C}$, the intersection is a cell of the form cW_J where $J = J_1 \cap \cdots \cap J_t$. This condition, which is a direct consequence of what has been said at the end of §1.2, gives rise to the construction of a residually connected geometry from the chamber system (cf. (Tits [1981])), which, in the case where M is a string diagram, is an abstract polytope.

2. The conjugacy problem

The conjugacy problem for Coxeter groups is the question of finding an algorithm that, upon input a Coxeter matrix M of rank n and two words $\mathbf{v}$, $\mathbf{w} \in R^*$, decides if $\rho(\mathbf{v})$ and $\rho(\mathbf{w})$ are conjugate in $W = W(M)$. The solution we present finds numbers $K, L \in \mathbf{N}$ such that, whenever $v = \rho(\mathbf{v})$ and $w = \rho(\mathbf{w})$ are conjugate in W, they are conjugate by an element $g \in W$ of length at most $n^{K \max(\ell(v), \ell(w)) + L}$. Thus, the solution is far from efficient, and leaves much in the direction of implementation to be desired. Yet, the setting of the proof, the Moussong complex to be defined below, gives hope for improvements.

We describe the construction of a metric complex that is based on (Moussong [1988]). It will lead to a solution of the conjugacy problem for Coxeter groups. In fact, in his PhD thesis, Moussong does almost all of the work leading to the solution, but does not draw the conclusion. I gratefully acknowledge the help of Michael Shapiro and Daan Krammer, to whom I owe much of the content of this section.

Throughout this section, we take M to be a fixed Coxeter diagram and (W, R) a Coxeter system of type M.

2.1. *Cells*

A *Euclidean cell* is a metric space (X, d_X) which is isometric to a compact subset of Euclidean m-space E^m (for some $m < \infty$) which is the intersection of finitely many closed halfspaces. Without loss of generality we may assume $X \subseteq E^m$ (with induced metric) and X spans E^m. A *face* of a Euclidean cell X is, by definition, either X or $X \cap H$ for H a hyperplane of E^m which is disjoint from the interior of X. Thus faces, with the induced metric, are again Euclidean cells. The faces do not depend on the isometric embedding of X in E^m.

We recall that a Coxeter matrix M is called *spherical* if W is finite. The spherical Coxeter matrices have been classified. A subset J of R is said to be *spherical* if $M|_{J\times J}$ is spherical. If W is finite and M is connected, the reflection representation in $\mathbf{R}^n$ is irreducible and leaves invariant a positive definite form $(\cdot,\cdot)$, which is a positive scalar multiple of B as in §1.4. Thus, the reflection representation can be seen as an isometric action on the Euclidean space $E^n = E(\mathbf{R}^n)$, with Euclidean distance $|x-y| = \sqrt{(x-y,x-y)}$ $(x,y \in \mathbf{R}^n)$. The origin of the linear reflection representation is recovered from the metric action as the unique fixed point.

2.2. Proposition. *Let M be a spherical Coxeter matrix of rank n, and (W,R) a Coxeter system of type M. Let W act on $E^n = E(\mathbf{R}^n)$ via the reflection representation on $\mathbf{R}^n$. Then there is a W-equivariant injective map $\kappa : W \to E^n$ such that:*

(i) $|\kappa(w) - \kappa(wr)| = 1$ for all $w \in W$, $r \in R$;

(ii) the metric space $E(W)$ consisting of the convex hull of $\kappa(W)$, with metric induced from E^n, is a Euclidean cell;

(iii) the nonempty faces of $E(W)$ are precisely the convex hulls of $\kappa(xW_J)$ for $x \in W$ and $J \subseteq R$.

The cell $E(W)$ is unique in $E(\mathbf{R}^n)$, the map κ is unique up to W-isometries.

Proof. Let ω_i $(i \in I)$ be the fundamental weights in the reflection representation space $\mathbf{R}^n$ in the sense that they form a dual basis to the normalised fundamental roots with respect to the W-invariant positive-definite form. Take $v \in \mathbf{R}^n$ to be half the sum of all ω_i for $i \in R$, and put $\kappa(w) = w \cdot v$. Then $|\rho_i(v) - v| = 1$ for all i. Since the stabiliser of v in W is trivial, κ embeds the chamber system $\mathcal{C}(W,R)$ into Euclidean space $E(\mathbf{R}^n)$. Moreover, vertices corresponding to adjacent chambers have distance 1, and, for any 2-set $\{i,j\} \subseteq J$, the subsystem $\mathcal{C}(W_{\{i,j\}}, \{i,j\})$ is a regular $2m_{ij}$-gon whose angles at the vertices are $\pi - \pi/m_{ij}$. Letting $E(W)$ be the convex hull of $\kappa(W)$, (i) and (ii) follow.

For $J \subseteq R$, let v_J be half the sum of all ω_i for $i \in R \setminus J$, so that $v_\emptyset = v$. Then $(wv - v, v_J) \leq 0$ with equality only if $w \in W_J$. This yields that the convex hull of $\kappa(W_J)$ is the face $v_J^\perp \cap E(W)$ of $E(W)$. A little more effort gives that each face on $\kappa(1)$ is of this form for some $J \subseteq R$. The rest of the proof follows from transitivity of W on the vertex set $\kappa(W)$. □

2.3. Corollary. *Each face of the Euclidean cell $E(W)$ is of the form $xE(W_J)$ for some Coxeter subsystem (W_J, J) of (W,R). Two such faces $xE(W_J)$ and $yE(W_K)$ intersect non-emptily if and only if xW_J and yW_K do, in which case there is $z \in W$ with $xW_J \cap yW_K = zW_{J\cap K}$. Such an element z also satisfies $xE(W_J) \cap yE(W_K) = zE(W_{J\cap K})$.*

Proof. By (i) and (iii) of the proposition, the convex hull of $\kappa(xW_J)$ equals $xE(W_J)$, and multiplication by x is an isometry between $E(W_J)$ and $xE(W_J)$. □

2.4. *Complexes*

A *Euclidean complex* is a pair $\mathcal{M} = (U, \mathcal{X})$ consisting of a set U and a collection $\mathcal{X}$ of metric spaces (X, d_X) where X is a subset of U, and is called a *cell* of $\mathcal{M}$, such that

(i) $U = \bigcup_{X \in \mathcal{X}} X$;

(ii) each member of $\mathcal{X}$ is a Euclidean cell;

(iii) if Y is a face of a cell (X, d_X), then Y is again a cell, and $d_Y = d_X|_{Y\times Y}$;

(iv) if (X, d_X) and (Y, d_Y) are cells, then $X \cap Y$ is a face of either cell.

Observe that, due to (iii), each cell has a unique metric. A Euclidean complex $\mathcal{M} = (U, \mathcal{X})$ has a natural topology in which N is open if and only if $N \cap X$ is open for each cell $X \in \mathcal{X}$.

The chamber system of a Coxeter system can be turned into a Euclidean complex by viewing its edges as Euclidean line segments of length 1.

2.5. *The Moussong complex*

We now construct a Euclidean complex from the Coxeter system (W, R) of type M. Let $\widehat{U}$ be the collection of all triples (x, J, u) with $x \in W$, J a spherical subset of R, and $u \in E(W_J)$. On $\widehat{U}$, define the relation $\sim$ by

$$(x, J, u) \sim (y, K, v) \iff \exists z \in W \quad \begin{cases} xW_J \cap yW_K = zW_{J\cap K} & \text{and} \\ (z^{-1}x)u = (z^{-1}y)v. \end{cases}$$

Observe that, whenever $xW_J \cap yW_K \neq \emptyset$, the point $(z^{-1}x)u$ lies in $E(W_{J\cap K})$ and does not depend on the choice of z. Using that $(x, J, u) \sim (y, K, v)$ implies

$$(x, J, u) \sim (z, J \cap K, (z^{-1}x)u) = (z, J \cap K, z^{-1}yv) \sim (y, K, v),$$

one can show that $\sim$ is an equivalence relation. Define U to be the quotient of $\widetilde{U}$ by this relation, and denote the $\sim$-class of (x, J, u) by $[x, J, u]$.

In order to define Euclidean cells on U, we let $\mathcal{X}$ be the collection of subsets of U consisting of the empty set and all

$$xE_J := \{[x, J, u] \mid u \in E(W_J)\}$$

for $x \in W$ and spherical $J \subseteq R$. Then, as a set, xE_J bijectively corresponds to $E(W_J)$, and $xE_J \cap yE_K$ is either empty, or of the form $zE_{J\cap K}$ for some $z \in W$. Using this correspondence, each member of $\mathcal{X}$ can be given the structure of a Euclidean cell. If $K \subset J$ and $y \in xW_J$ then the embedding of yW_K into xW_J corresponds to an embedding of the cell yE_K into xE_J via the mapping $[y, K, v] \mapsto [x, J, (x^{-1}y)v]$ $(v \in E(W_K))$.

We shall refer to the pair $(U, \mathcal{X})$ as the *Moussong complex of type* M or the *Moussong complex of* (W, R), and denote it by $\mathcal{M}$. This definition is justified by the following claim.

2.6. Lemma. *The Moussong complex $\mathcal{M} = (U, \mathcal{X})$ is a locally finite Euclidean complex. There are finitely many W-orbits on $\mathcal{X}$ and W acts discretely as a group of isometries on $\mathcal{M}$ with compact quotient and finite point stabilizers. The 1-skeleton of $\mathcal{M}$ coincides with the Euclidean complex defined by the chamber system $\mathcal{C}(W, R)$.*

Proof. Straightforward. For example, the quotient of U by W is the union of the cells E_J for $J \subseteq R$, J spherical, hence compact. □

The complex has been first constructed in (Moussong [1988]), whence its name. Prior to this, a non-metric version was given by (Davis [1983]).

2.7. *Paths*

Let $\mathcal{N}$ be a Euclidean complex. A *path* in $\mathcal{N}$ is a continuous map $\alpha : T \to U$ defined on a real interval T with the property that, for any $t \in T$, there is $\epsilon > 0$ such that $\alpha|_{T\cap[t,t+\epsilon]}$ and $\alpha|_{T\cap[t-\epsilon,t]}$ are differentiable paths with continuous derivatives, each contained in a cell. (In fact, it suffices for our purposes to restrict attention to paths that are piecewise linear.) Thus, if T is bounded and closed, the *speed* of α, that is, the absolute value of the derivative of α, is defined at all but a finite number of points. If $T = [0, 1]$ and the speed of α is constant, then α is said to be *normalised.* The *length* of the path $\alpha : T \to U$ is the integral of the speed of α over T. We shall write $\ell_{\mathcal{N}}(\alpha)$ for the length of α.

2.8. Proposition. *Let $\mathcal{N} = (U, \mathcal{X})$ be a locally finite connected Euclidean complex having only a finite number of isomorphism types of cells. Then the function $d : U \times U \to \mathbf{R}$ determined by*

$$d(x, y) = \inf \{\ell_{\mathcal{N}}(\gamma) \mid \gamma \text{ a path of } \mathcal{N} \text{ from } x \text{ to } y\}$$

is a metric on U.

2.9. *Geodesics*

A *geodesic* in a Euclidean complex $\mathcal{N}$ is a path $\alpha : T \to \mathcal{N}$ such that

$$\ell_{\mathcal{N}}(\alpha|_{[t,u]}) = d(\alpha(t), \alpha(u)) \quad \text{for all } t, u \in T.$$

If $\mathcal{N}$ is as in Proposition 2.8, then geodesics exist between any two points in the same connected component.

Now let $\mathcal{M}$ be the Moussong complex of the Coxeter system (W, R), and let $\mathcal{C}$ be the chamber system of (W, R). Then, by Lemma 2.6 and Proposition 2.8, both are metric spaces. In $\mathcal{C}$, the distance $d(x, y)$ between two chambers x, y is an integer coinciding with the graph-theoretic distance $d_{\mathcal{C}}(x, y)$. The path $x \cdot \mathbf{r}$, where $\mathbf{r} \in R^*$ has length q, can be viewed as a path $\alpha : [0, q] \to W$ of the Euclidean complex $\mathcal{C}$ by extending it to a path with constant speed on the edges. It occurs in the Moussong complex as its 1-skeleton, so, for any two elements $x, y \in W$,

$$d_{\mathcal{M}}(x, y) \le d_{\mathcal{C}}(x, y) = \ell_R(x^{-1}y).$$

Since W acts discretely on $\mathcal{M}$ with compact quotient, an argument of Milnor's gives a bound in the other direction, see below. In the metric space on $\mathcal{C}$, geodesics exist, but need not be unique. For instance, if $M = A_2$, the paths $1 \cdot [r, s, r]$ and $1 \cdot [s, r, s]$ are distinct geodesics, both starting at 1 and ending at rsr. In the Euclidean cell $E(W(A_2))$, there is a straight line from 1 to $rsr = srs$; it is the unique geodesic between the two chambers. This illustrates one reason why the metric space on the Moussong complex is useful.

The following result shows that geodesics can be "approximated" by paths in the chamber system.

2.10. Proposition. *Let $\mathcal{M}$ be the Moussong complex of type M, so that the chamber system $\mathcal{C}$ of (W, R) embeds in $\mathcal{M}$. Then the following hold.*

(i) Milnor's inequality (Milnor [1968]) *holds, i.e., there are non-negative constants K, L, such that, for every pair of chambers x, y of $\mathcal{C}$, we have*

$$d_{\mathcal{C}}(x, y) \le K d_{\mathcal{M}}(x, y) + L.$$

(ii) There exists a constant A such that for each normalised geodesic γ with end points $c = \gamma(0)$, $d = \gamma(1)$ in $\mathcal{C}$, a path $c \cdot \mathbf{r}$ of length $q \in \mathbf{N}$ and a strictly monotonous function $f : [0, 1] \to [0, q]$ can be found satisfying, for all $t \in [0, 1]$,

$$d_{\mathcal{M}}(c \cdot \mathbf{r}(f(t)), \gamma(t)) \le A.$$

2.11. *Links*

Instead of Euclidean cells, other kinds of cells could have been employed. For instance, if "Euclidean" is replaced by "spherical" in the above sections 2.1, 2.4, 2.7, 2.8 and 2.9, most constructions and properties continue to hold. We shall need the notion of spherical complex to cope with a metric space capturing the local structure around a point of $\mathcal{M}$. Here, as usual, the *star* of a point (or a face) is the union of all cells containing it.

Let x be a point of a Euclidean cell X of $\mathcal{M}$. By the definition of Euclidean cell, we may think of X as a convex part of a Euclidean space E. Then the *link of* x in X, notation $Lk(x, X)$, is the spherical cell obtained by taking all points y of the unit sphere around x in X such that xy contains a segment of X of positive length. If, for example, X is the regular m-gon with sides of length 1, the distance of the two vertices of the spherical cell is $\pi - \pi/m$. The link of a Euclidean complex $\mathcal{M}$ at a point x is the spherical complex whose cells are obtained by taking the union of all links $Lk(x, X)$ for X in the star of x, with proper identifications.

A *closed geodesic* is the image of an isometric embedding of a circle of strictly positive length. We have angular distances in spherical complexes. A Euclidean complex $\mathcal{M}$ is said to satisfy the *link condition* if there are no closed geodesics of length $< 2\pi$ in links.

2.12. Moussong's theorem. *For each Coxeter system* (W, R)*, the Euclidean complex* $\mathcal{M}$ *is simply connected and satisfies the link condition.*

Idea of proof. Tits' rewrite rules:

$$[\rho_i, \rho_i] \Rightarrow [\,]$$
$$\underbrace{[\rho_i, \rho_j, \rho_i, \ldots]}_{m_{ij}} \Rightarrow \underbrace{[\rho_j, \rho_i, \rho_j, \ldots]}_{m_{ij}}$$

are known to solve the word problem. Consequently, any closed path of $\mathcal{C}$ starting and ending at 1 is homotopic to the trivial path 1 in the 2-skeleton of $\mathcal{M}$. Since any closed path in $\mathcal{M}$ is homotopic to a path in $\mathcal{C}$, this gives simple connectedness. For the link condition, the main idea is that, in the link of a point, each closed geodesic goes via the 1-skeleton, where it can easily be seen to have length $\geq 2\pi$. Here an inductive argument on links is needed. See the examples below for an impression of what happens in the low-dimensional cases. □

2.13. *Examples*

(i). Let $n = 2$, say $R = \{r, s\}$. If $m_{rs} < \infty$, then $\mathcal{M}$ has a single 2-cell: the metric regular $2m_{rs}$-gon. The link at a point of $\mathcal{M}$ is a circle if the point is in the interior, an arc of length $\pi - \pi/m$ if the point is a chamber, and and arc of length π if the point is on an edge but not a chamber. If $m_{rs} = \infty$, then $\mathcal{M}$ has no 2-cells, and, as a complex, coincides with the chamber system, which is an infinite tree of valency 2.

(ii). For arbitrary n, with $m_{rs} = \infty$ whenever r, s are distinct, the complex $\mathcal{M}$ is the infinite tree of valency n.

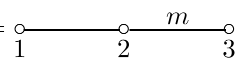
(iii). Let $n = 3$, say $R = \{i, j, k\}$. The sum of the lengths of the spherical arcs in the link of 1 in the three polygons on the chamber $1 \in W$ amounts to

$$\pi\left(3 - \left(\frac{1}{m_{ij}} + \frac{1}{m_{jk}} + \frac{1}{m_{ik}}\right)\right).$$

Now, as stated in §1.7, W is finite iff

$$\frac{1}{m_{ij}} + \frac{1}{m_{jk}} + \frac{1}{m_{ik}} > 1.$$

Thus, if the sum of the arc lengths is less than 2π, the three angles border a 2-cell. This is at the heart of the argument that $\mathcal{M}$ satisfies the link condition. For example, for $\mathcal{M}$ the Moussong complex of the Coxeter group of type

$M =$ 

,

the link at a chamber is

- a spherical 2-cell if $m \leq 5$ (because then M is spherical),
- a concatenation of two arcs, one of length $\pi/2$, one of length $2\pi/3$ if $m = \infty$, and
- a triangle with edges of length $\pi/2$, $2\pi/3$ and $(m-1)\pi/m$ if $5 < m < \infty$.

The latter triangle is a closed path of length $\geq 2\pi$. The case $m = 6$ shows that equality may occur.

2.14. *Unique geodesics and convex metric*

We say that a metric space has *unique geodesics* if any two of its points are connected by a unique normalised geodesic. A metric space (U, d_U) is said to have *convex metric* if, for any two normalised geodesics α, β,

$$d(\alpha(t), \beta(t)) \leq (1-t)d(\alpha(0), \beta(0)) + td(\alpha(1), \beta(1)) \quad \text{for all } t \in [0,1].$$

A metric space with convex metric in which geodesics exist, is easily seen to have unique geodesics.

The following result expresses the major step of this section; it is of a “local-to-gobal” nature.

2.15. Theorem. *Suppose $\mathcal{N}$ is a simply connected Euclidean complex satisfying the link condition. Then $\mathcal{N}$ has unique geodesics and convex metric.*

Proof. (Krammer [1993]) has given a fully elementary proof. More general results, using heavier machinery, are derived in (Bridson [1991]). □

One of the consequences of the theorem is that local geodesics are geodesics. Hence, for each cell X of $\mathcal{N}$, we have $d\,|_{X\times X} = d_X$.

For $v, w \in W$, set $N_W(v,w) := \{g \in W \mid gvg^{-1} = w\}$. Clearly, an algorithm to determine whether or not $N_W(v,w)$ is empty solves the conjugacy problem. The result below shows that an exhaustive search for an element of $N_W(v,w)$ among all elements of length exponentially bounded by $\max(\ell(v), \ell(w))$ provides such an algorithm.

2.16. Solution to the conjugacy problem. *Let M be a Coxeter matrix and (W,R) a Coxeter system of type M. Then there are constants K, $L \in \mathbf{N}$ such that, for all $v, w \in W$ with $N_W(v,w) \neq \emptyset$, there exists $g \in N_W(v,w)$ with*

$$\ell(g) \leq n^{K\max(\ell(v),\ell(w))+L}.$$

PROOF. Without loss of generality, we may take $\ell(w) \geq \ell(v)$. Suppose $N_W(v, w) \neq \emptyset$. Take $g \in N_W(v, w)$ of minimal length, denote by γ the normalised geodesic from 1 to g in the Moussong complex $\mathcal{M}$ of (W, R) guaranteed by Theorem 2.15 and choose a presentation $\mathbf{r} = [r_1, \ldots, r_q]$ of g such that the path $1 \cdot \mathbf{r}$ approximates γ in the sense of Proposition 2.10(ii) and such that q is minimal with respect to these requirements. Thus there are $A \in \mathbf{N}$ and $f : [0, 1] \to [0, q]$ independent of g, v, w such that, for all $t \in [0, 1]$,

$$d_{\mathcal{M}}(\mathbf{r}(f(t)), \gamma(t)) \leq A.$$

By the convex metric property applied to the geodesics γ from 1 to g and $w\gamma$ from w to $wg = gv$ and using the triangle inequality we see that, for any time $t \in [0, 1]$,

$$\begin{aligned} d_{\mathcal{M}}(1 \cdot \mathbf{r}(f(t)), w \cdot \mathbf{r}(f(t))) &\leq d_{\mathcal{M}}(\gamma(t), w\gamma(t)) + 2A \\ &\leq (1 - t) d_{\mathcal{M}}(1, w) + t d_{\mathcal{M}}(g, wg) + 2A \\ &\leq \ell_R(w) + 2A \end{aligned} \tag{1}$$

The last inequality follows from $wg = gv$ and the fact that g acts from the left as an isometry, so that $d_{\mathcal{M}}(g, wg) = d_{\mathcal{M}}(1, v) \leq \ell_R(v)$. Using Proposition 2.10(i) for times t at which $\mathbf{r}(f(t))$ represents an element of $\mathcal{C}$, we obtain the existence of constants K, L' such that, for all $k \in \{0, 1, \ldots, q\}$,

$$d_{\mathcal{C}}(r_1 \cdots r_k, wr_1 \cdots r_k) \leq K d_{\mathcal{M}}(r_1 \cdots r_k, wr_1 \cdots r_k) + L'.$$

Together with (1), putting $L = L' + 2AK$, we find

$$d_{\mathcal{C}}(r_1 \cdots r_k, wr_1 \cdots r_k) \leq K\ell_R(w) + L.$$

For each k, let P_k denote the label of a path in $\mathcal{C}$ of length at most $K\ell(w) + L$ connecting $r_1 \cdots r_k$ and $wr_1 \cdots r_k$. If $q > n^{K\ell(w)+L}$, there are k, k' such that $k < k'$ and $P_k = P_{k'}$. But then $g' = r_1 \cdots r_{k-1} r_{k'+1} \cdots r_q$, is an element of W with $g' \in N(v, w)$ and $\ell(g') \leq q - (k' - k) < q$, and so is a shorter expression than $\mathbf{r}$ for an element of $N(v, w)$, a contradiction. Hence $\ell(g) \leq q \leq n^{K\ell(w)+L}$, as required. □

2.17. *Remarks*

Corollary 4.5 of (Alonso & Bridson [1993]) states that if W acts properly and cocompactly by isometries on a space of non-positive curvature, then W is "semihyperbolic". Theorem 5.2 of (Alonso & Bridson [1993]) states that if W is semihyperbolic, then it has a solvable conjugacy problem. These results can be applied to give a proof that Coxeter groups have a solvable conjugacy problem. The method presented above follows similar lines (namely, those set out by Gromov, cf. (Gersten & Short [1990]; Gromov [1987])), but is simpler and hopefully leads to a more efficient solution.

A first step towards a better algorithm might be found by use of reductions of w of the form

$$w \mapsto sws \quad \text{whenever} \quad \ell(sws) \leq \ell(w). \tag{2}$$

We shall call w *conjugacy-reduced* if each series of reductions as in (2) starting with w leads to an element w' of W with $\ell(w') = \ell(w)$.

2.18. Conjecture. *Let C be a conjugacy class of W and put $\ell_C = \min\{\ell(w) \mid w \in C\}$. Then, for any $w \in C$, we have $\ell(w) = \ell_C$ if and only if w is conjugacy-reduced.*

By (Geck & Pfeiffer [1992]) the conjecture holds for Weyl groups. They use the result for Hecke algebra representations.

2.19. ***Conjugacy of parabolic subgroups.***

Let (W, R) be a Coxeter system. (Deodhar [1982]) describes how, for two subsets I, J of R, it can be decided if the corresponding subgroups W_I and W_J of W are conjugate. Subgroups of the form W_I are called *standard parabolic*; their conjugates are called *parabolic*. Thus, the algorithm helps to find out how many distinct conjugacy classes of parabolic subgroups W has.

Let $I \subset R$ and $s \in R \setminus I$. Write K for the connected component of the subgraph of M induced on $I \cup \{s\}$ that contains s. Suppose now that K is spherical. Then the longest elements w_K of W_K and $w_{K\setminus\{s\}}$ of $W_{K\setminus\{s\}}$ exist (they are involutions). In this situation, we set

$$\nu(I, s) = w_{K\setminus\{s\}} w_K.$$

This element of W has the property that, for some $s_0 \in K$,

$$I^{\nu(I,s)} = (I \cup \{s\}) \setminus \{s_0\}.$$

Thus, there is a subset J of R such that conjugation by $\nu(I, s)$ not only maps W_I to W_J, but even gives a bijection from I to J.

Now consider the directed graph $\mathcal{K}$ whose vertices are the subsets of R and in which two vertices I and J are connected by an edge pointing to J labelled s if the connected component of $I \cup \{s\}$ containing s is spherical (so $\nu(I, s)$ exists) and $I^{\nu(I,s)} = J$. For each $s \in I$, denote by e_s the fundamental root corresponding to s.

2.20. Theorem. *Let I and J be subsets of R. If $w \in W$ satisfies*

$$\{e_s \mid s \in I\} = \{we_t \mid t \in J\}$$

then there is a directed path $I = I_0 \overset{s_0}{\to} I_1 \overset{s_1}{\to} \dots \overset{s_{t-1}}{\to} I_t = J$ in $\mathcal{K}$ such that

$$w = \nu(I_0, s_0) \cdot \nu(I_1, s_1) \cdots \nu(I_{t-1}, s_{t-1})$$

and $\ell(w) = \sum_{i=0}^{t-1} \ell(\nu(I_i, s_i))$.

Proof. See Proposition 5.5 of (Deodhar [1982]). □

By arguments similar to those below, this leads to an effective method for deciding if two subsets of R generate conjugate parabolic subgroups of W. Rather than giving details, we shall be more elaborate on the determination of the normaliser of a parabolic subgroup.

2.21. Corollary. *Suppose I is a subset of R. Let $\mathcal{T}$ be a spanning tree of the connected component of $\mathcal{K}$ containing I. For each vertex J of $\mathcal{T}$, denote by $\mu(J)$ the element of W corresponding to the unique path in $\mathcal{T}$ from I to J. Then the normaliser of W_I in W is generated by I and all products of the form*

$$\mu(J)\nu(J,s)\mu(K)^{-1}$$

for edges $J \xrightarrow{s} K$ of the connected component of $\mathcal{K}$ containing I.

Proof. If $w \in W$ normalises W_I, then, up to left multiplication by elements of W_I, it can be assumed to satisfy $\ell(sw) \geq \ell(w)$ for all $s \in I$. It follows that $\{w^{-1}e_s \mid s \in I\}$ is a fundamental system contained in Φ^+ of the group $\langle w^{-1}Iw\rangle = I$, whence coincides with $\{e_t \mid t \in I\}$. Consequently, w satisfies the conditions of Theorem 2.20 with $J = I$. By the theorem, w is a product corresponding to a closed path in $\mathcal{K}$. The result follows as the fundamental group of $\mathcal{K}$ with base point I is generated by the paths $I \xrightarrow{\mathcal{T}} J \xrightarrow{s} K \xrightarrow{\mathcal{T}} I$, for all vertices J, K of $\mathcal{T}$ such that $J \xrightarrow{s} K$ is an edge of $\mathcal{K}$ but not of $\mathcal{T}$. Here, $\xrightarrow{\mathcal{T}}$ stands for a path in $\mathcal{T}$. □

2.22. *Implemented algorithms*

We note that the description of the finite generating set for the normalizer of W_I is effective. An algorithm to this effect is implemented in a new version of LiE, cf. (Leeuwen, Cohen & Lisser [1992]).

For Weyl groups, (Carter [1972]) has given very explicit methods to resolve the conjugacy problem. Recently, they have been implemented in LiE by (Pasqualucci [1992]). In the software package GAP, cf. (Geck & Pfeiffer [1992]), algorithms solving the conjugacy problem for finite Coxeter groups have been implemented by use of the permutation group algorithms available in the package.

3. The word problem

The word problem for Coxeter groups is the question of finding an algorithm that, upon input a Coxeter matrix M of rank n and a word $\mathbf{w} \in R^*$, decides if $\rho(\mathbf{w}) = 1$ in $W(M)$. Solutions to the word problem have been discussed in (Cohen [1991]). One of them was used in §2.12.

Another solution is based on the reflection representation. For, take $a = f_1 + \cdots + f_n$ with notation as in §1.6. Then, by §1.6, $\sigma^*(w)a = a$ if and only if $w = 1$, so we can test whether $\rho(\mathbf{w}) = 1$ for $\mathbf{w} = [r_1, \ldots, r_q] \in R^*$ by computing

$$\sigma_{r_1}(\sigma_{r_2}(\cdots\sigma_{r_q}a))$$

and verifying if this vector coincides with a.

This algorithm is implemented for Weyl groups in LiE, cf. (Leeuwen, Cohen & Lisser [1992]). To implement it on computer with exact arithmetic for arbitrary Coxeter groups, one would need to handle an extension field of $\mathbf{Q}$ containing the algebraic integers $2\cos(\pi/m)$ for each entry m of M, or an equivalent thereof.

3.1. *A combinatorial solution to the word problem*

In (du Cloux [1990]), a fast algorithm is presented that, given M, merely uses operations on words from R^* to rewrite an input word $[r_1, \ldots, r_q]$ (where $r_i \in R$) to the lexicographically minimal reduced word in $\rho^{-1}(r_1 \cdots r_q)$. Here a total ordering on R is used, e.g., $\rho_1 < \rho_2 < \cdots < \rho_n$.

For $r \in R$, let X_r denote the set of distinguished left coset representatives of the subgroup $\langle s \in R \mid s < r\rangle$ of $\langle s \in R \mid s \leq r\rangle$. Thus, if $r = \rho_j$,

$$X_r = \{w \in W_{\{1,\ldots,j\}} \mid \ell(sw) > \ell(w) \text{ for all } s \in R,\ s < r\}.$$

Then the multiplication map $X_1 \times X_2 \times \ldots \times X_n \to W$ is a bijection. The lexicographically minimal reduced word for any element of W respects this decomposition. Du Cloux describes

- how to obtain the lexicographically minimal reduced words for each X_r;
- how multiplication on the right by an element t from R affects an element $x \in X_r$: either produces the element of X_r again it is of the form sw with $s \in R$, $w \in X_r$, and $\ell(w) = \ell(x)$.

These facts suffice to process any expression for an element of W into the form $X_1X_2 \cdots X_n$.

3.2. *Regular languages*

The set of all lexicographically minimal reduced words turns out be remarkably well behaved, as the theorem below exhibits. If U and V are subsets of R^*, their *product* is the subset

$$UV := \{[u, v] \mid u \in U, v \in V\}$$

of all products of elements from U with elements from V. The *Kleene closure* of U is the subset

$$U^* := \{[u_1, \ldots, u_q] \mid q \in \mathbf{N},\quad \text{each } u_i \in U\}.$$

(Note that this notation is consistent with the earlier definition of R^*.) A subset of R^* is called a *regular language* if it can be obtained from the subsets $\{[\,]\}$ and $\{r\}$ where $r \in R$, by a finite number of applications of the operations union, product and Kleene closure.

An alternative definition of a regular language uses the notion of a finite state automaton: it is the set of words accepted by such an automaton. For every Coxeter group, Brink and Howlett provide a finite state automaton and construct additional multiplier automata, one for each $r \in R$. This proves that Coxeter groups are automatic (cf. (Epstein et al. [1992])). To be somewhat more precise, we define the notions automatic and bi-automatic.

3.3. *Automatic structures*

Extend R with a padding symbol $\$ \notin R$, and set

$$R_\$ = ((R \cup \{\$\}) \times (R \cup \{\$\})) \setminus \{(\$, \$)\}.$$

For $\mathbf{r} = [r_1, \ldots, r_t]$, $\mathbf{q} = [q_1, \ldots, q_s] \in R^*$, put

$$(\mathbf{r}, \mathbf{q}) = \begin{cases} [(r_1, q_1), \ldots, (r_s, q_s)] \text{ where } r_j = \$ & \text{if } t < j \leq s, \\ [(r_1, q_1), \ldots, (r_t, q_t)] \text{ where } q_j = \$ & \text{if } s < j \leq t. \end{cases}$$

As the notation suggests, this embeds $R^* \times R^*$ into $R^*_\$$. The triple (W, R, L) is said to be an *automatic structure* if L is a regular language in R^* with $\rho(L) = W$ such that

$$\{(\mathbf{r}, \mathbf{q}) \in R^*_\$ \mid \mathbf{r}, \mathbf{q} \in L, \quad \rho(\mathbf{r}) = \rho(\mathbf{q})\}$$

and, for each $s \in R$,

$$\{(\mathbf{r}, \mathbf{q}) \in R^*_\$ \mid \mathbf{r}, \mathbf{q} \in L, \quad \rho(\mathbf{r}) = \rho(\mathbf{q}s)\}$$

are regular languages in $R^*_\$$. If, in addition, for each $s \in R$, the language

$$\{(\mathbf{r}, \mathbf{q}) \in R^*_\$ \mid \mathbf{r}, \mathbf{q} \in L, \quad \rho(\mathbf{r}) = \rho(s\mathbf{q})\}$$

is regular in $R^*_\$$, then the triple (W, R, L) is called a *bi-automatic structure*. A group is called *automatic* if it has an automatic structure, and similarly for *bi-automatic*.

3.4. Theorem. (Brink & Howlett [1993]) *For each $w \in W$, let $\sigma(w) \in R^*$ be the lexicographically first reduced expression for w. Then*

$$L = \{\sigma(w) \in R^* \mid w \in W\}$$

is a regular language, and (W, R, L) is an automatic structure.

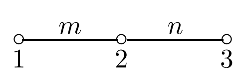
3.5. Theorem. (Brink & Howlett [1993]) *Coxeter groups (with finite diagram) are automatic.*

The proof heavily uses the reflection representation and the corresponding root system. If W is finite, the number of accepting states of the automaton they construct for L is close to the number of subsets of the set of positive roots. So, for the Coxeter group of type E_8 there are more than 10^{36} states.

If a group is bi-automatic, it has a solvable conjugacy problem. Thus one may ask:

3.6. Question. *Are Coxeter groups bi-automatic?*

Continuing the work in (Le-Chenadec [1986]) for some finite Coxeter groups, (Hermiller [1992]) has given finite complete sets of rewrite rules for all Coxeter groups of rank at most 3 and for all Coxeter groups whose types M have the property that all of its subdiagrams of the form



satisfy either $\infty \in \{m, n\}$ or $m = n = 2$. If such a group has a complete rewrite system, the set of canonical forms with respect to the rewrite system, is a regular language. Thus, for the class of Coxeter groups W covered by Hermiller's work, this result also gives a regular language L in R^* together with a computable section $\sigma : W \to R^*$ such that $\sigma(W) = L$.

4. Subgroup structure

In order to construct quasi-crystals, (Moody & Patera [1993]) have distinguished certain subgroups of Coxeter groups, which are Coxeter groups in their own right. Independently, (Mühlherr [1993]) has indicated similar subgroups in a more general approach, which we follow here. Both papers are inspired by (Scherbak [1988]). The subgroups found in (Dyer [1990]) are usually different from those Mühlherr pointed out, for the simple reason that the former are generated by reflections whereas the latter may not contain reflections at all. Earlier work in this direction, but from a polytopal point of view, can be found in (Monson [1987]).

For the remainder of this section, let M be a Coxeter matrix and (W, R) a Coxeter system of type M. Fix a partition Π of R whose parts are spherical (cf. §2.1), and let R^Π be the set of all longest elements w_J of W_J for $J \in \Pi$. Thus, $|R^\Pi| = |\Pi|$.

4.1. Theorem. (Mühlherr [1993]) *Suppose* Π *is a partition of* R *whose parts are spherical with the property that, for all* $w \in \langle R^\Pi \rangle$ *and* $J \in \Pi$, *either* $\ell(wr) = \ell(w) - 1$ *for each* $r \in J$ *or* $\ell(wr) = \ell(w) + 1$ *for each* $r \in J$. *Then* $(\langle R^\Pi \rangle, R^\Pi)$ *is a Coxeter system.*

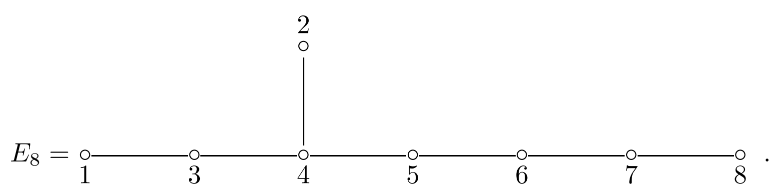
A further result in (Mühlherr [1993]) states that it suffices to verify the length condition for Coxeter subsystems based on the union of two parts from Π. More specifically, the length condition is equivalent to the requirement that, for each pair $J, K \in \Pi \times \Pi$ and each $w \in \langle w_J, w_K \rangle$ either $\ell(wr) = \ell(w) - 1$ for each $r \in J$ or $\ell(wr) = \ell(w) + 1$ for each $r \in J$.

4.2. *Applications*

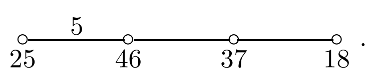
If α is an automorphism of W which leaves R invariant, then the orbits of α on R provide examples of partitions satisfying the hypotheses. With this method, for instance, the Coxeter group of classical type B_n can be seen to embed in the symmetric group $W(A_{2n-1})$. More generally, any Coxeter group can be shown to occur as a subgroup in the guise of the theorem of a Coxeter group whose type has labels ≤ 3 only. Other applications lead to a new proof of the classification of spherical Coxeter matrices.

4.3. *Example*

Consider the diagram



The partition $\{\{1, 8\}, \{2, 5\}, \{3, 7\}, \{4, 6\}\}$ satisfies the hypotheses of the theorem, so there exists a subgroup of $W(E_8)$ isomorphic to $W(H_4)$ as follows:



Here the nodes are labeled by the longest elements of the group W_J for J running over the parts of Π and the entry (J, K) of the corresponding Coxeter matrix equals the order of the element $w_J w_K$. For instance, for $J = \{2, 5\}$ and $K = \{4, 6\}$, the element $w_J w_K = 2546$ is a Coxeter element in the Coxeter group $\langle J \cup K \rangle$ of type A_4, whence of order 5.

This example occurs in both (Mühlherr [1993]) and (Moody & Patera [1993]). The latter paper exhibits beautiful configurations arising from projections of lattice points using the ring of "icosians".

4.4. *Maximal finite subgroups*

Using the Tits cone of the Coxeter system (W, R), it is easy to prove that every finite subgroup of W is contained in a spherical parabolic subgroup. Thus the conjugacy classes of maximal finite subgroups of Coxeter groups coincide with the conjugacy classes of maximal spherical subsets of R, a problem that can be algorithmically solved, cf. §2.19.

For example, from §1.3(iv) it is immediate that $PGL(2, \mathbf{Z})$ has precisely two conjugacy classes of finite maximal subgroups, one isomorphism type being the Klein fours group (corresponding to the spherical subset $\{1, 3\}$ of R) and the other isomorphism type being the dihedral group of order 6 (corresponding to the spherical subset $\{1, 2\}$ of R).

Conversely, the determination of conjugacy classes of finite maximal subgroups of $PGL(4, \mathbf{Z})$ can be employed to establish that the latter group is not a Coxeter group. For, the quotient of $W(F_4)$ by its center is a maximal finite subgroup of $PGL(4, \mathbf{Z})$ which is not a Coxeter group (this can be verified by a check of finite Coxeter groups). Since, by the above, maximal finite subgroups of Coxeter groups are Coxeter groups, it follows that $PGL(4, \mathbf{Z})$ is not a Coxeter group.

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