A theory of generalized functions based on one parameter groups of unbounded self-adjoint operators

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by

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OF UNBOUNDED SELF-ADJOINT OPERATORS

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Abstract

Let $A \geq 0$ be a self-adjoint unbounded operator in a Hilbert-space X . Then the operators e^{-tA} are well-defined for all $t \in \mathbb{C}$. For each $t > 0$ we introduce the sesquilinear form s_t by

$$s_t(x,y) =: (e^{-tA}x, e^{-tA}y) \quad , \quad x,y \in X .$$

The completion of X with respect to the norm $\|\circ\|_t$,

$$\|x\|_t =: \|e^{-tA}x\|_X \quad , \quad x \in X \quad ,$$

is denoted by X_t. The space $\sigma(X,A)$ of generalized functions is taken to be

$$\sigma(X,A) =: \bigcup_{t>0} X_t \quad .$$

The test function space, corresponding to $\sigma(X,A)$, is denoted by $\tau(X,A)$. The vector space $\tau(X,A)$ consists of trajectories, i.e. mappings $\mathbb{C} \to X$ which satisfy

$$\frac{du}{dt} = Au \; .$$

Such a trajectory is completely characterized by its "initial condition" $u(0) \in D((e^A)^\infty)$. Note that

$$\tau(X,A) = \bigcap_{t>0} D(e^{tA}) = D((e^A)^\infty) \; .$$

With respect to the space $\tau(X,A)$ and $\sigma(X,A)$, I discuss a pairing, topologies, morphisms, tensorproducts and Kernel theorems. Finally I mention some applications, a.o. to generalized functions in infinitely many dimensions.

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Introduction

Schwartz' space of tempered distributions $S'(\mathbb{R})$ may be regarded as the dual of the space $D(H^\infty) \subset L_2(\mathbb{R})$, with

$$H = \tfrac{1}{2}(-\frac{d^2}{dx^2} + x^2 + 1) ,$$

(for a proof see [Z1] or [K]). Note that $D(H^\infty)$ is the C^∞-domain of H, i.e. $f \in D(H^\infty)$ iff $H^k f \in D(H)$ for $k = 0,1,2,\ldots$.

The space $D(H^\infty)$ can be considered as a "trajectory space" in the following sense:

Let $u(0) \in D(H^\infty)$. Define the mapping $u : \mathbb{C} \to L_2(\mathbb{R})$ by

$$u(t) =: H^t u(0) = e^{t \log H} u(0), \ \log H \geq 0, \ t \in \mathbb{C} \ .$$

Then u is a so-called trajectory, i.e. u has the property $u(t + \tau) = H^\tau u(t)$ for all $t,\tau \in \mathbb{C}$. In this way each $u(0) \in D(H^\infty)$ is in one-to-one correspondence to a trajectory $u : t \to H^t u(0)$.

This observation led me to develop a theory of generalized functions which is a kind of reverse of a theory as developed by De Graaf in [G]. In [G] the generalized function space is a space of trajectories and the test function space is an inductive limit of Hilbert spaces. In the present paper the space of generalized functions is an inductive limit of Hilbert spaces and the test function space a trajectory space. It can be looked upon as very general theory on distributions of the tempered kind.

The results of this paper are inspired by and can be compared with the results in [G]. As in [G] generalized functions can be introduced on arbitrary measure spaces. I study the topologies of the spaces, their morphisms, necessary and sufficient conditions such that Kernel theorems hold. The notions "trajectory space" and "inductive limit space" are used in this paper as well as in [G]. However, we are forced to prove some important theorems with different techniques.

In this introduction I will illustrate the general theory by some examples I want to show that the theories of Judge, Zemanian and Korevaar (see [J], [Z2] and [K]) are special cases of ours.

Example 1

Let $A = -\frac{d^2}{dx^2}$, $X = L_2(\mathbb{R})$.

Consider the anti-diffusion equation

$$(1) \qquad \frac{\partial u}{\partial t} = -\frac{\partial^2}{\partial x^2} u \; .$$

A solution with the property that

$$\forall_{t \in \mathbb{C}} \; : \; u(\cdot,t) \in L_2(\mathbb{R})$$

will be called a trajectory. The set of trajectories is in one-to-one correspondence to the set of permitted initial conditions.This set of initial conditions consists precisely of all entire analytic functions u_0, satisfying

$$\forall_{\varepsilon>0} \quad : \int_{-\infty}^{\infty} |u_0(x+iy)|^2 \, dx = O(e^{\varepsilon y^2}) \; ,$$

The corresponding trajectory $u(z,t)$, $z \in \mathbb{R}$, $t \in \mathbb{C}$, is given by

$$(2) \qquad u(z,t) = \frac{1}{2\sqrt{-\pi t}} \int_K \exp\left(\frac{(z-w)^2}{4t}\right) u_0(w)\, dw$$

with contour $K : \xi e^{i\theta}$, $\xi \in \mathbb{R}$, $\theta = \arg\sqrt{-\pi t}$.

Note, that (2) defines an entire analytic function of two variables (cf. [BJS],[W]).

The complex vector space $\tau(X,A)$ consists of all trajectories generated by equation (1). For the dual space $\sigma(X,A)$ of $\tau(X,A)$ we take $\sigma(X,A) = \bigcup_{t>0} X_t$. Here X_t denotes the completion of $L_2(\mathbb{R})$ with respect to the sesquilinear form $(u,v)_t = (e^{-tA}u, e^{-tA}v)_{L_2(\mathbb{R})}$. It is clear that $X_t \subset X_\tau$ if $t \le \tau$. $\sigma(X,A)$ is called the space of generalized functions. Note that $F \in \sigma(X,A)$ iff there exists $t > 0$ such that $e^{-tA} F \in L_2(\mathbb{R})$.

The pairing between $\sigma(X,A)$ and $\tau(X,A)$ is defined by

$$(3) \qquad \langle u,F\rangle =: (u(\cdot,t) \; , \; e^{-tA} F)_{L_2(\mathbb{R})} \; ,$$

$u \in \tau(X,A)$, $F \in \sigma(X,A)$. This definition makes sense if $t > 0$ is taken sufficiently large. The definition does not depend on the choice of t, since $e^{-(t+\tau)A} = e^{-tA} e^{-\tau A}$, and e^{-tA} is a symmetric operator for all $t \in \mathbb{R}$.

Suppose P is a densely defined linear operator in $L_2(\mathbb{R})$ with a densely defined adjoint P^* which leaves $\tau(X,A)$ invariant, so $P^*(\tau(X,A)) \subset \tau(X,A)$. Then $\overline{P}$ defined by

$$< u,\overline{P}\,F > \; = \; < P^* u,F >$$

maps $\sigma(X,A)$ continuously into itself. $\overline{P}$ extends P to a continuous mapping in $\sigma(X,A)$. Examples of such operators P are e^{zA}, $T_b, R_a, Z_\lambda, \mathcal{D}$ and M_F, and compositions of these. Here $(T_b f)(x) = f(x+b)$, $(R_a f)(x) = e^{iax} f(x)$. $(Z_\lambda f)(x) = f(\lambda x)$, $(\mathcal{D} f)(\xi) = \frac{df}{dx}(\xi)$, $(M_F f)(x) = F(x)\, f(x)$ with $z \in \mathbb{C}$, $a,b \in \mathbb{R}$, $\lambda \in \mathbb{R}\backslash\{0\}$, and F an entirely analytic function satisfying

$$|F(x+iy)| \le c\, e^{\varepsilon y^2} \quad , \quad x,y \in \mathbb{R} \; ,$$

for $c > 0$ and all $\varepsilon > 0$.

Some strongly divergent Fourier integrals can be interpreted as elements of $\sigma(X,A)$. Let h be a measurable function in $\mathbb{R}$ such that for some $t > 0$ the function $x \to h(x)\, e^{-tx^2}$ is in $L_2(\mathbb{R})$. The possibly divergent integral

$$(\mathbb{F}\, h)(x) = \int_{\mathbb{R}} h(y)\, e^{iyx}\, dy$$

can be considered as an element of $\sigma(X,A)$, because for t sufficiently large the function

$$e^{-tA}\, (\mathbb{F}\, h) = \int_{\mathbb{R}} h(y)\, e^{-ty^2}\, e^{iyx}\, dy$$

is in $L_2(\mathbb{R})$.

Since there is no $t > 0$ such that e^{-tA} is a Hilbert-Schmidt operator on X, there is no Kernel theorem in this case. This means that there exist continuous linear mappings from $\tau(X,A)$ into $\sigma(X,A)$ which do not arise from a generalized function of two variables in the space $\sigma(L_2(\mathbb{R}^2), A \boxplus A)$ with $A \boxplus A = -(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. For instance, the natural injection $\tau(X,A) \hookrightarrow \sigma(X,A)$ is an operator of this type.

Example 2

Let $X = L_2([0,2\pi])$, $A_\alpha = (-\frac{\partial^2}{\partial x^2})^\alpha$, $\alpha > 0$

$$D\left(-\frac{\partial^2}{\partial x^2}\right) = \left\{u \mid u \in H^2([0,2\pi]) \ , \ u(0) = u(2\pi) \ , \ u'(0) = u'(2\pi)\right\} .$$

The functions $y \to e^{iny}$, $n \in \mathbb{Z}$, are eigenfunctions of the operator A_α with eigenvalues $n^{2\alpha}$, and they establish an orthonormal basis for $L_2([0,2\pi])$. Solutions of the equation

$$\frac{\partial u}{\partial t} = \left(-\frac{\partial^2}{\partial x^2}\right)^\alpha u$$

have the form

$$u(y,t) = \sum_{n\in\mathbb{Z}} e^{n^{2\alpha}t} c_n e^{iny} , \ y \in \mathbb{R} \ , \ t \in \mathbb{C} \ .$$

So we have

$$u \in \tau(X,A_\alpha) \ \text{ iff } \ u(y,t) = \sum_{n\in\mathbb{Z}} e^{n^{2\alpha}t} c_n e^{iny}$$

in which the sequence $(e^{n^{2\alpha}t} c_n)$ converges in ℓ_2-sense for all $t \in \mathbb{C}$. It is easily seen that every trajectory $u \in \tau(X,A)$ can be uniquely identified with a function on S^1, whose Fourier series has coefficients c_n satisfying

$$\sum_{n\in\mathbb{Z}} e^{n^{2\alpha}t} |c_n|^2 < \infty ,$$

for all $t \in \mathbb{C}$.

In the same way we can prove that the generalized function space $\sigma(X,A_\alpha)$ consists of possibly divergent Fourier series $\sum_{n\in\mathbb{Z}} g_n e^{iny}$ with coefficients g_n,

$$\sum_{n\in\mathbb{Z}} e^{-n^{2\alpha}t} |g_n|^2 < \infty$$

for some $t > 0$.

Since for some $t > 0$, even for all $t > 0$, the operator e^{-tA} is Hilbert-Schmidt, the Kernel theorems hold in this case. So all continuous linear mappings from $\tau(X,A)$ into $\sigma(X,A)$ arise from a generalized function of two variables on the torus $S^1 \times S^1$ in $\sigma(L_2([0,2\pi]^2), A_\alpha \boxplus A_\alpha)$ with

$$A_\alpha \boxplus A_\alpha = \left(-\frac{\partial^2}{\partial x^2}\right)^\alpha + \left(-\frac{\partial^2}{\partial y^2}\right)^\alpha .$$

Example 3

The operators of example 2 are of a special kind. Let A be a positive self-adjoint differential operator in $L_2(I)$ with $I = (a,b)$, $-\infty \le a < b \le \infty$. Suppose, that A has an orthonormal basis of eigenvectors ψ_n in $L_2(I)$ such that $A\psi_n = \lambda_n \psi_n$ with $1 \le \lambda_1 \le \lambda_2 \le \dots$. Then we have

$$u \in \tau(L_2(I),A) \quad \text{iff} \quad u(t) = \sum_{n=1}^{\infty} e^{t\lambda_n} c_n \psi_n \ , \quad t \in \mathbb{C} \ ,$$

and the sequence converges in ℓ_2-sense for all $t \in \mathbb{C}$. We can identify u with a Fourier series $\sum_{n=1}^{\infty} c_n \psi_n$ in which the c_n satisfy

$$\sum_{n=1}^{\infty} |e^{t\lambda_n}| \ |c_n|^2 < \infty$$

for all $t \in \mathbb{C}$.

The generalized function space $\sigma(L_2(I),A)$ consists of possibly divergent Fourier series $\sum_{n=1}^{\infty} g_n \psi_n$ with coefficients g_n, satisfying

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \ |g_n|^2 < \infty \ ,$$

for some $t > 0$.

Kernel theorems hold iff for some $t > 0$

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} < \infty.$$

Examples of such operators A are

(1) $$A_1 = \tfrac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + 1\right) \text{ defined in } L_2(\mathbb{R})\,.$$

The eigenfunctions of A_1 are the Hermite functions ψ_n with eigenvalues $\lambda_n = n + 1$, $n = 0,1,2,\ldots$. $\sigma(L_2(\mathbb{R}), A_1)$ is the class of generalized functions which was first introduced by Korevaar in [K]. In the last chapter of this paper this space is more extensively discussed.

(2) $$A_2 = -x\frac{d^2}{dx^2} - \frac{d}{dx} + \frac{x}{4} + \frac{1}{2} \text{ defined in } L_2((0,\infty))\,.$$

The eigenfunctions of A_2 are the Laguerre functions L_n with eigenvalues $\lambda_n = n+1$, $n = 0,1,2,\ldots$.

(3) $$A_3 = -\frac{d}{dx}(1-x^2)\frac{d}{dx} + 1 \text{ defined in } L_2((-1,1)).$$

The eigenfunctions of A_3, which form an orthonormal basis are the functions $\psi_n = \sqrt{n+\frac{1}{2}}\, P_n$ where the P_n are the Legendre polynomials. The eigenvalues λ_n are

$$\lambda_n = n^2 + n + 1 \qquad n = 0,1,2,\ldots\ .$$

Zemanian, in chapter 9 of [Z2], describes orthonormal series expansions for generalized functions. His test and generalized function spaces are precisely the spaces $\tau(L_2(I), \log A)$ and $\sigma(L_2(I), \log (A))$. Here A is a positive self-adjoint differential operator in $L_2(I)$, which has a complete system of eigenfunctions. We also refer to Judge ([J]) who generalizes Zemanian's theory to a class of diffential operators in $L_2(I)$.

For an application of our theory to distributions in infinitely many dimensions see chapter 7 of this paper.

Chapter 1

The space $\tau(X,A)$

Throughout this paper X denotes a Hilbert space with inner product $(\cdot,\cdot)_X$. If no confusion is likely to arise this inner product will also be denoted by $(\cdot,\cdot)$. Further A denotes an unbounded positive self-adjoint operator and we suppose that $(E_\lambda)_{\lambda\geq 0}$ is the spectral resolution of the identity belonging to A. Let ψ be a complex valued and everywhere finite Borel function on $\mathbb{R}$. We define formally

$$\psi(A) = \int_{-\infty}^{\infty} \psi(\lambda)\, d E_\lambda$$

on the domain $D(\psi(A)) = \{x \in X \mid \int_{-\infty}^{\infty} |\psi(\lambda)|^2 \, d(E_\lambda x, x) < \infty\}$.
Thus $(\psi(A)x, y) = \int_{-\infty}^{\infty} \psi(\lambda)\, d(E_\lambda x, y)$ for all $x \in D(\psi(A))$ and all $y \in X$, where $(E_\lambda x, y)$ is a finite Borel measure on $\mathbb{R}$. If ψ is real valued then $\psi(A)$ is self-adjoint. We have $(\psi \cdot \chi)(A) = \psi(A)\, \chi(A)$.
The notation

$$\int_a^b \psi(\lambda)\, d E_\lambda \quad , \quad 0 < a < b \leq \infty$$

is often employed in this paper. By this we mean

$$\int_{-\infty}^{\infty} \chi_{(a,b]}(\lambda)\, \psi(\lambda)\, d E_\lambda ,$$

where $\chi_{(a,b]}$ is the characteristic function of the interval $(a,b]$. And by

$$\int_0^b \psi(\lambda)\, d E_\lambda \quad , \quad 0 < b < \infty$$

we mean

$$\int_{-\infty}^{\infty} \chi_{(-1,b]}(\lambda)\ \phi(\lambda)\ dE_\lambda ,$$

For a detailed discussion of the operator calculus of a self-adjoint operator see [Y], ch. XI. For all $t \in \mathbb{C}$ the operator e^{tA} is well-defined and $D(e^{tA})$ consists of all $f \in X$ with $\int_{\omega}^{\infty} |e^{2\lambda t}|\ d(E_\lambda f, f) < \infty$.

We introduce the space of trajectories $\tau(X,A)$.

Definition 1.1

$\tau(X,A)$ denotes the complex vector space of all mappings $u : \mathbb{C} \to X$ with the property that

i) u is holomorphic

ii) $u(t) \in D(e^{\tau A})$ and $e^{\tau A}\ u(t) = u(t+\tau)$ for all $t,\tau \in \mathbb{C}$.

The mappings u of Definition 1.1 will be called trajectories. A trajectory u is uniquely determined by u(0), because $u_1(0) = u_2(0)$ implies

$$u_1(t) = e^{tA}\ u_1(0) = e^{tA}\ u_2(0) = u_2(t)$$

for all $t \in \mathbb{C}$. It is obvious that for all $u \in \tau(X,A), u(0) \in D((e^A)^\infty)$ and $u(t) = e^{tA}\ u(0)$, $t \in \mathbb{C}$.

Definition 1.2

In $\tau(X,A)$ we introduce the seminorms p_n, $(n \in \mathbb{N})$, by

$$p_n(u) =: \| u(n) \|_X ,$$

and the strong topology in $\tau(X,A)$ will be the corresponding locally convex topology.

Theorem 1.3

Endowed with the strong topology $\tau(X,A)$ is a Frêchet space.

Proof:

In $\tau(X,A)$ we define the metric d by

$$d(u) =: \sum_{k=1}^{\infty} 2^{-k}\, p_k(u)\, (1 + p_k(u))^{-1} \quad , u \in \tau(X,A).$$

For any $u \in \tau(X,A)$ we have $d(u) \geq 0$ and finite. By standard arguments we can prove that d is a metric in $\tau(X,A)$, which generates exactly the same topology as the seminorms p_n, $n \in \mathbb{N}$

We now prove the completeness of $\tau(X,A)$.

Suppose $(u_k)_{k\in\mathbb{N}}$ is a fundamental sequence in $\tau(X,A)$. Thus for any $n \in \mathbb{N}$ the sequence $(u_k(n))_{k\in\mathbb{N}}$ is fundamental in X. Using the trajectory property 1.1.ii) we find that for any $t > 0$, the sequence $(u_k(t))_{k\in\mathbb{N}}$ is fundamental in X. Let $u_t \in X$ be the limit of the sequence $(u_k(t))_{k\in\mathbb{N}}$. Then for each $\tau > 0$ and $h \in D(e^{\tau A})$

$$(u(t), e^{\tau A}h) = \lim_{k\to\infty} (u_k(t) , e^{\tau A}h) = (u(t+\tau),h) .$$

So $u(t) \in D(e^{\tau A})$ and $e^{\tau A}u(t) = u(t+\tau)$. It is clear that by $u : t \to u(t)$, $t > 0$, we define an element of $\tau(X,A)$, and that u is the limit of the fundamental sequence $(u_k)_{k\in\mathbb{N}}$.

□

For $\tau > 0$ we define the map $e^{\tau A} : \tau(X,A) \to \tau(X,A)$ by

$$e^{\tau A}f : t \to f(t + \tau) \quad , \quad f \in \tau(X,A) .$$

Lemma 1.4

For each $\tau > 0$ the map $e^{\tau A}$ is continuous from $\tau(X,A)$ into itself.

Proof:

Let $\tau > 0$. Then there is $n \in \mathbb{N}$ such that $n > \tau$.
The conclusion follows from the fact that $e^{(\tau-n)A}$ is a bounded operator on X and the fact that p_{k+n} is a continuous seminorm in $\tau(X,A)$ for all $k \in \mathbb{N}$.

□

Definition 1.5

We define the function-algebra $Fa(\mathbb{R})$. $Fa(\mathbb{R})$ consists of all everywhere finite, locally integrable functions ϕ on $\mathbb{R}$ satisfying

$$\sup_{x>0} |\phi(x)\, e^{tx}| < \infty \quad \text{for all } t > 0.$$

$Fa^{+}(\mathbb{R})$ is the subalgebra of $Fa(\mathbb{R})$ consisting of all positive functions in $Fa(\mathbb{R})$.

Lemma 1.6

If $u \in \tau(X,A)$, then there exists $\phi \in Fa^{+}(\mathbb{R})$ and $w \in X$ such that $u : t \to e^{tA}\phi(A)\,w$, $t \in \mathbb{C}$. In other words $u(0) = \phi(A)\,w$.

Proof:

Since $u \in \tau(X,A)$, we can take $N(0) = 0$, $N(n) > N(n-1)$, such that for all $n \in \mathbb{N}$

$$\int_{N(n)}^{\infty} d(E_\lambda\, u(n), u(n)) < \frac{1}{n^2} \quad .$$

Now define $\psi \in Fa^+(\mathbb{R})$ by

$$\psi(\lambda) \quad = \quad e^{-n\lambda} \quad \text{if} \quad \lambda \in (\,N(n\,),\ N(n+1)\,] \ .$$

Then $\int_0^\infty (\psi^{-1}(\lambda))^2\, d(E_\lambda\, u(0)\ ,\ u(0)) =$

$$= \sum_{n=0}^{\infty} \int_{N(n)}^{N(n+1)} d(E_\lambda\, u(n),\ u(n)) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} + \| u(0) \|^2 \ .$$

Hence $u(0) \in D(\psi^{-1}(A))$ and $u(t) = e^{tA}\psi(A)\ \psi(A)^{-1}\ u(0) = e^{tA}\psi(A)\,w$; with $w = \psi^{-1}(A)\ u(0)$ the proof is complete. □

Lemma 1.7

i) Suppose $\psi(A)$ is compact as an operator on X for all $\psi \in Fa^+(\mathbb{R})$. Then for all $t > 0$ the operator e^{-tA} is compact on X.

ii) Suppose $\psi(A)$ is Hilbert-Schmidt as an operator on X for all $\psi \in Fa^+(\mathbb{R})$. Then there exists $t > 0$ such that the operator e^{-tA} is Hilbert-Schmidt on X.

Proof:

i) By assumption e^{-A^2} is compact as an operator on X, because $(x \to e^{-x^2}) \in$ $\in Fa^+(\mathbb{R})$. Let (μ_i) be the eigenvalues of e^{-A^2}. Then $\mu_1 \geq \mu_2 \geq \dots$ and $\mu_i \to 0$. So for all i, $(-\log \mu_i)^{\frac{1}{2}}$ is well defined and $(-\log \mu_i)^{\frac{1}{2}} \to \infty$. The numbers $(-\log \mu_i)^{\frac{1}{2}}$ are just the eigenvalues of A. Especially for $t > 0$, we have

$$\exp(-t(-\log \mu_i)^{\frac{1}{2}}) \to 0 .$$

ii) We shall prove that there is $k \in \mathbb{N}$ so that e^{-kA} is HS on X. Suppose this were not true. Then there is a sequence (N_n) with $N_{n+1} > N_n$, $N_0 = 1$ and $N_n \to \infty$ such that $\sum_{j=N_{n-1}}^{N_n} e^{-2\lambda_j n} > 1$. Here the λ_j's are the eigenvalues of A.

If for some $k \in \mathbb{N}$ there does not exist $N_k \in \mathbb{N}$ such that

$$\sum_{j=N_{k-1}}^{N_k} e^{-2\lambda_j k} > 1 \text{ , then } \forall_{\ell \in \mathbb{N}} \sum_{j=N_{k-1}}^{N_{k-1}+\ell} e^{-2\lambda_j k} \leq 1 \text{ and } e^{-kA} \text{ would}$$

be Hilbert-Schmidt.

Now define $\varphi \in Fa^+(\mathbb{R})$ by

$$\varphi(\lambda) = e^{-n\lambda} , \lambda \in (\lambda_{N_{n-1}} , \lambda_{N_n}]$$

Then $\varphi(A)$ should be Hilbert-Schmidt by assumption. But

$$\sum_{j=1}^{N_n} |\varphi(\lambda_j)|^2 = \sum_{k=1}^{n} \sum_{j=N_{k-1}}^{N_k} e^{-2\lambda_j k} > n ;$$

$\sum_{j=1}^{\infty} |\varphi(\lambda_j)|^2$ is divergent, which is a contradiction. □

Theorem 1.8

A set $B \subset \tau(X,A)$ is bounded iff for every $t \in \mathbb{C}$ the set $\{u(t) \mid u \in B\}$ is bounded in X.

Proof:

$\Rightarrow$) Each continuous seminorm p_n has to be bounded on B. Therefore, for all $n \in \mathbb{N}$ the set

$$\{u(n) \mid u \in B\}$$

is bounded in X. Because of the boundedness of $e^{-\tau A}$ for each τ with $\text{Re}\,\tau > 0$, it follows that $\{u(t) \mid u \in B\}$ is a bounded set in X for each fixed $t \in \mathbb{C}$.

$\Leftarrow$) B is bounded in $\tau(X,A)$ iff every seminorm is bounded.

□

Theorem 1.9

A set $K \subset \tau(X,A)$ is compact iff for each $t \in \mathbb{C}$ the set $\{u(t) \mid u \in K\}$ is compact.

Proof:

$\Rightarrow$) Each sequence $(u_n) \subset K$ has a convergent subsequence. This means that in the set $K_t := \{u(t) \mid u \in K\}$, $t \in \mathbb{C}$ fixed, each sequence has a convergent subsequence. So K_t is compact in X.

$\Leftarrow$) Let (u_k) be a sequence in K. We shall prove the existence of a converging subsequence by a diagonal procedure. Consider the sequence $\{u_k(1)\} \subset K_1 \subset X$. K_1 is compact therefore a convergent subsequence in K_1 exists. We denote it by $(u_k^1(1))$. The sequence $u_k^1(2)$ has a convergent subsequence in K_2. We denote it by $(u_k^2(2))$. Proceeding in this way we get sequences $(u_k^m) \subset K$ such that $(u_k^m) \subset (u_k^\ell)$ for $m < \ell$ and $(u_k^m(m))$ converges in K_m. For the diagonal sequence (u_k^k) the sequence $(u_k^k(t))$ converges to $u(t) \in K_t$. So we conclude that $u_k \to u$ in the strong topology.

□

Without proof, but for the sake of completeness we mention the following lemma.

Lemma 1.10

If p is a continuous seminorm on $\tau(X,A)$, then there exists $k \in \mathbb{N}$ and $c > 0$, such that for all $u \in \tau(X,A)$

$$p(u) \leq c \, \| u(k) \| \, .$$

Theorem 1.11

I. $\tau(X,A)$ is bornological, i.e. every circled convex subset in $\tau(X,A)$, that absorbs every bounded subset in $\tau(X,A)$ contains an open neighbourhood of 0.

II. $\tau(X,A)$ is barreled, i.e. every barrel contains an open neighbourhood of the origin. A barrel is a subset which is radial, convex, circled and closed.

III. $\tau(X,A)$ is Montel, iff there exists $t > 0$ such that e^{-tA} is compact as a bounded operator on X.

IV. $\tau(X,A)$ is nuclear, iff there exists $t > 0$ such that the operator e^{-tA} is Hilbert-Schmidt on X.

Proof:

I,II $\tau(X,A)$ is bornological and barreled, because it is metrizable. For a simple proof see [SCH],II.8.

III $\Rightarrow$)

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in X, and let $\varphi \in \mathcal{F}a^+(\mathbb{R})$. Then $(\varphi(A)x_n)$ is a bounded sequence in $\tau(X,A)$. Since $\tau(X,A)$ is Montel there exists a converging subsequence of $(\varphi(A)x_n)$. So we observe that $\varphi(A)$ is compact as an operator on X. Since $\varphi \in \mathcal{F}a^+(\mathbb{R})$ was taken arbitrarily, this holds true for all $\varphi \in \mathcal{F}a^+(\mathbb{R})$. Following Lemma 1.7 the operator e^{-tA} is compact for each $t > 0$.

$\Leftarrow$)

Let e^{-tA} be compact. We use diagonal procedure. Let (u_n) be a bounded sequence in $\tau(X,A)$. For each $\tau > 0$ the sequence $u_n(\tau)$ is bounded in X. The sequence $(e^{-tA}\, u_n(t+1))$ has a converging subsequence, $(u_n^1(1))$, say. Analogously, the sequence $(u_n^1\,(2))$ has a converging subsequence $(u_n^2\,(2))$ We obtain subsequences $(u_n^k(k))$ that converge in X and have the property that $(u_n^k) \subset (u_n^\ell)$, $\ell < k$. Now define $\tilde{u}_n =: u_n^n$. Then the sequence $(\tilde{u}_n)$ is a subsequence of (u_n), and $(\tilde{u}_n)$ converges in $\tau(X,A)$. We conclude that $\tau(X,A)$ is Montel.

IV $\Leftarrow$)

Suppose that e^{-tA} is Hilbert-Schmidt for some $t > 0$. $\tau(X,A)$ is a nuclear space if and only if for each continuous seminorm p on $\tau(X,A)$ there is another seminorm $q \geq p$ such that the canonical injection $\hat{\tau}_q \to \hat{\tau}_p$ is a nuclear map. Here the Banach space $\hat{\tau}_p$ is defined as the completion of the quotient space $\tau(X,A)/\{p^{-1}(0)\}$. We have proved in Lemma 1.10 that there are $c > 0$ and $k \in \mathbb{N}$ such that

$$p(u) \leq c \,\|u\| \quad , \quad u \in \tau(X,A) \; .$$

Hence $\hat{\tau}_{p_k}$ can be mapped into $\hat{\tau}_p$ by a bounded operator. Since the composition of a bounded operator and a nuclear operator is also nuclear, we proceed.

Let $\ell \in \mathbb{N}$, $\ell > 2t$. $p_{k+\ell} \geq p_\ell$. Let J be the canonical injection $\hat{\tau}_{p_{k+\ell}} \hookrightarrow \hat{\tau}_{p_k}$, and let λ_j, $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of A belonging to the orthonormal system (e_j), with $Ae_j = \lambda_j e_j$, $(j \in \mathbb{N})$. Then

$$Ju = \sum_{j=1}^{\infty} e^{-\lambda_j \ell} \left(e^{(k+\ell)A} u \, , \, e^{(k+\ell)A} f_j\right) g_j$$

with $f_j =: e^{-(k+\ell)\lambda_j} e_j \in \hat{\tau}_{p_{k+\ell}}$, $\|f_j\|_{k+\ell} = 1$, and

$$g_j = e^{-k\lambda_j} e_j \in \hat{\tau}_{p_k} \qquad , \quad \|g_j\|_k = 1 \; .$$

Hence J is nuclear.

$\Rightarrow$)

Suppose $\tau(X,A)$ is nuclear. Take $p(u) = \|u(0)\|$, $u \in \tau(X,A)$. Then $\hat{\tau}_p = X$, since $\tau(X,A)$ is dense in X. Hence for some seminorm q the injection $\hat{\tau}_q \hookrightarrow X$ must be nuclear. Thus e^{-kA} is a nuclear map for $k \in \mathbb{N}$ such that $p_k \geq q$ (see Lemma 1.10).

e^{-kA} is a Hilbert-Schmidt operator in X.

$\square$

Chapter 2

The space $\sigma(X,A)$

For each $t > 0$ we define the sesquilinear form

$$(x,y)_t := (e^{-tA}x,\ e^{-tA}\ y)_X \quad ,$$

and the corresponding norm $\|x\|_t =: \|e^{-tA}x\|_X$. Let X_t be the completion of X with respect to the norm $\|\cdot\|_t$. Then X_t is a Hilbert space with inner product $(\cdot,\cdot)_t$ and $F \in X_t$ iff $\|e^{-tA}\ F\| < \infty$, with e^{-tA} the linear operator on X extended to X_t. Since $\|F\|_t \geq \|F\|_\tau$ if $\tau \geq t$ we have the natural embedding

$$X_t \subset X_\tau \quad , \quad \tau \geq t \quad .$$

We remark that $e^{tA} : X \to X_t$ establishes a unitary bijection. We now define the space $\sigma(X,A)$. X can be continuously embedded in $\sigma(X,A)$.

Definition 2.1

$$\sigma(X,A) =: \bigcup_{t>0} X_t = \bigcup_{n\geq m} X_n \ , \ n,m \in \mathbb{N} \ , \ m \text{ fixed.}$$

For the strong topology in $\sigma(X,A)$ we take the inductive limit topology generated by the spaces X_t, i.e. the finest locally convex topology on $\sigma(X,A)$ for which the injections $i_t : X_t \to \sigma(X,A)$ are all continuous. The inductive limit topology is not strict. We recall that the function-algebra $\mathcal{F}a(\mathbb{R})$ consists of all $\psi : \mathbb{R} \to \mathbb{R}$ satisfying $\sup_{x\geq 0}|\psi(x)|e^{tx} < \infty$

for all $t > 0$. (see Ch. 1). For each $\psi \in Fa(\mathbb{R})$ and each $F \in \sigma(X,A)$ we may consider $\psi(A)\,F$ as an element of X as follows

$$\psi(A)\,F = \psi(A)\; e^{tA}\; e^{-tA}\,F$$

with $t > 0$ sufficiently large.

We introduce the following seminorms on $\sigma(X,A)$.

$$p_\psi(F) =: \|\psi(A)\,F\|_X \quad , \quad F \in \sigma(X,A) \;,$$

for each $\psi \in Fa(\mathbb{R})$.

Next we define the sets $U_{\psi,\varepsilon}$, $\psi \in Fa(\mathbb{R})$, $\varepsilon > 0$, by

$$U_{\psi,\varepsilon} =: \{\, F \in \sigma(X,A) \mid p_\psi(F) < \varepsilon \}.$$

Before we formulate one of the fundamental theorems of this paper, we give some conventions:

Let $F \in \sigma(X,A)$. Then there exists $t > 0$ such that $e^{-tA}F \in X$ and the following expression is correct for each $\psi \in Fa(\mathbb{R})$

i) $$\psi(A)\,F = \int_0^\infty \psi(\lambda)\; e^{\tau\lambda}\; dE_\lambda(e^{-\tau A}F) \quad , \quad \tau \ge t \;,$$

(i) does not depend on the choice of $\tau \ge t$).

Hence

ii) $$\|\psi(A)F\|^2 = \int_0^\infty |\,\psi(\lambda)|^2\; e^{2\tau\lambda}\; d(E_\lambda(e^{-\tau A}\,F)\;,\; e^{-\tau A}\,F) \;.$$

In the sequel we shall denote formally

$$\psi(A)F = \int_0^\infty \psi(\lambda)\, dE_\lambda F$$

and

$$\|\psi(A)F\|^2 = \int_0^\infty |\psi(\lambda)|^2 (E_\lambda F, F)$$

The meaning of these expression is given by (i) and (ii).

Theorem 2.2

I. $U_{\psi,\varepsilon}$, $(\psi \in Fa(\mathbb{R}), \varepsilon > 0)$ is a convex, balanced and absorbing open set in the strong topology of $\sigma(X,A)$.

II. Let a convex set $\Omega \subset \sigma(X,A)$ be such that for each $t>0$, $\Omega \cap X_t$ contains a neighbourhood of 0 in X_t. Then Ω contains a set $U_{\psi,\varepsilon}$ with $\psi \in Fa^+(\mathbb{R})$.

So the family $\{U_{\psi,\varepsilon} | \psi \in Fa^+(\mathbb{R}), \varepsilon > 0\}$ establishes a basis of the neighbourhood system of 0 in $\sigma(X,A)$.

note: A set $\Omega \subset \sigma(X,A)$ is open iff $\Omega \cap X_t$ is open in X_t for all $t > 0$.

Proof:

I. By standard arguments it is easily shown, that $U_{\psi,\varepsilon}$ is convex, balanced and absorbing. We shall only prove that $U_{\psi,\varepsilon}$ is open.

Let $t > 0$. The seminorm p_ψ is continuous on X_t , because

$$\|\psi(A)F\| = \|\psi(A)\, e^{-tA} e^{-tA} F\| \quad \|\psi(A)e^{-tA}\| \; \|e^{-tA}F\| ,$$

for all $F \in X_t$. Hence the set $U_{\psi,\varepsilon} \cap X_t$ is open in X_t.

II. We proceed in four steps.

a) Let $P_n := \int_{n-1}^{n} d\, E_\lambda$, $n \in \mathbb{N}$. Then for each $F \in \sigma(X,A)$ we have

$P_n F = \int_{n-1}^{n} d E_\lambda F$ is an element of the Hilbert space X, because the characteristic function $\chi_{(n-1,n]}$ of the interval $(n-1,n]$ is an element of the algebra $Fa^+(\mathbb{R})$. Now let $r_{n,k}$ be the radius of the largest open ball within $P_n(X_k)$ that fits within $\Omega \cap P_n(X_k)$. Thus

$$r_{n,k} = \sup\left\{\rho > 0 \,\middle|\, [F \in P_n[\sigma(X,A)] \wedge \|P_n F\|_k = \int_{n-1}^{n} e^{-2k\lambda} (E_\lambda F,F) < \rho^2] \right.$$

$$\left. \Rightarrow [F \in \Omega \cap P_n[\sigma(X,A)] \cap X_k]\right\} .$$

We have

$$\|P_n F\|_k^2 = \int_{n-1}^{n} e^{-2\lambda k}\, d(E_\lambda F,F) \leq e^{2n\ell} \int_{n-1}^{n} e^{-2\lambda(k+\ell)}\, d(E_\lambda F,F)$$

$$\|P_n F\|_k^2 \geq e^{2(n-1)\ell} \int_{n-1}^{n} e^{-2\lambda(k+\ell)}\, d(E_\lambda F,F).$$

So

$$e^{(n-1)\ell}\|P_n F\|_{k+\ell} \leq \|P_n F\|_k \leq e^{n\ell}\|P_n F\|_{k+\ell} .$$

Let $\| P_n F \|_k \le e^{(n-1)\ell} \Gamma_{n,k+\ell}$. Then $\| P_n F \|_{k+\ell} \le \Gamma_{n,k+\ell}$. So $P_n F \in \Omega \cap X_k$, and $\Gamma_{n,k} \ge e^{(n-1)\ell}\Gamma_{n,k+\ell}$. Analogously let $P_n F \in X_k$ and $\| P_n F \|_{k+\ell} \le e^{-n\ell}\Gamma_{n,k}$. Then $\| P_n F \|_k \le e^{n\ell} e^{-n\ell}\Gamma_{n,k}$; so $\Gamma_{n,k} \le e^{n\ell}\Gamma_{n,k+\ell}$.

From the above calculation, we derive

$$e^{(n-1)\ell} \Gamma_{n,k+\ell} \le \Gamma_{n,k} \le e^{n\ell}\Gamma_{n,k+\ell}$$

for all $k,\ell \in \mathbb{N} \cup \{0\}$.

b) For any fixed $p > 0$ and $k \in \mathbb{N} \cup \{0\}$ the series $\sum_{n=1}^{\infty} n^p(\Gamma_{n,k})^{-1}$ is convergent. Let $p > 0$ and $k \in \mathbb{N} \cup \{0\}$. There exists an open ball in $X_{k+\ell}$, $\ell \in \mathbb{N}$, with sufficiently small radius $\varepsilon > 0$, centered at 0 which lies entirely within $\Omega \cap X_{k+\ell}$. Then for any $n \in \mathbb{N}$ we have $\Gamma_{n,k+\ell} \ge \varepsilon$. With the inequality in a) it follows that

$$(\Gamma_{n,k})^{-1} \le e^{-(n-1)\ell} (\Gamma_{n,k+\ell})^{-1} \le \frac{1}{\varepsilon} e^{-(n-1)\ell} ,$$

for all $n \in \mathbb{N}$. From this the assertion follows.

c) We define a function ν on $\mathbb{R}$ by

$$\nu(x) = 2n^2(\Gamma_{n,0})^{-1} \quad \text{for } x \in (n-1,n]$$

$$\nu(0) = \nu(1/2) , \quad \nu(x) = 0 \quad \text{for} \quad x < 0$$

Then $\nu \in Fa^+(\mathbb{R})$.

To show this, let $t > 0$, $n \in \mathbb{N}$ and let $x \in (n-1,n]$. Then

$$\nu(x)\, e^{tx} \le 2n^2 (\Gamma_{n,0})^{-1}\, e^{nt} \le 2n^2\, e^{-(n-1)(\ell - t)}\, e^{t} (\Gamma_{n,\ell})^{-1} .$$

Taking $\ell > t$ and invoking the estimate in b) the result follows.

d) We prove

$$(*_1) \qquad \|\nu(A)\, F\| < 1 \Rightarrow F \in \Omega.$$

Suppose $F \in X_k$ for some $k \in \mathbb{N}$. Then

$$\sum_{n=1}^{\infty} \|P_n\, F\|_k^2 < \infty \text{ , and for } \ell \in \mathbb{N}$$

$$(*_2) \qquad \|P_n\, F\|_{k+\ell} \le e^{-(n-1)\ell}\, \|P_n\, F\|_k \le e^{-(n-1)\ell}\, \|F\|_k .$$

We have

$$\|P_n\, F\|_X^2 = \int_{n-1}^{n} d(E_\lambda\, F,F) = \frac{1}{4n^4}\, \Gamma_{n,0}^2 \int_{n-1}^{n} \nu^2(\lambda)\, d(E_\lambda F,F) \le \frac{1}{4n^4}\, \Gamma_{n,0}^2 .$$

So $2n^2\, P_n\, F \in (\Omega \cap X) \subset (\Omega \cap X_{k+\ell})$ for every $n \in \mathbb{N}$, $\ell \in \mathbb{N}$. In $X_{k+\ell}$ we may represent F by

$$F = \sum_{n=1}^{N} \frac{1}{2n^2}\, (2n^2\, P_n\, F) + \left(\sum_{n=N+1}^{\infty} \frac{1}{2n^2} \right) F_N$$

with $$F_N = \left(\sum_{j=N+1}^{\infty} \frac{1}{2j^2} \right)^{-1} \sum_{n=N+1}^{\infty} P_n \, F \, .$$

With $(*_2)$ it follows that

$$\| F_N \|_{k+\ell} \le 4N^4 \sum_{n=N+1}^{\infty} \| P_n \, F \|^2_{k+\ell} \le 4N^4 \, e^{-2(N+1)\ell} \| F \|^2_k \, .$$

So $F_N \to 0$ in $X_{k+\ell}$. Since $\Omega \cap X_{k+\ell}$ contains an open neighbourhood of 0, there is $N_0 \in \mathbb{N}$ such that $F_{N_0} \in \Omega \cap X_{k+\ell}$. Now $F \in \Omega \cap X_{k+\ell}$ because F is a sub-convex combination of elements in $\Omega \cap X_{k+\ell}$.

A posteriori it is clear that $F \in \Omega \cap X_k$.

Definition 2.3

A subset $W \subset \sigma(X,A)$ is called bounded if for each neighbourhood U of 0 in $\sigma(X,A)$ there exists a complex number λ such that $W \subset \lambda U$. Cf.[SCH] .

In Theorem 2.4 we characterize bounded sets in $\sigma(X,A)$.

Theorem 2.4

A set $W \subset \sigma(X,A)$ is bounded iff

$$\exists_{t>0} \, \forall_{M>0} \, \forall_{F \in W} : \| F \|_t \le M \, .$$

Proof:

We remark that W is bounded iff $\forall_{\psi \in Fa^+(\mathbb{R})} \, \exists_{M>0} \, \forall_{F \in W} : \| \psi(A) \, F \| < M$.

⇒) If not then we have

$$(*) \qquad \forall_{k\in\mathbb{N}} \forall_{M>0} \exists_{F\in W} : \| e^{-kA} F \| > M .$$

Since the function $\lambda \to e^{-\lambda^2}$ belongs to $Fa^+(\mathbb{R})$ we have

$$(**) \qquad \forall_{F\in W} \forall_{M>0} \forall_{k\in\mathbb{N}} : \int_0^M e^{-k\lambda+2\lambda^2} d(E_\lambda e^{-A^2} F, e^{-A^2} F) \le e^{2M^2} \rho^2$$

with $\rho > 0$ such that

$$\| e^{-A^2} F \|^2 < \rho^2 \quad \text{for all } F \in W .$$

If $k = 1$, then following (*) we can take $M = 2$, $N_1 > 0$ and $F_1 \in W$ such that

$$\int_0^{N_1} e^{-2\lambda} d(E_\lambda F_1, F_1) > 1 .$$

We define inductively sequences (F_k) in W, (N_k) in $\mathbb{N}$. For $k < \ell + 1$, we assume that we have found N_{k+1} such that

$$\int_{N_k}^{N_{k+1}} e^{-2k\lambda} d(E_\lambda F_k, F_k) > k .$$

Now let $k = \ell + 1$, and suppose

$$\forall_{F\in W} \forall_{K>0} \int_{N_\ell}^{N_\ell + K} e^{-2(\ell+1)\lambda} d(E_\lambda F, F) \le \ell+1$$

is true. Then W is bounded in $X_{\ell+1}$, because with (**) we deduce

$$\int_0^\infty e^{-2(\ell+1)\lambda}\, d(E_\lambda F,F) = \left(\int_0^{N_\ell} + \int_{N_\ell}^\infty\right) \le e^{2(\ell+1)N_\ell^2} \rho^2 + \ell + 1$$

for all $F \in W$.

If not choose $N_{\ell+1} > N_\ell + 1$ and $F_{\ell+1} \in W$ such that

$$\int_{N_\ell}^{N_{\ell+1}} e^{-2(\ell+1)\lambda}\, d(E_\lambda F_{\ell+1}, F_{\ell+1}) > \ell+1 \quad .$$

If our sequence terminates for some $k \in \mathbb{N}$ then W is a bounded set in X_k. If that is not the case, then define

$$\psi(\lambda) = e^{-\lambda k} \quad , \quad \lambda \in (N_{k-1}, N_k] \quad , \quad k = 1,2,\ldots \quad , \quad \text{with } N_0 = -\infty .$$

Then $\psi \in Fa^+(\mathbb{R})$, and

$$\|\psi(A)\, F_n\|^2 = \int_0^\infty |\psi(\lambda)|^2\, d(E_\lambda F_n, F_n) \ge \int_{N_{n-1}}^{N_n} e^{-2\lambda n}\, d(E_\lambda F_n, F_n) > n.$$

Contradiction .

$\Rightarrow$)

Let $\psi \in Fa^+(\mathbb{R})$. Then for all $F \in W$

$$\|\psi(A)\, F\| = \|\psi(A)\, e^{tA}\, e^{-tA} F\| \le \|\psi(A)\, e^{tA}\|\, \|F\|_t \ .$$

$\square$

In the next theorem we characterize sequential convergence in $\sigma(X,A)$.

Theorem 2.5

Let $(F_n)_{n\in\mathbb{N}}$ be a sequence in $\sigma(X,A)$. Then we have $F_n \to 0$ in the strong topology iff there exists $t > 0$ such that $(F_n) \subset X_t$ and $\|F_n\|_t \to 0$.

Proof

$\Leftarrow$) $\quad \|\psi(A)\ F_n\| = \|\psi(A)\ e^{tA}\ e^{-tA}\ F_n\| \le \|\psi(A)\ e^{tA}\|\ \|F_n\|_t \to 0$.

$\Rightarrow$) Suppose $F_n \to 0$. Then for any $\psi \in Fa^+(\mathbb{R})$

$$\|\psi(A)\ F_n\| \to 0 \ .$$

Hence (F_n) is a bounded sequence in $\sigma(X,A)$. So there exist $M > 0$ such that $\|F_n\|_t < M$, $(n \in \mathbb{N})$, for some $t > 0$. Let $\tau > t$.

$$\|F_n\|_\tau^2 = \int_0^L e^{-2\tau\lambda}\ d(E_\lambda\ F_n,F_n) + \int_L^\infty e^{-2\tau\lambda}\ d(E_\lambda\ F_n,F_n) \ .$$

First, choose $L > 0$ so large that

$$(*) \qquad \int_L^\infty e^{-2\tau\lambda}\ d(E_\lambda\ F_n,F_n) \le e^{-2(\tau-t)L} \int_0^\infty e^{-2t\lambda}\ d(E_\lambda\ F_n,F_n) \le$$

$$\le e^{-2(\tau-t)L}\ M < \varepsilon^2/4$$

for all $n \in \mathbb{N}$, and $\varepsilon > 0$ fixed.

Next, observe that the function

$$\psi(\lambda) = \begin{cases} e^{-\tau\lambda} & \text{if } \lambda \in [0,L] \\ 0 & \text{elsewhere} \end{cases}$$

is in $Fa^+(\mathbb{R})$. So there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$.

(**) $\|\psi(A)\, F_n\| < \varepsilon/2$.

From (*) and (**) the assertion follows.

□

Theorem 2.6

i) Suppose $(F_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\sigma(X,A)$. Then there exists $t > 0$ with $(F_n) \subset X_t$ and (F_n) a Cauchy sequence in X_t.

ii) $\sigma(X,A)$ is sequentially complete.

Proof:

i) An argument similar to the proof of the preceding theorem.

ii) Follows from i) and the completeness of X_t.

□

Theorem 2.7

A subset $K \subset \sigma(X,A)$ is compact iff there exists $t > 0$ such that $K \subset X_t$ and K is compact in X_t.

Proof

$\Leftarrow$) let (Ω_α) be an open covering of K in $\sigma(X,A)$. Then $(\Omega_\alpha \cap X_t)$ is an open covering of K in X_t. So there exists a finite subcovering of (Ω_α), $(\Omega_{\alpha_i})_{i=1}^{N}$, say, with

$$K \subset \bigcup_{i=1}^{N} (\Omega_{\alpha_i} \cap X_t) \subset \bigcup_{i=1}^{N} \Omega_{\alpha_i} .$$

$\Rightarrow$) K is compact, hence a bounded set in $\sigma(X,A)$. So there is $t > 0$ such that $K \subset X_t$ is bounded in X_t, with bound M, say. We show that K is compact in $X_{t+\tau}$, $\tau > 0$. Let (F_n) be a sequence in K. Then there exists a converging subsequence $(F_{n_j}) \subset K$ with $F_{n_j} \to F$, convergence in $\sigma(X,A)$. So

$(F_{n_j} - F)$ is a bounded sequence in X_t and $\| \psi(A)(F_{n_j} - F) \| \to 0$ for all $\psi \in Fa^+(\mathbb{R})$ defined by

$$\psi(\lambda) = \begin{cases} e^{-(t+\ )} & \text{if } \lambda \in [0,T] \\ 0 & \text{elsewhere} \end{cases}$$

with arbitrary $T > 0$ and $\tau > 0$, fixed. We conclude (cf. the proof of Theorem 2.5) that

$$\| F_{n_j} - F \|_{t+\tau} \to 0 .$$

Thus K is compact in $X_{t+\tau}$. □

We define the following sesquilinear form in X

$$(x,y)_\psi = (\psi(A)x , \psi(A)y) \quad , \quad x,y \in X,$$

for $\psi \in Fa^+(\mathbb{R})$. Let X_ψ be the completion of X with respect to the norm $\| x \|_\psi =: \| \psi(A)x \|_X$. Then X_ψ is a Hilbert space with the sesquilinear form $(\cdot,\cdot)_\psi$ extended to X_ψ as an inner product. Note that X_ψ is naturally injected in X_χ if $\psi \geq \chi$.

Lemma 2.8

Let $H \in \bigcap_{\psi \in Fa^+(\mathbb{R})} X_\psi$. Then $H \in \sigma(X,A)$.

Proof

Suppose this were not true. Then for every $k \in \mathbb{N}$

$$\lim_{L\to\infty} \int_0^L e^{-2k\lambda} \, d(E_\lambda H,H) = \infty .$$

Thus there is a sequence (N_k), $N_0 = -\infty$, $N_{k-1} < N_k$, $(k \in \mathbb{N})$, and $N_k \to \infty$ such, that for all $k \in \mathbb{N}$

$$\int_{N_{k-1}}^{N_k} e^{-2k\lambda} \, d(E_\lambda H,H) > 1 .$$

Define χ on $(0,\infty)$ by $\chi(\lambda) = e^{-k\lambda}$, $\lambda \in (N_{k-1},N_k]$. Then $\chi \in \mathcal{F}a^+(\mathbb{R})$ and $\|\chi(A)H\| = \infty$.

Contradiction!

□

In the following theorem we use the standard terminology of topological vector spaces (see [SCH]) in order to make a link to the general literature about this subject.

<u>Theorem 2.9</u>

I. $\sigma(X,A)$ is complete.

II. $\sigma(X,A)$ is bornological.

III. $\sigma(X,A)$ is barreled.

IV. $\sigma(X,A)$ is Montel iff there exists $t > 0$ such that the operator e^{-tA} is compact on X.

V. $\sigma(X,A)$ is nuclear iff there exists $t > 0$ such that the operator e^{-tA} is Hilbert-Schmidt on X.

Proof:

I. Let (F_i) be a Cauchy net in $\sigma(X,A)$ with $i \in D$, D a directed set. Then for each $\varphi \in Fa^+(\mathbb{R})$, $(\varphi(A)F_i)$ is a Cauchy net in X. Since X is complete, there exists $F_\varphi \in X$ such that $\varphi(A)F_i \to F_\varphi$.
Let $\psi,\chi \in Fa^+(\mathbb{R})$. Then a simple calculation shows (T):$F_{\psi\cdot\chi} = \psi(A)\, F_\chi = \chi(A)\, F_\psi$. Define $F \in X_\psi$ by $F =: \psi^{-1}(A)\, F_\psi$. Let $\chi \in Fa^+(\mathbb{R})$. Then $\chi^{-1}(A)\, F_\chi \in X_\chi$ and with (T)

$$\chi^{-1}(A)\, F_\chi = \chi^{-1}(A)\psi^{-1}(A)\, F_{\chi\cdot\psi} = \psi^{-1}(A)\, F_\psi = F\ .$$

So $F \in \bigcap_{\varphi\in Fa^+(\mathbb{R})} X_\varphi$; thus $F \in \sigma(X,A)$. Finally, $\|\chi(A)(F_i - F)\| = \|\chi(A)F_i - F_\chi\| \to 0$ for all $\chi \in Fa^+(\mathbb{R})$. Thus $\sigma(X,A)$ is complete.

II. Bornological means that every circled convex subset $\Omega \subset \sigma(X,A)$ that absorbs every bounded subset $B \subset \sigma(X,A)$ contains an open neighbourhood of 0. Now let $\Omega \subset \sigma(X,A)$ be such a subset. Let U_t be the open unit ball in X_t , $t > 0$. U_t is bounded in $\sigma(X,A)$, so for some $\varepsilon > 0$ one has $\varepsilon\, U_t \subset \Omega \cap X_t$. We conclude that $\Omega \cap X_t$ contains an open neighbourhood of 0 for every $t > 0$. Following Theorem 2.2 Ω contains a set $U_{\psi,\varepsilon}$.

III. A barrel V is a subset which is radial, convex, circled and closed. We have to prove that every barrel contains an open neighbourhood of the origin. Because of the definition of the inductive limit topology $V \cap X_t$ has to be a barrel in X_t for each $t > 0$. Since X_t is a Hilbert space, X_t is barreled, and there exists an open neighbourhood of the origin, O, say with $O \subset V \cap X_t$. Again the conditions of Theorem 2.2 are satisfied so that V contains a set $U_{\psi,\varepsilon}$.

IV. ⇐ Suppose e^{-tA} is compact.

Let $W \subset \sigma(X,A)$ be closed and bounded. Then $W \subset X_{t_0}$ for some $t_0 > 0$ and W is closed and bounded in all $X_{t_0+\tau}$, $\tau > 0$. Let $\hookrightarrow$ denote the natural injection of X_{t_0} in X_{t_0+t}, and consider the diagram

$$\begin{array}{ccc} X_{t_0} & \hookrightarrow & X_{t_0+t} \\ \uparrow e^{t_0 A} & & \uparrow e^{(t+t_0)A} \\ X & \xrightarrow[e^{-tA}]{} & X \end{array}$$

Since the vertical arrows are isomorphisms, $\hookrightarrow$ is a compact map and W is compact in X_{t_0+t}. So W is compact in $\sigma(X,A)$.

⇒) Suppose $\sigma(X,A)$ is Montel. Let (u_n) be a bounded sequence in X. Then (u_n) is bounded in $\sigma(X,A)$. Consider the closure of the sequence (u_n) in $\sigma(X,A)$. This closure is a closed and bounded set in $\sigma(X,A)$. Thus (u_n) contains a $\sigma(X,A)$-converging subsequence, (u_{n_j}), say. So$(\psi(A)u_{n_j})$ is X convergent for all $\psi \in Fa^+(\mathbb{R})$. Thus $\psi(A)$ is compact as an operator on X for all $\psi \in Fa^+(\mathbb{R})$. Then by Lemma 1.7 the operator e^{-tA} is compact for each $t > 0$.

V. ⇐) Suppose e^{-tA} is Hilbert-Schmidt. Then there is an orthonormal sequence (e_n), which is a complete basis for X and

$$A e_n = \lambda_n e_n \quad , \quad \text{with } 0 < \lambda_1 \le \lambda_2 \le \dots \quad , \quad \lambda_n \to \infty,$$

and $\sum_{n=1}^{\infty} e^{-\lambda_n t} < \infty$.

$\sigma(X,A)$ is nuclear iff for every continuous seminorm p on $\sigma(X,A)$ there

is another seminorm $q \geq p$ such that the canonical injection $\hat{\sigma}_q \hookrightarrow \hat{\sigma}_p$ is a nuclear map. Here $\hat{\sigma}_p$ is the completion of $\sigma(X,A)/_{p^{-1}(\{0\})}$. Since the composition of a nuclear operator with a bounded operator is again nuclear, we may restrict ourselves to seminorms p_ψ, $\psi \in Fa^+(\mathbb{R})$.
Take $|\psi| \leq 1$. If $\psi \in Fa^+(\mathbb{R})$, then $e^{tA}\,\psi(A)$ is a bounded operator on X. So for each $\nu > 0$ the operator $(\psi(A))^\nu = e^{-tA}(e^{\frac{t}{\nu}A}\,\psi(A))^\nu$ is Hilbert-Schmidt. Now take $\psi \in Fa^+(\mathbb{R})$ and $\chi = \psi^{\frac{1}{2}}$. Then the canonical injection $J : \hat{\sigma}_\chi \hookrightarrow \hat{\sigma}_\psi$ can be written as

$$Ju = \sum_{n=1}^{\infty} \psi^{\frac{1}{2}}(\lambda_n)(\psi^{\frac{1}{2}}(A)\,u\;,\;\psi^{\frac{1}{2}}(A)(\psi^{-\frac{1}{2}}(\lambda_n)\,e_n))\psi^{-1}(\lambda_n)\,e_n$$

Since $\psi^{-\frac{1}{2}}(\lambda_n)\,e_n \in \hat{\sigma}_\chi$ and $\psi^{-1}(\lambda_n)\,e_n \in \hat{\sigma}_\psi$, with

$$\|\psi^{-\frac{1}{2}}(\lambda_n)\,e_n\|_\chi = 1 \quad \text{and} \quad \|\psi^{-1}(\lambda_n)\,e_n\| = 1 \quad , n \in \mathbb{N} ,$$

and since $\sum_{n=1}^{\infty} |\psi(\lambda_n)|^{\frac{1}{2}} < \infty$, J is a nuclear map

$\Rightarrow$) Suppose $\sigma(X,A)$ is nuclear. The Hilbert space X may be injected in every $\hat{\sigma}_\psi$ with $\psi \in Fa^+(\mathbb{R})$. Let $\psi \in Fa^+(\mathbb{R})$ and $\chi \in Fa^+(\mathbb{R})$ with $\chi \geq \psi$ such that $J_{\chi,\psi}$ is nuclear. The canonical injection $J_\psi : X \hookrightarrow \hat{\sigma}_\psi$ is equal to $J_\chi \cdot J_{\chi,\psi}$ with $J_\chi : X \hookrightarrow \hat{\sigma}_\chi$. Since J_χ is bounded, and $J_{\chi,\psi}$ is nuclear J_ψ is a nuclear mapping. So $\psi(A)$ is a Hilbert-Schmidt operator on X. Since this holds true for all $\psi \in Fa^+(\mathbb{R})$, by Lemma 1.7 the operator e^{-tA} is Hilbert-Schmidt on X for a well-chosen $t > 0$.

□

Chapter 3

The pairing of $\tau(X,A)$ and $\sigma(X,A)$

On $\tau(X,A) \times \sigma(X,A)$ we introduce a sesquilinear form by

$$< u , F > =: \quad (u(t) \ , \ e^{-tA} F)_X \ .$$

Note, that this definition makes sense for $t \geq 0$ sufficiently large, and that it does not depend on the choice of $t > 0$. We remark that $<g,F> = 0$ for all $F \in \sigma(X,A)$ implies $g = 0$ (use the fact that $X \subset \sigma(X,A)$), and also that $<u,G> = 0$ for all $u \in \tau(X,A)$ implies $G = 0$. We prove the last assertion. So suppose that $<u,G> = 0$ for all $u \in \tau(X,A)$. Then following Lemma 1.6 we have

$$< \varphi(A)\ w, \ G > = \quad (e^{tA}\varphi(A)\ w, \ e^{-tA}\ G)_X = \quad (w, \ \varphi(A)\ G)_X = 0 \ ,$$

for any $w \in X$ and $\varphi \in Fa^{+}(\mathbb{R})$. Hence $\varphi(A)\ G = 0$ for all $\varphi \in Fa^{+}(\mathbb{R})$. Thus $G = 0$.

Theorem 3.1

i) For each $F \in \sigma(X,A)$ the linear functional $g \to <g,F>$ is continuous in the strong topology of $\tau(X,A)$.

ii) For each strongly continuous linear functional ℓ on $\tau(X,A)$ there exists $G \in \sigma(X,A)$ such that $\ell(u) = <u,G>$ for all $u \in \tau(X,A)$.

iii) For each $v \in \tau(X,A)$ the linear functional $G \to \overline{<v,G>}$ is continuous in the strong topology of $\sigma(X,A)$.

iv) For each strongly continuous linear functional m on $\sigma(X,A)$ there exists $w \in \tau(X,A)$ such that $m(G) = \overline{\langle w,G \rangle}$ for all $G \in \sigma(X,A)$.

Proof:

i) Let $g_n \to 0$ in $\tau(X,A)$, and let $F \in \sigma(X,A)$. Then

$$|\langle g_n , F \rangle| = |(g_n(t) , e^{-tA} F)_X| \leq \| g_n(t) \| \| e^{-tA} F \| \to 0 ,$$

whenever $t > 0$ is large enough.

ii) Let ℓ be a continuous linear functional in $\tau(X,A)$. Let $\varphi \in Fa^+(\mathbb{R})$. Then the linear functional $\ell_\varphi(x) = \ell(\varphi(A)x)$, $(x \in X)$, is continuous on X, and so there exists $f_\varphi \in X$ such that $\ell_\varphi(x) = (x, f_\varphi)$ for all $x \in X$. We have

$$(*) \qquad f_{\varphi \bullet \psi} = \varphi(A)\, f_\psi = \psi(A)\, f_\varphi \quad , \quad \varphi,\psi \in Fa^+(\mathbb{R}) \quad ,$$

and

$$\varphi^{-1}(A)\, f_\varphi \in X_\varphi \quad , \quad \varphi \in Fa^+(\mathbb{R}) \quad .$$

Now let $F^\varphi = \varphi^{-1}(A)\, f_\varphi$ for each $\varphi \in Fa^+(\mathbb{R})$. Then

$$F^\varphi = \varphi^{-1}(A)\, f_\varphi = \varphi^{-1}(A)\, \psi^{-1}(A)\, \psi(A)\, f_\varphi = \psi^{-1}(A)\, f_\psi = F^\psi$$

with the aid of $(*)$. Take $\varphi \in Fa^+(\mathbb{R})$ fixed, and let $F = F^\varphi$. Then from the above paragraph we have

$$\forall_{\psi \in Fa^+(\mathbb{R})} : \quad F = F^\psi \quad \text{and} \quad \psi(A)\, F = f_\psi \in X$$

So $F \in \bigcap_{\varphi \in \mathcal{F}a^{+}(\mathbb{R})} X_{\varphi}$

Following Lemma 2.8 we have $F \in \sigma(X,A)$, and there exists $t > 0$ such that

$$\ell(h) = \ell(\varphi(A)\ \varphi^{-1}(A)h) = (\varphi^{-1}(A)h\ ,\ \varphi(A)F)_X =$$

$$= (\varphi^{-1}(A)h\ ,\ \varphi(A)\ e^{tA}\ e^{-t(A)}\ F)_X$$

$$= (h(t)\ ,\ e^{-tA}F)_X = \langle h\ ,\ F\rangle \quad ,\quad h \in \tau(X,A)\ .$$

iii)Let $v \in \tau(X,A)$, and let $G_n \to 0$ in the strong topology of $\sigma(X,A)$. Then there exists $t > 0$ such that $\| e^{-tA}\ G_n \| \to 0$. Hence

$$|\langle v, G_n\rangle| \leq \| v(t) \| \| e^{-tA}\ G_n \| \to 0\ .$$

iv) Let m be a continuous linear funtional in $\sigma(X,A)$. Then for each $t > 0$ the linear functional $m \circ e^{tA}$ is continuous on X.

So for all $t > 0$ there exists $x(t) \in X$ such that

$$m \circ e^{tA}\ (g) = (g,\ x(t)) \quad ,\quad g \in X\ .$$

If $g \in D(e^{\tau A})$, $\tau > 0$, then

$$m \circ e^{tA}\ \ (e^{\tau A}\ g) \ = \ (e^{\tau A}\ g,\ x(t))$$

and also $\quad m \circ e^{tA}\ (e^{\tau A}\ g) = (g,\ x(t+\tau)\ .$

Thus $x(t) \in D(e^{\tau A})$ for every $\tau > 0$, and $x(t + \tau) = e^{\tau A} x(t)$. Define $w \in \tau(X,A)$ by

$$w : t \to x(t) .$$

Then $m(G) = m \circ e^{tA} (e^{-tA} G) = \overline{(x(t) , e^{-tA} G)_X} = \overline{\langle w,G\rangle}$. □

Definition 3.2

The weak topology in $\tau(X,A)$ is the toplogy generated by the seminorms $|\langle u,F\rangle|$, $F \in \sigma(X,A)$. The weak topology in $\sigma(X,A)$ is the topology generated by the seminorms $|\langle u,F\rangle|$, $u \in \tau(X,A)$.

A standard argument, e.g.[CH] II, § 22, shows that the weakly continuous functionals on $\tau(X,A)$ are all obtained by pairing with elements of $\sigma(X,A)$, and vice versa. From this assertion and from Theorem 3.1 it then follows that $\sigma(X,A)$ and $\tau(X,A)$ are reflexive in the strong as well as in the weak topology.

Theorem 3.3

i) Let $Z \subset \sigma(X,A)$ be such that for each $g \in \tau(X,A)$ there exists $M_g > 0$ such that for every $H \in Z$

$$|\langle g,H\rangle| \le M_g \quad .$$

Then there exists $t > 0$ and $M > 0$ such that for every $H \in Z$

$$\| e^{-tA} H \|_X \le M \ .$$

ii) Let $P \subset \tau(X,A)$ be such that for each $F \in \sigma(X,A)$ there exists $M_F > 0$

such that for every $g \in P$

$$|\langle g,F\rangle| \leq M_F \quad .$$

Then for every $t > 0$, there exists $M_t > 0$ such that

$$\| g(t) \|_X \leq M_t \ .$$

Proof:

Let $\psi \in Fa^+(\mathbb{R})$. Then following Lemma 1.6

$$\forall_{\chi\in X} \ \exists_{M_{\psi,\chi}} : | < \psi(A)\, \chi \,,\, G > | \leq M_{\psi,\chi} \quad , \ G \in Z \ .$$

Hence, from the Banach-Steinhaus theorem in Hilbert spaces, we derive

$$\exists_{M_\psi>0} \ \|\psi(\ )G\| \leq M_\psi \quad , \qquad G \in Z \ .$$

Since $\psi \in Fa^+(\mathbb{R})$ arbitrary, the set $Z \subset \sigma(X,A)$ is bounded. With Theorem 2.4, the result follows.

ii) Let $t > 0$, $\chi \in X$. Following our assumption, there exists $M_{t,\chi} > 0$ such that

$$\forall_{g\in P} \ |\langle g, e^{tA}\, \chi\rangle| \ = \ |(g(t),\ \chi)| \ < \ M_{t,\chi} \quad .$$

Hence, there exists $M_t > 0$ such that

$$\| g(t) \| \leq M_t \quad ,$$

for all $g \in P$.

□

Theorem 3.4 (weak convergence in $\tau(X,A)$)

$g_n \to 0$ weakly in $\tau(X,A)$ iff

$$\forall_{t>0}\ \forall_{x\in X} \ : \ (g_n(t),\ x)_X \to 0 \ .$$

Proof:

$\Rightarrow$) For all $x \in X$: $\langle g_n, e^{tA} x\rangle = (g_n(t),\ x)_X \to 0$.

$\Leftarrow$) For all $G \in \sigma(X,A)$ there is $t > 0$, sufficiently large such that $e^{-tA}\ G \in X$. So

$$\langle g_n\ ,\ G\rangle = (g_n(t)\ ,\ e^{-tA}\ G)_X \to 0\ .$$

□

Corollary 3.5

i) Strong convergence in $\tau(X,A)$ implies weak convergence.

ii) Every bounded sequence in $\tau(X,A)$ has a weakly converging subsequence.

(with a diagonal argument!)

Theorem 3.6 (weak convergence in $\sigma(X,A)$)

$G_n \to 0$ weakly in $\sigma(X,A)$ iff

$$\exists_{t>0}\ :\ (G_n) \subset X_t\ ,\ \text{and}\ \forall_{\omega\in X_t}\ :\ (\omega,G_n)_t \to 0\ .$$

Proof:

$\Leftarrow$) Let $u \in \tau(X,A)$. Since $e^{-tA}\ G_n \to 0$ weakly in X, and $u(t) \in X$, it follows that $\langle g,G_n\rangle = (g(t),\ e^{-tA}\ G_n)_X \to 0$.

$\Rightarrow$) The set $\{G_n | n \in \mathbb{N}\} \subset \sigma(X,A)$ is bounded. So following Theorem 2.4

there exists $t > 0$ and $M > 0$ such that

$$\|e^{-tA}\, G_n\|_X \le M \ , \quad (n \in \mathbb{N}) .$$

Now let $x \in X$, and let $\tau > 0$. Then

$$(*) \qquad \left| \int_L^\infty e^{-\lambda(t+\tau)} \, d(E_\lambda \, G_n, x) \right| \le e^{-L\tau} \int_L^\infty e^{-\lambda t} \, d|(E_\lambda \, G_n, x)| \le M \, \|x\| \, e^{-L\tau} .$$

Since $\langle \psi(A)x \ , \ G_n \rangle \to 0$, $(n \to \infty)$, for all $\psi \in Fa^+(\mathbb{R})$, by assumption, we may take

$$\psi_L(\lambda) := \begin{cases} e^{-\lambda(t+\tau)} & \text{if } \lambda \in (0,L] \\ 0 & \text{elsewhere} \end{cases} .$$

Then $\psi_L \in Fa^+(\mathbb{R})$ for every $L > 0$, and

$$(**) \qquad \int_0^\infty \psi_L(\lambda) \quad d(E_\lambda \, G_n, x) \to 0 \ .$$

From (*) and (**) we obtain

$$\int_0^\infty e^{-\lambda(t+\tau)} \quad d(E_\lambda \, G_n, x) \to 0 \ .$$

So for all $\tau > 0$ and all $x \in X$

$$(G_n) \subset X_{t+\tau} \quad \text{and} \ (x, \ e^{-(t+\tau)A} \, G_n)_X \to 0$$

□

Corallary 3.7

i) Strong convergence of a sequence in $\sigma(X,A)$ implies its weak convergence.

ii) Every bounded sequence in $\sigma(X,A)$ has a weakly converging subsequence.

Theorem 3.8

The following three statements are equivalent.

i) There exists $t > 0$ such that e^{-tA} is a compact operator in X.

ii) Each weakly convergent sequence in $\tau(X,A)$ converges strongly in $\tau(X,A)$.

iii) Each weakly convergent sequence in $\sigma(X,A)$ converges strongly in $\sigma(X,A)$.

Proof:

i) $\Rightarrow$ ii) Let $(f_n) \subset \tau(X,A)$, and suppose $f_n \to 0$ weakly. Then $\forall_{x\in X} \forall_{t>0} : (f_n(\tau),x)_X \to 0$. So $f_n(\tau) \to 0$ weakly in X for all $\tau > 0$. Using the compactness of e^{-tA} we get

$$\forall_{\tau>t} : e^{-tA} f_n(\tau) = f_n(\tau - t) \to 0$$

strongly in X.

ii) $\Rightarrow$ i) Let $(x_n) \subset X$ with $x_n \to 0$ weakly in X, and let $\varphi \in Fa^+(\mathbb{R})$. Then $\varphi(A)\, x_n \to 0$ weakly in $\tau(X,A)$, and by assumption also strongly. We conclude that $\varphi(A)\, x_n \to 0$ strongly in X. So $\varphi(A)$ is compact as an operator in X. Following Lemma 1.7 there exists $t > 0$ such that e^{-tA} is compact.

i) $\Rightarrow$ iii) A weakly convergent sequence in $\sigma(X,A)$ converges weakly in some X_τ, $\tau > 0$. The natural injection $X_\tau \hookrightarrow X_{t+\tau}$ is compact. But then our sequence converges strongly in $X_{t+\tau}$.

iii) $\Rightarrow$ i) Let $x_n \to 0$ weakly in X. Then $x_n \to 0$ weakly is $\sigma(X,A)$. So for all

$\varphi \in Fa^{+}(\mathbb{R})$ we have

$$\varphi(A)\ x_n \to 0 \quad \text{strongly in } X ,$$

with the aid of iii). This implies that $\varphi(A)$ is compact s an operator in X for all $\varphi \in Fa^{+}(\mathbb{R})$. Hence following Lemma 1.7 there exists $t > 0$ such that e^{-tA} is compact.

□

Chapter 4

Characterization of continuous linear mappings between the spaces $\tau(X,A)$, $\tau(Y,B)$, $\sigma(X,A)$ and $\sigma(Y,B)$

Let $B \geq 0$ be a self-adjoint operator in a Hilbert space Y. In this chapter we shall derive some necessary and sufficient conditions such that the linear mappings $\tau(X,A) \to \tau(Y,B)$, $\tau(X,A) \to \sigma(Y,B)$, $\sigma(X,A) \to \tau(Y,B)$ and $\sigma(X,A) \to \sigma(Y,B)$ are continuous. First we prove some auxiliary results.

Theorem 4.1

Let L be a densely defined linear operator from $D(L) \subset X$ into Y, and let $L\psi(A) : X \to Y$ be defined and bounded for all $\psi \in Fa^+(\mathbb{R})$. Then there exists $t > 0$ such that the operator $L\, e^{-tA}$ is bounded.

Proof

Suppose the operator $L\, e^{-kA}$ is unbounded for all $k \in \mathbb{N}$. Then we have

$$(*) \qquad \forall_{k\in\mathbb{N}}\ \forall_{a>0}\ \forall_{C>0}\ \exists_{b>a} \ : \ \| L\, P_{(a,b]}\, e^{-kA} \| > C$$

Here we use the notation $P_{(a,b]} = \int_{-\infty}^{\infty} \chi_{(a,b]}(\lambda)\, d\, E_\lambda$, see Chapter 1. With the aid of $(*)$ we construct a sequence $(N_k) \subset \mathbb{R}^+$ with $N_0 = -\infty$, $0 < N_1 < N_2 < \ldots$ and $N_j \to \infty$, such that

$$\| L\, P_{(N_{k-1},N_k]}\, e^{-kA} \| > k \quad , \ (k \in \mathbb{N}) \ .$$

For each $k \in \mathbb{N}$ there exists $y_k \in Y$ with $\| y_k \| = 1$, and

$$\| (L\, P_{(N_{k-1},N_K]}\, e^{-kA})^*\, y_k \| > k\ .$$

(We note that $L\, P_{(N_{k-1},N_k]}$ is a bounded operator from X into Y.)

Now let $\varphi : \mathbb{R} \to \mathbb{R}$ be given by

$$\varphi(\lambda) = e^{-\lambda k} \quad \text{if} \quad \lambda \in (N_{k-1},N_k]\ .$$

Then $\varphi \in Fa^+(\mathbb{R})$, so $L\varphi(A)$ is bounded. But

$$\| L\ \varphi(A) \|^2 = \| \sum_{k=1}^{\infty} L\ e^{-kA}\ P_{(N_{k-1},N_k]} \|^2 =$$

$$= \| \sum_{k=1}^{\infty} P_{(N_{k-1},N_k]} (L\ e^{-kA}\ P_{(N_{k-1},N_k]})^* \|^2 \geq$$

$$\geq \| \sum_{k=1}^{\infty} P_{(N_{k-1},N_k]}\ (L\ e^{-kA}\ P_{(N_{k-1},N_k]})^* x \|^2 =$$

$$= \sum_{k=1}^{\infty} \| (L\ e^{-kA}\ P_{(N_{k-1},N_k]})^* x \|^2$$

for every $x \in X$ with $\| x \| = 1$.

Especially for $x = y_\ell$, we get

$$\| L\ \varphi(A) \|^2 \geq \sum_{k=1}^{\infty} \| (L\ e^{-kA}\ P_{(N_{k-1},N_k]})^*\ y_\ell \|^2 > \ell^2\ .$$

Contradiction!

$\square$

In the same way we can prove:

Corollary 4.2

Let $K : X \to Y$ be a densely defined linear operator such that $\varphi(B)\, K$ can be extended to a bounded operator in X for all $\varphi \in Fa^+(\mathbb{R})$. Then there exists $t > 0$ such that $e^{-tB}\, K$ can be extended to a continuous operator from X into Y.

Lemma 4.3

A linear mapping $L : \tau(X,A) \to Y$ is continuous in the strong topologies of $\tau(X,A)$ and Y iff there exists $t > 0$ such that $L\, e^{-tA}$ is a bounded operator from $\tau(X,A) \subset X$ into Y.

Proof:

$\Rightarrow$) Let $\varphi \in Fa^+(\mathbb{R})$. The mapping $\varphi(A) : X \to \tau(X,A)$ is continuous, because $\varphi(A)\, e^{tA}$ is bounded for all $t > 0$. Now suppose that $L : \tau(X,A) \to Y$ is continuous. Then the linear operator $L\, \varphi(A) : X \to Y$ is continuous. Since $\varphi \in Fa^+(\mathbb{R})$ is taken arbitrarily, we apply Theorem 4.1 and find that there is $t > 0$ such that $L\, e^{-tA}$ is a bounded operator in X.

$\Leftarrow$) Let (u_n) be a nullsequence in $\tau(X,A)$. Then $L\, u_n$ is a nullsequence in Y, because $L\, u_n = (L\, e^{-tA})\, u_n(t)$ and $L\, e^{-tA}$ is bounded for $t > 0$ sufficiently large.

Lemma 4.4

A linear mapping $K : X \to \sigma(Y,B)$ is continuous in the strong topology of both X and $\sigma(Y,B)$ iff there exists $t > 0$ such that $e^{-tB} K$ is a continuous operator from X into Y.

Proof:

$K : X \to \sigma(Y,B)$ is continuous iff $\varphi(B)\,K : X \to Y$ is continuous for all $\varphi \in Fa^+(\mathbb{R})$.

$\Rightarrow$) Follows form Corollary 4.2

$\Leftarrow$) Trivial, because each $\varphi(B)\ e^{tB}$ is bounded for each $\varphi \in Fa^+(\mathbb{R})$.

□

Lemma 4.5

A linear mapping $P : \sigma(X,A) \to V$, where V is an arbitrary locally convex topological vector space, is continuous

i) iff for each $t > 0$ the mapping $P\ e^{tA} : X \to V$ is continuous.

ii) iff for each nullsequence (G_n) in $\sigma(X,A)$ the sequence $(P\ G_n)$ is a nullsequence in V.

Proof:

i) $\sigma(X,A)$ has the inductive limit topology therefore P has to be continuous when restricted to X_t.

$\Rightarrow$) e^{tA} is a continuous isomorphism from X onto X_t, and X_t is continuously injected in $\sigma(X,A)$ if the latter has the inductive limit topology. So $P\ e^{tA}$ is continuous from X into V.

$\Leftarrow$) Let P_t denote the restriction of P to X_t. Since $P\ e^{tA}$ is continuous on X, P_t is continuous on X_t. Let Ω be an open-0-neighbourhood in V. Then for each $t > 0$, $P^{-1}(\Omega) \cap X_t = P_t^{-1}(\Omega)$ is an open-0-neigbourhood in X_t. Thus $P^{-1}(\Omega)$ is open in $\sigma(X,A)$.

ii) Trivial, because nullsequences in $\sigma(X,A)$ are nullsequences in some X_t and vice versa.

□

Linear mappings from $\tau(X,A)$ into $\tau(Y,B)$

Theorem 4.6

Let $R : \tau(X,A) \to \tau(Y,B)$ be a linear mapping. Then the following conditions are equivalent.

I. R is continuous with respect to the strong topologies of $\tau(X,A)$ and $\tau(Y,B)$.

II. For every $t > 0$ there is $\tau > 0$ such that the operator $e^{tB} R\, e^{-\tau A}$ is bounded in X.

III. For every $G \in \sigma(Y,B)$ the linear functional

$$f \to \langle Rf,G\rangle \;, \quad (f \in \tau(X,A)) \;,$$

is continuous.

Proof:

I $\Rightarrow$ II) For every $t > 0$, the operator $e^{tB} R$ is continuous from $\tau(X,A)$ into Y. Following Lemma 4.3, for each fixed $t > 0$, there exists $\tau > 0$ such that $e^{tB} R\, e^{-\tau A}$ is bounded.

II $\Rightarrow$ I) Let $u_n \to 0$ in $\tau(X,A)$ and let $t > 0$. Then there is $\tau > 0$ such that $e^{tB} R\, u_n = (e^{tB} R\, e^{-\tau A}) u_n(\tau) \to 0$.

I $\Rightarrow$ III) trivial.

III $\Rightarrow$ II) Let $t > 0$. For each $\varphi \in Fa^+(\mathbb{R})$ and $g \in Y$, we define a linear functional on X by $x \to (e^{tB} R\, \varphi(A)\, x, g)_Y$. This linear functional is continuous. So there exists $g_\varphi \in X$ such that

$$(*) \qquad (e^{tB} R\, \varphi(A)\, x, g)_Y = (x,\, g_\varphi)_X \;.$$

Replacing x by $\psi(A)y$, $\psi \in Fa^+(\mathbb{R})$, we have

$$g_{\varphi\cdot\psi} = \varphi(A)\, g_\psi = \psi(A)\, g_\varphi \quad .$$

So $g_\varphi \in \tau(X,A)$ following 1.6. From (*) we obtain

$$g_\varphi = (e^{tB}\, R\, \varphi(A))^* g \ , \quad g \in Y \quad ,$$

and $(e^{tB}\, R\, \varphi(A))^*$ is defined on the whole of Y. Since $e^{tB}\, R\, \varphi(A)$ is defined on the whole of X, $(e^{tB}\, R\, \varphi(A))^*$ is bounded. So $e^{tB}\, R\, \varphi(A)$ is bounded. With the aid of Theorem 4.1, we can conclude that there is $\tau > 0$ such that $e^{tB}\, R\, e^{-\tau A}$ is bounded.

□

Corollary 4.7

Suppose Q is a densely defined closable operator of X into Y. If $D(Q) \supset \tau(X,A)$ and $Q(\tau(X,A) \subset \tau(Y,B)$, then Q maps $\tau(X,A)$ continuously into $\tau(Y,B)$.

Proof:

Let $t > 0$ and let $\varphi \in Fa^+(\mathbb{R})$. Since $e^{tB}\, Q\varphi(A)$ is defined on the whole of X, its adjoint $(e^{tB}\, Q\, \varphi(A))^*$ is bounded. The adjoint is densely defined, because $e^{-tB}\, D(Q^*)$ is dense in Y, and on $e^{-tB}\, D(Q^*)$ one has

$$(e^{tB}\, Q\varphi(A))^* = \varphi(A)\; Q^*\; e^{tB} \ .$$

So $(e^{tB}\, Q\, \varphi(A))^*$ is defined on the whole of Y and bounded. Thus $e^{tB}\, Q\varphi(A)$ is bounded. Since $\varphi \in Fa^+(\mathbb{R})$ is taken arbitrarily, according to Theorem 4.1, there is $\tau > 0$ such that $e^{tB}\, Q\, e^{-\tau A}$ is bounded. According to Theorem 4.6 Q is a continuous mapping of $\tau(X,A)$ into $\tau(Y,B)$.

□

Continuous linear mappings $\tau(X,A) \to \sigma(Y,B)$

Theorem 4.8

Let $W : \tau(X,A) \to \sigma(Y,B)$ be a linear mapping. W is continuous with respect to the strong topologies of both $\tau(X,A)$ and $\sigma(Y,B)$ iff there exists $t > 0$ and $\tau > 0$ such that the operator $e^{-tB}\, W\, e^{-\tau A}$ is bounded as an operator from X into Y.

Proof:

First, note that both $W\varphi(A) : X \to \sigma(Y,B)$ and $\psi(B)W : \tau(X,A) \to Y$ are continuous mappings for all φ , $\psi \in Fa^+(\mathbb{R})$. So for all $\varphi \in Fa^+(\mathbb{R})$ there is $t > 0$ such that $e^{-tB}\, W\, \varphi(A)$ and $\varphi(B)\, W\, e^{-tA}$ are bounded on X. (see Corollary 4.3 and 4.4).

Now suppose the assertion is not true. Then we have

$$\forall_{t>0}\ \forall_{\tau>0}\ \forall_{K>0}\ \forall_{N>0}\ \exists_{M>N}\ \exists_{x,\|x\|=1} \ :$$

$$(*) \qquad \| Q_{(N,M]}\, e^{-tB}\, W\, P_{(N,M]}\, e^{-\tau A}\, x \| > K$$

with

$$P_{(N,M]} = \int_{-\infty}^{\infty} \chi_{(N,M]}(\lambda)\, d\, E_\lambda \quad \text{and}$$

$$Q_{(N,M]} = \int_{-\infty}^{\infty} \chi_{(N,M]}(\lambda)\, d\, E_\lambda \ , \quad \text{as usual.}$$

If this were not so, then there is $t > 0$ and $\tau > 0$, and $K > 0$ and $N > 0$ such that for all $M > N$ and for all $x, \|x\| = 1$

$$\| Q_{(N,M]}\, e^{-tB}\, W\, P_{(N,M]}\, e^{-\tau A}\, x \| > K \ .$$

Since φ_a defined by

$$\varphi_a(\lambda) = \chi_{(0,N]}(\lambda)\, e^{-a\lambda} \quad , \quad (\lambda \in \mathbb{R}) \ ,$$

is in $Fa^+(\mathbb{R})$ for each $a > 0$, there are $t_1, t_2 > 0$ such that

$$\| e^{-t_1 B}\, W\, \varphi_\tau(A) \| < \infty \quad \text{and} \quad \| \varphi_t(B)\, W\, e^{-t_2 A} \| < \infty \ .$$

So there exists $C > 0$ and $t', \tau' > 0$ such that for all $M > 0$

$$\| Q_{(0,M]}\, e^{-t'B}\, W\, e^{-\tau' A}\, P_{(0,M]} \| < C \, .$$

This implies the boundedness of $e^{-t'B}\, W\, e^{-\tau' A}$ and contradicts our assumption.

Following (*) there are sequences (t_k), (N_k) in $\mathbb{R}^+$ with $t_k \uparrow \infty$ and $N_k \uparrow \infty$, $N_{k+1} > N_k > 0 = N_0$, and there is a sequence (x_k) in X with $\| x_k \| = 1$ and $P_k x_k = x_k$ for all $k \in \mathbb{N}$, such that

$$\| Q_k\, e^{-t_k B}\, W\, P_k\, e^{-t_k A}\, x_k \| > k \ , \ k \in \mathbb{N} \, .$$

Here $Q_k = Q_{(N_{k-1}, N_k]}$ and $P_k = P_{(N_{k-1}, N_k]}$. Now define $\varphi \in Fa^+(\mathbb{R})$ by

$$\varphi(\lambda) = e^{-t_k \lambda} \quad \text{if } \lambda \in (N_{k-1}, N_k \ , \ k \in \mathbb{N}, \text{ else } \varphi(\lambda) = 0,$$

and let $x = \sum_{k=1}^{\infty} \frac{1}{k} x_k$. Then $P_k\, x = \frac{1}{k} x_k$, and

$$\| \varphi(\mathcal{B})\, \mathcal{W}\, \varphi(\mathcal{A}) x \|^2 = \sum_{k=1}^{\infty} \| Q_k\, e^{-t_k \mathcal{B}}\, \mathcal{W}\, e^{-t_k \mathcal{A}}\, P_k\, x \|^2 \geq$$

$$\geq \sum_{k=1}^{N} \frac{1}{k^2}\, k^2 = N \,,$$

for all $N \in \mathbb{N}$.

This is a contradiction, because $\varphi(\mathcal{B})\, \mathcal{W}\, \varphi(\mathcal{A})$ should be bounded.

$\Leftarrow$) Let $u_n \to 0$ in $\tau(X,\mathcal{A})$. Then $(e^{-t\mathcal{B}}\, \mathcal{W}\, e^{-\tau\mathcal{A}})\, u_n(\tau) \to 0$. Thus

$\mathcal{W} u_n \to 0$ in $\sigma(Y,\mathcal{B})$.

□

<u>Continuous linear mappings</u> $\sigma(X,\mathcal{A}) \to \tau(Y,\mathcal{B})$

<u>Theorem 4.9</u>

Let $\Gamma : \sigma(X,\mathcal{A}) \to \tau(Y,\mathcal{B})$ be a linear mapping . Γ is continuous with respect to the strong topologies of $\sigma(X,\mathcal{A})$ and $\tau(Y,\mathcal{B})$ iff for each $t > 0$ and for each $\tau > 0$ the operator $e^{t\mathcal{B}}\, \Gamma\, e^{\tau\mathcal{A}}$ is a bounded operator from X into Y.

<u>Proof:</u>

$\Leftarrow$) Let (G_n) be a nullsequence in $\sigma(X,\mathcal{A})$. Then there is $\tau > 0$ such that (G_n) is a nullsequence in X_τ. So for all $t > 0$

$$(e^{t\mathcal{B}}\, \Gamma)\, G_n = (e^{t\mathcal{B}}\, \Gamma\, e^{\tau\mathcal{A}})\, e^{-\tau\mathcal{A}}\, G_n \to 0 \quad .$$

$\Rightarrow$) For each $\tau > 0$, the operator $\Gamma\, e^{\tau\mathcal{A}}$ is continuous from X into $\tau(Y,\mathcal{B})$. So for each $t > 0$ the operator $e^{t\mathcal{B}}\, \Gamma\, e^{\tau\mathcal{A}}$ is a bounded mapping from X into Y.

□

Continuous linear mappings $\sigma(X,A) \to \sigma(Y,B)$

Theorem 4.10

Let $V : \sigma(X,A) \to \sigma(Y,B)$ be a linear mapping. Then the following conditions are equivalent.

I. V is continuous with respect to the strong topologies of $\sigma(X,A)$ and $\sigma(Y,B)$.

II. For each $t > 0$ there exists $\tau > 0$ such that $e^{-\tau B} V e^{tA}$ is a bounded operator from X into Y.

III. For every $g \in \tau(Y,B)$ the linear functional

$$F \to \overline{< g, VF >}$$

is continuous in $\sigma(X,A)$.

Proof:

I $\Leftarrow$ II) Let $G_n \to 0$ in $\sigma(X,A)$. Then there is $t > 0$ such that $e^{-tA}G_n \to 0$ in X. So

$$e^{-\tau B}V G_n = (e^{-\tau B} V e^{tA})(e^{-tA} G_n) \to 0 \text{ in } X.$$

I $\Rightarrow$ II) Let $t > 0$. Then $V e^{tA}$ maps X continuously into $\sigma(Y,B)$. According to Corollary 4.2 there exists $\tau > 0$ such that $e^{-\tau B} V e^{tA}$ is bounded from X into Y.

I $\Rightarrow$ III) Trivial.

III $\Rightarrow$ II) Let $t > 0$. For each $\varphi \in Fa^+(\mathbb{R})$ and $g \in Y$ we define a linear functional on X by

$$(*) \qquad x \to \overline{(g, \varphi(B)\, V\, e^{tA} x)_Y} \, .$$

By assumption, this linear functional is continuous. So there exists

$g_\varphi \in X$ such that

$$\overline{(g,\ \varphi(B)\ \vee\ e^{tA}\ x)_Y} = (x, g_\varphi)_X \ .$$

So $g_\varphi = (\varphi(B) \vee e^{tA})^* g$, and $(\varphi(B) \vee e^{tA})^*$ is defined on the whole of Y. Since $\varphi(B) \vee e^{tA}$ can be extended to an everywhere defined operator on X by (*), $(\varphi(B) \vee e^{tA})^*$ is bounded. So $\varphi(B) \vee e^{tA}$ can be extended to a bounded operator on X. With the aid of Corollary 4.2, we conclude that there is $\tau > 0$ such that $e^{-\tau B} \vee e^{tA}$ is bounded.

□

Theorem 4.11

(Extension Theorem)

Let E be a linear mapping $X \supset D(E) \to Y$ with $\overline{D(E)} = X$. E can be extended to a continuous linear mapping $\overline{E} : \sigma(X,A) \to \sigma(Y,B)$ iff E has a densely defined adjoint $E^*: Y \supset D(E^*) \supset \tau(Y,B) \to X$ with $E^*(\tau(Y,B) \subset \tau(X,A)$.

Proof:

$\Leftarrow$)E^* is densely defined,closable from Y into X, and $\tau(Y,B) \subset D(E^*)$ with $E^*(\tau(Y,B)) \subset \tau(X,A)$. Following Corollary 4.7 E^* maps $\tau(Y,B)$ continuously into $\tau(X,A)$. So following condition 4.10 III, the dual $(E^*)'$ of E^* is a continuous linear mapping of $\sigma(X,A)$ into $\sigma(Y,B)$ and for $x \in D(E)$ we have $(E^*)'x = Ex$ because

$$\forall_{y \in \tau(Y,B)} : \langle E^* y, x\rangle = \langle y, (E^*)'x\rangle$$

and $$: \langle E^* y, x\rangle = (E^* y, x)_X = (y, Ex)_Y \ .$$

With $\overline{E} =: (E^*)'$ the proof is complete.

$\Rightarrow$) Let $\overline{E} : \sigma(X,A) \to \sigma(Y,B)$ exist and be continuous. For each $x \in D(E)$ and $g \in \tau(Y,B)$ one has

$$(g,Ex)_Y = \langle g,\overline{E}x\rangle = \langle E'g,x\rangle = (\overline{E}'g,x)_X \; .$$

It follows that $E^* \supset \overline{E}'$ and $E^*(\tau(Y,B)) \subset \tau(X,A)$.

□

<u>Corollary 4.9</u>

A continuous linear mapping $T : \tau(X,A) \to \tau(Y,B)$ can be extended to a continuous linear mapping $\overline{T} : \sigma(X,A) \to \sigma(Y,B)$ iff T has an adjoint T^* with $D(T^*) \supset \tau(Y,B)$ and $T^*(\tau(Y,B) \subset \tau(X,A)$.

Chapter 5

Topological tensor products of the spaces $\tau(X,A)$ and $\sigma(X,A)$

Let X and Y be two Hilbert spaces. By $X\otimes_a Y$ we denote the algebraic tensor product of X and Y, i.e. all finite linear combinations $\sum_{j=1}^{N} \xi_j \otimes \eta_j$ with $\xi_j \in X$ and $\eta_j \in Y$ and usual identifications. In $X\otimes_a Y$ we define the following sesquilinear form:

$$(Z,K)_{X\otimes Y} =: \sum_{i=1}^{\infty} (Ze_i, Ke_i)_Y \quad , \quad Z,K \in X\otimes_a Y$$

in which (e_i) is an orthonormal basis for X. The sesquilinear form does not depend on the choice of the orthonormal basis (e_i) in X. Along with this sesquilinear form we define a norm in $X\otimes_a Y$ by

$$\| Z \|^2_{X\otimes Y} = \sum_{i=1}^{\infty} \| Ze_i \|^2_Y \quad , \quad Z \in X\otimes_a Y .$$

Let $X \otimes Y$ be the completion of $X\otimes_a Y$ with respect to this norm. If we take $(\cdot,\cdot)_{X\otimes Y}$ as an inner product then $X\otimes Y$ becomes a Hilbert space. Note that the space $X \otimes Y$ consists of all Hilbert-Schmidt operators Z from X into Y. $X \otimes Y$ is sometimes called the topological tensor product of X and Y. Without proof we mention the following properties (see [RS], ch VIII).

Properties 5.1

a) $\forall_{x,\xi\in X} \forall_{y,\eta\in Y}$: $(\xi \otimes \eta, x \otimes y)_{X\otimes Y} = (x,\xi)_X \, (\eta,y)_Y$.

b) $\forall_{\lambda\in\mathbb{C}} \forall_{\xi\in X} \forall_{\eta\in Y}$: $\lambda(\xi \otimes \eta) = (\overline{\lambda}\xi) \otimes \eta = \xi \otimes \lambda\eta$.

c) $\forall_{Z\in X\otimes Y} \forall_{x\in X} \forall_{y\in Y}$: $(Z, x \otimes y)_{X\otimes Y} = (Zx,y)_Y$.

Let $C, \mathcal{D}$ denote bounded linear operators from X, resp. Y into themselves. Then $C \otimes \mathcal{D}$ is the linear operator of $X \otimes Y$ into itself defined by $C \otimes \mathcal{D}\,(x \otimes y) = Cx \otimes \mathcal{D}y$, and linear extension followed by continuous extension. We have

d) $\| C \otimes \mathcal{D} \| = \| C \| \, \| \mathcal{D} \|$

e) $\forall_{Z \in X \otimes Y} : (C \otimes \mathcal{D}) Z = \mathcal{D} Z C^*$

f) $C, \mathcal{D}$ injective $\Rightarrow$ $C \otimes \mathcal{D}$ injective.

Let A with domain $D(A)$, and B with domain $D(B)$ be positive self-adjoint operators in X resp. Y. In the sequel we denote the spectral resolutions of A and B by $(E_\lambda)_{\lambda \geq 0}$ resp. $(F_\lambda)_{\lambda \geq 0}$. On $D(A) \otimes_a D(B)$ we introduce the operator $A \otimes_a I + I \otimes_a B$ by

$$(A \otimes_a I + I \otimes_a B)(x \otimes y) = A\,x \otimes y + x \otimes By \ ,$$

and linear extension. We have

<u>Theorem 5.2</u>

i) $A \otimes_a I + I \otimes_a B$ is essentially self-adjoint in $X \otimes Y$. We denote the unique self-adjoint extension by $A \boxplus B$.

ii) $A \boxplus B \geq 0$.

iii) On $X \otimes Y$ we have for $t \geq 0$

$$e^{-t(A \boxplus B)} = e^{-tA} \otimes e^{-tB} \ .$$

Proof:

See [W], ch. 8.5, or [G] ch.V. □

Since $A \boxplus B$ is self-adjoint and positive, we may apply the theory of the preceeding chapters and introduce the spaces

$$\tau(X \otimes Y,\ A \boxplus B),\ \sigma(X \otimes Y,\ A \boxplus B),\ \tau(X \otimes Y,\ A \otimes I),\ \text{etc.}$$

Definition 5.3

The canonical sesquilinear mapping $\otimes : \tau(X,A) \times \tau(Y,B) \to \tau(X \otimes Y, A \boxplus B)$ is defined by

$$(u,v) \to u \otimes v : t \to u(t) \otimes v(t)$$

This definition is consistent, because for $u \in \tau(X,A)$ and $v \in \tau(Y,B)$, and $t,\tau > 0$ we have

$$u(t) \otimes v(t) \;=\; (e^{-\tau A} \otimes e^{-\tau B})\, u(t+\tau) \otimes v(t+\tau) \quad .$$

So $u(t+\tau) \otimes v(t+\tau) = e^{\tau(A \boxplus B)}\,(u(t) \otimes v(t))$ by Theorem 5.2 iii).

Theorem 5.4

$\tau(X \otimes Y,\ A \boxplus B)$ is a complete topological tensor product of $\tau(X,A)$ and $\tau(Y,B)$, by this we mean

i) $\tau(X \otimes Y,\ A \boxplus B)$ is complete.

ii) The mapping $\otimes : \tau(X,A) \times \tau(Y,B) \to \tau(X \otimes Y, A \boxplus B)$ is continuous.

iii) $\tau(X,A) \otimes_a \tau(Y,B)$ is dense in $\tau(X \otimes Y, A \boxplus B)$.

Proof:

i) The completeness follows from Theorem 1.3.

ii) Since $\otimes$ is sesquilinear it is enough to check the continuity at $[0,0]$. Let $t > 0$. Then

$$\| u(t) \otimes v(t) \| = \| u(t) \| \, \| v(t) \| .$$

From this the continuity at $[0,0]$ follows.

iii) Let $P_N =: P_{(-1,N]} = \int_{-\infty}^{\infty} \chi_{(-1,N]}(\lambda)\, dE_\lambda$ and $Q_N =: Q_{(-1,N]} = \int_{-\infty}^{\infty} \chi_{(-1,N]}(\lambda)\, dF_\lambda$ for all $N \in \mathbb{N}$.

The set $\{P_N x \mid x \in X,\ N \in \mathbb{N}\}$ is dense in $\tau(X,A)$

To show this let $u \in \tau(X,A)$. Then there is $\varphi \in Fa^+(\mathbb{R})$, $w \in X$ such that $u(t) = e^{tA} \varphi(A) w$. We have for all $t > 0$

$$\| e^{tA} \varphi(A)(w - P_N w) \| \to 0 \quad , \quad N \to \infty .$$

The assertion is proved, because $\varphi(A)w \in X$.

Similarly we have that

$$\{Q_N y \mid y \in Y,\ N \in \mathbb{N}\} \text{ is dense in } \tau(Y,B)$$

and

$$\{(P_N \otimes Q_N) W \mid W \in X \otimes Y,\ N \in \mathbb{N}\} \text{ is dense in } \tau(X \otimes Y,\ A \boxplus B) .$$

Since $\{(P_N x) \otimes (Q_N y) \mid x \in X, y \in Y,\ N \in \mathbb{N}\} \subset \tau(X,A) \otimes_a \tau(Y,B)$, and since $(P_N \otimes Q_N)(X \otimes_a Y)$ is dense in $(P_N \otimes Q_N)(X \otimes Y)$, we have proved that $\tau(X,A) \otimes_a \tau(Y,B)$ is dense in $\tau(X \otimes Y, A \boxplus B)$.

□

Definition 5.5

The canonical sesquilinear mapping $\otimes : \sigma(X,A) \times \sigma(Y,B) \to \sigma(X \otimes Y, A \boxplus B)$ is defined by

$$[G,H] \to G \otimes H = e^{t(A \boxplus B)}(e^{-tA} G \otimes e^{-tB} H) ,$$

if $t > 0$ sufficiently large. This definition is consistent, since $e^{-tA} G \otimes e^{-tB} H \in X \otimes Y$ if $t > 0$ is chosen sufficiently large, and does not depend on the specific choice of $t > 0$.

Theorem 5.6

$\sigma(X \otimes Y, A \boxplus B)$ is a complete topological tensor product:

i) $\sigma(X \otimes Y, A \boxplus B)$ is complete.

ii) The sesquilinear mapping $\otimes$ of Definition 5.5 is continuous.

iii) $\sigma(X,A) \otimes_a \sigma(Y,B)$ is dense in $\sigma(X \otimes Y, A \boxplus B)$.

Proof:

i) The completeness follows from Theorem 2.10.

ii) We check the continuity of $\otimes$ at $[0,0]$ Let W be a convex open neighbourhood of 0 in $\sigma(X \otimes Y, A \boxplus B)$. Then for each $t > 0$, the set $W \cap (X \otimes Y)_t$ is an open neighbourhood of 0 in $(X \otimes Y)_t$; thus it contains an open ball, centered at 0, with radius Γ_t, $0 < \Gamma_t < 1$, say. In X_t and Y_t we introduce open balls A_t resp. B_t, centered at 0 and both with radius Γ_t.

Let $A =: \bigcup_{t>0} A_t$ and $B =: \bigcup_{t>0} B_t$. Then $A \times B$ contains an open neighbourhood in $\sigma(X,A) \times \sigma(Y,B)$, and

$$\| x \otimes y \|_{\max(t,\tau)} \leq \| x \|_t \, \| y \|_\tau \leq \Gamma_{\max(t,\tau)} \quad ,$$

whenever $x \in A_t$ and $y \in B_\tau$. So $\otimes$ maps $A \times B$ in W. Let $\hat{A}$ and $\hat{B}$ denote the convex hulls of A resp. B. Then $\otimes$ maps $\hat{A} \times \hat{B}$ into W. From Theorem 2.2 we conclude that $\hat{A}$ contains an open set $U_{\psi,\varepsilon}$, and similarly $\hat{B}$ an open set $V_{\chi,\delta}$. So $\otimes$ maps $U_{\psi,\varepsilon} \times V_{\chi,\delta}$ into W.

iii) For each $t > 0$ the tensor product $X_t \otimes_a Y_t$ is dense in $(X \otimes Y)_t$, from which the assertion follows.

□

Definition 5.7.a

We introduce the following space of trajectories: $\tau(\sigma(X \otimes Y, I \otimes B), A \otimes I)$; for shortness we denote the space by τ_B^A in the sequel. τ_B^A is the complex vector space of trajectories Φ

$$\Phi : \mathbb{R}^+ \to \sigma(X \otimes Y, I \otimes B)$$

satisfying $\Phi(t) = (e^{-\tau A} \otimes I)\, \Phi(t+\tau)$ for all $t,\ \tau \geq 0$.

The map $e^{-\tau A} \otimes I$ is well-defined in $\sigma(X \otimes Y, I \otimes B)$, because for each $F \in \sigma(X \otimes Y, I \otimes B)$ there is $t > 0$ such that $(I \otimes e^{-tB})F$ in $X \otimes Y$, and

$$(e^{-\tau A} \otimes I)\, F = (I \otimes e^{tB}) \left[(e^{-\tau A} \otimes I)(I \otimes e^{-tB}) \right] F$$

is in $\sigma(X \otimes Y, I \otimes B)$.

Definition 5.7.b

On τ_B^A we introduce the seminorms

$$p_{t,\psi}(\Phi) =: \| (I \otimes \psi(B))\, \Phi(t) \|_{X \otimes Y}, \quad t > 0, \psi \in Fa^+(\mathbb{R}),$$

and the strong topology in $\tau_{\mathcal{B}}^{A}$ is the locally convex topology generated by the seminorms $\rho_{t,\phi}$

Definition 5.8

The sesquilinear mapping $\otimes : \tau(X,A) \times \sigma(Y,\mathcal{B}) \to \tau_{\mathcal{B}}^{A}$ is defined by

$$\otimes : (f,G) \to f \otimes G : t \to f(t) \otimes G.$$

The span of the image of $\otimes$ is denoted by $\tau(X,A) \otimes_a \sigma(Y,\mathcal{B})$. This definition is consistent because for all $\tau, t > 0$ we have

$$(f(t) \otimes G) \in \sigma(X \otimes Y, I \otimes \mathcal{B}),$$

and

$$f(t+\tau) \otimes G = (e^{\tau A} f(t)) \otimes G = (e^{\tau A} \otimes I)(f(t) \otimes G).$$

Theorem 5.9

$\tau_{\mathcal{B}}^{A}$ is a complete topological tensor product of $\tau(X,A)$ and $\sigma(Y,\mathcal{B})$. By this we mean

i) $\tau_{\mathcal{B}}^{A}$ is complete.

ii) The sesquilinear mapping $\otimes$ of Definition 5.8 is continuous.

iii) $\tau(X,A) \otimes_a \sigma(Y,\mathcal{B})$ is dense $\tau_{\mathcal{B}}^{A}$.

Proof:

i) Let (Φ_i) be a Cauchy net in $\tau_{\mathcal{B}}^{A}$. Then for each $t > 0$ the net $(\Phi_i(t))$ is Cauchy in $\sigma(X \otimes Y, I \otimes \mathcal{B})$. So there exists $\Phi(t) \in \sigma(X \otimes Y, I \otimes \mathcal{B})$ with $\Phi_i(t) \to \Phi(t)$, following Theorem 2.10.
Now $\Phi_i(t) = (e^{-\tau A} \otimes I)\, \Phi_i(t+\tau) \to (e^{-\tau A} \otimes I)\, \Phi(t+\tau)$.

Take $\Phi : t \to \Phi(t)$. Then $\Phi \in \tau_{\mathcal{B}}^{A}$ and $\Phi_i \to \Phi$.

ii) We check the continuity at $[0,0]$. Let $W_{t,\varphi,\varepsilon}$ be the set of elements Φ in $\tau_{\mathcal{B}}^{A}$ for which the seminorm $\rho_{t,\varphi}(\Phi) < \varepsilon$, with $\varepsilon > 0$. Then $W_{t,\varphi,\varepsilon}$ is an open neighbourhood of 0 in $\tau_{\mathcal{B}}^{A}$. Let V be the set in $\sigma(Y,\mathcal{B})$ the elements of which satisfy

$$\|\varphi(\mathcal{B})\, G\|_Y < \sqrt{\varepsilon}\,, \qquad G \in V,$$

and let W be the set in $\tau(X,A)$ the elements of which satisfy

$$\|f(t)\|_X < \sqrt{\varepsilon}\,, \quad f \in W\,.$$

V and W are open neighbourhoods of 0 in $\sigma(Y,\mathcal{B})$ and $\tau(X,A)$.

We have

$$\rho_{t,\varphi}(f \otimes G) = \|f(t) \otimes (\varphi(\mathcal{B})G)\|_{X\otimes Y} = \|f(t)\|_X \,\|\varphi(\mathcal{B})\, G\|_Y < \varepsilon,$$

from which continuity at $[0,0]$ follows.

iii) First we shall prove that $\{(P_N \otimes Q_N)\, \Xi \mid \Xi \in \tau_{\mathcal{B}}^{A}\,,\ N \in \mathbb{N}\}$ is dense in $\tau_{\mathcal{B}}^{A}$, where $P_N = P_{(-1,N]}$ and $Q_N = Q_{(-1,N]}$. Therefore, let $\Xi \in \tau_{\mathcal{B}}^{A}$. Then $\Xi(t) \in \sigma(X \otimes Y, I \otimes \mathcal{B})$, for all $t \in \mathbb{R}^+$, and

$$\rho_{t,\varphi}(\Xi - (P_N \otimes Q_N)\,\Xi) = \|(I \otimes I - P_N \otimes Q_N)(I \otimes \varphi(\mathcal{B}))\, \Xi(t)\| \to 0$$

because $(I \otimes \varphi(\mathcal{B}))\, \Xi(t) \in X \otimes Y$.

Next observe that $\{(P_N \otimes Q_N)\, \Xi \mid \Xi \in \tau_{\mathcal{B}}^{A},\ N \in \mathbb{N}\}$ can be embedded in $\tau(X \otimes Y, A \boxplus \mathcal{B})$. This is a consequence of the following estimate

$$\|(e^{tA} \otimes e^{t\mathcal{B}})(P_N \otimes Q_N)\, \Xi\|_{X\otimes Y} \leq \|(e^{tA}P_N \otimes e^{(t+\tau)\mathcal{B}}Q_N)(I \otimes e^{-\tau\mathcal{B}})\, \Xi(0)\|_{X\otimes Y}$$

$$\leq \|(e^{tA}P_N) \otimes (e^{(t+\tau)\mathcal{B}}Q_N)\| \ \|I \otimes e^{-\tau\mathcal{B}}\, \Xi(0)\|_{X\otimes Y}$$

$$< \infty$$

if $\tau > 0$ is taken large enough. Thus $\tau(X,A) \otimes_a \tau(Y,\mathcal{B})$ is dense in $\tau_{\mathcal{B}}^{A}$. A posteriori it is clear that $\tau(X,A) \otimes_a \sigma(Y,\mathcal{B})$ is dense in $\tau_{\mathcal{B}}^{A}$. □

Definition 5.10.a

We introduce the following space of trajectories:

$$\tau(\sigma(X \otimes Y, A \otimes I)\ ,\ I \otimes \mathcal{B}).$$

In the sequel we shall denote this space by $\tau_A^{\mathcal{B}}$.

$\tau_A^{\mathcal{B}}$ is the complex vector space of linear mappings

$$\omega : \mathbb{R}^+ \to \sigma(X \otimes Y, A \otimes I)$$

which satisfy $\omega(t) = I \otimes e^{-t\mathcal{B}}\, \omega(t+\tau)$ for all $t,\tau \geq 0$.

On $\tau_A^{\mathcal{B}}$ we introduce the seminorms

$$V_{k,\psi}(\omega) =: \|(\psi(A) \otimes I)\, \omega(t)\|_{X\otimes Y}\ ,$$

and the corresponding locally convex topology.

Definition 5.10.b

The canonical mapping $\otimes : \sigma(X,A) \times \tau(Y,B) \to \tau_A^B$ is defined by

$$\otimes : (H,g) \to H \otimes g : t \to H \otimes g(t)$$

The span of the image of $\otimes$ is denoted by $\sigma(X,A) \otimes_a \tau(Y,B)$.

Theorem 5.11

τ_A^B is a complete topological tensor product of $\sigma(X,A)$ and $\tau(Y,B)$.

Proof:

As for Theorem 5.8. □

Since $\tau(X \otimes Y, I \otimes B) \subset X \otimes Y$, the linear subspace $(e^{tA} \otimes I) \; \tau(X \otimes Y, I \otimes B)$ of $(e^{tA} \otimes I)(X \otimes Y)$ is well-defined. In $(e^{tA} \otimes I) \; \tau(X \otimes Y, I \otimes B)$ we introduce the metric d_t by

$$d_t \; (\theta) = d_B \; ((e^{-tA} \otimes I)\theta)$$

with d_B the metric in $\sigma(X \otimes Y, I \otimes B)$.

Definition 5.12

We introduce the locally convex topological vector space

$$\sigma(\tau(X \otimes Y, I \otimes B), A \otimes I) = \bigcup_{t>0} \; (e^{tA} \otimes I) \; \tau(X \otimes Y, I \otimes B).$$

with the inductive limit topology. In the sequel we shall denote this space by σ_B^A . Note that for all $t > 0$ we have

$$(e^{tA} \otimes I)\ \tau(X \otimes Y,\ I \otimes B) \subset (e^{tA} \otimes I)(X \otimes Y) \subset \sigma(X \otimes Y, A \otimes I).$$

So $\sigma_B^A = \bigcup_{t>0} (e^{tA} \otimes I)\ \tau(X \otimes Y, I \otimes B) \subset \sigma(X \otimes Y,\ A \otimes I)$.

On $\tau_B^A \times \sigma_B^A$ we introduce the sesquilinear form

$$\langle V,\theta\rangle_B^A = \langle V(t)\ ,\ (e^{-tA} \otimes I)\ \theta\ \rangle_{\tau(X\otimes Y, I\otimes B)}\ ,$$

for $t > 0$ sufficiently large.

<u>Theorem 5.13</u>

i) $\sigma(X,A) \otimes_a \tau(Y,B)$ is dense in σ_B^A.

ii) σ_B^A is continuously embedded in τ_A^B by the mapping

$$\mathrm{emb} : \theta \to \mathrm{emb}(\theta) : t \to (I \otimes e^{tB})(\theta)\ .$$

iii) For each fixed $V \in \tau_B^A$ the linear functional

$$\sigma_B^A \to \mathbb{C} \text{ defined by } \theta \to \overline{\langle V,\theta\rangle_B^A}$$

is continuous on σ_B^A.

<u>Proof:</u>

i) Following Theorem 5.4 we have $X \otimes_a \tau(Y,B)$ is dense in $\tau(X \otimes Y, I \otimes B)$.

Therefore for all $t > 0$ the set $X_t \otimes_a \tau(Y,B)$ is dense in

$(e^{tA} \otimes I)\ \tau(X \otimes Y, I \otimes B)$. We conclude that

$$\bigcup_{t>0} X_t \otimes_a \tau(Y,B) = \sigma(X,A) \otimes_a \tau(Y,B)$$

is dense in σ_B^A.

ii) Let $\theta \in \sigma_B^A$.

Then $t \to (I \otimes e^{tB})\theta \in \tau_A^B$, because for some $\tau > 0$ and all $t > 0$

$$(e^{-\tau A} \otimes I)(I \otimes e^{tB})\theta \in X \otimes Y .$$

iii) We have $|\langle V,\theta\rangle_B^A| \le c_t\ d_B\ ((e^{-tA} \otimes I)\theta) = c_t\ d_t(\theta)$,

for all $\theta \in (e^{tA} \otimes I)\ \tau(X \otimes Y, I \otimes B)$.

□

Definition 5.14

Similarly to Definition 5.12 we introduce the locally convex topological vector space

$$\sigma(\tau(X \otimes Y, A \otimes I),\ I \otimes B) = \bigcup_{t>0} (I \otimes e^{tB})\ \tau(X \otimes Y, A \otimes I).$$

We shall denote this space by σ_A^B. We remark that $\sigma_A^B \subset \sigma(X \otimes Y, I \otimes B)$.
In σ_A^B we introduce the metric Δ_t by

$$\Delta_t(Q) = d_A((I \otimes e^{-tB})Q)\ ,\ Q \in \sigma_A^B\ ,$$

with d_A the metric of $\tau(X \otimes Y, A \otimes I)$.
For the strong topology in σ_A^B we take the inductive limit topology. On $\tau_A^B \times \sigma_A^B$ we introduce the sesquilinear form

$$\langle L,Q\rangle_A^B =: \langle L(t)\ ,(I \otimes e^{-tB})Q\rangle_{\tau(X\otimes Y,A\otimes I)}$$

for $t > 0$ sufficiently large.

Theorem 5.15

i) $\tau(X,A) \otimes_a \sigma(Y,B)$ is dense in σ_A^B.

ii) σ_A^B is continuously embedded in τ_B^A by the mapping

$$\text{emb} : Q \to \text{emb}(Q) : t \to (e^{tA} \otimes I)\, Q.$$

iii) For each fixed $L \in \tau_A^B$ the linear functional $\sigma_A^B \to \mathbb{C}$ defined by $Q \to \overline{\langle L,Q\rangle_A^B}$ is continuous on σ_A^B.

Proof:

Similar to Theorem 5.13.

□

Chapter 6

Kernel theorems

In chapter 4 we discussed four types of linear mappings. In this chapter we show that elements of the topological tensor products which are introduced in chapter 5, can be interpreted as continuous linear mappings of one of the types in chapter 4. Necessary and sufficient conditions are given which ensure that the topological tensor products comprise all continuous linear mappings under consideration, i.e. the Kernel theorem holds We want to remind that every element of the topological tensor product $X \otimes Y$ represents a bounded linear mapping $X \to Y$.

Case (a). Continuous linear mappings $\sigma(X,A) \to \tau(Y,B)$.

We consider an element $\theta \in \tau(X \otimes Y, A ⊞ B)$ as a linear mapping $\sigma(X,A) \to \tau(X,B)$ in the following way. Let $G \in \sigma(X,A)$

$$\text{(a)} \qquad \theta G : t \to e^{(t-\tau)B}\, \theta(\tau)\, e^{-\tau A}\, G\,,$$

$\tau > 0$ sufficiently large. Definition (a) does not depend on the choice of $\tau' > 0$. This definition is correct, because $\theta(\tau) \in X \otimes Y$ for all $\tau > 0$, and for each $t > 0$,

$$\| e^{(t-\tau)B}\, \theta(\tau)\, e^{-\tau A}\, G \|_Y \le \| e^{(t-\tau)B}\, \theta(\tau) \| \; \| e^{-\tau A}\, G \|_X < \infty\,,$$

if $\tau > 0$ is taken sufficiently large.

Theorem 6.1

I. For each $\theta \in \tau(X \otimes Y, A \boxplus B)$ the linear mapping $\sigma(X,A) \to \tau(Y,B)$ as defined in (a) is continuous.

II. For each $\theta \in \tau(X \otimes Y, A \boxplus B)$, $F \in \sigma(X,A)$ and $G \in \sigma(Y,B)$

$$<\theta\, F, G>_Y = <\theta, F \otimes G>_{X\otimes Y} .$$

III. If e^{-tA} or e^{-tB} is HS for some $t > 0$, then $\tau(X \otimes Y, A \boxplus B)$ comprises all continuous linear mappings $\sigma(X,A) \to \tau(Y,B)$.

IV. $\tau(X \otimes Y, A \boxplus B)$ comprises all continuous linear mappings $\sigma(X,A) \to \tau(X,A)$ iff e^{-tA} is HS for some $t > 0$.

Proof:

I. Let $\theta \in \tau(X \otimes Y, A \boxplus B)$. Then for all $t,\tau > 0$

$$e^{tB}\, \theta\, e^{\tau A} = e^{-(t'-t)B}\, \theta(t')\, e^{-(t'-\tau)A} .$$

So $e^{tB}\, \theta\, e^{\tau A}$ is bounded as an operator $X \to Y$ when $t' > 0$ sufficiently large. Following Theorem 4.11 θ maps $\sigma(X,A)$ continuously into $\tau(Y,B)$.

II. $<\theta\, F, G>_Y = (e^{(t-\tau)B}\, \theta(\tau)\, e^{-\tau A}\, F,\ e^{-tB}\, G)_Y =$

$$= (\theta\ (\tau),\ e^{-\tau A}\, F \otimes e^{-\tau B}\, G)_{X\otimes Y}$$

$$= <\theta,\ F \otimes G>_{X\otimes Y} .$$

III. Let $\Gamma : \sigma(X,A) \to \tau(Y,B)$ be continuous. Then for all $t > 0$ the mapping

$e^{tB}\,\Gamma\,e^{tA}$ is bounded from X into Y. Since $e^{-\tau A}$ or $e^{-\tau B}$ is HS for some $\tau > 0$, $e^{tB}\,\Gamma\,e^{tA} = e^{-\tau B}(e^{(t+\tau)B}\,\Gamma\,e^{(t+\tau)A})\,e^{-\tau A}$ is HS. So $\Gamma : t \to e^{tB}\,\Gamma\,e^{tA}$ is in $\tau(X \otimes Y, A \boxplus B)$.

IV. The if-part is a special case of III. Let $\varphi \in Fa^{+}(\mathbb{R})$. Then $\varphi(A)$ maps $\sigma(X,A)$ continuously in $\tau(X,A)$. So $\varphi(A)$ is HS for all $\varphi \in Fa^{+}(\mathbb{R})$. Following Lemma 1.7 there is $t > 0$ such that e^{-tA} is HS.

□

Case (b). Continuous linear mappings $\tau(X,A) \to \sigma(Y,B)$.

Let $K \in \sigma(X \otimes Y, A \boxplus B)$. For $h \in \tau(X,A)$ we define $Kh \in \sigma(Y,B)$ by

(b) $$K\,h = e^{tB}\,(e^{-tB}\,K\,e^{-tA})\;h(t).$$

This definition makes sense for $t > 0$ sufficiently large and does not depend on the specific choice of t.

Theorem 6.2

I. For each $K \in \sigma(X \otimes Y, A \boxplus B)$ the linear mapping $K : \tau(X,A) \to \sigma(Y,B)$ as defined in (b) is continuous.

II. For each $K \in \sigma(X \otimes Y, A \boxplus B)$, $f \in \tau(X,A)$ and $g \in \tau(Y,B)$

$$\langle g, K\,f\rangle_Y = \langle f \otimes g\,,\,K\rangle_{X\otimes Y}\,.$$

III. If for some $t > 0$ the operator e^{-tA} or e^{-tB} is HS, then $\sigma(X \otimes Y, A \boxplus B)$ comprises all continuous linear mappings from $\tau(X,A)$ into $\sigma(Y,B)$.

IV. $\sigma(X \otimes X, A \boxplus A)$ comprises all continuous linear mappings from $\tau(X,A)$ into $\sigma(X,A)$ iff e^{-tA} is HS for some $t > 0$.

Proof:

I. Let $K \in \sigma(X \otimes Y, A \boxplus B)$. Then $e^{-tB} K e^{-tA}$ is a bounded operator from X into Y for sufficiently large $t > 0$. Following Theorem 4.10 is a continuous mapping from $\tau(X,A)$ into $\sigma(Y,B)$.

II. $$\langle g, Kf\rangle_Y = (g(t), e^{-tB} K e^{-tA} f(t))_Y$$
$$= (f(t) \otimes g(t), e^{-tB} K e^{-tA})_{X\otimes Y}$$
$$= \langle f \otimes g, K\rangle_{X\otimes Y}$$

III. Let $L : \tau(X,A) \to \sigma(Y,B)$ be continuous. Then there are $t, \tau > 0$ such that $e^{-\tau B} L e^{-tA}$ is bounded. Now suppose $e^{-t_0 A}$ or $e^{-t_0 B}$ is HS, then

$$L = e^{(t_0+\tau)B} (e^{-(t_0+\tau)B} L e^{-(\tau+t_0)A}) e^{(\tau+t_0)A} .$$

So $L \in \sigma(X \otimes Y, A \boxplus B)$, because $e^{-(\tau+t_0)B} L e^{-(\tau+t_0)A}$ is HS and $L = e^{(\tau+t_0)(A \boxplus B)} (e^{-t_0+\tau)B} L e^{-(t_0+\tau)A})$.

IV. The if-part is a special case of III.

Let $J : \tau(X,A) \hookrightarrow \sigma(X,A)$. J is continuous and can be considered as an element of $\sigma(X \otimes X, A \boxplus A)$ if $e^{-tA} J e^{-tA} = e^{-2tA}$ is HS for some $t > 0$.

□

Case (c). Continuous linear mappings $\tau(X,A) \to \tau(Y,B)$.

Let $P \in \tau_A^B$. For $f \in \tau(X,A)$ we define $Pf \in \tau(Y,B)$ by

$$\text{(c)} \qquad P f : t \to P(t) e^{-\varepsilon(t)A} f(\varepsilon(t)),$$

Here we take $\varepsilon(t) > 0$ such that $P(t) e^{-\varepsilon(t)A}$ is a HS operator on X.

Theorem 6.3

I. For each $P \in \tau_A^B$, the linear operator $P : \tau(X,A) \to \tau(Y,B)$ as defined in (c) is continuous.

II. For each $P \in \tau_A^B$, $f \in \tau(X,A)$ and $G \in \sigma(Y,B)$

$$\langle Pf, G\rangle_Y = \langle P, f \otimes G\rangle_A^B .$$

III. If for some $t > 0$ the operator e^{-tA} or e^{-tB} is HS then τ_A^B comprises all continuous linear mappings from $\tau(X,A)$ into $\tau(Y,B)$.

IV. τ_A^B comprises all continuous linear mappings from $\tau(X,A)$ into $\tau(X,A)$ iff e^{-tA} is HS for some $t > 0$.

Proof:

I. Let $P \in \tau_A^B$. Then for each $t > 0$, the operator $(I \otimes e^{tB})P = P(t)$ $\in \sigma(X \otimes Y, A \otimes I)$. So there exists $\tau > 0$ such that $P(t)\, e^{-\tau A}$ is a bounded linear operator in X. Following Theorem 4.4. P is a continuous mapping form $\tau(X,A)$ into $\tau(Y,B)$.

II.
$$\begin{aligned} \langle Pf,G\rangle_Y &= (P(t)\, e^{-\tau A} f(\tau),\, e^{-tA} G)_Y \\ &= (P(t)\, e^{-\tau A},\, f(\tau) \otimes e^{-tB} G)_{X\otimes Y} \\ &= \langle P, f \otimes G\rangle_A^B . \end{aligned}$$

III. Let $T : \tau(X,A) \to \tau(Y,B)$ be continuous.

Then $T(t) =: (e^{tB}\, T\, e^{-\varepsilon(t)A})\, e^{\varepsilon(t)A}$, $t > 0$.

Suppose $e^{-t_0 A}$ or $e^{-t_0 B}$ is a Hilbert-Schmidt operator. Then

$T(t) = e^{-t_0 B} (e^{(t+t_0)B}\, T\, e^{-\varepsilon(t+t_0)A}\, e^{-t_0 A})\, e^{(\varepsilon(t+t_0)+t_0)A}$.

So for all $t > 0$ we have $T(t) \in \sigma(X \otimes Y, A \otimes I)$, and the mapping T is represented by the element

$$t \to T(t)$$

in τ_{A}^{B}.

IV. The if-part is a special case of III.

The identity $I : \tau(X,A) \to \tau(X,A)$ is continuous.

Let $t > 0$. Then for some $\tau > 0$ we have

$$e^{tA}\, I\, e^{-\tau A} \in X \otimes X\ .$$

Thus $e^{(t-\tau)A}$ is Hilbert-Schmidt.

□

Case (d). Continuous linear mappings $\sigma(X,A) \to \sigma(Y,B)$. .
Let $\Phi \in \tau_{B}^{A}$. For $F \in \sigma(X,A)$ we define ΦF by

$$\text{(d)} \qquad \Phi\, F = e^{\varepsilon(t)B}\, (e^{-\varepsilon(t)B}\, \Phi(t))\, e^{-tA}\, F$$

$t > 0$ sufficiently large. The definition does not depend on the choice of $t > 0$. The mapping Φ is well-defined, because $e^{-\varepsilon(t)B}\, \Phi(t)$ is HS for some $\varepsilon(t) > 0$, and $e^{-tA}\, F \in X$ for $t > 0$ sufficiently large.

Theorem 6.4

I. For each $\Phi \in \tau_{B}^{A}$ the linear operator $\Phi : \sigma(X,A) \to \sigma(Y,B)$ as defined in (d) is continuous.

II. For each $\Phi \in \tau_{B}^{A}$, $F \in \sigma(X,A)$ and $g \in \tau(Y,B)$

$$<g, \Phi F>_{Y} = \overline{<\Phi, F \otimes g>_{B}^{A}}$$

III. If for some $t > 0$ the operator e^{-tA} or e^{-tB} is HS, then τ_{B}^{A} comprises all continuous linear operators from $\sigma(X,A)$ in $\sigma(Y,B)$.

IV. τ_B^A comprises all continuous linear operators from $\sigma(X,A)$ into itself iff for some $t > 0$ the operator e^{-tA} is HS.

Proof:

I. Let $\Phi \in \tau_B^A$. Then for all $t > 0$, there is $\tau > 0$, such that $e^{-\tau B}\,\Phi(t)$ is HS and therefore bounded. Following Theorem 4.12 Φ is a continuous mapping from $\sigma(X,A)$ into $\sigma(Y,B)$.

II. $$\langle g, \Phi F\rangle_Y = (g(\tau)\ ,\ e^{-\tau B}\,\Phi(t)\ e^{-tA}\,F)_Y$$
$$= (e^{-tA}\,F \otimes g(\tau),\ e^{-\tau B}\,\Phi(t))_{X\otimes Y}$$
$$= \overline{\langle \Phi,\ F \otimes g\rangle_B^A}\ .$$

III. Let $\beta : \sigma(X,A) \to \sigma(Y,B)$ be a continuous linear mapping. Then $\beta(t) =: e^{\varepsilon(t)B}\,(e^{-\varepsilon(t)B}\,\beta\,e^{tA})$. Suppose that e^{-t_0A} or e^{-t_0B} is a Hilbert-Schmidt operator. Then $\beta(t) =$
$$e^{\varepsilon(t+t_0)B+t_0B}\,(e^{-t_0B}\,e^{-\varepsilon(t+t_0)B}\,\beta\,e^{t+t_0)A}\,)\,e^{-t_0A}\ .$$
So $\beta(t) \in \sigma(X \otimes Y, I \otimes B)$, and the mapping β is represented by the element
$$t \to \beta(t)$$
in τ_B^A.

IV. The if-part is a special case of III.

The identity $I: \sigma(X,A) \to \sigma(X,A)$ is continuous. Let $t > 0$. Then for some $\tau > 0$ we have $e^{-\tau A}\ I\ e^{tA} \in X \otimes X$. Thus $e^{(t-\tau)A} \in X \otimes X$.

□

Chapter 7

Two illustrations

This chapter contains two illustrations. In Illustration I an important example of the general theory is investigated, and in Illustration II we construct a space of generalized functions in infinitely many dimensions. We prove that this space is nuclear.

Illustration I

We investigate the space $\tau(L_2(\mathbb{R}), H)$, in which

$$H = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + 1\right),$$

the Hamiltonian operator of the harmonic oscillator, and $L_2(\mathbb{R})$ is the Lebesque space of square integrable functions. The eigenfunctions of H are the Hermite functions,

$$\psi_n(x) = e^{-\frac{1}{2}x^2} H_n(x) \quad (\pi^{n/2}\, 2^n\, n!)^{\frac{1}{2}} \quad , \qquad n = 0,1,2,\ldots ,$$

where H_n is the n-th Hermite polynomial, and we have

$$H\,\psi_k = (k+1)\psi_k \quad , \qquad k = 0,1,2,\ldots .$$

The ψ_k's establish an orthonormal basis in $L_2(\mathbb{R})$.

In the Introduction we gave the following equivalences

(7.1.a) $\quad f \in \tau(L_2(\mathbb{R}), H) \Leftrightarrow f = \sum_{k=1}^{\infty} a_k\, \psi_k,$

where the a_k's satisfy

$$a_k = O(e^{-tk})$$

for all $t > 0$.

b) $G \in \sigma(L_2(\mathbb{R})H) \Leftrightarrow G = \sum_{k=1}^{\infty} b_k \psi_k$,

where the b_k's satisfy

$$b_k = O(e^{k\tau})$$

for a fixed $\tau > 0$

In his remarkable paper [B], see p.260, De Bruijn allready mentions the space that we denote by $\sigma(L_2(\mathbb{R}), H)$. He notes that it would not be hard to extend S^* to the space of Hermite pansions, introduced by Korevaar in [K]. The space he aims at is our space $\sigma(L_2(\mathbb{R}), H)$. There is a small notational difference. Instead of the ψ_k's , he takes the functions φ_k, defined as

$$\varphi_k(x) =: e^{-\pi x^2} H_k \ (x\sqrt{2\pi}) \ / \ (2^{n-\frac{1}{2}} \ n!)^{\frac{1}{2}} =$$

$$= \pi^{n/4} \ 2^{-\frac{1}{4}} \ \psi_k(x\sqrt{2\pi}) \quad , \qquad k = 0,1,2,\ldots \quad .$$

In De Graaf's terminology (see [G]), the generalized function space S^*, introduced by De Bruijn ([B]), is the space $S'_{L_2(\mathbb{R}),H}$. In general we have the inclusion

$$S'_{X,A} \subset \sigma(X,A) \ .$$

De Bruijn's test function space is characterized by

$f \in S \Leftrightarrow$: f is entirely analytic, and there are A,B,C > 0:

$$|f(x + iy)| \leq C \exp(-Ax^2 + By^2) .$$

Here we shall derive a similar characterization for elements in $\tau(L_2(\mathbb{R}),H)$. Let $f \in \tau(L_2(\mathbb{R}),H)$. Then for each $\alpha > 0$ there exists $g_\alpha \in L_2(\mathbb{R})$ such that $f = N_\alpha g_\alpha$, where $(N_\alpha)_{\alpha>0}$ is the semigroup generated by H. Taking into account the modification that we mentioned before, it follows from Theorem 6.3 in [B]

$$|f(x + iy)| \leq C_\alpha \exp\left(y^2 \frac{\coth \alpha}{2} - x^2 \frac{\tanh \alpha}{2}\right)$$

for $C_\alpha > 0$, only depending g_α .

Since $\alpha > 0$ can be taken arbitrarily

$$f \in \tau(L_2(\mathbb{R}),H) \Rightarrow f \text{ is entirely analytic,}$$

and for all a, $0 < a < 1$, there is $C_a > 0$ such that

$$|f(x + iy)| \leq C_a \exp\left(\frac{a^{-1}}{2} y^2 - \frac{a}{2} x^2\right).$$

Suppose f is entirely analytic and f satisfies the inequalities given above. Then for each $\alpha > 0$ there is b, $0 < b < 1$ with $\coth \alpha > b^{-1}$. From Theorem 10.1 in [B] it follows that there exists $g_\alpha \in L_2(\mathbb{R})$ with $f = N_\alpha g_\alpha, \alpha > 0$. So $f \in \tau(L_2(\mathbb{R}),H)$, and we have proved

Theorem 7.2

$f \in \tau(L_2(\mathbb{R}),H) \Leftrightarrow f$ is entirely analytic, and

$$\forall_{a,0<a<1}\ \exists_{C_a}: |f(x+iy)| \le C_a \exp\left(\frac{a^{-1}}{2} y^2 - \frac{a}{2} x^2\right) .$$

Let F_1 denote the Hilbert space of entirely analytic functions as introduced by Bargmann in [Ba 1]. The inner product in F_1 is given by

$$(f,g) = \int_{\mathbb{C}} f(z)\ \overline{g(z)}\ d\mu_{\mu_1}(z) ,$$

with $z = x + iy$, $d\mu_1(z) = \frac{1}{\pi} \exp(-|z|^2)\ dx\ dy$, so that f belongs to F_1 if and only if $(f,f) < \infty$.

An orthonormal basis of F_1 is given by

$$u_m(z) = \frac{z^m}{\sqrt{m!}} , \qquad m \in \mathbb{N} \cup \{0\} .$$

Thus every $f \in F_1$ can be written as

$$f(z) = \sum_{m=0}^{\infty} a_m\ u_m(z) ,$$

and we have $|\underline{f}(z)| \le \|\underline{f}\|\ e^{\frac{1}{2}|z|^2}$. The unitary operator $\hat{A} : L_2(\mathbb{R}) \to F_1$ is given by

$$(\hat{A}\ g)\ (z) = \int^{\infty} A(z,x)\ g(x)\ dx ,$$

with

$$A(z,x) = \pi^{-\frac{1}{4}} \exp -\tfrac{1}{2}(z^2 + x^2) + \sqrt{2}\ \ zx) .$$

The operator $\hat{A}$ has the following properties

$$\hat{A}\,\psi_m = u_m \quad \text{and} \quad (\hat{A}\,H\psi_m)(\zeta) = u_m(\zeta) + \zeta\frac{du_m}{dz}(\zeta) \;.$$

The positive self adjoint operator $1 + z\frac{d}{dz}$, defined in F_1, generates a semigroup $(M_t)_{t>0}$, whose action is simply given by

$$(M_t\, f)(z) = e^{-t}\, f(e^{-t}\, z) \quad , \quad t > 0.$$

In the following theorems we characterize the elements in $\tau(F_1, 1 + z\frac{d}{dz})$ and $\sigma(F_1, (1 + z\frac{d}{dz}))$.

Theorem 7.3

$$f \in \tau(F_1,\ 1 + z\frac{d}{dz}) \Leftrightarrow \forall_{c>0}\ \exists_{D>0} : |f(z)| \le D\, e^{c|z|^2} \;.$$

Proof:

$\Rightarrow$) Let $c > 0$. Take $t > 0$ zo large that $\frac{1}{2}e^{-2t} < c$. Then

$$|f(z)| = e^{-t}|(\underline{M}_t\, f)(e^{-t}\, z)| \le e^{-t}\|\underline{M}_t\, f\|\exp(\tfrac{1}{2}e^{-2t}\, |z|^2) \;.$$

$\Leftarrow$) Let $t > 0$. Take $0 < c < \frac{1}{2}e^{-2t}$. Then

$$|f(e^t\, z)| \le D\exp(c\, e^{2t}\, |z|^2)$$

So
$$\int_{\mathbb{C}} |f(e^t\, z)|^2\, d\mu_1(z) < \infty \;.$$

Theorem 7.4

$F \in (F_1,\ 1 + z\frac{d}{dz}) \Leftrightarrow F$ is entirely analytic, and

$$\exists_{c>0}\ \exists_{D>0} : |F(z)| \le D\, e^{c|z|^2} .$$

Proof:

$\Rightarrow$) There is $\tau > 0$ such that $M_\tau F \in F_1$. Thus for all $t \ge \tau$, there are $a_m(t)$, $m \in \mathbb{N} \cup \{0\}$, such that

$$(M_t F)(z) = \sum_{m=0}^{\infty} a_m(t)\, u_m(z) .$$

Note that $a_m(t)$ satisfies $a_m(t_1 + t_2) = e^{-mt_1}\, a_m(t_2)$ for all $t_1, t_2 > 0$ with $t_1 + t_2 > \tau$. It follows that there are a_m which satisfy

$$a_m(t) = e^{-mt}\, a_m\ , \quad m \in \mathbb{N} \cup \{0\},\ t > 0.$$

We have

$$F(z) = e^{t}(M_t F)(e^{t} z) = \sum_{m=0}^{\infty} a_m(t)\, e^{mt}\, u_m(z)$$

and the series converge for all $z \in \mathbb{C}$.

Thus F is entirely analytic.

Since $M_\tau F \in F_1$ we have

$$|F(z)| = e^{\tau} |(M_\tau F)(e^{\tau} z)| \le e^{\tau} \| M_\tau F \| \exp(\tfrac{1}{2}(z)^2\, e^{2t}) .$$

$\Leftarrow$) Take $\tau > 0$ so large that $c\, e^{-2\tau} < \frac{1}{2}$. Then

$$|(M_\tau F)(z)| = |e^{-\tau} F(e^{-\tau} z)| \le D\, e^{-\tau} \exp(c\, e^{-2\tau} |z|^2) .$$

□

In the introduction we showed that the space of tempered distributions $S'(\mathbb{R})$ is equal to $\sigma(L_2(\mathbb{R}), \log(H))$. In [Ba 2], Bargmann proves that the mapping $\hat{A}$ can be uniquely etended to $S'(\mathbb{R})$. As a simple consequence of the general theory, given in this paper, we easily derive that $\hat{A}$ can be uniquely extended to a unitary mapping from $\sigma(L_2(\mathbb{R}), H)$ onto $\sigma(F_1, (1 + z\frac{d}{dz}))$. In [Ba 2], Bargmann extensively studies the image of $S'(\mathbb{R})$ under the unitary mapping A. Similarly, $\sigma(F_1, 1 + z\frac{d}{dz})$ could be subject for further investigation.

Finally, note that the case of several dimensions

$$\sigma(L_2(\mathbb{R}^k), H_k) \quad \text{and} \quad \sigma\left(F_k, k + \sum_{i=1}^{k} z_i \frac{d}{dz_i}\right)$$

with

$$H_k = \tfrac{1}{2}\left(\sum_{i=1}^{k}\left(-\frac{\partial^2}{\partial x_i^2} + x_i^2\right) + k\right),$$

can be treated in the same way.

Illustration II

We will construct a nuclear space of generalized functions in infinitely many dimensions. This space can be regarded as a direct sum of generalized function spaces. Similar constructions are given in [KMP] and in [BLT]. Our construction suits perfectly well in the frame-work of the general theory as given in this paper.

First we give some general results concerning direct sums. Let X_i, $i \in \mathbb{N}$, be Hilbert spaces, and A_i, $i \in \mathbb{N}$, be unbounded positive self-adjoint operators in X_i with domains $D(A_i)$. By X we denote the countable direct sum $\bigoplus_{i=1}^{\infty} X_i$, and the elements of X will be denoted by (f_i). Thus $(f_i) \in X$ if and only if $f_i \in X_i$, $i \in \mathbb{N}$, and $\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$. With the inner product $((f_i),(g_i)) = \sum_{i=1}^{\infty} (f_i,g_i)_{X_i}$, X becomes a Hilbert space.
Let L_i be a linear operator in X_i, $i \in \mathbb{N}$. Then formally we define the linear operator $\mathrm{diag}(L_i)$ in X by

$$\mathrm{diag}(L_i)(f_i) = (L_i\, f_i) \ .$$

According to this definition the operator $\mathrm{diag}(e^{-tA_i})$, $t > 0$ is bounded from X into X , and the operators form a holomorphic semigroup in the right half plane $\mathrm{Re}\ t > 0$.
We define

$$D(A) := \{(f_i) \in X \mid f_i \in D(A_i) \text{ and } (A_i f_i) \in X\}.$$

Then $A(f_i) = \mathrm{diag}(A_i)(f_i)$ is well-defined for $(f_i) \in D(A)$.

Lemma 7.5

A with domain $D(A)$ is self-adjoint.

Proof:

$D(A)$ is dense in X. Let $(g_i) \in X$. Suppose there is $(u_i) \in X$ such that for all $(f_i) \in D(A)$

$$(A(f_i),(g_i)) = ((f_i),(u_i)).$$

Let $j \in \mathbb{N}$. For all $f_j \in D(A_j)$ we have

$$(A_j f_j, g_j) = (f_j, u_j) ,$$

by taking $(f_i) = (0,0,\ldots 0,f_j,0,\ldots)$. From the self-adjointness of A_j it follows that $g_j \in D(A_j)$ and $A_j g_j = u_j$. Since $j \in \mathbb{N}$ is arbitrarily chosen $(g_i) \in D(A)$ and $A(g_i) = (u_i)$

□

Let Y be a Hilbert space. Then we define

$$X_k =: Y^{\otimes k} = \underbrace{Y \otimes Y \otimes \ldots \otimes Y}_{k \text{ times}}$$

We can identify $Y^{\otimes k}$ and $Y^{\otimes k-1} \otimes Y$ following the general theory about tensor products of Hilbert spaces cf[RS I] , p. 49. Now let T be a positive self-adjoint operator in Y. Then we take $A_1 = T = T^{\boxplus 1}$ with $D(A_1) = D(T)$ and for $k > 1$.

$$A_k = T^{\boxplus k} =: T^{\boxplus k-1} \boxplus T$$

with its domain the algebraic tensor product.

$$D(A_k) = D(A_{k-1}) \otimes_a D(T) \ .$$

Note that $A_k = \underbrace{T \boxplus T \boxplus \ldots \boxplus T}_{k \text{ times}} = T \otimes I \otimes \ldots \otimes I + I \otimes T \otimes I \otimes \ldots \otimes I + \ldots$

$$\ldots + I \otimes \ldots \otimes I \otimes T \ .$$

With the aid of Theorem 5.3 it can be proved inductively, that all the A_k's are essentially self-adjoint in X_k. We denote the unique self-adjoint extension by $\tilde{A}_k$; $\tilde{A}_k$ is positive.

The space $X =: \bigoplus_{i \in \mathbb{N}} X_i$ is a well-defined Hilbert space and $A = \text{diag}(\tilde{A}_1, \tilde{A}_2, \ldots)$ is a self-adjoint operator in X.

We are now going to define a space of generalized functions in infinitely many dimensions. For X we take the Lebesgue space $L_2(\mathbb{R})$ and

$$H = \tfrac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + 1\right) ,$$

the Hamiltonian operator of the harmonic oscillator, satisfying $H\psi_k = (k+1)\psi_k$, $k = 0,1,2,\ldots$ with ψ_k the k-th Hermite function (see Illustration I).

Then $X_k = L_2(\mathbb{R}^k)$ and H_k is the Hamiltonian operator of the harmonic oscillator in k dimensions,

$$H_k = \tfrac{1}{2} \sum_{i=1}^{k} \left(-\frac{\partial^2}{\partial x_i^2} + x_i^2 + 1\right)$$

We denote $\bigoplus_{k=1}^{\infty} L_2(\mathbb{R}^k)$ by $F(\mathbb{R})$ and the positive self-adjoint operator $\text{diag}(H_i)$ by H_∞. The eigenvalues of H_∞ are the natural numbers $\mathbb{N} = 1,2,\ldots$.

Using some combinatorics we can easily show that the multiplicity $M^{(N)}$ of the eigenvalue N is just 2^{N-1}. We assert that e^{-tA} is a Hilbert-Schmidt operator if $t > \frac{1}{2} \log 2$. To this end we compute the sum of the squares of the eigenvalues of e^{-tA},

$$\sum_{N=1}^{\infty} M^{(N)} e^{-2Nt} = \sum_{N=1}^{\infty} 2^{N-1} e^{-2Nt} =$$

$$= \frac{1}{2} \sum_{N=1}^{\infty} e^{-2N(t-\frac{1}{2}\log 2)} < \infty.$$

From Theorem 1.11 and Theorem 2.9 we conclude that $\tau(F(\mathbb{R}), H_\infty)$ and $\sigma(F(\mathbb{R}), H_\infty)$ are nuclear, and following Chapter VI the Kernel theorems hold true.

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