Skew-Hermitean representations of Lie algebras of vectorfields on the unit-sphere

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by

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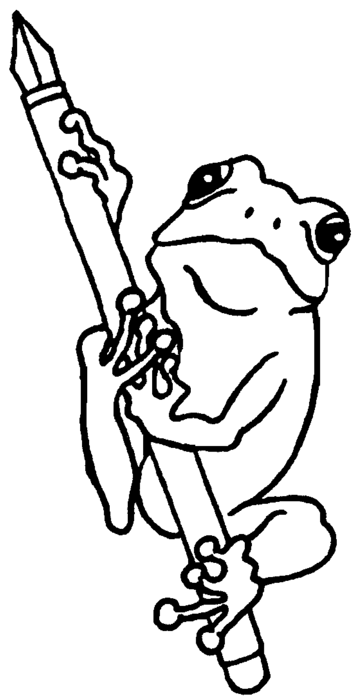

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1. Summary

On the unit sphere S^{q-1} in $\mathbb{R}^q$ we consider (infinite dimensional) Lie algebras of analytic vectorfields. By adding suitable "potentials" to these vectorfields the Lie algebras are represented by a Lie algebra of skew-Hermitean operators in $L_2(S^{q-1})$.
This leads to q-dimensional analogues of the algebras discussed in [KR].
We point at some connections with harmonic functions with representations of classical Lie groups and also at the regularity properties of the one-parameter groups generated by the representations of our Lie algebra.

2. Some function spaces and operators

By S^{q-1} we denote the unit sphere in $\mathbb{R}^q$. The solid open ball with radius R, centered at $\underline{0} \in \mathbb{R}^q$ is denoted by B_R. If $R = 1$ we denote B instead of B_1. On S^{q-1} we introduce the Hilbert space $L_2(S^{q-1})$ with inner product

$$(f,g) = \frac{1}{\omega_q} \int_{|\underline{\xi}|=1} f(\underline{\xi})\, \overline{g(\underline{\xi})}\, d\sigma, \quad \omega_q = \int_{|\underline{\xi}|=1} d\sigma$$

and norm

$$\|f\| = (f,f)^{\frac{1}{2}} .$$

It is well known that each $f \in L_2(S^{q-1})$ can be decomposed in spherical harmonics $f = \sum_{k=0}^{\infty} f_k$. Here f_k is the restriction to S^{q-1} of a harmonic homogeneous polynomial of degree k on $\mathbb{R}^q$. In short, cf. [Mü],

$$L_2(S^{q-1}) = \bigoplus_{k=0}^{\infty} HP_k(\mathbb{R}^q) .$$

Here $HP_k(\mathbb{R}^q)$ denotes the space of homogeneous harmonic polynomials of degree k on $\mathbb{R}^q$.

We are especially interested in dense subspaces of "very smooth" functions in $L_2(S^{q-1})$: Let $R > 1$,

$$HA(B_R) = \{f \mid \forall r,\ 0 < r < R,\ \sum_{k=0}^{\infty} r^{2k} \|f_k\|^2 < \infty\} .$$

Note that $HA(B_R)$ consists precisely of those functions f in $L_2(S^{q-1})$ which extend to a harmonic function f_e on B_R. Cf. [G].

If we want to emphasize that we consider the (unique) harmonic extension of a certain $f \in L_2(S^{q-1})$ as a function on B_R, we write f_e instead of f. From [G] we quote

2.1. THEOREM (harmonic multiplication/dilatation)

(i) Let $f, g \in HA(B_R)$, $R > 1$, then also the pointwise product fg, defined on S^{q-1}, has a harmonic extension to B_R. So $fg \in HA(B_R)$.

(ii) Let $f \in HA(B_R)$, $R > 1$. Let $A \in \mathbb{R}^{q\times q}$, $\|A\| = R_1 < R$. Define $g \in L_2(S^{q-1})$ by $g(\underline{\xi}) = f_e(A\underline{\xi})$, $|\underline{\xi}| = 1$. Then $g \in HA(B_\rho)$, with $\rho = R/R_1$.

Finally, we introduce the space of harmonic functions on the closed unit ball and the space of entire harmonic functions on $\mathbb{R}^q$:

$$HA(\bar{B}) = \bigcup_{R>1} HA(B_R), \quad HA(\mathbb{R}^q) = \bigcap_{R>1} HA(B_R) .$$

Next we turn to some classes of operators in $L_2(S^{q-1})$ which all have a geometric flavour.

2.2. DEFINITION

On the domain $HA(B_R)$, fixed $R > 1$, we define the operators L_A ($A \in \mathbb{R}^{q\times q}$, $\|A\| = R_1 < R$), Z_λ $(0 < \lambda < R)$, P_i $(1 \le i \le q)$, Q_i $(1 \le i \le q)$ and N as follows. Take $|\underline{\xi}| = 1$ and $|\underline{x}| < R$

$$(L_A f)(\underline{\xi}) = f_e(A\underline{\xi}) \quad , \quad (Z_\lambda f)(\underline{\xi}) = f_e(\lambda\underline{\xi}) ,$$

$$(P_i f)(\underline{\xi}) = \frac{\partial f_e}{\partial x_i}(\underline{\xi}) \quad , \quad (Q_i f)(\underline{\xi}) = \xi_i f(\underline{\xi})$$

$$(Nf)(\underline{\xi}) = (\underline{\xi}^T \nabla f_e)(\underline{\xi}) = \frac{\partial f_e}{\partial n}(\underline{\xi}) .$$

Note that

$$(N+\alpha)^{-1} f_e(\underline{x}) = \int_0^1 s^{\alpha-1} f_e(s\underline{x})\, ds \quad , \quad \mathrm{Re}\,\alpha > 0 .$$

The following properties follow from Theorem 2.1 and checking the harmonicity.

2.3. PROPERTIES

(i) $L_A(HA(B_R)) \subset HA(B_\rho)$ with $\rho = R/R_1$
$Z_\lambda(HA(B_R)) \subset HA(B_r)$ with $r = R/\lambda$.

(ii) $(Z_\lambda f)_e(\underline{x}) = f_e(\lambda \underline{x})$, $(P_i f)_e(\underline{x}) = \frac{\partial f_e}{\partial x_i}(\underline{x})$,
$(N f)_e(\underline{x}) = (\underline{x}^T \nabla f_e)(\underline{x})$.

(iii) $((N+\alpha)^{-1} f)_e(\underline{x}) = \int_0^1 s^{\alpha-1} f_e(s\underline{x})\, ds$.

(iv) $(Q_i f)_e(\underline{x}) = x_i f_e(\underline{x}) + (1 - |\underline{x}|^2)(2N+q)^{-1} \frac{\partial f}{\partial x_i}(\underline{x})$.

(v) If $f_k \in HP_k(\mathbb{R}^q)$ then $N f_k = k f_k$ and

$$(Q_j f_k)_e(\underline{x}) = \left[x_j f_k(\underline{x}) - \frac{1}{2j-2+q} |\underline{x}|^2 \frac{\partial f}{\partial x_i}(\underline{x}) \right] + \left[\frac{1}{2j-q+q} \frac{\partial f}{\partial x_i}(\underline{x}) \right] .$$

The first term $[\cdot] \in HP_{k+1}(\mathbb{R}^q)$, the second term $[\cdot] \in HP_{k-1}(\mathbb{R}^q)$.

Note that both $HA(\bar{B})$ and $HA(\mathbb{R}^q)$ are invariant dense subspaces for all mentioned operators. The operators $R_j = P_j - Q_j N$, $1 \le j \le q$, will frequently occur. They correspond to the tangent vector fields $\underline{r}_j(\underline{x}) = \underline{e}_j - x_j \underline{x} = \underline{e}_j - x_j \underline{n}$ on S^{q-1}.
From straightforward calculations, using the explicit expression for the harmonic extension $(Q_i f)_e$ we find the following algebraic relations.

2.4. THEOREM

- $P_k Q_l - Q_l P_k = \delta_{kl} I - 2\, Q_k (2N+q)^{-1} P_l$
- $P_k N - N P_k = P_k$
- $N Q_k - Q_k N = Q_k - 2\,(2N+q)^{-1} P_k$
- $Q_k P_l - Q_l P_k = R_k R_l - R_l R_k = Q_k R_l - Q_l R_k$

The operators P_j and R_j are not skew-hermitean in $L_2(S^{q-1})$. In the next theorem we construct skew-hermitean operators from the building blocks P_j, Q_j and N.

2.5. THEOREM

The operators $P_j - Q_j N + \alpha Q_j = R_j + \alpha Q_j$, with $\alpha = -\frac{q-1}{2}$, $1 \le j \le q$, are skew-hermitean on each domain $HA(B_R)$, $R > 1$.

Proof. [M]
The result follows from the equality

$$M = (P_j f, g) + (f, P_j g) = (Q_j N f, g) + (f, Q_j N g) + (q-1)(Q_j f, g).$$

The latter equality follows from Gauss' theorem, viz.

$$M = \int_{S^{q-1}} \left[\frac{\partial}{\partial x_j} (f\bar{g}) \right] d\sigma = \sum_{k=1}^{q} \int_{S^{q-1}} \frac{\partial f\bar{g}}{\partial x_j} x_k^2 \, d\sigma =$$

$$\sum_{k=1}^{q} \int_B \frac{\partial}{\partial x_k} \left[\frac{\partial}{\partial x_j} (f\bar{g}) \right] x_k \, d\underline{x} + q \int_B \frac{\partial}{\partial x_j} (f\bar{g}) \, d\underline{x} =$$

$$\sum_{k=1}^{q} \int_B \frac{\partial}{\partial x_j} \left[\frac{\partial f\bar{g}}{\partial x_k} x_k \right] d\underline{x} + (q-1) \int_B \frac{\partial f\bar{g}}{\partial x_j} \, d\underline{x} =$$

$$\int_{S^{q-1}} \left[\frac{\partial f}{\partial n} \bar{g} + f \frac{\partial \bar{g}}{\partial n} \right] x_j \, d\sigma + (q-1) \int_{S^{q-1}} f\bar{g} x_j \, d\sigma .$$

□

3. Lie algebras of tangent vector fields on S^{q-1} and their skew-adjoint representations in $L_2(S^{q-1})$

If we put a non-constant coefficient a in front of $(R_k + \alpha Q_k)$, the operator $a\,(R_k + \alpha Q_k)$ will no longer be skew-hermitean. However,

3.1. THEOREM

Let $a \in HA(B_R)$, $R > 1$. The operators $a R_k + \alpha a Q_k + \frac{1}{2}(R_k a)$, with $1 \le k \le q$, $\alpha = -\frac{q-1}{2}$ and $(R_k a) = \frac{\partial a}{\partial x_k} + x_k \frac{\partial a}{\partial n}$ are skew-Hermitean on the invariant domain $HA(B_R) \subset L_2(S^{q-1})$.

Further, for the commutator of two such operators, we have

$$[a R_k + \alpha a Q_k + \tfrac{1}{2}(R_k a),\; b R_l + \alpha b Q_l - \tfrac{1}{2}(R_l b)] =$$

$$= -(c R_k + \alpha c Q_k + \tfrac{1}{2}(R_k c)) + (d R_l + \alpha d Q_l - \tfrac{1}{2}(R_l d))$$

with

$$c = b(R_l a) + x_l a \quad \text{and} \quad d = a(R_k b) + x_k b .$$

Proof.
For the first part of the theorem we apply a general trick: Let A, B and S be operators on a

common invariant domain W in a Hilbert space H.
Suppose that on W one has $A^* = A$, $B^* = -B$, $S^* = S$ and $AB = S + BA$, then

$$(AB - \tfrac{1}{2} S)^* + (AB - \tfrac{1}{2} S) = 0 \quad \text{on} \quad W .$$

In our case take A = multiplication by a in $L_2(S^{q-1})$, take $B = R_k + \alpha Q_k$. Let $u \in HA(B_R)$. Then, since R_k corresponds to a tangent vector field,

$$a(P_k - Q_k N + \alpha Q_k) u - (P_k - Q_k N + \alpha Q_k) au =$$

$$-(P_k a - Q_k N a) u = -(R_k a) u .$$

By taking S = multiplication by $-(R_k a)$ the first result follows. The second result follows from a straightforward calculation. □

Remark: If a and b extend to harmonic polynomials then also c and d extend to harmonic polynomials.
Now we are ready to present the main definition of this paper.

3.2. DEFINITION

Let $\underline{a} = \sum_{k=1}^{q} a^k \underline{e}_k = a^k \underline{e}_k$, $a^k \in HA(B_R)$, be a vector field (not necessarily tangent) on S^{q-1}.
With $\underline{a}$ we associate the operators ($\alpha = -\frac{q-1}{2}$)

$$G(\underline{a}) = a^k R_k + \alpha a^k Q_k + \tfrac{1}{2}(R_k a^k) = \underline{a}^T \underline{R} + \alpha \underline{a}^T \underline{Q} + \tfrac{1}{2}(\underline{R}^T \underline{a}) .$$

Note that $\underline{a}^T \underline{R}$ corresponds to the vector field $a^k(\underline{e}_k - x_k \underline{x}) = \underline{a} - (\underline{a}^T \underline{n}) \underline{n}$ which is tangent to S^{q-1}.

3.3. THEOREM

(i) The operator $G(\underline{a})$ maps $HA(B_R) \subset L_2(S^{q-1})$ into itself.

(ii) If $\underline{b} = b^j \underline{e}_j$ is another vector field with coefficients $b^j \in HA(B_R)$ then we have for the commutator

$$[G(\underline{a}), G(\underline{b})] = G(\underline{c})$$

with

$$\underline{c} = (\underline{R}\, \underline{b}^T)^T \underline{a} - (\underline{R}\, \underline{a}^T)^T \underline{b} + (\underline{b}\, \underline{a}^T - \underline{a}\, \underline{b}^T) \underline{x} .$$

(iii) The Lie algebra of tangent vector fields to S^{q-1} with coefficients in $HA(B_R)$ has $\underline{a} \mapsto G(\underline{a})$ as a skew-Hermitean representation in $L_2(S^{q-1})$.

(iv) A sub Lie algebra of (iii) is obtained if one takes the coefficients in the spherical harmonics. Again $\underline{a} \mapsto G(\underline{a})$ is a skew-Hermitean representation.

3.4. SPECIAL CASES

I. $\underline{a} = K\underline{x}$, $\underline{b} = L\underline{x}$, $K, L \in \mathbb{R}^{q\times q}$, $K^T = -K$, $L^T = -L$.
Substitution leads to $G(\underline{a}) = \underline{x}^T K \underline{R} = \underline{x}^T K^T \underline{P}$ which is a "moment of momentum" operator. (We used $R\,\underline{x}^T = \underline{T} - \underline{x}\,\underline{x}^T$ and $\underline{x}^T K\,\underline{x} = 0$.)
For the commutator of $G(\underline{a})$ and $G(\underline{b})$ we find $G(\underline{c})$ with $\underline{c} = (KL - LK)\underline{x}$.
Operators of this type establish a Lie-algebra which corresponds to $so(\mathbb{R}, q)$.

II. $\underline{a}$ = constant $\underline{b}$ = constant.
$[G(\underline{a}), G(\underline{b})] = G(\underline{c})$ with $\underline{c} = (\underline{b}\,\underline{a}^T - \underline{a}\,\underline{b}^T)\underline{x}$.
Note that $(\underline{b}\,\underline{a}^T - \underline{a}\,\underline{b}^T) \in so(\mathbb{R}, q)$.
Operators of the form $G(\underline{a} + K\underline{x})$, $\underline{a} \in \mathbb{R}^q$, $K \in so(\mathbb{R}, q)$ establish a Lie algebra.
A $(q+1)\times(q+1)$-matrix representation is $\begin{bmatrix} K & \underline{a} \\ \underline{a}^T & 0 \end{bmatrix}$.
This is an extension of $so(\mathbb{R}, q)$. Cf. also III.

III. $\underline{a} = K\underline{x}$ $\underline{b} = L\underline{x}$ $K, L \in \mathbb{R}^{q\times q}$.

$$G(\underline{a}) = \underline{x}^T K^T(\nabla - \underline{x}\,\underline{x}^T\nabla) + \alpha\underline{x}^T K^T \underline{x} + \tfrac{1}{2}(\nabla - \underline{x}\,\underline{x}^T\nabla)^T K\,\underline{x}$$

$$= \underline{x}^T K^T (I - \underline{x}\,\underline{x}^T)\nabla + \frac{q-1}{2}\,\underline{x}^T K\,\underline{x} - \tfrac{1}{2}\underline{x}^T K\,\underline{x}$$

$$G(\underline{a})\,u = \underline{x}^T K^T \nabla u - (\underline{x}^T K\,\underline{x})\underline{x}^T\nabla u - \frac{q}{2}(\underline{x}^T K\,\underline{x}) + \tfrac{1}{2}\,tr\,K\,.$$

So, we have another extension of $so(\mathbb{R}, q)$ which corresponds both to $sl(\mathbb{R}, q)$ and to Martens' representation of $SL(\mathbb{R}, q)$, cf. [M2]. The extensions of II and III cannot be combined into one single small Lie-algebra in a simple way. Note that the matrices $\left\{ \begin{bmatrix} A & \underline{a} \\ \underline{a}^T & 0 \end{bmatrix} \right\}$ establish a Lie-algebra only if $A^T = -A$.
Note also that, taking $\underline{a}$ = constant and $\underline{b} = L\underline{x}$ we find a commutator with

$$\underline{c} = ((\nabla - \underline{x}\,\underline{x}^T\nabla)\underline{x}^T L^T)^T \underline{a} + L\,\underline{x}\,\underline{a}^T\underline{x} - \underline{a}\,\underline{x}^T L^T \underline{x}$$

$$= L\,\underline{a} - \underline{a}\,\underline{x}^T L^T \underline{x}\,.$$

Only if $L^T = -L$ this reduces to $L\,\underline{a}$.

IV. $\underline{a} = f(\underline{x})\underline{x}$. Then $G(\underline{a}) = 0$.

FINAL REMARKS

- The spectra of the operators $G(\underline{a})$ can be very different from each other. For example, the moment of momentum operators (case I) have discrete spectra while the spectral properties of the $R_j + \alpha Q_j$ are the same as those of d/dx in $L_2(\mathbb{R})$.

- In order to study Lie groups of operators generated by the $G(\underline{a})$ one needs a Hille-Yosida-Ouchi-theorem adapted for inductive limits of Hilbert spaces. Cf. [L]. One expects the analyticity domain for the whole Lie group to be $HA(\bar{B})$.

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