A Gelfand triple approach to Wigner and Husimi representations

A GELFAND TRIPLE APPROACH TO WIGNER AND HUSIMI REPRESENTATIONS

R.BEUKEMA, W.M. DE MUYNCK, AND J. DE GRAAF

ABSTRACT. The notion of Gelfand triples is applied to interpret mathematically a family of phase-space representations of quantum mechanics interpolating between the Wigner and Husimi representations. Gelfand triples of operators on Hilbert space, and Gelfand triples of functions on phase-space are introduced in order to get isomorphic correspondences between operators and their phase-space representations. The phase-space Gelfand triples are characterized by means of growth conditions on the analytic continuation of the functions. We give integral expressions for the sesquilinear forms belonging to the phase-space Gelfand triples. This provides mathematically rigorous phase-space analogues for quantum mechanical expectation values of bounded operators.

1. INTRODUCTION

It is shown in [9] and in [1] that quantum mechanical expectation values can be expressed as phase-space averages. However, little care has been taken to provide a mathematically rigorous formulation.

For the special case of the Wigner distribution several mathematically rigorous formulations of the idea to express expectation values in terms of 'averages' of functions on phase-space have been carried out using appropriate theories of generalized functions. For example [12] and more recently [14]. The generalized functions in [12] are introduced by means of a special choice for a one-parameter semi-group of 'smoothing' operators. In [27] the approach taken in [12] is generalized and used to give a mathematically rigorous version of the Dirac formalism.

The concept of Dirac basis presented in [27] puts the concept of (continuous) bases of (generalized) eigenvectors of (unbounded) self-adjoint operators into an elegant functional analytic framework. We show that the Wigner representation is generated by a family of bounded operators which are the generalized simultaneous eigenvectors of two commuting self-adjoint super operators. These super operators constitute a complete set of commuting self-adjoint operators. By (some version of) the spectral theorem this implies that the Wigner distribution gives a unitary correspondence between

Date: January 8, 2002.

Hilbert-Schmidt operators and square-integrable functions on phase-space. We construct an isomorphic correspondence between a Geldfand triple of operators centered around the Hilbert-Schmidt space and a Gelfand triple of (generalized) functions centered around the space of square-integrable functions on phase-space. This is similar to the approach taken in [14] where, however, a different Gelfand triple is used. Our system of Gelfand triples (which is related to the theory in [12]) makes it possible to treat Wigner and Husimi representations in a similar way.

The Husimi representation is generated by the family of one-dimensional projection operators on coherent state vectors. This family is a subset of the set of all simultaneous eigenvectors of two commuting (unbounded) non-normal super operators. We use a system of Gelfand triples as described (for the case of the Wigner representation) above to get an isomorphic correspondence between the operator Geldfand triple and a phase-space Gelfand triple.

The phase-space Gelfand triples are characterized by means of growth conditions on the analytic continuation of the functions. We give integral expressions for the sesquilinear forms belonging to the phase-space Gelfand triples. This provides a mathematically rigorous phase-space analogue for quantum mechanical expectation values of bounded operators.

2. Outline

States of a quantum mechanical system are described by density operators on a separable complex Hilbert space H. A density operator is a non-negative self-adjoint trace-class operator ρ on H normalized by the condition $\mathrm{Tr}(\rho) = 1$. Quantum mechanical measurements are described by positive operator valued measures. A positive operator valued measure (POVM) is a set function M defined on a σ-algebra and taking its values in the space of bounded operators on H, such that for all f in the unit sphere of H, the set function M_f, defined by $M_f(\Delta) = (f, M(\Delta)f)_{\mathsf{H}}$, is a probability measure. (See [7].) A POVM whose range contains projection operators only is called a PVM (projection valued measure) or a spectral measure. The range of a PVM is a commutative algebra, and, conversely, a POVM with commutative range is a probability average over PVM's taking their values in the (commutative) von Neumann algebra generated by the range of the POVM. (See [2], [3] and [21].) A POVM is sometimes called a generalized spectral family (see e.g. [23]). POVM's M are related to subnormal operators $\mathcal{A}$ (and their adjoints $\mathcal{A}^*$) by relations $\mathcal{A}^{*k}\mathcal{A}^{\ell} = \int_{\mathbb{C}} \bar{z}^k z^\ell \, dM(z)$, $k, \ell \in \mathbb{N}_0$ (see [11]). (See [7] for integration theory w.r.t. POVM's). PVM's are related to normal operators in this way. The outcomes of a quantum mechanical measurement of a POVM M are distributed according to the (classical) probability law

$S \mapsto \mathrm{Prob}_\rho(\Delta) = E^Q_\rho(M(\Delta))$, where $E^Q_\rho(M(\Delta)) = \mathrm{Tr}(M(\Delta)\rho)$, the quantum mechanical expectation value of the operator $M(\Delta)$. (See [16], chapter I, for the interpretations of density operators as (non-commutative) generalizations of classical probability distributions, of bounded self-adjoint operators as (non-commutative) generalizations of classical random variables, and of $E^Q_\rho(B)$ as a generalized expectation value of classical probability theory.)

From this it is clear that we have to deal with the following triple of linear spaces:

$$\mathsf{B}_1(\mathsf{H}) \subset \mathsf{B}_2(\mathsf{H}) \subset \mathsf{B}_\infty(\mathsf{H}), \tag{2.1}$$

where $\mathsf{B}_1(\mathsf{H})$ denotes the space of trace-class operators on H, $\mathsf{B}_2(\mathsf{H})$ the space of Hilbert-Schmidt operators on H and $\mathsf{B}_\infty(\mathsf{H})$ the space of bounded operators on H. It is also clear that we have to consider the sesquilinear form $(\mathcal{B}, \rho) \mapsto \mathrm{Tr}(\mathcal{B}^*\rho)$ on $\mathsf{B}_\infty(\mathsf{H}) \times \mathsf{B}_1(\mathsf{H})$. If we want to use phase-space representations of quantum mechanics then we need a phase-space analog for this sesquilinear form. In this paper we will provide, for a family of phase-space representations interpolating between the Wigner representation and Husimi representation, such a phase-space analog. However, for mathematical reasons (one reason is that we have no characterization for the space of Wigner representations of trace-class operators) we extend in this paper the sesquilinear form w.r.t. the first argument and restrict it w.r.t. the second argument. The set of density operators that is considered in our setup is thus not equal to the set of all density operators; However, it is large enough to contain some interesting (from a physical point of view) density operators such as projections on number states and projections on coherent states. More precisely, we consider the density operators in a linear space which we denote by B_+. This space is characterized by theorem 10.3. We will now give a short description of our setup:

In section 6 we define an operator Gelfand triple $\mathsf{B}_+^{(\tau)} \subset \mathsf{B}_2(\mathsf{H}) \subset \mathsf{B}_-^{(\tau)}$ where $\mathsf{B}_+^{(\tau)} \subset \mathsf{B}_1$ and $\mathsf{B}_-^{(\tau)} \supset \mathsf{B}_\infty$. In sections 7, 8 and 9 we define the Wigner operator $\mathbf{W}\colon \mathsf{B}_2(\mathsf{H}) \to \mathsf{L}_2(\mathbb{R}^2)$, the Husimi operator $\mathbf{H}\colon \mathsf{B}_2(\mathsf{H}) \to \mathsf{L}_2(\mathbb{R}^2)$ and a family of operators $\mathbf{W}_s\colon \mathsf{B}_2(\mathsf{H}) \to \mathsf{L}_2(\mathbb{R}^2)$ such that $\mathbf{W}_0 = \mathbf{W}$ and $\mathbf{W}_1 = \sqrt{2\pi}\mathbf{H}$. In section 10 we define for each $s \geq 0$ a Gelfand triple centered around $\mathsf{L}_2(\mathbb{R}^2)$ denoted by $\mathsf{W}_+^{(s,\tau)} \subset \mathsf{L}_2(\mathbb{R}^2) \subset \mathsf{W}_-^{(s,\tau)}$. This is done in such a way that $\mathbf{W}_s\colon \mathsf{B}_+^{(\tau)} \to \mathsf{W}_+^{(s,\tau)}$ is unitary. In section 8 we prove that for all bounded operators $\mathcal{B}$, $\mathrm{Tr}(\rho\mathcal{B})$ can be approximated uniformly on bounded subsets of B_1 by finite sums of the form $\sum_{n=1}^N b_n \mathbf{H}[\rho](q_n, p_n)$, $N \in \mathbb{N}$, $b_n \in \mathbb{C}$ and $q_n, p_n \in \mathbb{R}$. In section 12 we give an explicit formula for the approximation of $\mathrm{Tr}(\rho\mathcal{B})$, with $\rho \in \mathsf{B}_+ = \cup_{\tau>0}\mathsf{B}_+^{(\tau)}$, by integrals over of $\mathbf{H}[\rho]$.

The Gelfand triples and their interrelations are depicted in the following diagram.

$$\begin{array}{ccccc} \mathsf{B}_1 & \subset & \mathsf{B}_2 & \subset & \mathsf{B}_\infty \\ \cup & & & & \cap \\ \mathsf{B}_+^{(\tau)} & \subset & \mathsf{B}_2 & \subset & \mathsf{B}_-^{(\tau)} \\ \mathbf{w}_s \big\downarrow & & & & \big\downarrow \mathbf{w}_s^{-*} \\ \mathsf{W}_+^{(s,\tau)} & \subset & \mathsf{L}_2(\mathbb{R}^2) & \subset & \mathsf{W}_-^{(s,\tau)} \end{array}$$

$$\mathbf{W}^{-*} = \mathbf{W}$$
$$\mathbf{W}_s = \exp\{-\tfrac{s}{4}|\triangle|\}\mathbf{W}$$
$$\mathbf{W}_s^{-*} = \exp\{\tfrac{s}{4}|\triangle|\}\mathbf{W}^{-*}$$
$$<\mathbf{W}_s^{-*}[\mathcal{B}], \mathbf{W}_s[\mathcal{A}]> = <\mathcal{B}, \mathcal{A}>$$
$$<\mathcal{B}, \mathcal{A}> = \mathrm{Tr}(\mathcal{B}^*\mathcal{A}) \text{ when } \mathcal{B} \in \mathsf{B}_\infty$$

(See section 9 for the definition of $|\triangle|$ in the Gaussian convolution operator $\exp\{-\frac{s}{4}|\triangle|\}$.)

3. Preliminaries and notations

For a Hilbert space H we identify the sesquilinear tensor product $\mathsf{H} \otimes \mathsf{H}$ with the space of Hilbert-Schmidt operators on H. For example: $f \otimes g$, where $f, g \in \mathsf{H}$, is identified with the operator $f \otimes g$ on H defined by $f \otimes g[h] = (g, h)_{\mathsf{H}} f$.

For $n \in \mathbb{N}$, the inner product on $\mathsf{L}_2(\mathbb{R}^n)$ is defined by $(f, g) = \int_{\mathbb{R}^n} \overline{f(\underline{x})} g(\underline{x}) d\underline{x}$. Define $\mathbf{K}\colon \mathsf{L}_2(\mathbb{R}) \otimes \mathsf{L}_2(\mathbb{R}) \to \mathsf{L}_2(\mathbb{R}^2)$ by $\mathbf{K}[f \otimes g](x, y) = f(x)\overline{g(y)}$ and linear and isometric extension. Hilbert-Schmidt operators are integral operators with an integral kernel from $\mathsf{L}_2(\mathbb{R}^2)$. The operator $\mathbf{K}$ maps Hilbert-Schmidt operators onto their integral kernels. We will identify $\mathsf{L}_2(\mathbb{R}) \otimes \mathsf{L}_2(\mathbb{R})$ with $\mathsf{L}_2(\mathbb{R}^2)$ through $\mathbf{K}$. We will denote the integral kernel of a Carleman operator $\mathcal{A}$ on $\mathsf{L}_2(\mathbb{R})$ (see e.g. [28]) by $\mathbf{K}[\mathcal{A}]$ even if $\mathcal{A} \notin \mathsf{B}_2$.

The space of Hilbert-Schmidt operators on H is denoted by $\mathsf{B}_2(\mathsf{H})$. This is a Hilbert space with inner product defined by $(\mathcal{A}_1, \mathcal{A}_2)_{\mathsf{B}_2} = \mathrm{Tr}(\mathcal{A}_1^*\mathcal{A}_2)$. The space of trace-class operators on H will be denoted by $\mathsf{B}_1(\mathsf{H})$ and the space of bounded operators on H by $\mathsf{B}_\infty(\mathsf{H})$. We will denote $\mathsf{B}_1(\mathsf{L}_2(\mathbb{R}))$ by B_1, $\mathsf{B}_2(\mathsf{L}_2(\mathbb{R}))$ by B_2 and $\mathsf{B}_\infty(\mathsf{L}_2(\mathbb{R}))$ by B_∞.

For operators $\mathcal{A}_1, \mathcal{A}_2$ on H with domains $\mathcal{D}(\mathcal{A}_1)$ and $\mathcal{D}(\mathcal{A}_2)$, the operator $\mathcal{A}_1 \bigotimes \mathcal{A}_2$ is defined on $\mathcal{D}(\mathcal{A}_1) \otimes_a \mathcal{D}(\mathcal{A}_2)$ (the algebraic sesquilinear tensor product) by $\mathcal{A}_1 \bigotimes \mathcal{A}_2[f \otimes g] = \mathcal{A}_1[f] \otimes \mathcal{A}_2[g]$ and linear extension. (See [28].) It acts as a super operator on B_2-operators (contained in its domain) by

$$(\mathcal{A}_1 \bigotimes \mathcal{A}_2)[\mathcal{B}] = \mathcal{A}_1 \mathcal{B} \mathcal{A}_2^*.$$

If $\mathcal{A}_1, \mathcal{A}_2$ are bounded (or symmetric or self-adjoint) operators on H then the operators $\mathcal{A}_1 \bigotimes \mathcal{A}_2$ and $\mathcal{A}_1 \bigotimes \mathcal{I} + \mathcal{I} \bigotimes \mathcal{A}_2$ on $\mathsf{B}_2(\mathsf{H})$ are bounded (or symmetric or self-adjoint). (See [28].) For two operators $\mathcal{A}_1, \mathcal{A}_2$, operators $\mathcal{A}_1 \bigotimes \mathcal{I}$ and $\mathcal{I} \bigotimes \mathcal{A}_2$ commute. For self-adjoint operators $\mathcal{A}_1$ and $\mathcal{A}_2$,

$$\exp\{i(\mathcal{A}_1 \bigotimes \mathcal{I} + \mathcal{I} \bigotimes \mathcal{A}_2)\} = \exp\{i\mathcal{A}_1\} \bigotimes \exp\{-i\mathcal{A}_2\}. \tag{3.1}$$

(See [28] theorem 8.35. Note the sign in this formula. Note also that $i(\mathcal{A}_1 \bigotimes \mathcal{A}_2) = (i\mathcal{A}_1) \bigotimes \mathcal{A}_2 = \mathcal{A}_1 \bigotimes (-i\mathcal{A}_2)$. For skew-adjoint operators $\mathcal{A}_1, \mathcal{A}_2$, (3.1) can be written as $\exp\{\mathcal{A}_1 \bigotimes \mathcal{I} + \mathcal{I} \bigotimes \mathcal{A}_2\} = \exp\{\mathcal{A}_1\} \bigotimes \exp\{\mathcal{A}_2\}$. If $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$ this implies $e^{[\mathcal{A},\cdot]}\mathcal{B} = e^{\mathcal{A}}\mathcal{B}e^{-\mathcal{A}}$.)

Define the self-adjoint operators $\mathcal{P}$ and $\mathcal{Q}$ on their usual domains in $\mathsf{L}_2(\mathbb{R})$ by $\mathcal{P}[f](x) = -if'(x)$ and $\mathcal{Q}[f](x) = xf(x)$. We have $[\mathcal{Q}, \mathcal{P}] = \mathcal{Q}\mathcal{P} - \mathcal{P}\mathcal{Q} = i\mathcal{I}$, where $\mathcal{I}$ is the identity operator. Let $\phi_n \in \mathsf{L}_2(\mathbb{R})$ be the n'th Hermite basis function. We define ϕ_n as follows: $\phi_0(q) = \pi^{-1/4}e^{-q^2/2}$ and $\phi_n = (n!)^{-1/2}\mathcal{S}^n\phi_0$, where $\mathcal{S} = (\mathcal{Q} - i\mathcal{P})/\sqrt{2}$. We have $\mathcal{S}\phi_n = \sqrt{n+1}\phi_{n+1}$, $\mathcal{S}^*\phi_n = \sqrt{n}\phi_{n-1}$ and $\mathcal{N}\phi_n = n\phi_n$ where $\mathcal{N} = \mathcal{S}\mathcal{S}^* = \frac{1}{2}(\mathcal{Q}^2 + \mathcal{P}^2 - \mathcal{I})$. The fractional Fourier transform $\mathcal{F}_\theta$ is defined by $\mathcal{F}_\theta = \exp(-i\theta\mathcal{N})$, and satisfies $\mathcal{F}_\theta\phi_n = e^{-i\theta n}\phi_n$ (see e.g. [6]). The ordinary Fourier transform $\mathcal{F} = \mathcal{F}_{\pi/2}$ satisfies

$$\mathcal{F}[f](p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(q)e^{-ipq}dq.$$

We have $\mathcal{F}_\theta^*\mathcal{Q}\mathcal{F}_\theta = \cos(\theta)\mathcal{Q} + \sin(\theta)\mathcal{P}$. (This is a consequence of $[i\mathcal{N}, \mathcal{Q}] = \mathcal{P}$ and $[i\mathcal{N}, \mathcal{P}] = -\mathcal{Q}$, as is explained in the proof of lemma 7.8 below.) In particular, $\mathcal{P} = \mathcal{F}^*\mathcal{Q}\mathcal{F} = -\mathcal{F}\mathcal{Q}\mathcal{F}^*$. From the canonical commutation relation, $[\mathcal{S}^*, \mathcal{S}] = \mathcal{I}$, follows the generalized Leibniz rule (see [20]):

$$f(\mathcal{S}^*)g(\mathcal{S}) = \sum_{n=0}^{\infty} \frac{1}{n!} g^{(n)}(\mathcal{S})f^{(n)}(\mathcal{S}^*). \tag{3.2}$$

for real analytic functions f, g. (Proof: The general case can be reduced to the case where f and g are of the form $x \mapsto e^{tx}$, $t \in \mathbb{R}$. For this special case, (3.2) follows from the identities $e^{t\mathcal{S}^*}e^{r\mathcal{S}} = e^{tr}e^{r\mathcal{S}}e^{t\mathcal{S}^*}$, by expanding e^{tr} as a Taylor series around $tr = 0$.) We have $\mathcal{F} = e^{-i(\frac{\pi}{2})\mathcal{N}}$. The parity operator Π, defined on $\mathsf{L}_2(\mathbb{R})$ by $\Pi[f](x) = f(-x)$, satisfies $\Pi = \mathcal{F}^2$, hence $\Pi\phi_n = (-1)^n\phi_n$.

For a bounded operator $\mathcal{A}$ on $\mathsf{L}_2(\mathbb{R})$ we will denote by $\mathcal{A}^{(k)}$ the operator on $\mathsf{L}_2(\mathbb{R}^n)$ acting as $\mathcal{A}$ on the k'th variable only. For example $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$ denote the partial Fourier transforms on $\mathsf{L}_2(\mathbb{R}^2)$ in the first and second variable.

The 'squeezing' operator $\mathcal{Z}_\lambda\colon \mathsf{L}_2(\mathbb{R}) \to \mathsf{L}_2(\mathbb{R})$ is defined by $(\mathcal{Z}_\lambda f)(x) = \lambda^{-1/2}f(\lambda^{-1}x)$.

Definition 3.1. For an entire analytic function $\varphi\colon \mathbb{C}^2 \to \mathbb{C}$ let $\mathbf{D}[\varphi]\colon \mathbb{R}^2 \to \mathbb{C}$ be defined by $\mathbf{D}[\varphi](x, y) = \varphi(x + iy, x - iy)$.

Lemma 3.2. ***D*** *is injective.*

Proof. Let $\varphi(w_1, w_2) = \sum_{k,\ell=0}^{\infty} c_{k,\ell}w_1^k w_2^\ell$ be absolutely convergent for every $w_1, w_2 \in \mathbb{C}$. Then $k!\ell!\, c_{k\ell} = (\frac{\partial}{\partial z})^k(\frac{\partial}{\partial \bar{z}})^\ell \mathbf{D}[\varphi](x, y)|_{x=y=0}$ where $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. □

Lemma 3.3. *Let* $\varphi\colon \mathbb{C}^2 \to \mathbb{C}$ *be entire analytic. Let* $f = \boldsymbol{D}[\varphi]$. *Then* f *has a continuation to an entire analytic function of two complex variables which satisfies*

$$f(z_1, z_2) = \boldsymbol{D}^{-1}[f](z_1 + iz_2, z_1 - iz_2)$$

and

$$f(\frac{z_1 + z_2}{2}, \frac{z_1 - z_2}{2i}) = \boldsymbol{D}^{-1}[f](z_1, z_2).$$

4. Gelfand triples

In the theory of rigged Hilbert spaces (See [15], [8]) or Gelfand triples, Hilbert space is replaced by a triple of topological vector spaces: The Hilbert space itself, a dense linear subspace continuously embedded in the Hilbert space, and the space of continuous linear forms on this subspace. Examples are given by (the classical) Sobolev spaces, considered as dense linear subspace of the corresponding Hilbert spaces of square integrable function classes (see e.g. [25]). Another example is given by distribution theory (e.g. of L. Schwartz): Distributions are the continuous linear forms on a space of test-functions continuously embedded in a space of square integrable function classes. In [14] Gelfand triples of Banach spaces are used for investigations related to generalized coherent states and in [19] these Gelfand triples are used for investigations related to the Wigner and the Husimi representations. In this paper the test-space is always a Hilbert space or a union of an increasing family of Hilbert spaces equipped with the inductive limit topology related to this family (see e.g. [10]). In the Gelfand triple (2.1), B_∞ takes the place of the topological dual of B_1 even though this is only a particular representation thereof. Gelfand triples are defined below accordingly.

We will use the following notation: $\mathsf{X} \hookrightarrow \mathsf{Y}$, for topological vector spaces X and Y, means X is a linear subspace of Y and the canonical inclusion operator is continuous.

Definition 4.1. A Gelfand triple is a triple of locally convex topological $\mathbb{C}$-vector spaces $\mathsf{X}_+ \hookrightarrow \mathsf{H} \hookrightarrow \mathsf{X}_-$ centered around a complex Hilbert space H; This triple is equipped with a sesquilinear form $<\cdot,\cdot>\colon \mathsf{X}_- \times \mathsf{X}_+ \to \mathbb{C}$ which reduces on $\mathsf{H} \times \mathsf{X}_+$ to the inner product of H, is separating on both sides, separately continuous and provides an anti-linear representation of X_- as the strong topological dual of X_+.

Because the anti-linear forms $<\cdot, w>$ for $w \in \mathsf{X}_+$ separate X_-, the linear space X_+ is weak*-dense in X_-. In particular, X_+ is dense in H.

4.0.1. *Extension of operators.* For $k = 1, 2$ let $\mathsf{X}_+^{(k)} \subset \mathsf{H}^{(k)} \subset \mathsf{X}_-^{(k)}$ be a Gelfand triple. Let $\mathcal{A}\colon \mathsf{H}^{(1)} \to \mathsf{H}^{(2)}$ be an operator satisfying $\mathcal{A}^*(\mathsf{X}_+^{(2)}) \subset \mathsf{X}_+^{(1)}$.

Define the operator $\mathcal{A}^{\text{ext}}\colon \mathsf{X}_-^{(1)} \to \mathsf{X}_-^{(2)}$ by $<\mathcal{A}^{\text{ext}}[F], g> = <F, \mathcal{A}^*[g]>$ for $F \in \mathsf{X}_-^{(1)}$ and $g \in \mathsf{X}_+^{(2)}$.

4.1. **Gelfand triples of Banach spaces.** If X_+ is a Banach space then X_- is also a Banach space. We will assume in this case that the norm of X_- is related to that of X_+ by

$$||F||_{\mathsf{X}_-} = \sup\{|<F, w>| : w \in \mathsf{X}_+,\ ||w||_{\mathsf{X}_+} \leq 1\}. \tag{4.1}$$

4.2. **Gelfand triples of Hilbert spaces.** In the following definition we introduce a Gelfand triple corresponding to a fixed bounded operator $\mathcal{R}$ on a Hilbert space H.

Definition 4.2. Let H be a Hilbert space. Let $\mathcal{R}$ be a closed densely defined injective operator on H with dense range. Let H_+ be the Hilbert space completion of the range of $\mathcal{R}$, equipped with the inner product $(\cdot,\cdot)_+$ defined by $(v, w)_+ = (\mathcal{R}^{-1}v, \mathcal{R}^{-1}w)_{\mathsf{H}}$. Let H_- be the Hilbert space completion of H, equipped with the inner product $(\cdot,\cdot)_-$ defined by $(f, g)_- = (\mathcal{R}^*[f], \mathcal{R}^*[g])_{\mathsf{H}}$. Define $(\mathcal{R}^*)^{\text{ext}}\colon \mathsf{H}_- \to \mathsf{H}$ by isometric extension of $\mathcal{R}^*\colon (\mathsf{H}, (\cdot,\cdot)_-) \to \mathsf{H}$. Define the sesquilinear form $<\cdot,\cdot>\colon \mathsf{H}_- \times \mathsf{H}_+ \to \mathbb{C}$ by

$$<F, g> = ((\mathcal{R}^*)^{\text{ext}}[F], \mathcal{R}^{-1}g)_{\mathsf{H}}.$$

The operators $\mathcal{R}\colon \mathsf{H} \to \mathsf{H}_+$ and $(\mathcal{R}^*)^{\text{ext}}\colon \mathsf{H}_- \to \mathsf{H}$ are unitary. If $\mathcal{R}$ is bounded, then $\mathsf{H}_+ \hookrightarrow \mathsf{H} \hookrightarrow \mathsf{H}_-$ is a Gelfand triple. This Gelfand triple is referred to as 'the Gelfand triple corresponding to the operator $\mathcal{R}$ on H' and denoted by $\mathcal{R}(\mathsf{H}) \subset \mathsf{H} \subset \mathcal{R}^{-1}(\mathsf{H})$.

4.2.1. *Extension of certain operators.* Let $\mathsf{H}_+^{(k)} \subset \mathsf{H}^{(k)} \subset \mathsf{H}_-^{(k)}$, $k \in \{0, 1\}$, be the Gelfand triples related to the operators $\mathcal{R}_k\colon \mathsf{H}^{(k)} \to \mathsf{H}^{(k)}$.

If the bounded operator $\mathcal{A}\colon \mathsf{H}^{(1)} \to \mathsf{H}^{(2)}$ intertwines $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ (i.e. $\mathcal{R}_2^*\mathcal{A} = \mathcal{A}\mathcal{R}_1^*$) then $\mathcal{A}^*(\mathsf{H}_+^{(2)}) \subset \mathsf{H}_+^{(1)}$ and the extension $\mathcal{A}^{\text{ext}}\colon \mathsf{H}_-^{(1)} \to \mathsf{H}_-^{(2)}$, defined in 4.0.1, satisfies $\mathcal{A}^{\text{ext}} = \mathcal{R}_2^{-*}\mathcal{A}\mathcal{R}_1^*$.

4.3. **Gelfand triples constructed using semigroups of operators.** In this section we introduce a Gelfand triple corresponding to a semigroup $(\mathcal{R}_\tau)_{\tau>0}$ of operators on a Hilbert space H.

Let $\mathcal{R}_\tau = \exp\{-\tau\mathcal{A}\}$, where $\mathcal{A}$ is a non-negative self-adjoint operator on Hilbert space H. For each $\tau > 0$, let $\mathsf{H}_+^{(\tau)} \subset \mathsf{H} \subset \mathsf{H}_-^{(\tau)}$ be the Gelfand triple corresponding to the operator $\mathcal{R}_\tau$ on H. Let $\mathsf{H}_+ = \cup_{\tau>0}\mathsf{H}_+^{(\tau)}$ and $\mathsf{H}_- = \cap_{\tau>0}\mathsf{H}_-^{(\tau)}$. There is a useful way to topologize such unions of increasing sequences or intersections of decreasing sequences of Hilbert spaces. These topologies, called inductive and projective limit topologies, respectively, are defined in a context similar to ours in [13]. In this paper we will not provide the definitions of these topologies.

Let $g \in \mathsf{H}_+$ and $F \in \mathsf{H}_-$. There is a $\tau > 0$ such that $g \in \mathsf{H}_+^{(\tau)}$ and $F \in \mathsf{H}_-^{(\tau)}$. This number τ is not unique, but $<F, g> = ((\mathcal{R}_\tau^*)^{\mathrm{ext}} F, \mathcal{R}_\tau^{-1} g)_{\mathsf{H}}$ does not depend on the particular choice of τ. Hence the following definition makes sense.

Definition 4.3. Let the sesquilinear form $<\cdot,\cdot>\colon \mathsf{H}_- \times \mathsf{H}_+ \to \mathbb{C}$ be defined on $\mathsf{H}_- \times \mathsf{H}_+^{(\tau)}$ by the previously defined sesquilinear form $<\cdot,\cdot>\colon \mathsf{H}_-^{(\tau)} \times \mathsf{H}_+^{(\tau)} \to \mathbb{C}$.

The triple of linear spaces $\mathsf{H}_+ \subset \mathsf{H} \subset \mathsf{H}_-$ equipped with the sesquilinear form $<\cdot,\cdot>$ is a Gelfand triple.

5. Complete sets of commuting self-adjoint operators

Definition 5.1. Let H be a Hilbert space. Let $\mathsf{X} \subset \mathsf{H} \subset \mathsf{Y}$ be a Gelfand triple. A family $\{F_x : x \in M\}$, with $F_x \in \mathsf{Y}$, indexed by a measure space (M, μ) is called Dirac basis if

$$\int_M \overline{<F_x, v>} <F_x, w> \, d\mu(x) = (v, w)_{\mathsf{H}}$$

for all $v, w \in \mathsf{X}$.

If $\mathsf{X} \subset \mathsf{H} \subset \mathsf{Y}$ is the Gelfand triple of Hilbert spaces corresponding to operator $\mathcal{R}$ on H and the unitary operator $\mathcal{U}\colon \mathsf{H} \to \mathsf{L}_2(M,\mu)$ is defined on the dense subset $\mathsf{X} \subset \mathsf{H}$ by $\mathcal{U}[w](x) = <F_x, w>$, for μ-almost every $x \in M$, then the operator $\mathcal{U}\mathcal{R}\colon \mathsf{H} \to \mathsf{L}_2(M,\mu)$ is a (bounded) Carleman operator (see [28]): $\mathcal{U}\mathcal{R}[f](x) = (e_x, f)_{\mathsf{H}}$, where $e_x = \mathcal{R} F_x$. This correspondence between Dirac bases and bounded Carleman operators is bijective.

This Dirac basis concept, introduced in [27], generalizes the concept of orthonormal bases in three distinct ways: The index set is generalized, the operators $M(\Delta)$, defined by $M(\Delta)w = \int_\Delta <F_x, w> F_x \, d\mu(x)$, constitute a POVM in stead of a PVM, and the Hilbert space is enlarged. Because of the enlargement of the Hilbert space, this also generalizes the notion of a tight generalized frame (see e.g. [17] or [4]). The generalized simultaneous eigenvectors of a complete set of commuting self-adjoint operators constitute a Dirac basis with the special property that it generates a PVM $\Delta \mapsto M(\Delta)$.

In [24], the generalized eigenvalue problem for a finite set of commuting self-adjoint operators is solved. Here we state a theorem about the existence of a Dirac basis consisting of the generalized eigenvectors of a complete set of commuting self-adjoint operators. A commuting set of self-adjoint operators is called complete if it is cyclic (i.e. if it has a cyclic vector).

Theorem 5.2. *Let* H *be a Hilbert space and* $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ *a set of commuting self-adjoint operators on* H. *The following conditions are equivalent:*

(a) $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$ *is cyclic.*

(b) *There is a finite non-negative Borel measure* μ *on* $\mathbb{R}^n$ *and a unitary operator* $\mathcal{U}\colon \mathsf{H} \to \mathsf{L}_2(\mathbb{R}^n, \mu)$ *such that* $\mathcal{U}\mathcal{A}_k\mathcal{U}^* = \mathcal{Q}_k$*, where the operator* $\mathcal{Q}_k$ *is defined by* $(\mathcal{Q}_k f)(x_1, \cdots, x_n) = x_k f(x_1, \cdots, x_n)$.

(c) *There is a Gelfand triple* $\mathsf{X} \subset \mathsf{H} \subset \mathsf{Y}$ *such that for all* k, $\mathcal{A}_k\colon \mathsf{X} \to \mathsf{X}$ *is continuous. There is a non-negative finite Borel measure* μ *on* $\mathbb{R}^n$ *such that* $supp(\mu) = \sigma(\mathcal{A}_1, \ldots, \mathcal{A}_n)$*, the joint spectrum of* $(\mathcal{A}_1, \ldots, \mathcal{A}_n)$*. There is a Dirac basis* $\{F_x : x \in supp(\mu)\}$ *in* Y *such that*

$$\mathcal{A}_k^{\mathrm{ext}} F_x = x_k F_x, \quad \mu\text{-almost all } x,\ 1 \le k \le n.$$

6. Some operator Gelfand triples

6.1. **Trace-class and bounded operators.** The triple (2.1) of Banach spaces is an example of a Gelfand triple. In particular: The sesquilinear form $(\mathcal{B}, \mathcal{T}) \mapsto \mathrm{Tr}(\mathcal{B}^*\mathcal{T})$ on $\mathsf{B}_1 \times \mathsf{B}_\infty$ provides an anti-linear representation of B_∞ as the strong topological dual of B_1. Apart from the usual Banach space topology on B_∞, we use the weak* (or the weak dual) topology. This locally convex topology is generated by the seminorms $\mathcal{B} \mapsto |\,\mathrm{Tr}(\mathcal{B}\mathcal{T})|$, $\mathcal{T} \in \mathsf{B}_1$. (See e.g. [10].)

6.2. **Harmonic oscillator.**

Definition 6.1. Let $\mathsf{B}_+^{(\tau)} \subset \mathsf{B}_2 \subset \mathsf{B}_-^{(\tau)}$ be the Gelfand triple related to the Hilbert space B_2 and the operator $e^{-\tau\mathcal{N}} \bigotimes e^{-\tau\mathcal{N}}$ on B_2.

We have the following inclusions:

$$\mathsf{B}_+^{(\tau)} \subset \mathsf{B}_1 \subset \mathsf{B}_2 \subset \mathsf{B}_\infty \subset \mathsf{B}_-^{(\tau)}.$$

For $\mathcal{B} \in \mathsf{B}_\infty$, $(e^{-\tau\mathcal{N}} \bigotimes e^{-\tau\mathcal{N}})^{\mathrm{ext}}[\mathcal{B}] = e^{-\tau\mathcal{N}}\mathcal{B}e^{-\tau\mathcal{N}}$. Hence $<\cdot,\cdot>$ corresponds with the sesquilinear form $\mathsf{B}_\infty \times \mathsf{B}_1 \ni (\mathcal{B}, \mathcal{A}) \mapsto \mathrm{Tr}(\mathcal{B}^*\mathcal{A})$ on $\mathsf{B}_\infty \times \mathsf{B}_+^{(\tau)}$.

Proposition 6.2. *For every* $\tau > 0$, $\mathsf{B}_+^{(\tau)}$ *is dense in* B_2 *and also in* B_1.

Proof. More generally, if $\mathcal{R}_1, \mathcal{R}_2 \in \mathsf{B}_\infty$ and $\mathcal{R}_k \in \mathsf{B}_2$ for some $k \in \{1, 2\}$ and $\mathcal{R}_1, \mathcal{R}_2$ both have dense range then $\mathcal{R}_1 \bigotimes \mathcal{R}_2(\mathsf{B}_2)$ is dense in B_1 and hence in B_2. Proof: Let $\mathcal{B} \in \mathsf{B}_\infty$. Assume that $\mathrm{Tr}(\mathcal{B}^*(\mathcal{R}_1\mathcal{A}\mathcal{R}_2^*)) = 0$ for all $\mathcal{A} \in \mathsf{B}_2$. Then $\mathrm{Tr}((\mathcal{R}_1^*\mathcal{B}\mathcal{R}_2)^*\mathcal{A}) = 0$ for all $\mathcal{A} \in \mathsf{B}_2$. This implies that $\mathcal{R}_1^*\mathcal{B}\mathcal{R}_2 = 0$. Hence $(\mathcal{R}_1 f, \mathcal{B}\mathcal{R}_2 g) = 0$ for all $f, g \in \mathsf{L}_2(\mathbb{R})$. Because $\mathcal{R}_1, \mathcal{R}_2$ both have dense range this implies that $\mathcal{B} = 0$. Let $\mathbf{R} = \mathcal{R}_1 \bigotimes \mathcal{R}_2$. We have shown that every continuous linear form on B_1 that is zero on the linear subspace $\mathbf{R}(\mathsf{B}_2)$ of B_1, is identically zero. This implies that the linear subspace $\mathbf{R}(\mathsf{B}_2)$ is dense in B_1. □

Definition 6.3. Let $\mathsf{B}_+ \subset \mathsf{B}_2 \subset \mathsf{B}_-$ be the Gelfand triple related to the Hilbert space B_2 and the semigroup $(e^{-\tau\mathcal{N}} \bigotimes e^{-\tau\mathcal{N}})_{\tau>0}$.

Proposition 6.4. *The self-adjoint operators $\mathcal{Q}\bigotimes\mathcal{I}, \mathcal{I}\bigotimes\mathcal{Q}, \mathcal{P}\bigotimes\mathcal{I}$ and $\mathcal{I}\bigotimes\mathcal{P}$ are continuous operators on* B_+

Proof. Proposition 13.7 and theorem 13.9. □

7. Wigner representation

Definition 7.1. The Wigner function $\mathbf{W}[\mathcal{A}]$ of a trace-class operator $\mathcal{A}$ is defined by

$$\mathbf{W}[\mathcal{A}](q,p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbf{K}[\mathcal{A}](q+\frac{y}{2}, q-\frac{y}{2}) e^{-ipy}\, dy.$$

Let $\mathbf{G}_{\mathcal{Q}} = \frac{1}{2}(\mathcal{Q}\bigotimes\mathcal{I} - \mathcal{I}\bigotimes\mathcal{Q})$ and $\mathbf{G}_{\mathcal{P}} = \frac{1}{2}(\mathcal{P}\bigotimes\mathcal{I} - \mathcal{I}\bigotimes\mathcal{P})$. This is a pair of commuting self-adjoint operators on B_2. Let $\mathbf{G}_{u,v} = \exp\{2i(v\mathbf{G}_{\mathcal{Q}} - u\mathbf{G}_{\mathcal{P}})\}$. By (3.1), $\mathbf{G}_{u,v} = \exp\{i(v\mathcal{Q} - u\mathcal{P})\}\bigotimes\exp\{i(v\mathcal{Q} - u\mathcal{P})\} = e^{iv\mathcal{Q}}e^{-iu\mathcal{P}}\bigotimes e^{iv\mathcal{Q}}e^{-iu\mathcal{P}}$. The order of composition of the operators $e^{iv\mathcal{Q}}$ and $e^{-iu\mathcal{P}}$ in the last identity may be interchanged on both sides of the $\bigotimes$ symbol simultaneously; on both sides this must be compensated with an extra factor, but the two factors cancel each other out.

Let $\tilde{\mathbf{G}}_{\mathcal{Q}} = \frac{1}{2}(\mathcal{Q}\bigotimes\mathcal{I} + \mathcal{I}\bigotimes\mathcal{Q})$ and $\tilde{\mathbf{G}}_{\mathcal{P}} = \frac{1}{2}(\mathcal{P}\bigotimes\mathcal{I} + \mathcal{I}\bigotimes\mathcal{P})$. This is a pair of commuting self-adjoint operators on B_2. Let $\tilde{\mathbf{G}}_{u,v} = \exp\{2i(v\tilde{\mathbf{G}}_{\mathcal{Q}} - u\tilde{\mathbf{G}}_{\mathcal{P}})\}$. By (3.1), $\tilde{\mathbf{G}}_{u,v} = \exp\{i(v\mathcal{Q} - u\mathcal{P})\}\bigotimes\exp\{-i(v\mathcal{Q} - u\mathcal{P})\} = e^{iv\mathcal{Q}}e^{-iu\mathcal{P}}\bigotimes e^{-iv\mathcal{Q}}e^{iu\mathcal{P}}$. Again, the order of composition of the two operators on both sides of the $\bigotimes$ symbol may be interchanged on both sides simultaneously.

Proposition 7.2. *The self-adjoint operators $\boldsymbol{G}_{\mathcal{Q}}$, $\boldsymbol{G}_{\mathcal{P}}$, $\tilde{\boldsymbol{G}}_{\mathcal{Q}}$ and $\tilde{\boldsymbol{G}}_{\mathcal{P}}$ and the unitary operators $\boldsymbol{G}_{u,v}$ and $\tilde{\boldsymbol{G}}_{u,v}$ map B_+ into itself. Hence the extensions of these operators to B_- exist. They will be denoted by $\boldsymbol{G}^{\text{ext}}_{\mathcal{Q}}$, $\boldsymbol{G}^{\text{ext}}_{\mathcal{P}}$, $\tilde{\boldsymbol{G}}^{\text{ext}}_{\mathcal{Q}}$, $\tilde{\boldsymbol{G}}^{\text{ext}}_{\mathcal{P}}$, $\boldsymbol{G}^{\text{ext}}_{u,v}$ and $\tilde{\boldsymbol{G}}^{\text{ext}}_{u,v}$.*

Proposition 7.3. *Let $\mathcal{U}_{u,v} = \frac{1}{\sqrt{2\pi}}\tilde{\boldsymbol{G}}^{\text{ext}}_{u/2,v/2}[\mathcal{I}] = \frac{1}{\sqrt{2\pi}}\exp\{i(v\mathcal{Q} - u\mathcal{P})\}$. Then $\boldsymbol{G}^{\text{ext}}_{\mathcal{Q}}\mathcal{U}_{q,p} = \frac{q}{2}\mathcal{U}_{q,p}$ and $\boldsymbol{G}^{\text{ext}}_{\mathcal{P}}\mathcal{U}_{q,p} = \frac{p}{2}\mathcal{U}_{q,p}$.*

Proof. From $[\tilde{\mathbf{G}}_{\mathcal{Q}}, \mathbf{G}_{Q}] = 0$ it follows that $\tilde{\mathbf{G}}^{-1}_{u,v}\mathbf{G}_{\mathcal{Q}}\tilde{\mathbf{G}}_{u,v} = e^{2iu[\tilde{\mathbf{G}}_{\mathcal{P}},\cdot]}\mathbf{G}_{\mathcal{Q}}$. Using $[\tilde{\mathbf{G}}_{P}, \mathbf{G}_{\mathcal{Q}}] = \frac{1}{2i}\mathbf{I}$, we get $\tilde{\mathbf{G}}^{-1}_{u,v}\mathbf{G}_{\mathcal{Q}}\tilde{\mathbf{G}}_{u,v} = \mathbf{G}_{\mathcal{Q}} + u\mathbf{I}$. The first result follows from $\mathbf{G}^{\text{ext}}_{\mathcal{Q}}[\mathcal{I}] = 0$. The proof of the second result is similar. □

Proposition 7.4. *Let $\mathcal{W}_{q,p} = \sqrt{\frac{2}{\pi}}\boldsymbol{G}^{\text{ext}}_{q,p}[\Pi]$. Then*

$$\tilde{\boldsymbol{G}}^{\text{ext}}_{\mathcal{Q}}\mathcal{W}_{q,p} = q\mathcal{W}_{q,p}, \qquad \tilde{\boldsymbol{G}}^{\text{ext}}_{\mathcal{P}}\mathcal{W}_{q,p} = p\mathcal{W}_{q,p}.$$

Proof. From $[\mathbf{G}_{\mathcal{Q}}, \tilde{\mathbf{G}}_{Q}] = 0$ it follows that $\mathbf{G}^{-1}_{u,v}\tilde{\mathbf{G}}_{\mathcal{Q}}\mathbf{G}_{u,v} = e^{2iu[\mathbf{G}_{\mathcal{P}},\cdot]}\tilde{\mathbf{G}}_{\mathcal{Q}}$. Using $[\mathbf{G}_{P}, \tilde{\mathbf{G}}_{\mathcal{Q}}] = \frac{1}{2i}\mathbf{I}$, we get $\mathbf{G}^{-1}_{u,v}\tilde{\mathbf{G}}_{\mathcal{Q}}\mathbf{G}_{u,v} = \tilde{\mathbf{G}}_{\mathcal{Q}} + u\mathbf{I}$. The first result follows from $\tilde{\mathbf{G}}^{\text{ext}}_{\mathcal{Q}}[\Pi] = 0$. The proof of the second result is similar. □

Proposition 7.5.

$$\tilde{\boldsymbol{G}}_{\mathcal{Q}}^{\text{ext}}\mathcal{U}_{q,p} = \frac{1}{i}\frac{\partial}{\partial p}\mathcal{U}_{q,p}, \quad \tilde{\boldsymbol{G}}_{\mathcal{P}}^{\text{ext}}\mathcal{U}_{q,p} = -\frac{1}{i}\frac{\partial}{\partial q}\mathcal{U}_{q,p}, \tag{7.1}$$

$$\boldsymbol{G}_{\mathcal{Q}}^{\text{ext}}\mathcal{W}_{q,p} = \frac{1}{2i}\frac{\partial}{\partial p}\mathcal{W}_{q,p}, \quad \boldsymbol{G}_{\mathcal{P}}^{\text{ext}}\mathcal{W}_{q,p} = -\frac{1}{2i}\frac{\partial}{\partial q}\mathcal{W}_{q,p}. \tag{7.2}$$

Proof. (7.1) follows from $\tilde{\mathbf{G}}_{\mathcal{Q}}^{\text{ext}}[\mathcal{I}] = \mathcal{Q}$ and $\tilde{\mathbf{G}}_{\mathcal{P}}^{\text{ext}}[\mathcal{I}] = \mathcal{P}$. (7.2) follows from $\mathbf{G}_{\mathcal{Q}}^{\text{ext}}[\Pi] = \mathcal{Q}\Pi$ and $\mathbf{G}_{\mathcal{P}}^{\text{ext}}[\Pi] = \mathcal{P}\Pi$. ☐

Proposition 7.6. *We have* $\mathcal{W}_{q,p}^* = \mathcal{W}_{q,p}$ *and* $\boldsymbol{W}[\mathcal{A}](q,p) = Tr(\mathcal{W}_{q,p}\mathcal{A})$ *for* $\mathcal{A} \in \mathsf{B}_1$.

Proof. The first result follows directly from the definitions. From proposition 7.5 it follows that $\mathbf{W}[\mathbf{G}_{u,v}\mathcal{A}](q,p) = \mathbf{W}[\mathcal{A}](q-u, p-v)$ for $\mathcal{A} \in \mathsf{B}_1$. From definition 7.1 it follows that $\mathbf{W}[f \otimes g](0,0) = (\mathcal{W}_{0,0}[g], f)$. This proves the second result of the proposition. ☐

Remark 7.7. (7.2) implies the following: If $\mathcal{A}\mathcal{Q} = \mathcal{Q}\mathcal{A}$ then $\frac{\partial}{\partial p}\mathbf{W}[\mathcal{A}](q,p) = 0$. If $\mathcal{A}\mathcal{P} = \mathcal{P}\mathcal{A}$ then $\frac{\partial}{\partial q}\mathbf{W}[\mathcal{A}](q,p) = 0$.

Lemma 7.8. *Let* $\boldsymbol{R}_\theta = \exp\{i\theta(\mathcal{Q}\bigotimes\mathcal{P} + \mathcal{P}\bigotimes\mathcal{Q})\}$. *Then*

$$\boldsymbol{K}[\boldsymbol{R}_\theta[\mathcal{A}]](x,y) = \boldsymbol{K}[\mathcal{A}](x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta)$$

and

$$\begin{aligned} \tilde{\boldsymbol{G}}_{\mathcal{Q}} &= 2^{-1/2}\boldsymbol{R}_{\pi/4}(\mathcal{Q}\bigotimes\mathcal{I})\boldsymbol{R}_{\pi/4}^{-1}, & \tilde{\boldsymbol{G}}_{\mathcal{P}} &= 2^{-1/2}\boldsymbol{R}_{\pi/4}(\mathcal{I}\bigotimes\mathcal{P})\boldsymbol{R}_{\pi/4}^{-1}, \\ \boldsymbol{G}_{\mathcal{Q}} &= 2^{-1/2}\boldsymbol{R}_{\pi/4}(-\mathcal{I}\bigotimes\mathcal{Q})\boldsymbol{R}_{\pi/4}^{-1}, & \boldsymbol{G}_{\mathcal{P}} &= 2^{-1/2}\boldsymbol{R}_{\pi/4}(\mathcal{P}\bigotimes\mathcal{I})\boldsymbol{R}_{\pi/4}^{-1}. \end{aligned} \tag{7.3}$$

Proof. Let $\theta \in \mathbb{R}$. Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be operators. We have $e^{\theta\mathcal{C}}\mathcal{A}e^{-\theta\mathcal{C}} = e^{\theta[\mathcal{C},\cdot]}\mathcal{A}$. If $[\mathcal{C},\mathcal{A}] = \mathcal{B}$ and $[\mathcal{C},\mathcal{B}] = -\mathcal{A}$ then $e^{\theta\mathcal{C}}\mathcal{A}e^{-\theta\mathcal{C}} = \cos(\theta)\mathcal{A} + \sin(\theta)\mathcal{B}$. (This follows from the Taylor series around 0 of the sin and cos functions respectively.) The commutation relations

$$[i(\mathcal{Q}\bigotimes\mathcal{P} + \mathcal{P}\bigotimes\mathcal{Q}), \mathcal{Q}\bigotimes\mathcal{I}] = \mathcal{I}\bigotimes\mathcal{Q},$$
$$[i(\mathcal{Q}\bigotimes\mathcal{P} + \mathcal{P}\bigotimes\mathcal{Q}), \mathcal{I}\bigotimes\mathcal{Q}] = -\mathcal{Q}\bigotimes\mathcal{I},$$

imply that $\mathcal{R}_\theta(\mathcal{Q}\bigotimes\mathcal{I})\mathcal{R}_\theta^{-1} = \cos\theta(\mathcal{Q}\bigotimes\mathcal{I}) + \sin\theta(\mathcal{I}\bigotimes\mathcal{Q})$ and $\mathcal{R}_\theta(\mathcal{I}\bigotimes\mathcal{Q})\mathcal{R}_\theta^{-1} = \cos\theta(\mathcal{I}\bigotimes\mathcal{Q}) - \sin\theta(\mathcal{Q}\bigotimes\mathcal{I})$. This implies the left hand sides of (7.3) and also that $\mathbf{K}[\mathbf{R}_\theta[\mathcal{Q}f \otimes g]](x,y) = (x\cos\theta + y\sin\theta)\mathbf{K}[\mathbf{R}_\theta[f \otimes g]](x,y)$ and $\mathbf{K}[\mathbf{R}_\theta[f \otimes \mathcal{Q}g]](x,y) = (y\cos\theta - x\sin\theta)\mathbf{K}[\mathbf{R}_\theta[f \otimes g]](x,y)$, for $f, g \in \mathsf{L}_2(\mathbb{R})$. By applying this repeatedly with $f = g = \phi_0$, it follows that $\mathbf{K}[\mathbf{R}_\theta[f \otimes g]](x,y) = \mathbf{K}[f \otimes g](x\cos\theta + y\sin\theta, y\cos\theta - x\sin\theta)$ for functions f, g which are polynomials multiplied by ϕ_0 and in particular for all Hermite basis functions.

The proofs of the right hand sides of (7.3) are similar to those of the left hand sides. ☐

Proposition 7.9. *The operators $\boldsymbol{G}_\mathcal{Q}$ and $\boldsymbol{G}_\mathcal{P}$ form a complete set of commuting self-adjoint operators on* B_2. *The operators $\tilde{\boldsymbol{G}}_\mathcal{Q}$ and $\tilde{\boldsymbol{G}}_\mathcal{P}$ also form a complete set of commuting self-adjoint operators on* B_2.

Proof. The operators $\mathcal{Q}\bigotimes\mathcal{I}$ and $\mathcal{I}\bigotimes\mathcal{P}$ form a complete set of commuting self-adjoint operators on B_2. By lemma 7.8 this is also the case with the pair of operators $\mathbf{G}_\mathcal{Q}$ and $\mathbf{G}_\mathcal{P}$ and the pair $\tilde{\mathbf{G}}_\mathcal{Q}$ and $\tilde{\mathbf{G}}_\mathcal{P}$. □

Theorem 7.10. *The families $(\mathcal{U}_{q,p})_{q,p\in\mathbb{R}}$ and $(\mathcal{W}_{q,p})_{q,p\in\mathbb{R}}$ are Dirac bases for the Gelfand triple* $\mathsf{B}_+ \subset \mathsf{B}_2 \subset \mathsf{B}_-$:

$$Tr(\mathcal{A}^*\mathcal{B}) = \int_{\mathbb{R}^2} \overline{Tr(\mathcal{W}_{q,p}\mathcal{A})}\, Tr(\mathcal{W}_{q,p}\mathcal{B})\, dqdp, \quad \mathcal{A},\mathcal{B} \in \mathsf{B}_+$$

and similar for $\mathcal{U}_{q,p}$ i.s.o. $\mathcal{W}_{q,p}$. The operator $\mathcal{A} \mapsto \boldsymbol{W}[\mathcal{A}]$ extends to a unitary mapping $\boldsymbol{W}\colon \mathsf{B}_2 \to \mathsf{L}_2(\mathbb{R}^2)$.

Proof. $\mathcal{W}_{q,p} \in \mathsf{B}_\infty$ hence $\mathcal{W}_{q,p} \in \mathsf{B}_-$. By theorem 5.2 and proposition 7.9 there is a non-negative Borel measure μ on $\mathbb{R}^2$ such that

$$\mathrm{Tr}(\mathcal{A}^*\mathcal{B}) = \int_{\mathbb{R}^2} \overline{\mathrm{Tr}(\mathcal{W}_{q,p}\mathcal{A})}\, \mathrm{Tr}(\mathcal{W}_{q,p}\mathcal{B})\, d\mu(q,p).$$

By proposition 7.5, μ is translation invariant, hence a constant multiple of the ordinary Lebesgue measure on $\mathbb{R}^2$. By calculating $(\phi_0, \mathcal{W}_{q,p}\phi_0)$ we see that this constant is 1: Using $e_{q,p} = e^{i(p\mathcal{Q}-q\mathcal{P})}\phi_0$, $\Pi e_{q,p} = e_{-q,-p}$ and the definition of $\mathcal{W}_{q,p}$, we get $(\phi_0, \mathcal{W}_{q,p}\phi_0) = \sqrt{\frac{2}{\pi}}\exp\{-q^2 - p^2\}$. Hence $\int_{\mathbb{R}^2} |\,\mathrm{Tr}(\phi_0, \mathcal{W}_{q,p}\phi_0)|^2 dqdp = 1$. The proof of the statement about $\mathcal{U}_{q,p}$ is similar. □

Proposition 7.11. *The Dirac bases $(\mathcal{U}_{q,p})_{q,p\in\mathbb{R}}$ and $(\mathcal{W}_{q,p})_{q,p\in\mathbb{R}}$ are related to each other by Fourier transform:*

$$\mathcal{W}_{q,p} = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i(vq-up)}\mathcal{U}_{u,v}\, dudv. \tag{7.4}$$

Proof. From $\mathcal{F}\bigotimes\mathcal{F}^*\tilde{\mathbf{G}}_\mathcal{Q} = -\mathbf{G}_\mathcal{P}\mathcal{F}\bigotimes\mathcal{F}^*$ and $\mathcal{F}\bigotimes\mathcal{F}^*\tilde{\mathbf{G}}_\mathcal{P} = \mathbf{G}_\mathcal{Q}\mathcal{F}\bigotimes\mathcal{F}^*$ it follows that $\mathcal{F}\bigotimes\mathcal{F}^*\tilde{\mathbf{G}}_{q,p} = \mathbf{G}_{p,-q}\mathcal{F}\bigotimes\mathcal{F}^*$. This, together with $\Pi = \mathcal{F}\bigotimes\mathcal{F}^*[\mathcal{I}]$, implies that $\mathcal{F}\mathcal{U}_{\sqrt{2}q,\sqrt{2}p}\mathcal{F} = \frac{1}{2}\mathcal{W}_{p/\sqrt{2},-q/\sqrt{2}}$. Hence (7.4) is equivalent to

$$\mathcal{F}\mathcal{W}_{\frac{q}{\sqrt{2}},\frac{p}{\sqrt{2}}}\mathcal{F} = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i(uq+vp)}\mathcal{W}_{\frac{u}{\sqrt{2}},\frac{v}{\sqrt{2}}}\, dudv. \tag{7.5}$$

By (3.1), $\mathcal{F}\bigotimes\mathcal{F}^* = \exp\{-i(\frac{\pi}{2})(\mathcal{N}\bigotimes\mathcal{I}+\mathcal{I}\bigotimes\mathcal{N})\}$. We have $\mathcal{N}\bigotimes\mathcal{I}+\mathcal{I}\bigotimes\mathcal{N} = \mathbf{G}_\mathcal{Q}^2 + \tilde{\mathbf{G}}_\mathcal{Q}^2 + \mathbf{G}_\mathcal{P}^2 + \tilde{\mathbf{G}}_\mathcal{P}^2 - \mathbf{I}$. By proposition 7.4 and (7.2),

$$(\mathcal{N}\bigotimes\mathcal{I}+\mathcal{I}\bigotimes\mathcal{N})\mathcal{W}_{\frac{u}{\sqrt{2}},\frac{v}{\sqrt{2}}} = \frac{1}{2}(u^2+v^2-\frac{\partial^2}{\partial u^2}-\frac{\partial^2}{\partial v^2}-2)\mathcal{W}_{\frac{u}{\sqrt{2}},\frac{v}{\sqrt{2}}}. \tag{7.6}$$

This implies (7.5). □

Proposition 7.12. *We have*

$$\mathcal{Z}^{(1)}_{\sqrt{2}}\mathcal{Z}^{(2)}_{\sqrt{2}}\boldsymbol{W}[e^{-\tau\mathcal{N}}\mathcal{A}e^{-\tau\mathcal{N}}] = e^{-\tau\mathcal{N}^{(1)}}e^{-\tau\mathcal{N}^{(2)}}\mathcal{Z}^{(1)}_{\sqrt{2}}\mathcal{Z}^{(2)}_{\sqrt{2}}\boldsymbol{W}[\mathcal{A}].$$

Proof. This follows from (7.6). □

Proposition 7.13. *Let* $\mathcal{A} \in \mathsf{B}_2$*. Then* $\boldsymbol{W}[\boldsymbol{G}_{u,v}\mathcal{A}](q,p) = \boldsymbol{W}[\mathcal{A}](q-u, p-v)$ *and* $\boldsymbol{W}[\mathcal{F}_\theta\mathcal{A}\mathcal{F}^*_\theta](q,p) = \boldsymbol{W}[\mathcal{A}](q\cos\theta - p\sin\theta, q\sin\theta + p\cos\theta)$.

Proof. The first identity follows directly from the definition of $\mathcal{W}_{q,p}$ and the group properties of $\mathbf{G}_{u,v}$, $u, v \in \mathbb{R}$. We have $2(\tilde{\mathbf{G}}_\mathcal{Q}\mathbf{G}_\mathcal{Q} + \tilde{\mathbf{G}}_\mathcal{P}\mathbf{G}_\mathcal{P}) = \mathcal{N}\otimes\mathcal{I} - \mathcal{I}\otimes\mathcal{N}$. Hence $\mathcal{F}_\theta \bigotimes \mathcal{F}_\theta = \exp\{-2i\theta(\tilde{\mathbf{G}}_\mathcal{Q}\mathbf{G}_\mathcal{Q} + \tilde{\mathbf{G}}_\mathcal{P}\mathbf{G}_\mathcal{P})\}$. The second identity in the proposition follows from proposition 7.4 and (7.2). □

8. Husimi representation

The Husimi representation of quantum mechanics is related to the squeezed coherent states. For simplicity we will consider in this paper only the case with no squeezing.

Definition 8.1. The Husimi operator $\mathbf{H}\colon \mathsf{B}_1 \to \mathsf{L}_2(\mathbb{R}^2)$ is defined by

$$\mathbf{H}[\mathcal{A}](q,p) = (2\pi)^{-1}(e_{q,p}, \mathcal{A}e_{q,p}), \tag{8.1}$$

where $e_{q,p} \in \mathsf{L}_2(\mathbb{R})$ is the coherent state vector defined by

$$e_{q,p}(x) = \pi^{-1/4}e^{-iqp/2}\exp\{-(x-q)^2/2 + ipx\}.$$

Theorem 8.2. $(2\pi)^{-1}\int_{\mathbb{R}^2} e_{q,p}\otimes e_{q,p}\,dqdp = \mathcal{I}$ *as a continuous linear form on* B_1*. In other words:* $\boldsymbol{H}[\mathcal{A}] \in \mathsf{L}_1(\mathbb{R}^2)$ *and* $\int_{\mathbb{R}^2}\boldsymbol{H}[\mathcal{A}](q,p)dqdp = Tr(\mathcal{A})$ *for* $\mathcal{A} \in \mathsf{B}_1$.

Proof. It is easily seen that $\mathcal{S}^*e_{q,p} = \frac{q+ip}{\sqrt{2}}\,e_{q,p}$ and $(e_{q,p}, \phi_0) = e^{-(q^2+p^2)/4}$. We have $\mathcal{S}\phi_k = \sqrt{k+1}\phi_{k+1}$. Let $f_z = \sum_{k=0}^\infty (k!)^{-1/2}z^k\phi_k$. Then $\mathcal{S}^*f_z = z\,f_z$ and $(f_z, \phi_0) = 1$. Because $\ker(\mathcal{S}^*) = \operatorname{span}\{\phi_0\}$ we see that

$$e_{q,p} = e^{-(q^2+p^2)/4}\sum_{k=0}^\infty \frac{z^k}{\sqrt{2^k k!}}\,\phi_k. \tag{8.2}$$

Denoting the polar decomposition of points $q + ip \in \mathbb{C}$ by $re^{i\theta}$, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e_{q,p}\otimes e_{q,p}\,d\theta = \sum_{n=0}^\infty p_{\frac{1}{2}r^2}(n)\phi_n\otimes\phi_n, \quad p_\lambda(n) = \frac{\lambda^n}{n!}\,e^{-\lambda}. \tag{8.3}$$

For all $n \in \mathbb{N}_0$, $\int_0^\infty p_\lambda(n)d\lambda = 1$. Hence $\frac{1}{2\pi}\int_{\mathbb{R}^2}(f, e_{q,p})(e_{q,p}, g)\,dqdp = (f,g)$ for $f, g \in \mathsf{L}_2(\mathbb{R})$.

Using the polar decomposition of $\mathcal{A}$ and spectral decomposition of $(\mathcal{A}^*\mathcal{A})^{1/2}$ we see that there are orthonormal bases $\{\psi_k^{(\ell)} : k \in \mathbb{N}\}$, $\ell \in \{1,2\}$, and a summable sequence (λ_k) of non-negative integers such that $\mathcal{A} = \sum_{k=1}^{\infty} \lambda_k \psi_k^{(1)} \otimes \psi_k^{(2)}$. This reduces the general case to the case where $\mathcal{A}$ is of the form $\mathcal{A} = f \otimes g$.

Remark 8.3. Using the results from the proof of theorem 8.2 it is easily seen that $(e_{q,p}, \phi_k) = e^{-(q^2+p^2)/4}(k!)^{-1/2}(\frac{q-ip}{\sqrt{2}})^k$, and that $e^{-\tau\mathcal{N}}[e^{(q^2+p^2)/4}e_{q,p}] = e^{(u^2+v^2)/4}e_{u,v}$, where $u = e^{-\tau}q$ and $v = e^{-\tau}p$, and that $\mathcal{F}_\theta[e_{q,p}] = e_{q',p'}$, where $q', p' \in \mathbb{R}$ are defined by $q' + ip' = e^{-i\theta}(q + ip)$.

Using $x + d/dx = e^{-x^2/2}(d/dx)e^{x^2/2}$ and

$$\begin{aligned} &\mathbf{K}[e_{q,p} \otimes e_{q,p}](x,y) = \\ &= \pi^{-1/2} \exp\{-(q+ip-x)^2/2 - (q-ip-y)^2/2 - p^2\} \end{aligned} \tag{8.4}$$

it is easily seen that

$$\begin{aligned} \mathcal{S}^*\bigotimes\mathcal{I}[e_{q,p} \otimes e_{q,p}] &= \frac{q+ip}{\sqrt{2}}\, e_{q,p} \otimes e_{q,p}, \\ \mathcal{I}\bigotimes\mathcal{S}^*[e_{q,p} \otimes e_{q,p}] &= \frac{q-ip}{\sqrt{2}}\, e_{q,p} \otimes e_{q,p}. \end{aligned} \tag{8.5}$$

The operators $\mathcal{S}^*\bigotimes\mathcal{I}$ and $\mathcal{I}\bigotimes\mathcal{S}^*$ (densely defined on B_2) commute. They are not self-adjoint and not normal. The eigenvectors $e_{q,p} \otimes e_{q,p}$, $q, p \in \mathbb{R}$ are not orthogonal w.r.t. the inner product of B_2. The vectors $e_{q,p} \otimes e_{q',p'}$ with $q \neq q'$ and $p \neq p'$ are also simultaneous eigenvectors of $\mathcal{S}^*\bigotimes\mathcal{I}$ and $\mathcal{I}\bigotimes\mathcal{S}^*$. However, the eigenvectors with $q = q'$ and $p = p'$ suffice to separate trace-class operators. We will show this in theorem 8.5 below.

A C^*-subalgebra A of $\mathsf{B}_\infty(\mathsf{H})$ is called non-degenerate if for every non-zero $f \in \mathsf{H}$ there exists an $\mathcal{A} \in \mathsf{A}$ such that $\mathcal{A}f \neq 0$. For a subset $\mathsf{B} \subset \mathsf{B}_\infty$, let $\mathrm{ball}(\mathsf{B}) = \{\mathcal{A} \in \mathsf{B} : ||\mathcal{A}||_\infty \leq 1\}$. The following proposition follows from the Kaplansky density theorem (see [22]).

Proposition 8.4. *Let* A *be a non-degenerate* C^**-subalgebra of* $\mathsf{B}_\infty(\mathsf{H})$*. Then* $\mathrm{ball}(\mathsf{A}'')$ *is the weak***-closure of* $\mathrm{ball}(\mathsf{A})$*.*

Theorem 8.5. *If* $\mathcal{T} \in \mathsf{B}_1$ *and* $(e_{q,p}, \mathcal{T}e_{q,p}) = 0$ *for all* $q, p \in \mathbb{R}$ *then* $\mathcal{T} = 0$*. Moreover, for every* $\mathcal{B} \in \mathsf{B}_\infty$*,* $Tr(\mathcal{BT})$ *can be approximated, uniformly for* $\mathcal{T}$ *in bounded subsets of* B_1*, by finite sums of the form* $\sum_{n=1}^{N} b_n(e_{q_n,p_n}, \mathcal{T}e_{q_n,p_n})$*, with* $N \in \mathbb{N}$*,* $b_n \in \mathbb{C}$ *and* $q_n, p_n \in \mathbb{R}$*.*

Proof. Let $\mathsf{M} = \mathrm{span}\{e_{q,p} \otimes e_{q,p} : q, p \in \mathbb{R}\}$. It suffices to prove that $\mathrm{ball}(\mathsf{M})$ is weak*-dense in $\mathrm{ball}(\mathsf{B}_\infty)$. Let

$$\mathsf{A} = \operatorname*{span}_{\tau>0,k,\ell\in\mathbb{N}_0} \{\iint_{\mathbb{R}^2} (q-ip)^k (q+ip)^\ell\, e_{q,p} \otimes e_{q,p}\, e^{-\tau(q^2+p^2)} dq dp\}.$$

We will show that A is a non-degenerate C^*-subalgebra of B_∞ and that $\mathsf{A}'' = \mathsf{B}_\infty$. By proposition 8.4 this implies that ball(A) is weak*-dense in ball(B_∞). This suffices to prove the theorem, because $\mathsf{A} \subset \mathrm{cl}(\mathsf{M})$, the closure of M in B_∞, equipped with its usual Banach space topology.

Using (8.5) and theorem 8.2 we see that

$$\mathcal{S}^{*k}\mathcal{S}^\ell = \int_{\mathbb{C}} (\frac{q+ip}{\sqrt{2}})^k (\frac{q-ip}{\sqrt{2}})^\ell \, e_{q,p} \otimes e_{q,p} \, dqdp, \tag{8.6}$$

By remark 8.3,

$$\mathsf{A} = \mathrm{span}\{e^{-\tau\mathcal{N}}\mathcal{S}^{*k}\mathcal{S}^\ell e^{-\tau\mathcal{N}} : \tau > 0, k, \ell \in \mathbb{N}_0\}.$$

Using (3.2) and $e^{-\tau\mathcal{N}}\mathcal{S} = e^{-\tau}\mathcal{S}e^{-\tau\mathcal{N}}$, it is easily seen that A is a C^*-subalgebra of B_∞. A is non-degenerate, because it contains the injective operators $e^{-\tau\mathcal{N}}$, $\tau > 0$.

Let E be the range of the spectral measure of $\mathcal{S}^*\mathcal{S}$. Let $\tau > 0$. Because $e^{-\tau\mathcal{N}}\mathcal{S}^*\mathcal{S}e^{-\tau\mathcal{N}} \in \mathsf{A}$, $\mathsf{A}' \subset \{e^{-\tau\mathcal{N}}\mathcal{S}^*\mathcal{S}e^{-\tau\mathcal{N}}\}'$ which is equal to $\mathsf{E}' = \mathsf{E}''$. Hence $\mathsf{A}' \subset \mathsf{E}''$. Hence the representation corresponding to the o.n.b. (ϕ_k) diagonalizes the operators in A' (simultaneously). Because the operators in A' commute with $\mathcal{S}$, which is a (weighted) shift w.r.t. the orthonormal basis (ϕ_k),this implies that $\mathsf{A}' = \mathrm{span}\{\mathcal{I}\}$. Hence $\mathsf{A}'' = \mathsf{B}_\infty$. □

9. A family of representations interpolating between the Wigner and Husimi representations

By proposition 7.5, $\mathbf{W}[e^{-s(\mathbf{G}_\mathcal{Q}^2+\mathbf{G}_\mathcal{P}^2)}[\mathcal{A}]] = e^{-\frac{s}{4}|\triangle|}\mathbf{W}[\mathcal{A}]$, where $|\triangle| = -\partial^2/\partial q^2 - \partial^2/\partial p^2$. Let $\mathbf{W}_s[\mathcal{A}] = e^{-\frac{s}{4}|\triangle|}\mathbf{W}[\mathcal{A}]$. Then $\mathbf{W}_s[\mathcal{A}](q,p) = \mathrm{Tr}(\mathcal{W}^{(s)}_{q,p}\mathcal{A})$, where $\mathcal{W}^{(s)}_{q,p} = e^{-s(\mathbf{G}_\mathcal{Q}^2+\mathbf{G}_\mathcal{P}^2)}\mathcal{W}_{q,p}$. From the properties of the Gaussian convolution operator $\exp\{-\frac{s}{4}|\triangle|\}$ it follows that $\mathbf{W}_s \colon \mathsf{B}_1 \to \mathsf{L}_2(\mathbb{R}^2)$ is injective. Thus we have a family of phase-space representations containing the Wigner representation as a special case. We will show that the Husimi representation is also a special case: $\mathbf{H} = (2\pi)^{-1/2}\mathbf{W}_1$.

Proposition 9.1. *If* $\mathcal{A} \in \mathsf{B}_+$ *then*

$$\boldsymbol{W}_s[\tilde{\boldsymbol{G}}^{(s)}_\mathcal{Q}\mathcal{A}](q,p) = q\,\boldsymbol{W}_s[\mathcal{A}](q,p), \quad \boldsymbol{W}_s[\tilde{\boldsymbol{G}}^{(s)}_\mathcal{P}\mathcal{A}](q,p) = p\,\boldsymbol{W}_s[\mathcal{A}](q,p)$$

where $\tilde{\boldsymbol{G}}^{(s)}_\mathcal{Q} = \tilde{\boldsymbol{G}}_\mathcal{Q} - is\boldsymbol{G}_\mathcal{P}$ *and* $\tilde{\boldsymbol{G}}^{(s)}_\mathcal{P} = \tilde{\boldsymbol{G}}_\mathcal{P} + is\boldsymbol{G}_\mathcal{Q}$.

Proof. Let $\mathbf{A} = \exp\{s(\mathbf{G}_\mathcal{Q}^2 + \mathbf{G}_\mathcal{P}^2)\}\tilde{\mathbf{G}}_\mathcal{Q}\exp\{-s(\mathbf{G}_\mathcal{Q}^2 + \mathbf{G}_\mathcal{P}^2)\}$. Because the operators $\mathbf{G}_\mathcal{Q}$ and $\tilde{\mathbf{G}}_\mathcal{Q}$ commute, we have $\mathbf{A} = \exp\{s\mathbf{G}_\mathcal{P}^2\}\tilde{\mathbf{G}}_\mathcal{Q}\exp\{-s\mathbf{G}_\mathcal{P}^2\} = \exp\{s[\mathbf{G}_\mathcal{P}^2,\cdot]\}\tilde{\mathbf{G}}_\mathcal{Q}$. We have $[\mathbf{G}_\mathcal{P}, \tilde{\mathbf{G}}_\mathcal{Q}] = \frac{1}{2i}\mathbf{I}$ hence $[\mathbf{G}_\mathcal{P}^2, \tilde{\mathbf{G}}_\mathcal{Q}] = -i\mathbf{G}_\mathcal{P}$. Hence $\mathbf{A} = \tilde{\mathbf{G}}^{(s)}_\mathcal{Q}$. This proves the first identity. The proof of the second identity is similar. □

Proposition 9.2. *Let* $q, p, \theta \in \mathbb{R}$, $s \geq 0$ *and* $\mathcal{A} \in \mathsf{B}_2$. *Then* $\boldsymbol{W}_s[\boldsymbol{G}_{u,v}\mathcal{A}](q,p) = \boldsymbol{W}_s[\mathcal{A}](q-u, p-v)$ *and* $\boldsymbol{W}_s[\mathcal{F}_\theta \mathcal{A} \mathcal{F}_\theta^*](q,p) = \boldsymbol{W}_s[\mathcal{A}](q\cos\theta - p\sin\theta, q\sin\theta + p\cos\theta)$.

Proof. This follows from proposition 7.13 and the properties of the Gaussian convolution operator. □

Proposition 9.3. $\boldsymbol{H} = (2\pi)^{-1/2}\,\boldsymbol{W}_1$.

Proof. In the proof of theorem 7.10 we saw that $\mathbf{W}[\phi_0 \otimes \phi_0](q,p) = \sqrt{\frac{2}{\pi}} e^{-q^2-p^2}$. Hence $\mathbf{W}_s[\phi_0 \otimes \phi_0](q,p) = \frac{\sqrt{2\pi}}{\pi(1+s)} \exp\{-(q^2+p^2)/(1+s)\}$. We have $\mathbf{H}[\phi_0 \otimes \phi_0](q,p) = \frac{1}{2\pi}|(e_{q,p}, \phi_0)|^2 = \frac{1}{2\pi}\exp\{-(q^2+p^2)/2\}$. The operators $\mathcal{W}^{(1)}_{q,p}$ and $e_{q,p} \otimes e_{q,p}$ are simultaneous eigenvectors of $\mathcal{S}^* \bigotimes \mathcal{I}$ and $\mathcal{I} \bigotimes \mathcal{S}^*$ with corresponding eigenvalues, and $\ker(\mathcal{S}^* \bigotimes \mathcal{I}) \cap \ker(\mathcal{I} \bigotimes \mathcal{S}^*) = \operatorname{span}\{\phi_0 \otimes \phi_0\}$, and the orthonormal projections on $\operatorname{span}\{\phi_0 \otimes \phi_0\}$ of $\mathcal{W}^{(1)}_{q,p}$ and $\frac{e_{q,p} \otimes e_{q,p}}{\sqrt{2\pi}}$ are equal. Hence $\mathcal{W}^{(1)}_{q,p} = \frac{e_{q,p} \otimes e_{q,p}}{\sqrt{2\pi}}$. □

Proposition 9.4. *If* $\mathcal{A} \in \mathsf{B}_+$ *and* $s \geq 0$ *then* $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^2} \boldsymbol{W}_s[\mathcal{A}](q,p)\,dq\,dp = Tr(\mathcal{A})$.

Proof. For $s = 1$, this follows from proposition 9.3 and theorem 8.2. The general case can be reduced to this case by the properties of the Gaussian convolution operator. □

Proposition 9.5. *The expression of the operator* $\mathcal{W}^{(s)}_{q,p}$ *in terms of the Dirac basis* $\mathcal{U}_{u,v}$, $u, v \in \mathbb{R}$ *is given by*

$$\mathcal{W}^{(s)}_{q,p} = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-\frac{s}{4}(u^2+v^2)} e^{-i(vq-up)} \mathcal{U}_{u,v}\,du dv.$$

Proof. This follows from (7.4) and the properties of the Gaussian convolution operator $e^{-\frac{s}{4}|\triangle|}$. □

Remark 9.6. In the following proposition we will introduce for every $s > 0$ and $z_1, z_2 \in \mathbb{C}$ an operator $\tilde{\mathcal{W}}^{(s)}_{z_1,z_2} \in \mathsf{B}_2$ such that $\mathbf{D}^{-1}\mathbf{W}_s[\mathcal{A}](z_1,z_2) = (\tilde{\mathcal{W}}^{(s)}_{z_1,z_2}, \mathcal{A})_{\mathsf{B}_2}$. We can express $\tilde{\mathcal{W}}^{(1)}_{z_1,\bar{z}_2}$ in terms of the coherent state vectors at z_1 and z_2:

$$\tilde{\mathcal{W}}^{(1)}_{z_1,\bar{z}_2} = \frac{1}{\sqrt{2\pi}} \frac{e_{x_2,y_2} \otimes e_{x_1,y_1}}{(e_{x_1,y_1}, e_{x_2,y_2})},$$

if $z_k = x_k + iy_k$, $x_k, y_k \in \mathbb{R}$ for $k \in \{1,2\}$.

Proposition 9.7. *Let*

$$\tilde{\mathcal{W}}^{(s)}_{z_1,z_2} = \exp\{-\bar{z}_1(\boldsymbol{G}_{\mathcal{Q}} + i\,\boldsymbol{G}_{\mathcal{P}})\} \exp\{\bar{z}_2(\boldsymbol{G}_{\mathcal{Q}} - i\,\boldsymbol{G}_{\mathcal{P}})\} \mathcal{W}^{(s)}_{0,0}. \tag{9.1}$$

Then $\mathcal{W}^{(s)}_{q,p} = \tilde{\mathcal{W}}^{(s)}_{z,\bar{z}}$ *for* $z = q + ip$ *and*

$$\boldsymbol{D}^{-1}\,\boldsymbol{W}_s[\mathcal{A}](z_1, z_2) = (\tilde{\mathcal{W}}^{(s)}_{z_1,z_2}, \mathcal{A})_{\mathsf{B}_2}. \tag{9.2}$$

The operators $\tilde{\mathcal{W}}^{(s)}_{z_1,z_2}$, $z_1, z_2 \in \mathbb{C}$ *are the simultaneous eigenvectors of the super operators* $(\tilde{\boldsymbol{G}}^{(s)}_{\mathcal{Q}})^*$ *and* $(\tilde{\boldsymbol{G}}^{(s)}_{\mathcal{P}})^*$:

$$(\tilde{\boldsymbol{G}}^{(s)}_{\mathcal{Q}})^*\tilde{\mathcal{W}}^{(s)}_{z_1,z_2} = \frac{\bar{z}_1 + \bar{z}_2}{2}\,\tilde{\mathcal{W}}^{(s)}_{z_1,z_2}, \quad (\tilde{\boldsymbol{G}}^{(s)}_{\mathcal{P}})^*\tilde{\mathcal{W}}^{(s)}_{z_1,z_2} = \frac{\bar{z}_1 - \bar{z}_2}{-2i}\,\tilde{\mathcal{W}}^{(s)}_{z_1,z_2}. \tag{9.3}$$

For $\mathcal{A}, \mathcal{B} \in \mathsf{B}_2$ *we have*

$$(\mathcal{B}, \mathcal{A})_{\mathsf{B}_2} = \iint_{\mathbb{C}^2} (\mathcal{B}, \tilde{\mathcal{W}}^{(s)}_{z_1,\bar{z}_2})_{\mathsf{B}_2} (\tilde{\mathcal{W}}^{(s)}_{z_1,\bar{z}_2}, \mathcal{A})_{\mathsf{B}_2}\, \frac{e^{-|z_1 - z_2|^2/(2s)}}{2\pi s} dz_1 dz_2. \tag{9.4}$$

A proof of this proposition can be found in appendix 13.6.

10. Phase-space Gelfand triple related to $\mathbf{W}_s$

First we will introduce a Gelfand triple $\mathsf{W}^{(s,\tau)}_+ \subset \mathsf{L}_2(\mathbb{R}^2) \subset \mathsf{W}^{(s,\tau)}_-$ such that $\mathbf{W}_s \colon \mathsf{B}^{(\tau)}_+ \to \mathsf{W}^{(s,\tau)}_+$ is unitary. Then we will introduce a Gelfand triple $\mathsf{W}^{(s)}_+ \subset \mathsf{L}_2(\mathbb{R}^2) \subset \mathsf{W}^{(s)}_-$ such that $\mathbf{W}_s \colon \mathsf{B}_+ \to \mathsf{W}^{(s)}_+$ is bijective. We will state a theorem characterizing this Gelfand triple.

Definition 10.1. For $s, \tau \geq 0$, let $\mathsf{W}^{(s,\tau)}_+ \subset \mathsf{L}_2(\mathbb{R}^2) \subset \mathsf{W}^{(s,\tau)}_-$ be the Gelfand triple corresponding to operator $e^{-\frac{s}{4}|\triangle|}\mathbf{S}_\tau$ on $\mathsf{L}_2(\mathbb{R}^2)$ where the operator $\mathbf{S}_\tau$ on $\mathsf{L}_2(\mathbb{R}^2)$ is defined by $\mathbf{S}_\tau = \mathcal{Z}^{(1)}_{1/\sqrt{2}} \mathcal{Z}^{(2)}_{1/\sqrt{2}} e^{-\tau\mathcal{N}^{(1)}} e^{-\tau\mathcal{N}^{(2)}} \mathcal{Z}^{(1)}_{\sqrt{2}} \mathcal{Z}^{(2)}_{\sqrt{2}}$.

Define the operator $\mathbf{W}^{-*}_s \colon \mathsf{B}^{(\tau)}_- \to \mathsf{W}^{(s,\tau)}_-$ by $<\mathbf{W}^{-*}_s[\mathcal{B}], \mathbf{W}_s[\mathcal{A}]> = <\mathcal{B}, \mathcal{A}>$ for $\mathcal{A} \in \mathsf{B}^{(\tau)}_+$ and $\mathcal{B} \in \mathsf{B}^{(\tau)}_-$. The operators $\mathbf{W}_s \colon \mathsf{B}^{(\tau)}_+ \to \mathsf{W}^{(s,\tau)}_+$ and $\mathbf{W}^{-*}_s \colon \mathsf{B}^{(\tau)}_- \to \mathsf{W}^{(s,\tau)}_-$ are unitary. By the unitarity of the Wigner operator we have $\mathbf{W}^{-*}_0 = \mathbf{W}_0$.

Definition 10.2. For $s \geq 0$, let $\mathsf{W}^{(s)}_+ = \cup_{\tau>0}\mathsf{W}^{(s,\tau)}_+$ and $\mathsf{W}^{(s)}_- = \cap_{\tau>0}\mathsf{W}^{(s,\tau)}_-$.

Using proposition 7.12, we see that $\mathbf{W}_s \colon \mathsf{B}_+ \to \mathsf{W}^{(s)}_+$ and $\mathbf{W}^{-*}_s \colon \mathsf{B}_- \to \mathsf{W}^{(s)}_-$ are bijections.

In the following theorem we give a characterization of the spaces $\mathsf{W}^{(s)}_+$ and $\mathsf{W}^{(s)}_-$.

Theorem 10.3. *Let* $s \geq 0$. $g \in \mathsf{W}^{(s)}_+$ *iff* $g \in$ *range*$(\boldsymbol{D})$ *and there exists* $\tau, M > 0$ *such that*

$$|\boldsymbol{D}^{-1}[g](z_1, \bar{z}_2)| \leq M \exp\{\frac{-|z_1 + z_2|^2}{4s + 4\coth\tau}\} \exp\{\frac{|z_1 - z_2|^2}{4s + 4\tanh\tau}\}.$$

Let $s>0$. $F\in \mathsf{W}_-^{(s)}$ *iff* $f=\exp\{-\frac{s}{2}|\triangle|\}F\in$ *range*$(\boldsymbol{D})$, *and for every* $\tau>0$ *such that* $0<\tanh\tau<\min(s,\frac{1}{s})$ *there is an* $M>0$ *such that*

$$|\boldsymbol{D}^{-1}[f](z_1,\bar z_2)|\le M\exp\{\frac{-|z_1+z_2|^2}{4s-4\coth\tau}\}\exp\{\frac{|z_1-z_2|^2}{4s-4\tanh\tau}\}.$$

Proof. This follows from theorem 13.13, lemma 3.3 and the following facts: If $f(z_1,z_2)=\mathrm{Im}(z_1)^2+\mathrm{Im}(z_2)^2$ then $f(\frac{z_1+\bar z_2}{2},\frac{z_1-\bar z_2}{2i})=|z_1-z_2|^2/4$. If $f(z_1,z_2)=\mathrm{Re}(z_1)^2+\mathrm{Re}(z_2)^2$ then $f(\frac{z_1+\bar z_2}{2},\frac{z_1-\bar z_2}{2i})=|z_1+z_2|^2/4$. □

11. Integral expression for $<\cdot,\cdot>\colon \mathsf{W}_-^{(s)}\times\mathsf{W}_+^{(s)}\to\mathbb{C}$

We can express $<\cdot,\cdot>\colon \mathsf{W}_-\times\mathsf{W}_+\to\mathbb{C}$ by phase-space integrals:

$$<F,g>=\lim_{\tau\searrow 0}\int_{\mathbb{R}^2}\overline{F_\tau(q,p)}g(q,p)dqdp,$$

where $F_\tau=e^{-\tau\mathcal{N}^{(1)}}e^{-\tau\mathcal{N}^{(2)}}F$. In this section we will show that we can approximate an $F\in\mathsf{W}_-^{(s)}$ by continuous functions $F_R\in\mathsf{W}_-^{(s)}$ such that

$$<F,g>=\lim_{R\to\infty}\int_{\mathbb{R}^2}\overline{F_R(q,p)}g(q,p)dqdp,$$

for $g\in\mathsf{W}_+^{(s)}$. We will also show that we can identify an element $F\in\mathsf{W}_-^{(s)}$ with the (possibly) divergent integral

$$F(q,p)=\int_{\mathbb{C}}\mathbf{D}^{-1}[f](z+w,\bar z-\bar w)e^{-|w|^2/(2s)}dw,\quad z=q+ip$$

where $f=\exp\{-\frac{s}{2}|\triangle|\}F$. By changing the order of integration in the formal expression $\int_{\mathbb{R}^2}\overline{F(q,p)}g(q,p)dqdp$, where $F(q,p)$ represents the (possibly) divergent integral, we get a meaningful expression for $<F,g>\colon \mathsf{W}_-^{(s)}\times\mathsf{W}_+^{(s)}\to\mathbb{C}$. (This is related to the deconvolution formula in appendix 13.5.) Proofs of the following two lemma's can be found in appendix 13.6.

Lemma 11.1. *Let* $s>0$. *Let* $g\in\mathsf{W}_+^{(s)}$, $F\in\mathsf{W}_-^{(s)}$. *Then*

$$<F,g>=\frac{1}{2\pi s}\iint_{\mathbb{C}^2}\overline{\boldsymbol{D}^{-1}[f_s](z+w,\bar z)}\boldsymbol{D}^{-1}[g](z+w,\bar z)\,e^{-|w|^2/(2s)}dzdw$$

where $f_s=\exp\{-\frac{s}{2}|\triangle|\}F$.

Lemma 11.2. *Let* $s>0$ *and let* $g\in\mathsf{W}_+^{(s)}$ *and* $F\in\mathsf{W}_-^{(s)}$. *For every* $w\in\mathbb{C}$,

$$\begin{aligned}&\int_{\mathbb{C}}\overline{\boldsymbol{D}^{-1}[f_s](z+w,\bar z)}\boldsymbol{D}^{-1}[g](z+w,\bar z)dz=\\ &=\int_{\mathbb{R}^2}\overline{\boldsymbol{D}^{-1}[f_s](z+w,\bar z-\bar w)}g(q,p)dqdp\end{aligned}\tag{11.1}$$

where $z=q+ip$ *and* $f_s=\exp\{-\frac{s}{2}|\triangle|\}F$.

Example 11.3. We have $\mathbf{H}[\phi_m \otimes \phi_n](x,y) = (2\pi)^{-1}e^{-|z|^2}z^n\bar{z}^m$, where $z = (x+iy)/\sqrt{2}$. The identity (11.1) with $f = \mathbf{H}[\phi_m \otimes \phi_n]$ and $g = \mathbf{H}[\phi_k \otimes \phi_\ell]$ is

$$\int_{\mathbb{C}} \frac{(\bar{z}+\bar{w})^n z^m}{\exp\{(\bar{z}+\bar{w})z\}} \frac{(z+w)^\ell \bar{z}^k}{\exp\{(z+w)\bar{z}\}}\, dz =$$
$$= \int_{\mathbb{C}} \frac{(\bar{z}+\bar{w})^n (z-w)^m}{\exp\{(\bar{z}+\bar{w})(z-w)\}}\, z^\ell \bar{z}^k\, e^{-|z|^2}\, dz.$$

After we have replaced on both sides $(\bar{z}+\bar{w})^n\bar{z}^k$ by $e^{\beta(\bar{z}+\bar{w})}e^{\alpha\bar{z}}$ and subsequently by $e^{\alpha\bar{z}}$, where α and β are arbitrary complex numbers, the integrals can be calculated easily by using the fact that $(z_1,z_2) \mapsto e^{\bar{z}_1 z_2}$ is a reproducing kernel for the functional Hilbert space $\mathsf{L}_2^a(\mathbb{C}, \pi^{-1}e^{-|z|^2}dz)$, the closed linear subspace of entire analytic functions in $\mathsf{L}_2(\mathbb{C}, \pi^{-1}e^{-|z|^2}dz)$. (See [6])

Theorem 11.4. *Let $g \in \mathsf{W}_+^{(s)}$, $F \in \mathsf{W}_-^{(s)}$. Then*

$$<F,g> = \frac{1}{2\pi s}\int_{\mathbb{C}}\Big(\int_{\mathbb{C}} \overline{\boldsymbol{D}^{-1}[f_s](z+w,\bar{z}-\bar{w})}g(q,p)\,dqdp\Big)e^{-|w|^2/(2s)}dw$$

where $z = q+ip$ and $f_s = \exp\{-\frac{s}{2}|\triangle|\}F$.

Proof. Lemma 11.1 and lemma 11.2. □

Theorem 11.5. *Let $\mathcal{B} \in \mathsf{B}_-$ and $F_s = \boldsymbol{W}_s^{-*}[\mathcal{B}]$. For $R > 0$ let*

$$\mathcal{B}_R^{(s)} = \frac{1}{2\pi s}\iint_{|z-w|\le R} \overline{<\mathcal{B}, \tilde{\mathcal{W}}_{z,\bar{w}}^{(s)}>}\tilde{\mathcal{W}}_{z,\bar{w}}^{(s)}\, e^{-|z-w|^2/(2s)}dzdw. \tag{11.2}$$

and

$$F_R^{(s)}(q,p) = \frac{1}{2\pi s}\int_{|w|\le R}\boldsymbol{D}^{-1}[f_s](z+w,\bar{z}-\bar{w})e^{-|w|^2/(2s)}dw$$

where $z = q+ip$ and $f_s = \exp\{-\frac{s}{2}|\triangle|\}F_s = \boldsymbol{W}_s[\mathcal{B}]$. Then $\mathcal{B}_R^{(s)} \to \mathcal{B} \in \mathsf{B}_-$, $F_R^{(s)} = \boldsymbol{W}_s^{-*}[\mathcal{B}_R^{(s)}]$ and

$$\mathcal{B}_R^{(s)} = \int_{\mathbb{R}^2}\overline{F_R^{(s)}(q,p)}\mathcal{W}_{q,p}^{(s)}\, dqdp$$

as a continuous linear form on B_+.

A proof of this theorem can be found in appendix 13.6.

By the following proposition, the Dirac basis $\mathcal{U}_{u,v}$, $u,v \in \mathbb{R}$ consists of generalized eigenvectors of the smoothing operator $\mathcal{B} \mapsto \mathcal{B}_R^{(s)}$.

Proposition 11.6. *Let $s, R > 0$ and $u,v \in \mathbb{R}$. We have*

$$(\mathcal{U}_{u,v})_R^{(s)} = f_R^{(s)}(u^2+v^2)\,\mathcal{U}_{u,v},$$

where $f_R^{(s)}(x) = e^{-sx/2}\int_0^{R^2/(2s)} I_0(\sqrt{2sx\lambda})e^{-\lambda}d\lambda$, where I_n is the n'th modified Bessel function of the first kind.

Proof. We have $f_R^{(s)}(x) = \sum_{n=0}^{\infty} p_{sx/2}(n) \int_0^{R^2/(2s)} p_\lambda(n) d\lambda$, where $p_\lambda(n)$ is defined in (8.3). Hence

$$f_R^{(s)}(u^2+v^2) = \frac{1}{2\pi s} \int_{|w|\le R} \exp\{w \frac{u-iv}{2} + \bar{w}\frac{u+iv}{2} - \frac{|w|^2}{2s}\} \, dw.$$

By proposition 9.5, $\mathbf{W}_s[\mathcal{U}_{u,v}](q,p) = \frac{1}{2\pi}\exp\{-\frac{s}{4}(u^2+v^2) - i(vq-up)\}$. Hence

$$\mathbf{D}^{-1}\mathbf{W}_s[\mathcal{U}_{u,v}](z_1,z_2) = \frac{1}{2\pi}\exp\{-\frac{s}{4}(u^2+v^2) + z_1\frac{u-iv}{2} - z_2\frac{u+iv}{2}\}.$$

The result follows from the definition of $\mathcal{B} \mapsto \mathcal{B}_R^{(s)}$ as given in theorem 11.5.

□

11.1. **Some properties of the smoothing process $\mathcal{B} \mapsto \mathcal{B}_R^{(s)}$.** Let $s > 0$, $\mathcal{B} \in \mathsf{B}_-$ and $\mathcal{A} \in \mathsf{B}_+$. By definition (11.2),

$$<\mathcal{B}_R^{(s)}, \mathcal{A}> = \frac{1}{2\pi s} \iint_{|z-w|\le R} <\mathcal{B}, \tilde{\mathcal{W}}_{z,\bar{w}}^{(s)}><\tilde{\mathcal{W}}_{z,\bar{w}}^{(s)}, \mathcal{A}> e^{-|z-w|^2/(2s)} dz dw.$$

It is easily seen that $<\mathcal{B}_R^{(s)}, \mathcal{A}> = <\mathcal{B}, \mathcal{A}_R^{(s)}>$. For every $R > 0$, the subspace $\mathrm{span}\{\tilde{\mathcal{W}}_z^{(s)} : |z-w| \le R\}$ is dense in B_+. Hence the operators $\mathcal{A}_R^{(s)}$, $\mathcal{A} \in \mathsf{B}_+$ form a dense linear subspace of B_+ and hence $\mathcal{B} \mapsto \mathcal{B}_R^{(s)}$ is injective on B_-. If $\mathcal{B} \in \mathsf{B}_2$ then $0 \le \mathrm{Tr}(\mathcal{B}^*\mathcal{B}_R^{(s)}) \le \mathrm{Tr}(\mathcal{B}^*\mathcal{B})$. Hence for every $R > 0$, the operator $\mathcal{B} \mapsto \mathcal{B}_R^{(s)}$ is non-negative, self-adjoint and bounded on B_2. Because the domain of integration in (12.3) is not of the form $\Delta \times \Delta$ with $\Delta \subset \mathbb{C}$, the smoothed operator $\mathcal{B}_R^{(s)}$ is not necessarily non-negative whenever $\mathcal{B}$ is. From proposition 11.6 it follows that $(\mathbf{G}_{u,v}^{\mathrm{ext}}[\mathcal{B}])_R^{(s)} = \mathbf{G}_{u,v}[\mathcal{B}_R^{(s)}]$ and $(\mathcal{F}_\theta \mathcal{B} \mathcal{F}_\theta^*)_R^{(s)} = \mathcal{F}_\theta \mathcal{B}_R^{(s)} \mathcal{F}_\theta^*$. Hence the smoothing operator $\mathbf{W}_s^{-*}[\mathcal{B}] \mapsto \mathbf{W}_s^{-*}[\mathcal{B}_R^{(s)}]$ commutes with translations and rotations around any point $(q,p) \in \mathbb{R}^2$. In particular, if $\mathbf{W}_s^{-*}[\mathcal{B}]$ is invariant under translations in a particular direction or invariant under rotations around a particular point (as a generalized function), then the same is true for $\mathbf{W}_s^{-*}[\mathcal{B}_R]$ (as an ordinary function).

11.2. **Mode of convergence.** Let $\mathcal{B} \in \mathsf{B}_-$ and $\mathcal{A} \in \mathsf{B}_+$. By lemma 13.29 (in the appendix), $\mathcal{B}_R^{(s)} \to \mathcal{B} \in \mathsf{B}_-$ and $\mathcal{A}_R^{(s)} \to \mathcal{A} \in \mathsf{B}_+$. Hence the approximation of $<\mathcal{B}, \mathcal{A}>$ by $<\mathcal{B}_R^{(s)}, \mathcal{A}> = <\mathcal{B}, \mathcal{A}_R^{(s)}>$ is uniform for $\mathcal{A}$ in bounded subsets of B_+ and uniform for $\mathcal{B}$ in bounded subsets of B_-.

Example 11.7. By proposition 9.5, $\mathbf{D}^{-1}\mathbf{W}_s[\mathcal{W}_{0,0}^{(s)}](z_1,z_2) = \frac{e^{-z_1 z_2/(2s)}}{2\pi s}$. Hence, if $z = q + ip$ then

$$\mathbf{W}_s^{-*}[(\mathcal{W}_{0,0}^{(s)})_R](q,p) = \frac{e^{-|z|^2/(2s)}}{(2\pi s)^2} \int_{|w|\le R} e^{(z\bar{w}-\bar{z}w)/(2s)} dw.$$

Let J_n be the n'th Bessel function. Using $rJ_0(r|z|) = \frac{1}{2\pi}\int_0^{2\pi} e^{(\bar{z}w - z\bar{w})/2}\, r\, d\theta$, where $w = re^{i\theta}$, we see that

$$\mathbf{W}_s^{-*}[(\mathcal{W}_{0,0}^{(s)})_R](q,p) = \frac{R\, e^{-(q^2+p^2)/2}}{2\pi s\sqrt{q^2+p^2}} J_1(\frac{R}{s}\sqrt{q^2+p^2}).$$

Hence

$$\mathbf{W}_s^{-*}[(\mathcal{W}_{q',p'}^{(s)})_R](q+q',p+p') = \frac{R\, e^{-(q^2+p^2)/2}}{2\pi s\sqrt{q^2+p^2}} J_1(\frac{R}{s}\sqrt{q^2+p^2}).$$

(This converges to $\delta_{0,0}$ in $\mathsf{W}_-^{(s)}$.)

12. Expression of quantum mechanical expectation values as generalized phase-space averages

We know from theorem 8.5 that we can approximate $\mathrm{Tr}(\rho\mathcal{B})$, for $\mathcal{B} \in \mathsf{B}_\infty$ and $\rho \in \mathsf{B}_1$ by finite sums of the form $\sum_{n=1}^N b_n \mathbf{H}[\rho](q_n,p_n)$, $N \in \mathbb{N}$, $b_n \in \mathbb{C}$ and $q_n, p_n \in \mathbb{R}$. The proof of this theorem is however not constructive: it does not give us the points z_k and the corresponding values b_k. In this section we will apply the results of the previous section for the special case of $s = 1$. Therefore we make the assumption that $\rho \in \mathsf{B}_+$. Remember that $\mathbf{W}_1 = \sqrt{2\pi}\mathbf{H}$. The results of the previous section imply that we can approximate $\mathrm{Tr}(\mathcal{B}\rho)$ by integrals over $\mathbf{H}^{-*}[\mathcal{B}_R]$, where $\mathcal{B}_R = \mathcal{B}_R^{(1)}$ as defined in the previous section, with respect to the weight function $\mathbf{H}[\rho]$. First we will consider the approximation of $\mathrm{Tr}(\rho\mathcal{B})$ in terms of the Wigner distribution $\mathbf{W}[\rho]$ of ρ.

For $\rho \in \mathsf{B}_1$ and $\mathcal{B} \in \mathsf{B}_\infty$, the quantum mechanical expectation value $E_\rho^{\mathcal{Q}}(\mathcal{B}) = \mathrm{Tr}(\rho\mathcal{B})$ can be approximated by phase-space integrals containing Wigner functions:

$$\mathrm{Tr}(\rho\mathcal{B}) = \lim_{\tau\searrow 0}\int_{\mathbb{R}^2} \mathbf{W}^{-*}[\mathcal{B}_\tau](q,p)\mathbf{W}[\rho](q,p)dqdp, \tag{12.1}$$

where $\mathcal{B}_\tau = e^{-\tau\mathcal{N}}\mathcal{B}e^{-\tau\mathcal{N}}$. The integral in (12.1) is sometimes called a generalized phase-space average of the function $\mathbf{W}^{-*}[\mathcal{B}_\tau]$ w.r.t. the generalized density function $\mathbf{W}[\rho]$.

Theorem 12.1. *For $\mathcal{A} \in \mathsf{B}_+$ and $\mathcal{B} \in \mathsf{B}_-$ we can approximate $<\mathcal{B},\mathcal{A}>$ by phase-space averages of functions $\boldsymbol{H}^{-*}[\mathcal{B}_R]$ w.r.t. the density function $\boldsymbol{H}[\mathcal{A}]$:*

$$<\mathcal{B},\mathcal{A}> = \lim_{R\to\infty}\int_{\mathbb{R}^2} \boldsymbol{H}^{-*}[\mathcal{B}_R](q,p)\boldsymbol{H}[\rho](q,p)dqdp \tag{12.2}$$

where $\mathcal{B}_R \in \mathsf{B}_\infty$ and $\mathcal{B}_R \to \mathcal{B}$ in B_- as $R \to \infty$. This can be accomplished by taking

$$\mathcal{B}_R = (2\pi)^{-2}\iint_{|z-w|\le R} (e_z, \mathcal{B}e_w)e_z \otimes e_w dzdw \tag{12.3}$$

where $e_z = e_{q,p}$ *if* $z = q + ip$. *We have*

$$\boldsymbol{H}^{-*}[\mathcal{B}_R](q,p) = \int_{|w|\le R} \boldsymbol{D}^{-1}\boldsymbol{H}[\mathcal{B}](z+w, \bar{z}-\bar{w})e^{-|w|^2/2}dw. \tag{12.4}$$

$\mathcal{B}_R$, considered as a continuous linear form on B_+, admits the following expansion in terms of projections on coherent state vectors:

$$\mathcal{B}_R = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathbf{H}^{-*}[\mathcal{B}_R](q,p)e_{q,p}\otimes e_{q,p}\,dqdp.$$

In other words $\mathrm{Tr}(\mathcal{B}_R\mathcal{A}) = \int_{\mathbb{R}^2}\mathbf{H}^{-*}[\mathcal{B}_R](q,p)\mathbf{H}[\mathcal{A}](q,p)dqdp$ for $\mathcal{A}\in\mathsf{B}_+$. For arbitrary $\mathcal{B}\in\mathsf{B}_-$ we may formally write $\mathcal{B} = \int_{\mathbb{R}^2}\mathbf{H}^{-*}[\mathcal{B}](q,p)e_{q,p}\otimes e_{q,p}\,dqdp$ where $\mathbf{H}^{-*}[\mathcal{B}](q,p) = \int_{\mathbb{C}}\mathbf{D}^{-1}\mathbf{H}[\mathcal{B}](z+w, z-w)e^{-|w|^2/2}dw$. However, in general, the integral defining $\mathbf{H}^{-*}[\mathcal{B}](q,p)$ does not converge and $\mathbf{H}^{-*}[\mathcal{B}]$ is an element of $\mathsf{W}_-^{(1)}$ and not necessarily a genuine function. However, the integral expression for $\mathcal{B}$ makes sense after a change in the order of the integration:

Proposition 12.2. *Let* $\mathcal{A}\in\mathsf{B}_+$ *and* $\mathcal{B}\in\mathsf{B}_-$. *Then*

$$<\mathcal{B},\mathcal{A}> = \int_{\mathbb{C}}\Big(\int_{\mathbb{C}}\overline{\boldsymbol{D}^{-1}[\boldsymbol{H}[\mathcal{B}]](z+w,\bar{z}-\bar{w})}\boldsymbol{H}[\mathcal{A}](z)\,dz\Big)e^{-|w|^2/2}dw,$$

where $\boldsymbol{H}[\mathcal{A}](z)$ *is defined as* $\boldsymbol{H}[\mathcal{A}](z) = \boldsymbol{H}[\mathcal{A}](q,p)$ *if* $z = q+ip$.

This proposition is similar to theorem 4 in [5] where $\mathrm{Tr}(\mathcal{B}\mathcal{A})$ is expressed as a sum over integrals containing the functions $\mathbf{H}[\mathcal{A}]$ and $\triangle^n\mathbf{H}[\mathcal{B}]$, for $n\in\mathbb{N}_0$.

12.1. **Practice.** Let ρ be a density operator in B_+. To get a good approximation of $<\mathcal{B},\rho>$ one has to use large values of R in general. However, especially for large R, the size of $|\mathbf{H}^{-*}[\mathcal{B}_R](q,p)|$ for certain $q,p\in\mathbb{R}$ and the growth behaviour (as $q,p\to\infty$) of the function $\mathbf{H}^{-*}[\mathcal{B}_R](q,p)$ can, in general, be such that the determination of $\mathrm{Tr}(\mathcal{B}_R^*\rho)$ from the Husimi function $\mathbf{H}[\rho]$ by means of the phase-space integral is problematic in practice because of the strong dependence of the result on minor variations in $\mathbf{H}[\rho]$. Let for example $\mathcal{B}^{(n)} = \phi_n\otimes\phi_n$ and assume that we want to calculate the number $(\phi_n,\rho\phi_n)$ from the Husimi function $\mathbf{H}[\rho]$. The anti-Husimi function $\mathbf{H}^{-*}[\mathcal{B}^{(n)}](q,p)$ is not a genuine function, hence we cannot calculate $(\phi_n,\rho\phi_n)$ with a phase-space integral. However, we can approximate $(\phi_n,\rho\phi_n)$ by $\mathrm{Tr}(\mathcal{B}_R^{(n)}\rho)$, which has an expression as a phase-space integral in terms of the functions $\mathbf{H}[\rho]$ and $\mathbf{H}^{-*}[\mathcal{B}_R^{(n)}]$. The approximation can be made arbitrarily good by taking R sufficiently large. By 11.2, the choice of R does not depend on the value of n or on the density operator ρ as long as ρ stays in a bounded subset of B_+. (In a quantum mechanical measurement, the density operator ρ results from a preparation procedure, as is described e.g. in [16]. Uncertainties in the preparation procedure lead to uncertainties in ρ. Our approximation of

$\mathrm{Tr}(\mathcal{B}\rho)$ by $\mathrm{Tr}(\mathcal{B}_R\rho)$ is stable under small (in the B_+ sense) variations in ρ.) However, the behaviour of the functions $\mathbf{H}^{-*}[\mathcal{B}_R^{(n)}]$, for a fixed value of R, depends on n. If $\mathbf{H}[\rho]$ is obtained by a measurement then the phase-space integral for $\mathrm{Tr}(\mathcal{B}_R\rho)$ has to be approximated from imperfect knowledge of $\mathbf{H}[\rho]$. How problematic this is, depends on the behaviour of $\mathbf{H}^{-*}[\mathcal{B}_R^{(n)}]$.

Example 12.3. Examples of operators $\mathcal{B}$ such that $\mathbf{H}^{-*}[\mathcal{B}]$ is a genuine function are: $\mathcal{U}_{u,v}$ (see proposition 11.6), $(\mathcal{S}^*)^k\mathcal{S}^\ell$ (see (8.6)), $\mathcal{S}^\ell(\mathcal{S}^*)^k$ (by the previous example and (3.2)), $e^{-\tau\mathcal{N}}$.

Example 12.4. (Rotation invariance) For $0 < a < b$ let $\mathcal{B} = \int_a^b \mathcal{M}_\lambda^{(q,p)}\, d\lambda$, where

$$\mathcal{M}_\lambda^{(q,p)} = \sum_{n=0}^{\infty} p_\lambda(n)\, \phi_n^{(q,p)} \otimes \phi_n^{(q,p)},$$

where $\phi_n^{(q,p)} = \exp\{i(p\mathcal{Q} - q\mathcal{P})\}\phi_n$ and p_λ is defined in (8.3). Because

$$\mathcal{M}_{\frac{1}{2}r^2}^{(q,p)} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \mathcal{W}^{(1)}_{q+q',p+p'}\, d\theta,$$

where $q' + ip' = re^{i\theta}$, $\mathbf{H}^{-*}[\mathcal{B}] = 1_\Delta$, the indicator function of the annulus Δ around $(q,p) \in \mathbb{R}^2$ with inner diameter $\sqrt{a}$ and outer diameter $\sqrt{b}$. Clearly $\mathbf{H}^{-*}[\mathcal{B}]$ is invariant under rotations around (q,p). By 11.1, this is also true for $\mathbf{H}^{-*}[\mathcal{B}_R]$.

Example 12.5. If $\mathbf{H}^{-*}[\mathcal{B}]$ is rotation invariant, then

$$<\mathcal{B},\mathcal{A}> = \lim_{R\to\infty} \int_0^\infty \overline{\mathbf{H}^{-*}[\mathcal{B}_R](r,0)} (\mathcal{M}_{\frac{1}{2}r^2},\mathcal{A})\, rdr.$$

From $\mathcal{F}_\theta \otimes \mathcal{F}_\theta[\phi_n \otimes \phi_n] = \phi_n \otimes \phi_n$, it follows that $\mathbf{H}^{-*}[\phi_n \otimes \phi_n]$ is rotation invariant around $(0,0)$. From this, the identity $\mathbf{H} = \exp\{-\frac{1}{2}|\triangle|\}\mathbf{H}^{-*}$, the properties of Gaussian convolution and proposition 9.2, it follows that for every $q,p \in \mathbb{R}$, $\mathbf{H}^{-*}[\phi_n^{(q,p)} \otimes \phi_n^{(q,p)}]$ is rotation invariant around (q,p). Hence, for every $R > 0$, the function $\mathbf{H}^{-*}[(\phi_n^{(q,p)} \otimes \phi_n^{(q,p)})_R]$ is rotation invariant around (q,p). For $\lambda, R > 0$, let $f_R^{(n)}(\lambda) = \mathbf{H}^{-*}[(\phi_n^{(q,p)} \otimes \phi_n^{(q,p)})_R](\sqrt{2\lambda},0)e^{-\lambda}$. Then

$$\lim_{R\to\infty} \int_0^\infty f_R^{(n)}(\lambda) \frac{\lambda^{n'}}{n'!}\, d\lambda = (\phi_n^{(q,p)} \otimes \phi_n^{(q,p)}, \phi_{n'}^{(q,p)} \otimes \phi_{n'}^{(q,p)}) = \delta_{nn'}.$$

Hence $\lim_{R\to\infty} f_R = \delta_0^{(n)}$, the n'th derivative of the Dirac measure δ_0 on $[0,\infty)$ at 0.

12.2. **Comparison with** [18]**.** In [18] it was shown that trace-class operators $\mathcal{T}$ have a diagonal representation

$$\mathcal{T} = \lim_{n\to\infty} \int_{\mathbb{R}^2} t_n(q,p) e_{q,p} \otimes e_{q,p}\, dqdp \in \mathsf{B}_1,$$

where $t_n \in \mathsf{S}(\mathbb{R}^2)$, the Schwartz test space of infinitely differentiable functions on $\mathbb{R}^2$ with rapid decay at infinity. Also explicit constructions for obtaining the t_n were given. The coefficients $F_R(q,p)$ in our diagonal representation of $\mathcal{A} \in \mathsf{B}_\pm$;

$$\mathcal{A} = \lim_{R\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}^2} F_R(q,p)\, e_{q,p} \otimes e_{q,p}\, dqdp \in \mathsf{B}_\pm, \tag{12.5}$$

are defined (in comparison with the obviously cumbersome construction of the coefficients t_n) in a more natural and straightforward way by $F_R = \mathbf{H}^{-*}[\mathcal{A}_R]$ and (12.4).

12.3. **The space B_+.** The space B_+ does not contain all possible density operators. However, $\phi_0 \otimes \phi_0 \in \mathsf{B}_+$ and the creation operators $\mathcal{S} \bigotimes \mathcal{I}$ and $\mathcal{I} \bigotimes \mathcal{S}$ and the displacement operators $\exp\{i(v\mathcal{Q}-u\mathcal{P})\}$, $u, v \in \mathbb{R}$ map B_+ into itself. Hence B_+ contains a.o. all projections on the number states and all projections on the coherent states. It contains the operator $\int_{\mathbb{R}^2} f(q,p)\, e_{q,p} \otimes e_{q,p}\, dqdp$ if an $\epsilon > 0$ exists such that $(q,p) \mapsto e^{\epsilon(q^2+p^2)} f(q,p)$ is square integrable. The space B_+ is characterized by theorem 10.3 in terms of the analytic continuations on the Wigner functions ($s = 0$) or the Husimi functions ($s = 1$) of the operators.

13. Appendices

13.1. **Generalized functions on $\mathsf{L}_2(\mathbb{R})$ corresponding to the Harmonic oscillator.** In [12] a new theory of generalized functions was introduced. First a space of 'smooth' functions related to the semigroup of 'smoothing' operators $e^{-\tau\mathcal{N}}$, $\tau > 0$ is introduced. This space will be denoted in the following by S_+. Then a related space of generalized functions, which we will denote by S_-, is introduced. Elements of B_- are 'quasibounded' operators of S_+ into S_-, which are related bijectively in [12] with elements of a two dimensional version of S_- through the Weyl correspondence. In this section we will give the definitions of S_+ and S_- and the relations with B_+ and B_-. We will identify the bounded sets of S_+ and S_- and show that the 'quasibounded' operators of S_+ into S_- are precisely the bounded operators (i.e. the operators that map bounded sets into bounded sets).

Definition 13.1. Let $\mathsf{S}_+^{(\tau)} \subset \mathsf{L}_2(\mathbb{R}) \subset \mathsf{S}_-^{(\tau)}$ be the Gelfand triple corresponding to $\mathsf{L}_2(\mathbb{R})$ and the operator $e^{-\tau\mathcal{N}}$. Let $\mathsf{S}_+ \subset \mathsf{L}_2(\mathbb{R}) \subset \mathsf{S}_-$ be the Gelfand triple corresponding to $\mathsf{L}_2(\mathbb{R})$ and the semigroup $(e^{-\tau\mathcal{N}})_{\tau>0}$.

Lemma 13.2. *Analytic continuation provides a unitary mapping from $\mathsf{S}_+^{(\tau)}$ onto $\mathsf{L}_2^a(\mathbb{C}, \rho_\tau(z)dz)$, the Hilbert space of $\rho_\tau(z)dz$-square-integrable entire functions on $\mathbb{C}$, where*

$$\rho_\tau(z) = C_\tau \exp\{\tanh(\tau)x^2 - \coth(\tau)y^2\}$$

for some $C_\tau > 0$. *For all* $\varphi \in \mathsf{L}_2^a(\mathbb{C}, \rho_\tau(z)dz)$ *there is an* $M_\tau > 0$ *such that*

$$|\varphi(x+iy)|^2 \leq M_\tau \exp\{-\tanh(\tau)x^2 + \coth(\tau)y^2\}. \tag{13.1}$$

Conversely, if a function φ *has a continuation to an entire analytic function that satisfies this estimate, then* $\varphi \in \mathsf{S}_+^{(\sigma)}$ *for every* $\sigma < \tau$.

Proof. Let $p = r = \coth(\tau)$, $q = -1/\sinh(\tau)$. Then

$$\mathbf{K}[e^{-\tau\mathcal{N}}](x,y) = c\exp\{-\frac{1}{2}(px^2 + ry^2 + 2qxy)\}$$

for some $c > 0$. It is easily seen that

$$e^{-\tau\mathcal{N}}\mathcal{Z}_{1/\sqrt{\tau}}\phi_n = u_n \quad \text{where} \quad u_n(x) = ac(x)\frac{(bx)^n}{\sqrt{n!}},$$

where $a > 0$, $b = (\frac{q^2}{2r})^{1/2}$ and $c(z) = \exp\{-(p - q^2/(2r))\frac{z^2}{2}\}$. We will use the same notation for u_n and its analytic continuation. Using

$$\rho_\tau(z) = C_\tau \exp\{\frac{pr - q^2}{r}\operatorname{Re}(z)^2 - p\operatorname{Im}(z)^2\}$$

($pr - q^2 = 1$) it is easily seen that we can choose C_τ such that u_n is an orthonormal basis for $\mathsf{L}_2^a(\mathbb{C}, \rho_\tau(z)dz)$. Hence $e^{-\tau\mathcal{N}}\colon \mathsf{L}_2(\mathbb{R}) \to \mathsf{L}_2^a(\mathbb{C}, \rho_\tau(z)dz)$ is unitary. Because $e^{-\tau\mathcal{N}}\colon \mathsf{L}_2(\mathbb{R}) \to \mathsf{S}_+^{(\tau)}$ is also unitary, analytic continuation provides a unitary mapping from $\mathsf{S}_+^{(\tau)}$ onto $\mathsf{L}_2^a(\mathbb{C}, \rho_\tau(z)dz)$.

Let $P_z(w) = \sum_{n=0}^{\infty} \overline{u_n(z)}u_n(w)$. Then $P_z \in \mathsf{L}_2^a(\mathbb{C}, \rho_\tau(z)dz)$ and $\varphi(z) = (P_z, \varphi)_{\rho_\tau}$ for every $\varphi \in \mathsf{L}_2^a(\mathbb{C}, \rho_\tau(z)dz)$ and $z \in \mathbb{C}$. By the CSB-inequality, $|\varphi(z)| \leq ||\varphi||_{\rho_\tau}||P_z||_{\rho_\tau}$. We have $||P_z||_{\rho_\tau}^2 = P_z(z) = |a|^2|c(z)|^2\exp(|bz|^2)$ and this is $|a|^2\exp\{-\tanh(\tau)x^2 + \coth(\tau)y^2\}$.

Assume that the function φ has a continuation to an entire analytic function which satisfies (13.1). Then $\varphi \in \mathsf{L}_2(\mathbb{C}, \rho_\sigma(z)dz)$, for every $\sigma < \tau$. Hence $\varphi \in \mathsf{S}_+^{(\sigma)}$, for every $\sigma < \tau$. □

Theorem 13.3. *Let* $S_{M,A,B}$ *be the set of functions* $f\colon \mathbb{R} \to \mathbb{C}$ *that have an analytic continuation to an entire function which satisfies*

$$|f(x+iy)| \leq M\exp\{-Ax^2 + By^2\}, \quad x, y \in \mathbb{R} \tag{13.2}$$

for some $M, A, B > 0$. *Then*

$$\mathsf{S}_+ = \cup_{A,B>0}\mathsf{S}_{A,B} \quad \textit{where} \quad \mathsf{S}_{A,B} = \cup_{M>0}S_{M,A,B}.$$

Proof. This follows from lemma 13.2. □

Proposition 13.4. *A set is* $\mathsf{S}_+^{(\tau)}$*-bounded for some* $\tau > 0$ *iff its contained in a set* $S_{M,A,B}$ *for some* $M, A, B > 0$.

Proof. This follows from lemma 13.2 □

Definition 13.5. A set is S_+-bounded iff it is $\mathsf{S}_+^{(\tau)}$-bounded for some $\tau > 0$. A set is S_--bounded iff for all $\tau > 0$ its image under $e^{-\tau\mathcal{N}}$ is S_+-bounded. An operator $\mathcal{A}\colon \mathsf{S}_+ \to \mathsf{S}_\pm$ or $\mathcal{A}\colon \mathsf{S}_- \to \mathsf{S}_\pm$ is continuous iff it is bounded (i.e. if it maps bounded sets onto bounded sets).

Remark 13.6. That this definition is consistent with the inductive limit topology of S_+ and the projective limit topology on S_- alluded to in 4.3 follows from the considerations in [13].

Proposition 13.7. *The following operators are continuous on* S_+:

- *The Fourier transform* $\mathcal{F}$.
- *The operators* $\mathcal{Q}$ *and* $\mathcal{P}$.
- $\mathcal{R}_z = \exp(iz\mathcal{Q})$ *for* $z \in \mathbb{C}$,
- *Complex translation:* $\mathcal{T}_z = e^{iz\mathcal{P}}$; $\mathcal{T}_z[f](x) = f(x+z)$, *with* $z \in \mathbb{C}$.

We have $\int_{\mathbb{R}} \mathcal{T}_z[f](x)dx = \int_{\mathbb{R}} f(x)dx$.

Proof. $\mathcal{F}$ commutes with $e^{-\tau\mathcal{N}}$ hence $\mathcal{F}\colon \mathsf{S}_+^{(\tau)} \to \mathsf{S}_+^{(\tau)}$ is unitary for all $\tau > 0$. Hence $\mathcal{F}\colon \mathsf{S}_+ \to \mathsf{S}_+$ is continuous. The operators $\mathcal{Q}, \mathcal{P}, \mathcal{R}_z$ and $\mathcal{T}_z$ map each $S_{M,A,B}$ into some $S_{M',A',B'}$, and hence are continuous. The last property is a consequence of Cauchy's theorem as was remarked in [12]. It follows also from $\mathcal{R}_z[f](0) = f(0)$ and $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)dx = \mathcal{F}[f](0)$ for $f \in \mathsf{S}_+$. □

We will define the set of continuous operators on B_+ analogous to the set of continuous operators on S_+:

Definition 13.8. A set is B_+-bounded iff it is $\mathsf{B}_+^{(\tau)}$-bounded for some $\tau > 0$. An operator $\mathcal{A}\colon \mathsf{B}_+ \to \mathsf{B}_+$ is continuous iff it is bounded.

Theorem 13.9. *The space* B_+ *consists of the bounded operators on* B_2 *that have an extension to a continuous operator from* S_- *to* S_+. *The space* B_- *consists of the densely defined operators on* B_2 *that are continuous as operator from* S_+ *to* S_-. *If* $\mathcal{A}, \mathcal{B}\colon \mathsf{S}_+ \to \mathsf{S}_+$ *are continuous then* $\mathcal{A} \bigotimes \mathcal{B}\colon \mathsf{B}_+ \to \mathsf{B}_+$ *is continuous.*

Proof. By the kernel theorems in [13] or [27] or [26], this is a consequence of $e^{-\tau\mathcal{N}} \in \mathsf{B}_1$, for $\tau > 0$. □

Remark 13.10. Unlike $\mathcal{D}'(\mathbb{R})$, the space of distributions of L.Schwartz, or $\mathcal{S}'(\mathbb{R})$, the space of tempered distributions, the space S_- contains the exponentials $x \mapsto e^{zx}$, for $z \in \mathbb{C}$, and the complex delta's δ_z defined by $<\delta_z, f> = f(z)$, for $f \in \mathsf{S}_+$.

13.2. **Characterization of $\mathsf{W}_+^{(s,\tau)}$.** In the following definition we define $\mathsf{W}_+^{(s,\tau)}$ for negative τ:

Definition 13.11. For $s > 0$ and $\tau < 0$ let $\mathsf{W}_+^{(s,\tau)}$ be the Hilbert space H_+ (see definition 4.2) corresponding to the operator $\mathcal{R} = e^{-\frac{s}{4}|\triangle|}\mathbf{S}_\tau$ on the Hilbert space $\mathsf{H} = \mathsf{L}_2(\mathbb{R}^2)$.

For $\tau < 0$, $\mathsf{W}_+^{(s,\tau)} = \exp\{-\frac{s}{2}|\triangle|\}\mathsf{W}_-^{(s,-\tau)}$. We denote the norm and inner product of $\mathsf{W}_+^{(s,\tau)}$ by $||\cdot||_{s,\tau}$ and $(\cdot,\cdot)_{s,\tau}$.

Lemma 13.12. *Let* $s > 0$ *and* $\tanh\tau > -\min(s, \frac{1}{s})$. *Then* $e^{-\tau\mathcal{N}}\mathcal{W}_{q,p}^{(s)}e^{-\tau\mathcal{N}} \in \mathsf{B}_2$, $q, p \in \mathbb{R}$. *We have* $g \in \mathsf{W}_+^{(s,\tau)}$ *iff*

$$g(q,p) = (e^{-\tau\mathcal{N}}\mathcal{W}_{q,p}^{(s)}e^{-\tau\mathcal{N}}, \mathcal{A}), \quad \forall\, q,p \in \mathbb{R}, \quad \textit{for an } \mathcal{A} \in \mathsf{B}_2. \tag{13.3}$$

In that case, the function g has a continuation to an entire analytic function for which

$$||g||_{s,\tau}^2 = \iint_{\mathbb{C}} |g(z_1,\bar{z}_2)|^2 d\mu_{s,\tau}(z_1,z_2),$$

where

$$d\mu_{s,\tau}(z_1,z_2) = C_{s,\tau}\exp\{2\frac{\mathrm{Re}(z_1)^2 + \mathrm{Re}(z_2)^2}{s+\coth\tau} - 2\frac{\mathrm{Im}(z_1)^2 + \mathrm{Im}(z_2)^2}{s+\tanh\tau}\}dz_1dz_2$$

and $C_{s,\tau} > 0$. *The analytic continuation of g satisfies*

$$|g(z_1,z_2)| \le M_{s,\tau}||g||_{s,\tau}\exp\{-\frac{\mathrm{Re}(z_1)^2 + \mathrm{Re}(z_2)^2}{s+\coth\tau} + \frac{\mathrm{Im}(z_1)^2 + \mathrm{Im}(z_2)^2}{s+\tanh\tau}\} \tag{13.4}$$

for some positive number $M_{s,\tau}$ that does not depend on g. Conversely, if $g\colon \mathbb{R}^2 \to \mathbb{C}$ has a continuation to an entire analytic function that satisfies (13.4) (for any $M_{s,\tau}$), then $g \in \mathsf{W}_+^{(s,\sigma)}$ for every $\sigma < \tau$ such that $\tanh(\sigma) > -\min(s, \frac{1}{s})$.

Proof. The assumptions on s and τ imply that $s + \tanh(\tau) > 0$ and that $\mathrm{sign}(s + \coth\tau) = \mathrm{sign}(\tau)$ when $\tau \neq 0$. Let

$$p = \frac{\cosh\tau}{s\cosh\tau + \sinh\tau}, \quad r = \frac{s\sinh\tau + \cosh\tau}{s\cosh\tau + \sinh\tau}, \quad q = \frac{-1}{s\cosh\tau + \sinh\tau}.$$

The assumptions on s and τ imply that $p, r, -q > 0$.

We have $\mathbf{K}[\exp(\frac{s}{2}\frac{d^2}{dx^2})\exp(-\tau\mathcal{N})](x,y) = c_{s,\tau}\exp\{-\frac{1}{2}(px^2 + ry^2 + 2qxy)\}$ for some constant $c_{s,\tau}$. Hence $\exp(\frac{s}{2}\frac{d^2}{dx^2})\exp(-\tau\mathcal{N})\mathcal{Z}_{1/\sqrt{r}}\phi_n = u_n$, where $u_n(x) = ac(x)\frac{(bx)^n}{\sqrt{n!}}$, where $a > 0$ and b, c are related to p, q, r as in the proof of lemma 13.2. Hence $e^{-\frac{s}{4}|\triangle|}\mathbf{S}_\tau\mathcal{Z}_{1/\sqrt{2r}}^{(1)}\mathcal{Z}_{1/\sqrt{2r}}^{(2)}[\phi_k\cdot\phi_\ell] = \mathcal{Z}_{1/\sqrt{2}}[u_k]\cdot\mathcal{Z}_{1/\sqrt{2}}[u_\ell]$, where we used the symbol $\cdot$ to denote the pointwise product which transforms a pair of functions of one variable into one function of two variables. We will use the same notation for u_n and its analytic continuation. The family $\mathcal{Z}_{1/\sqrt{2}}[u_n]$, $n \in \mathbb{N}_0$ is, up to a constant factor (depending on s and τ), an orthonormal basis of $\mathsf{L}_2^a(\mathbb{C}, \rho(z)dz)$, where

$$\rho(z) = \exp\{2\frac{pr - q^2}{r}\operatorname{Re}(z)^2 - 2p\operatorname{Im}(z)^2\}.$$

Using $pr - q^2 = \frac{\sinh\tau}{s\cosh\tau + \sinh\tau}$ we see that

$$\rho(z) = \exp\{\frac{2\operatorname{Re}(z)^2}{s + \coth\tau} + \frac{-2\operatorname{Im}(z)^2}{s + \tanh\tau}\}.$$

Hence we can choose $C_{s,\tau}$ such that the analytic continuations of the functions $u_k \cdot u_\ell$, $k, \ell \in \mathbb{N}_0$ form an orthonormal basis of $\mathsf{L}_2^a(\mathbb{C}^2, \mu_{s,\tau})$.

Now we will prove (13.4). Let

$$P_{z_1,z_2}(w_1, w_2) = \sum_{k,\ell=0}^{\infty} \overline{u_k(z_1)u_\ell(\bar{z}_2)}u_k(w_1)u_\ell(\bar{w}_2).$$

Then $(P_{z_1,z_2}, f)_{\mu_{s,\tau}} = f(z_1, z_2)$ for $f \in \mathsf{L}_2^a(\mathbb{C}^2, \mu_{s,\tau})$. By the CSB-inequality, $|f(z_1, \bar{z}_2)| \le ||f||_{\mu_{s,\tau}}||P_{z_1,\bar{z}_2}||_{\mu_{s,\tau}}$. We have $||P_{z_1,\bar{z}_2}||^2_{\mu_{s,\tau}} = P_{z_1,\bar{z}_2}(z_1, \bar{z}_2)$ and this is $|\varphi(z_1)|^2\,|\varphi(z_2)|^2$, where $\varphi(z) = \sum_{n=0}^{\infty}|u_n(z)|^2 = |a|^2|c(z)|^2\exp(|bz|^2)$. We have

$$\begin{aligned}|c(z)|^2\exp\{|bz|^2\} &= \exp\{-2r^{-1}(pr - q^2)\operatorname{Re}(z)^2 + 2p\operatorname{Im}(z)^2\}\\ &= \exp\{\frac{-2\operatorname{Re}(z)^2}{s + \coth\tau} + \frac{2\operatorname{Im}(z)^2}{s + \tanh\tau}\}.\end{aligned}$$

Hence every $g \in \mathsf{W}_+^{(s,\tau)}$ has a continuation to an entire analytic function for which

$$(13.5)\quad |g(z_1, z_2)| \le M||g||_{s,\tau}\exp\{-\frac{\operatorname{Re}(z_1)^2 + \operatorname{Re}(z_2)^2}{s + \coth\tau} + \frac{\operatorname{Im}(z_1)^2 + \operatorname{Im}(z_2)^2}{s + \tanh\tau}\}$$

for some $M > 0$. Conversely, if g has a continuation to an entire analytic function that satisfies this estimate, then the analytic continuation $g \in \mathsf{L}_2(\mathbb{C}^2, \mu_{s,\sigma})$ for every $\sigma < \tau$ such that $\tanh(\sigma) > -\min(s, \frac{1}{s})$. Hence $g \in \mathsf{W}_+^{(s,\sigma)}$ for every $\sigma < \tau$ such that $\tanh(\sigma) > -\min(s, \frac{1}{s})$. □

Theorem 13.13. *Let $s \ge 0$. $g \in \mathsf{W}_+^{(s)}$ iff $g\colon \mathbb{R}^2 \to \mathbb{C}$ has a continuation to an entire analytic function that satisfies*

$$|g(z_1, z_2)| \le M\exp\{-\frac{\operatorname{Re}(z_1)^2 + \operatorname{Re}(z_2)^2}{s + \coth\tau} + \frac{\operatorname{Im}(z_1)^2 + \operatorname{Im}(z_2)^2}{s + \tanh\tau}\}$$

for some $\tau, M > 0$. Let $s > 0$. $F \in \mathsf{W}_-^{(s)}$ iff the analytic continuation of $f = \exp\{-\frac{s}{2}|\triangle|\}F$ satisfies the following condition: For every $\tau > 0$ such *that* $0 < \tanh\tau < \min(s, \frac{1}{s})$ *there exist an* $M > 0$ *such that*

$$|f(z_1, z_2)| \leq M \exp\{-\frac{\mathrm{Re}(z_1)^2 + \mathrm{Re}(z_2)^2}{s - \coth\tau} + \frac{\mathrm{Im}(z_1)^2 + \mathrm{Im}(z_2)^2}{s - \tanh\tau}\}.$$

Remark 13.14. Let $\mathcal{B} \in \mathsf{B}_-$ and $F = \mathbf{W}_s^{-*}[\mathcal{B}]$. Then $\mathbf{W}_s[\mathcal{B}] = \exp\{-\frac{s}{2}|\triangle|\}F$.

Proof. If $F \in \mathsf{W}_-^{(s)}$ and $g \in \mathsf{W}_+^{(s)}$ then $f \in \mathsf{W}_+^{(s,-\sigma)}$ and $g \in \mathsf{W}_+^{(s,\sigma)}$ for some $\sigma > 0$. Lemma 13.12 implies that f and g satisfy the given estimates. Assume now these estimates are satisfied by some functions f and g. By lemma 13.12, $f \in \mathsf{W}_+^{(s,-\sigma)}$ and $g \in \mathsf{W}_+^{(s,\sigma)}$ for some $\sigma > 0$. Hence $F \in \mathsf{W}_-^{(s)}$ and $g \in \mathsf{W}_+^{(s)}$. □

13.3. **The operators** $e^{w(\mathcal{Q}\pm i\mathcal{P})}$**.**

Lemma 13.15. *For all* $f \in \mathsf{S}_+$, $b > 0$ *there is an* $a > 0$ *such that for every* $n \in \mathbb{N}_0$, $||\mathcal{B}^n f|| \leq ab^n n!$, *where* $\mathcal{B} \in \{\mathcal{Q}, \mathcal{P}\}$.

Proof. Let $\tau > 0$. $\mathcal{N}_{2\tau}(\mathsf{L}_2(\mathbb{R})) = e^{-\tau(\mathcal{Q}^2+\mathcal{P}^2)}(\mathsf{L}_2(\mathbb{R}))$. Using $||\mathcal{A}||_\infty^2 = ||\mathcal{A}^*\mathcal{A}||_\infty$ we see that $||\mathcal{B}^n e^{-\tau(\mathcal{Q}^2+\mathcal{P}^2)}||_\infty^2 \leq ||e^{-\tau(\mathcal{Q}^2+\mathcal{P}^2)}\mathcal{B}^{2n}e^{-\tau(\mathcal{Q}^2+\mathcal{P}^2)}||_\infty$. Because the norm of a Hermitian sesquilinear form equals that of the corresponding quadratic form and hence that $||\cdot||_\infty$ is monotone on the cone of non-negative operators, this is $\leq ||e^{-2\tau(\mathcal{Q}^2+\mathcal{P}^2)}(\mathcal{Q}^2+\mathcal{P}^2)^n||_\infty$. We have $\sigma(\mathcal{Q}^2+\mathcal{P}^2) \subset [0,\infty)$. By the spectral mapping theorem $\sigma(e^{-2\tau(\mathcal{Q}^2+\mathcal{P}^2)}(\mathcal{Q}^2+\mathcal{P}^2)^n) \subset \{e^{-2\tau x}x^n : x > 0\}$. For positive numbers x, $e^{-2\tau x}x^n \leq (2\tau)^{-n}n!$. The norm of a bounded normal operator is equal to its spectral radius. Hence $||e^{-2\tau(\mathcal{Q}^2+\mathcal{P}^2)}(\mathcal{Q}^2 + \mathcal{P}^2)^n||_\infty \leq (2\tau)^{-n}n!$. Hence $||\mathcal{B}^n e^{-\tau(\mathcal{Q}^2+\mathcal{P}^2)}||_\infty \leq (2\tau)^{-n/2}\sqrt{n!}$. For every $\tau, b > 0$ there is an $a > 0$ such that for every $n \in \mathbb{N}_0$, $(2\tau)^{-n/2} \leq ab^n\sqrt{n!}$. □

Proposition 13.16. *Let* $\exp\{w(\mathcal{Q} \pm i\mathcal{P})\}[f] = \sum_{n=0}^\infty \frac{w^n}{n!}(\mathcal{Q} \pm i\mathcal{P})^n[f]$ *for* $f \in \mathsf{S}_+$. *The sum converges in* S_+. *The operator* $\exp\{w(\mathcal{Q} \pm i\mathcal{P})\}\colon \mathcal{S}_+ \to \mathcal{S}_+$ *is continuous. We have* $\exp\{w(\mathcal{Q} \pm i\mathcal{P})\} = e^{\pm w^2/2}e^{w\mathcal{Q}}e^{\pm iw\mathcal{P}}$ *on* S_+.

Proof. Let $f \in \mathsf{S}_+$. Let $\mathcal{A}_\pm(w) = e^{\pm w^2/2}e^{w\mathcal{Q}}e^{\pm iw\mathcal{P}}$. Then $\mathcal{A}_\pm(w)[f] \in \mathsf{S}_+$, $w \mapsto \mathcal{A}_\pm(w)[f]$ is entire analytic and $\mathcal{A}_\pm'(w) = (\mathcal{Q} \pm i\mathcal{P})\mathcal{A}_\pm(w)$ on S_+. We have $\mathcal{A}_\pm^{(n)}(0) = (\mathcal{Q}\pm i\mathcal{P})^n$. Hence the Taylor series around 0 of $\mathcal{A}_\pm(\cdot)[f]$ is given by $\mathcal{A}_\pm(w)[f] = \sum_{n=0}^\infty \frac{w^n}{n!}(\mathcal{Q}\pm i\mathcal{P})^n[f]$. The Taylor series is absolutely convergent. We have $\mathcal{A}_\pm(w) = \exp\{w(\mathcal{Q} \pm i\mathcal{P})\}$. By proposition 13.5, $e^{w\mathcal{Q}}$ is continuous on S_+ for all $w \in \mathbb{C}$. We have $e^{iw\mathcal{P}} = \mathcal{F}^*e^{iw\mathcal{Q}}\mathcal{F}$ and $\mathcal{F}, \mathcal{F}^*\colon \mathsf{S}_+ \to \mathsf{S}_+$ are continuous. Hence $e^{iw\mathcal{P}}$ is continuous on S_+. Hence $\exp\{w(\mathcal{Q} \pm i\mathcal{P})\}\colon \mathsf{S}_+ \to \mathsf{S}_+$ is continuous. Hence its adjoint $\exp\{\bar{w}(\mathcal{Q}\mp i\mathcal{P})\}\colon \mathsf{S}_- \to \mathsf{S}_-$ is continuous. □

13.4. **The operators** $\exp\{w(\mathbf{G}_{\mathcal{Q}} \pm i\mathbf{G}_{\mathcal{P}})\}$**.** For $z \in \mathbb{C}$ let $\mathbf{W}_s[\mathcal{A}](z) = \mathbf{W}_s[\mathcal{A}](x,y)$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

Proposition 13.17. *For* $\mathcal{A} \in \mathsf{B}_+$, $s > 0$, $z, w \in \mathbb{C}$,

$$\begin{aligned} \boldsymbol{D}^{-1}\,\boldsymbol{W}_s[\mathcal{A}](z-w,\bar z) &= \boldsymbol{W}_s[\exp\{w(\boldsymbol{G}_{\mathcal{Q}} - i\,\boldsymbol{G}_{\mathcal{P}})\}\mathcal{A}](z),\\ \boldsymbol{D}^{-1}\,\boldsymbol{W}_s[\mathcal{A}](z,\bar z+\bar w) &= \boldsymbol{W}_s[\exp\{\bar w(\boldsymbol{G}_{\mathcal{Q}} + i\,\boldsymbol{G}_{\mathcal{P}})\}\mathcal{A}](z),\\ \boldsymbol{D}^{-1}\,\boldsymbol{W}_s[\mathcal{A}](z-w,\bar z-\bar w) &= \boldsymbol{W}_s[\boldsymbol{G}_{u,v}[\mathcal{A}]](z), \end{aligned}$$

where $u = \mathrm{Re}(w)$ *and* $v = \mathrm{Im}(w)$.

Proof. Proposition 9.7. The last identity follows from the first two by

$$\mathbf{G}_{u,v} = \exp\{-\bar w(\mathbf{G}_{\mathcal{Q}} + i\mathbf{G}_{\mathcal{P}})\}\exp\{w(\mathbf{G}_{\mathcal{Q}} - i\mathbf{G}_{\mathcal{P}})\}.$$

□

Lemma 13.18. *For all* $\mathcal{A} \in \mathsf{B}_+$, $b > 0$ *there is an* $a > 0$ *such that* $||\boldsymbol{B}^n[\mathcal{A}]||_{\mathsf{B}_2} \le ab^n n!$ *for all* $n \in \mathbb{N}$, *where* $\boldsymbol{B} \in \{\boldsymbol{G}_{\mathcal{Q}}, \boldsymbol{G}_{\mathcal{P}}, \boldsymbol{G}_{\mathcal{Q}} \pm i\,\boldsymbol{G}_{\mathcal{P}}\}$.

Proof. The proof is similar to that of lemma 13.15. □

Proposition 13.19. *Let* $\mathcal{A} \in \mathsf{B}_+$. *Then*

$$\begin{aligned} \exp\{w(\boldsymbol{G}_{\mathcal{Q}} + i\,\boldsymbol{G}_{\mathcal{P}})\}[\mathcal{A}] &= \exp\{\frac{w}{\sqrt2}\mathcal{S}\}\mathcal{A}\exp\{-\frac{w}{\sqrt2}\mathcal{S}\},\\ \exp\{w(\boldsymbol{G}_{\mathcal{Q}} - i\,\boldsymbol{G}_{\mathcal{P}})\}[\mathcal{A}] &= \exp\{\frac{w}{\sqrt2}\mathcal{S}^*\}\mathcal{A}\exp\{-\frac{w}{\sqrt2}\mathcal{S}^*\}. \end{aligned}$$

Proof. If $\mathcal{X}\colon \mathcal{S}_+ \to \mathcal{S}_+$ is continuous and $\mathcal{A}_w[f] = \sum_{n=0}^{\infty}(n!)^{-1}w^n\mathcal{X}^n[f]$ is an $\mathcal{S}_+$-convergent power series, then $(d/dw)\mathcal{A}_w[f] = \mathcal{X}\mathcal{A}_w[f]$. By lemma 13.18 and proposition 13.16 the exponentials in the proposition can be expanded in terms of $\mathcal{S}_+$-convergent power series when we apply both sides of the identities in the proposition to an element in $\mathcal{S}_+$. We have

$$\begin{aligned} \mathbf{G}_{\mathcal{Q}} + i\mathbf{G}_{\mathcal{P}} &= 2^{-1/2}(\mathcal{S}^*\bigotimes\mathcal{I} - \mathcal{I}\bigotimes\mathcal{S}),\\ \mathbf{G}_{\mathcal{Q}} - i\mathbf{G}_{\mathcal{P}} &= 2^{-1/2}(\mathcal{S}\bigotimes\mathcal{I} - \mathcal{I}\bigotimes\mathcal{S}^*) \end{aligned} \tag{13.6}$$

These identities imply the identities of the proposition by differentiation w.r.t. w. □

Proposition 13.20. *The operator* $\exp\{w(\boldsymbol{G}_{\mathcal{Q}} \pm i\,\boldsymbol{G}_{\mathcal{P}})\}\colon \mathsf{B}_+ \to \mathsf{B}_+$ *is continuous. For* $\mathcal{A} \in \mathsf{B}_+$ *and* $\mathcal{B} \in \mathsf{B}_-$ *we have*

$$<\mathcal{B}, \exp\{w(\boldsymbol{G}_{\mathcal{Q}} \pm i\,\boldsymbol{G}_{\mathcal{P}})\}\mathcal{A}> = <(\exp\{\bar w(\boldsymbol{G}_{\mathcal{Q}} \mp i\,\boldsymbol{G}_{\mathcal{P}})\})^{\mathrm{ext}}\mathcal{B}, \mathcal{A}>.$$

Proof. Proposition 13.19. □

Proposition 13.21. *Let* $\mathcal{A} \in \mathsf{B}_+$. *Then* $\mathit{Tr}(\exp\{w(\boldsymbol{G}_{\mathcal{Q}} \pm i\,\boldsymbol{G}_{\mathcal{P}})\}\mathcal{A}) = \mathit{Tr}(\mathcal{A})$, *for all* $w \in \mathbb{C}$.

Proof. This follows from proposition 13.19 and proposition 13.16 by applying the following rule: If for some $\tau > 0$, $\mathcal{B}e^{-\tau\mathcal{N}}, e^{-\tau\mathcal{N}}\mathcal{B}^{-1} \in \mathsf{B}_\infty$ and $e^{\tau\mathcal{N}}\mathcal{A}e^{\tau\mathcal{N}} \in \mathsf{B}_1$ then $\mathrm{Tr}(\mathcal{B}\mathcal{A}\mathcal{B}^{-1}) = \mathrm{Tr}(\mathcal{A})$. □

Corollary 13.22. *Let $\mathcal{A} \in \mathsf{B}_+$. Then $Tr(\mathcal{A}) = \int_{\mathbb{C}} \boldsymbol{D}^{-1}\boldsymbol{H}[\mathcal{A}](z+w,\bar{z})dz$ for every $w \in \mathbb{C}$.*

Proof. This follows from the previous proposition using theorem 8.2. □

13.5. **Deconvolution formula.** The principle behind theorems 11.4 and 11.5 is deconvolution of the Husimi function in order to obtain the anti-Husimi function. This deconvolution is done by analytic continuation and integrating along the imaginary axes. In this section this will be explained in a one dimensional setting.

Lemma 13.23. *Analytic continuation provides a unitary mapping from the range of $\exp(-\frac{\gamma}{2}\mathcal{P}^2)$ onto $\mathsf{L}_2^a(\mathbb{C}, \rho_\gamma(z)dz)$, the Hilbert space of $\rho_\gamma(z)dz$-square-integrable entire functions on $\mathbb{C}$, where*

$$\rho_\gamma(z) = (\pi\gamma)^{-1/2}\exp\{-\operatorname{Im}(w)^2/\gamma\}.$$

For all $\varphi \in \mathsf{L}_2^a(\mathbb{C}, \rho_\gamma(z)dz)$ there is an $M_\gamma > 0$ such that

$$|\varphi(x+iy)|^2 \leq M_\gamma \exp\{y^2/\gamma\}. \tag{13.7}$$

Conversely, if a function φ has a continuation to an entire analytic function that satisfies this estimate, then $\varphi \in range(\exp\{-\frac{\tilde{\gamma}}{2}\mathcal{P}^2\})$ for every $\tilde{\gamma} < \gamma$.

Proof. The proof is the same as the proof of lemma 13.2 except that now $p = r = -q = 1/\gamma$ and $pr - q^2 = 0$. □

Theorem 13.24. *(Deconvolution formula) Let $f \in \mathsf{S}_+$. Let $g = \exp\{-\frac{\gamma}{2}\mathcal{P}^2\}[f]$, where $\gamma > 0$. Then g has a continuation to an entire analytic function and*

$$f(x) = \frac{1}{\sqrt{2\pi\gamma}}\int_{\mathbb{R}} g(x+iy)e^{-y^2/(2\gamma)}dy.$$

Proof. Let $h_u(x) = e^{iux}$. Then $\frac{1}{\sqrt{2\pi s}}\int_{\mathbb{R}} h_u(x+iy)e^{-y^2/(2s)}dy = e^{su^2/2}h_u(x)$. By lemmas 13.23 and 13.2 there exists and $\epsilon > 0$ and a $\tilde{f} \in \mathsf{L}_2(\mathbb{R})$ such that $f = \exp\{-\epsilon\mathcal{P}^2\}[\tilde{f}]$. We have $g(x) = \int_{\mathbb{R}} \mathcal{F}[\tilde{f}](u)h_u(x)\,e^{-(\gamma+\epsilon)u^2}du$. The result follows by interchanging the integrals involved which can be justified by Fubini's theorem. □

Example 13.25. For $n \in \mathbb{N}$ let H_n be the n'th Hermite polynomial defined by $H_n(x) = n!\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^k(2x)^{n-2k}}{k!(n-2k)!}$. From the generating formula for the Hermite polynomials $\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x) = \exp\{2zx - z^2\}$ it follows that

$$\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-(x-y)^2}H_n(y)\,dy = (2x)^n.$$

Although $H_n \notin \mathsf{S}_+$, the conclusion of theorem 13.24 is valid in this case:

$$H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{\mathbb{R}} (x+iy)^n e^{-y^2} dy$$

as follows from the binomial formula and the identity $\int_{\mathbb{R}} y^k e^{-y^2} dy = \Gamma(\frac{k}{2} + \frac{1}{2}) = \sqrt{\pi}(2k)!/(4^k k!)$ for $k = 0, 2, \cdots$ and 0 otherwise. Remark: The Hermite basis functions satisfy $\phi_n(x) = \pi^{-1/4} e^{-x^2/2} (n! 2^n)^{-1/2} H_n(x)$.

13.6. **Some proofs.** For $z \in \mathbb{C}$ let $\mathbf{W}_s[\mathcal{A}](z) = \mathbf{W}_s[\mathcal{A}](x, y)$, where $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$.

Proof of proposition 9.7. $\mathcal{W}^{(s)}_{q,p} = \tilde{\mathcal{W}}^{(s)}_{z,\bar{z}}$ follows from $\mathcal{W}^{(s)}_{q,p} = \mathbf{G}_{q,p}[\mathcal{W}^{(s)}_{0,0}]$. This implies (9.2). (9.3) follows from proposition 9.1 by analytic continuation. In order to prove (9.4), we use

$$\tilde{\mathcal{W}}^{(s)}_{q,p} = \frac{e^{-(q^2+p^2)/(2s)}}{\sqrt{2\pi s}} \sum_{k,\ell=0}^{\infty} \frac{q^k p^\ell}{q_k p_\ell} \phi_{k\ell}, \tag{13.8}$$

where $\phi_{k\ell} = \mathbf{R}_{\pi/4}[\mathcal{Z}_{\sqrt{s}}\phi_k \otimes \mathcal{Z}_{1/\sqrt{s}}\phi_\ell]$, $q_k = \sqrt{s^k k!}$ and $p_\ell = (-i)^\ell \sqrt{s^\ell \ell!}$. First we prove (13.8): The integral kernel of the left hand side of (13.8) is

$$\mathbf{K}[\mathcal{W}^{(s)}_{q,p}](x,y) = \frac{1}{\pi\sqrt{2s}} \exp\{-\frac{1}{s}(q - \frac{x+y}{2})^2 - s(\frac{x-y}{2})^2 + ip(x-y)\}$$

By lemma 7.8,

$$\mathbf{K}[\mathcal{Z}_{\frac{1}{\sqrt{s}}} \bigotimes \mathcal{Z}_{\sqrt{s}} \mathbf{R}^{-1}_{\pi/4} \mathcal{W}^{(s)}_{q,p}](x,y) =$$
$$= \frac{1}{\pi\sqrt{2s}} \exp\{-\frac{q^2}{s} - \frac{x^2+y^2}{2} + \sqrt{\frac{2}{s}}(qx + ipy)\}.$$

By (8.2) this is $\frac{e^{-(q^2+p^2)/(2s)}}{\sqrt{2\pi s}} \sum_{k,\ell=0}^{\infty} \frac{q^k p^\ell}{q_k p_\ell} \phi_k(x)\phi_\ell(y)$. Thus (13.8) is proved. (9.4) follows by analytic continuation from (13.8) and

$$\iint_{\mathbb{C}^2} \overline{(\frac{z_1+\bar{z}_2}{2})^k (\frac{z_1-\bar{z}_2}{2i})^\ell} (\frac{z_1+\bar{z}_2}{2})^{k'} (\frac{z_1-\bar{z}_2}{2i})^{\ell'} \frac{\exp\{-\frac{|z_1|^2+|z_2|^2}{2s}\}}{(2\pi s)^2} dz_1 dz_2$$
$$= \delta_{kk'}\delta_{\ell\ell'} \overline{q_k p_\ell} q_{k'} p_{\ell'}.$$

Lemma 13.26. *If $\mathcal{A} \in \mathsf{B}_+$ and $\mathcal{B} \in \mathsf{B}_-$ then*

$$< \boldsymbol{W}_s[\mathcal{B}], \boldsymbol{W}_s[\mathcal{A}]> = \int_{\mathbb{C}} \overline{\boldsymbol{W}_s[\mathcal{B}](z)}\, \boldsymbol{W}_s[\mathcal{A}](z)\, dz.$$

Proof. This is true for $\mathcal{A}, \mathcal{B} \in \mathsf{B}_2$. If $\mathcal{B} \in \mathsf{B}_-$ then we can approximate $\mathcal{B}$ by $e^{-\tau\mathcal{N}}\mathcal{B}e^{-\tau\mathcal{N}}$, with $\tau > 0$ and use the dominated convergence theorem in combination with theorem 10.3. □

Lemma 13.27. *Let* $s > 0$ *and* $\tanh\tau > -\min(s, \frac{1}{s})$. *There exists an* $M_{s,\tau} > 0$ *such that*

$$||e^{-\tau\mathcal{N}}\tilde{\mathcal{W}}^{(s)}_{z,\bar{w}}e^{-\tau\mathcal{N}}||_2 \leq M_{s,\tau}\exp\{\frac{-|z+w|^2}{4s+4\coth\tau} + \frac{|z-w|^2}{4s+4\tanh\tau}\}.$$

Proof. This follows from lemma 13.12. □

Lemma 13.28. *Let* $s > 0$. *For* $\mathcal{A} \in \mathsf{B}_+$ *and* $\mathcal{B} \in \mathsf{B}_-$ *we have*

$$<\mathcal{B},\mathcal{A}> = \frac{1}{2\pi s}\iint_{\mathbb{C}^2} <\mathcal{B},\tilde{\mathcal{W}}^{(s)}_{z_1,\bar{z}_2}><\tilde{\mathcal{W}}^{(s)}_{z_1,\bar{z}_2},\mathcal{A}>e^{-|z_1-z_2|^2/(2s)}dz_1dz_2.$$

Proof. By proposition 9.7, this is true for $\mathcal{A},\mathcal{B} \in \mathsf{B}_+$. If $\mathcal{B} \in \mathsf{B}_-$ then we can approximate $\mathcal{B}$ by $e^{-\tau\mathcal{N}}\mathcal{B}e^{-\tau\mathcal{N}}$, with $\tau > 0$ and use the dominated convergence theorem in combination with lemma 13.27. □

Proof of lemma 11.1. This is a reformulation of lemma 13.28 using (9.2).

Proof of lemma 11.2. Let $\mathcal{B} \in \mathsf{B}_-$ and $\mathcal{A} \in \mathsf{B}_+$. By theorem 13.20,

$$<\mathbf{W}[(\exp\{w(\mathbf{G}_\mathcal{Q} - i\mathbf{G}_\mathcal{P})\})^{\text{ext}}\mathcal{B}], \mathbf{W}[\exp\{w(\mathbf{G}_\mathcal{Q} - i\mathbf{G}_\mathcal{P})\}\mathcal{A}]> =$$
$$= <\mathbf{W}[(\exp\{\bar{w}(\mathbf{G}_\mathcal{Q} + i\mathbf{G}_\mathcal{P})\})^{\text{ext}}(\exp\{w(\mathbf{G}_\mathcal{Q} - i\mathbf{G}_\mathcal{P})\})^{\text{ext}}\mathcal{B}], \mathbf{W}[\mathcal{A}]>.$$

It follows from theorem 10.3 that $\mathsf{W}^{(s)}_+ \subset \mathsf{W}_+$. Hence $\exp\{-\frac{s}{4}|\triangle|\}$ maps W_+ into itself. Hence the selft-adjoint operator $\exp\{-s(\mathbf{G}^2_\mathcal{Q} + \mathbf{G}^2_\mathcal{P})\}$ maps B_+ into itself. Because $\exp\{w(\mathbf{G}_\mathcal{Q} - i\mathbf{G}_\mathcal{P})\}$ commutes with $\exp\{-s(\mathbf{G}^2_\mathcal{Q} + \mathbf{G}^2_\mathcal{P})\}$ we get

$$<\mathbf{W}_s[(\exp\{w(\mathbf{G}_\mathcal{Q} - i\mathbf{G}_\mathcal{P})\})^{\text{ext}}\mathcal{B}], \mathbf{W}_s[\exp\{w(\mathbf{G}_\mathcal{Q} - i\mathbf{G}_\mathcal{P})\}\mathcal{A}]> =$$
$$= <\mathbf{W}_s[(\exp\{\bar{w}(\mathbf{G}_\mathcal{Q} + i\mathbf{G}_\mathcal{P})\})^{\text{ext}}(\exp\{w(\mathbf{G}_\mathcal{Q} - i\mathbf{G}_\mathcal{P})\})^{\text{ext}}\mathcal{B}], \mathbf{W}_s[\mathcal{A}]>.$$

when we apply the previous considerations to the operators $\exp\{-s(\mathbf{G}^2_\mathcal{Q} + \mathbf{G}^2_\mathcal{P})\}[\mathcal{A}]$ and $(\exp\{-s(\mathbf{G}^2_\mathcal{Q} + \mathbf{G}^2_\mathcal{P})\})^{\text{ext}}[\mathcal{B}]$ i.s.o. $\mathcal{A}$ and $\mathcal{B}$.

The result follows from lemma 13.26 and proposition 13.17.

Lemma 13.29. *Let* $\mathcal{B} \in \mathsf{B}_-$ *and* $\mathcal{A} \in \mathsf{B}_+$. *Then* $\mathcal{B}^{(s)}_R \to \mathcal{B} \in \mathsf{B}_-$ *and* $\mathcal{A}^{(s)}_R \to \mathcal{A} \in \mathsf{B}_+$ *as* $R \to \infty$.

Proof. Let $R, \tau > 0$. By theorem 11.5,

$$||e^{-\tau\mathcal{N}}(\mathcal{B} - \mathcal{B}^{(s)}_R)e^{-\tau\mathcal{N}}||_2 \leq$$
$$\iint_{|z-w|\geq R} |(\tilde{\mathcal{W}}^{(s)}_{z,\bar{w}},\mathcal{B})|\, ||e^{-\tau\mathcal{N}}\tilde{\mathcal{W}}^{(s)}_{z,\bar{w}}e^{-\tau\mathcal{N}}||_2\, e^{-|z-w|^2/(2s)}dzdw.$$

By the CSB-inequality, $|(\tilde{\mathcal{W}}^{(s)}_{z,\bar{w}},\mathcal{B})| \leq ||e^{-\sigma\mathcal{N}}\mathcal{B}e^{-\sigma\mathcal{N}}||_2||e^{\sigma\mathcal{N}}\tilde{\mathcal{W}}^{(s)}_{z,\bar{w}}e^{\sigma\mathcal{N}}||_2$, for some $\sigma > 0$. The function $(z,w) \mapsto ||e^{\sigma\mathcal{N}}\tilde{\mathcal{W}}^{(s)}_{z,\bar{w}}e^{\sigma\mathcal{N}}||_2||e^{-\tau\mathcal{N}}\tilde{\mathcal{W}}^{(s)}_{z,\bar{w}}e^{-\tau\mathcal{N}}||_2$ is integrable if

$$\tanh\sigma < \frac{\tanh\tau}{1+2\max(s,\frac{1}{s})\tanh\tau}$$

as follows from lemma 13.27, a straightforward calculation and the fact that for $a, b \in \mathbb{R}$ the function $(z,w) \mapsto \exp\{-a|z+w|^2 + b|z-w|^2\}$ is integrable whenever $a > 0$ and $b < 0$. Hence $\mathcal{B}_R^{(s)} \to \mathcal{B} \in \mathsf{B}_-$. The proof of $\mathcal{A}_R^{(s)} \to \mathcal{A} \in \mathsf{B}_+$ is similar (take $\tau < 0$ and $\sigma > 0$). □

Proof of theorem 11.5. Let $\mathcal{A} \in \mathsf{B}_+$ and $g_s = \mathbf{W}_s[\mathcal{A}]$. The function $f_s \in \mathsf{W}_+^{(s,\tau)}$ for every $\tau < 0$. By theorem 10.3 there is a $\sigma > 0$ such that for every $\epsilon > 0$ that is small enough, there is an $M > 0$ such that

$$|\mathbf{D}^{-1}[f_s](z+w,\bar{z}-\bar{w})|\ |g_s(z)| \leq M\exp\{-\frac{|z|^2}{s+\sigma} + \frac{|w|^2}{s-\epsilon}\}.$$

By lemma 13.26, $<F_R^{(s)}, g_s> = \int_{\mathbb{C}} \overline{F_R^{(s)}(z)} g_s(z)dz$. By Fubini's theorem this is

$$\frac{1}{2\pi s}\int_{|w|\leq R}\Big(\int_{\mathbb{C}} \overline{\mathbf{D}^{-1}[f_s](z+w,\bar{z}-\bar{w})} g_s(z)dz\Big)e^{-|w|^2/(2s)}dw.$$

By lemma 11.2 this is

$$\frac{1}{2\pi s}\int_{|w|\leq R}\Big(\int_{\mathbb{C}} \overline{\mathbf{D}^{-1}[f_s](z+w,\bar{z})}\mathbf{D}^{-1}[g_s](z+w,\bar{z})dz\Big)e^{-|w|^2/(2s)}dw.$$

By (9.2) this is

$$\frac{1}{2\pi s}\iint_{|w-z|\leq R} <\mathcal{B}, \tilde{\mathcal{W}}_{z,\bar{w}}^{(s)}><\tilde{\mathcal{W}}_{z,\bar{w}}^{(s)}, \mathcal{A}>\, e^{-|z-w|^2/(2s)}dzdw.$$

This is $<\mathcal{B}_R^{(s)}, \mathcal{A}>$. Hence if $F_s = \mathbf{W}_s^{-*}[\mathcal{B}]$ and $g_s = \mathbf{W}_s[\mathcal{A}]$ then

$$\int_{\mathbb{C}} \overline{F_R^{(s)}(z)} g_s(z)dz = <\mathcal{B}_R^{(s)}, \mathcal{A}>.$$

Hence $<F_R^{(s)}, g_s> = <\mathcal{B}_R^{(s)}, \mathcal{A}>$. Hence $F_R^{(s)} = \mathbf{W}_s^{-*}[\mathcal{B}_R^{(s)}]$. By lemma 13.29, $\mathcal{B}_R^{(s)} \to \mathcal{B} \in \mathsf{B}_-$ as $R \to \infty$.

Acknowledgement. This work is part of the research programme of the 'Stichting voor Fundamenteel Onderzoek der Materie (FOM)', which is financially supported by the 'Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO)'.

References

1. G.S. Agarwal and E. Wolf. Calculus for functions of noncommuting operators and general phase-space methods in quantum mechanics II. *Physical Review D*, 2(10):2187–2205, 1970.
2. S.T. Ali. A geometrical property of POV-measures and systems of covariance. In H.D. Doebner, S.I. Andersson, and H.R.Petry, editors, *Differential geometric methods in mathematical physics*. springer, 1982.
3. S.T. Ali. Commutative systems of covariance and a generalization of Mackey's imprimitivity theorem. *Canad.Math.Bull.*, 27(4):390–197, 1984.
4. S.T. Ali, J.P. Antoine, and J.P. Gazeau. Continuous frames in Hilbert space. *Ann. Physics 222, no.1*, pages 1–37, 1993.
5. D.M. Appleby. Husimi transform of an operator product. Los Alamos e-print quant-ph/0001103, 2000.
6. V. Bargmann. On a Hilbert space of analytic functions and an associated integral transform, part I. *Comm. Pure Appl. Math.*, 14:187–214, 1961.
7. S.K. Berberian. *Notes on spectral theory.* Van Nostrand, 1966.
8. A. Böhm. *The rigged Hilbert space and quantum mechanics*. springer, 1978.
9. K.E. Cahill and R.J. Glauber. Density operators and quasiprobability distributions. *physical review*, 177(5):1882–1902, 1969.
10. J.B. Conway. *A course in functional analysis.* Springer, 1990.
11. J.B. Conway. *The theory of subnormal operators.* AMS, 1991.
12. N.G. de Bruijn. A theory of generalized functions, with applications to Wigner distribution and Weyl correspondence. *Nieuw Archief voor Wiskunde (3)*, 21:205–280, 1973.
13. J. de Graaf. A theory of generalized functions based on holomorphic semi-groups. *Nederl. Akad. Wetensch. Indag. Math.*, 46(2):155–187, 1984.
14. H.G. Feichtinger and W. Kozek. Quantization of tf lattice-invariant operators on elementary lca groups. In H.G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms.* Birkhauser, 1998.
15. I.M. Gelfand and N.Y. Vilenkin. *Generalized functions*, volume 4. Academic Press, 1964.
16. A.S. Holevo. *Probabilistic and statistical aspects of quantum theory.* North-Holland, 1982.
17. G. Kaiser. *Quantum physics, relativity, and complex spacetime.* North-Holland, 1990.
18. J.R. Klauder and E.C.G. Sudarshan. *Fundamentals of quantum optics.* Benjamin, 1968.
19. W. Kozek. Adaptation of Weyl-Heisenberg frames to underspread environments. In H.G.Feichtinger and T.Strohmer, editors, *Gabor Analysis and Algorithms.* Birkhauser, 1998.
20. T.T. Nielsen. *Bose algebras: the complex and real wave representations.* Springer, 1991.
21. R.R. Phelps. *Lectures on Choquet's theorem.* Van Nostrand, 1966.
22. H.H. Schaefer and M.P. Wolff. *Topological vector spaces.* Springer, 1999.
23. Béla Sz.-Nagy. *Prolongements des transformations de l'espace de Hilbert qui sortent de cet espace.* Akademiai Kiado, 1955.
24. S.J.L. van Eijndhoven and J. de Graaf. A fundamental approach to the generalized eigenvalue problem for self-adjoint operators. *J.Funct.Anal.*, 60(1):74–85, 1985.

25. S.J.L. van Eijndhoven and J. de Graaf. A measure theoretical Sobolev lemma. *J.Funct.Anal*, 60(1):95–101, 1985.
26. S.J.L. van Eijndhoven and J. de Graaf. *Trajectory spaces, generalized functions and unbounded operators*. Springer, 1985.
27. S.J.L. van Eijndhoven and J. de Graaf. *A mathematical introduction to Dirac's formalism*. North-Holland, 1986.
28. J. Weidmann. *Linear operators in Hilbert spaces*. Springer, 1980.