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Subexponential asymptotics of hybrid fluid and ruin models

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Abstract

This paper is concerned with the tail asymptotics of the supremum of the superposition of two stochastic processes, of which at least one has subexponential characteristics. Canonical examples of such processes include Levy processes and random walks with subexponential jumps, On-Off processes with subexponential activity periods, and Gaussian processes exhibiting long-range dependence; these processes routinely arise in queueing and risk theory. In contrast to previous work, we allow combinations of any of the above processes as input in the same model.

We give general necessary as well as sufficient conditions for a so-called reduced-load equivalence. In this case, one of the two processes can be replaced by its mean. It is shown that this property holds whenever certain structural properties are satisfied.

If the reduced-load equivalence does not hold, then the asymptotics are qualitatively different. This is illustrated by a number of examples, which show that the well-behaved process may contribute to the asymptotics by its moderate deviations, large deviations, or oscillatory behavior.

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1 Introduction

Consider two independent stochastic processes $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$. This paper is concerned with the tail asymptotics of the supremum of the superposition of X and Y , i.e., we are interested in the behavior of

$$\mathbb{P}\{\sup_{t \geq 0} [X(t) + Y(t) - ct] > u\}, \quad u \rightarrow \infty. \quad (1.1)$$

This probability may be interpreted as an overflow probability in queueing theory, but also as a ruin probability. Motivated by applications in both queueing and ruin problems, we are especially interested in the case where at least one of the processes X and Y has *subexponential* characteristics. Typical examples of such processes include:

- Processes with subexponential jumps, like the input process of an ordinary single-server queue or the standard risk model. See Asmussen [5] for an excellent textbook treatment.
- Processes with gradual input, where at least one of the driving random variables is heavy-tailed. A celebrated queueing example is the On-Off model, but there are also models in insurance risk with this property, like the Björk-Grandell model (see Section 5 of Asmussen *et al.* [6]).
- Processes where none of the driving random variables are heavy-tailed, but where subexponentiality is a consequence of the strong dependence structure. Specifically, one could think of Gaussian processes like fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

The above list is, of course, not exhaustive. For instance, heavy tails arise in a completely different way in stochastic difference equations, see e.g. Kesten [30], Goldie [23], and Kalashnikov & Norberg [29]. There are also processes exhibiting both heavy tails and long-range dependence, see e.g. Mikosch & Samorodnitsky [35] for related recent work.

Although the sample paths of the above processes look qualitatively different, they share an important property: The tail of their supremum distribution is subexponential. Key references for the three respective cases are Veraverbeke [45], Jelenković & Lazar [27], and Hüsler & Piterbarg [26].

This brings us to the motivation of the present paper, which is in fact two-fold. First of all, in applications like insurance or telecommunications, models can arise that have two qualitatively different input streams. A well-known example in insurance is the class of *perturbed risk models*, which is reviewed in Schmidli [43]. Another example comes from telecommunications, where several traffic streams, each having a completely different level of aggregation, may share a buffer. This may be modeled by the superposition of a Gaussian process and an On-Off process. In particular, Gaussian processes can arise as the (heavy-traffic) limit of a large number of On-Off sources, see e.g. Kurtz [32] and Dębicki & Palmowski [15]. A similar phenomenon arises in modeling Internet traffic. Files are often distinguished by their size, which can show extreme variability, leading to a distinction between *mice* and *elephants*. These two traffic classes may be modeled by Gaussian processes and On-Off processes.

From a methodological perspective, this paper may be viewed as an attempt to combine the analysis of different classes of processes into a single framework. As indicated above, in

isolation, the above-mentioned processes share the property that the tail of their supremum distribution can be subexponential. This fact, with some further structural assumptions on the processes involved, is sometimes sufficient to reach conclusions on the tail asymptotics of the superposition, see Section 3. As we shall see, in other cases, the situation is more complicated.

There exists a vast literature on the tail asymptotics of the above-mentioned processes in isolation. The tail asymptotics of the supremum of random walks and queues have been analyzed by, among others, Pakes [38], Veraverbeke [45], Korshunov [31], and Asmussen *et al.* [6]. For work on On-Off models, see e.g. Agrawal *et al.* [2], Boxma [12], Jelenković & Lazar [27], Jelenković *et al.* [28], Likhanov & Mazumdar [33], Rolski *et al.* [41], Zwart *et al.* [46], and Zwart [47]. The exact tail asymptotics of the supremum of a class of self-similar Gaussian processes have been derived by Hüsler & Piterbarg [26]. Related results may be found in, for example, Dębicki [13, 14], Duffield & O’Connell [18], Narayan [36], and Norros [37]. A survey is provided in Dębicki & Rolski [16].

We now turn to a discussion of the results of the present paper. The first question we investigate is under what conditions a so-called *reduced-load equivalence* (RLE) holds, i.e.,

$$\mathbb{P}\{\sup_{t \geq 0} [X(t) + Y(t) - ct] > u\} \sim \mathbb{P}\{\sup_{t \geq 0} [Y(t) - ct] > u\}, \quad (1.2)$$

or a similar equivalence with the role of X and Y interchanged. (The symbol \sim denotes that the ratio of both sides tends to 1 as $u \rightarrow \infty$.) We give general necessary as well as sufficient conditions for the above equivalence in Section 3, extending recent results of Agrawal *et al.* [2] and Jelenković *et al.* [28]. It is shown that the ‘threshold point’ of Weibullian tails with exponent $1/2$ found in these references as well as in Asmussen *et al.* [4] does not always apply. In fact, the main theme of Section 3 is that this threshold is determined by the order of growth of (i) the variance of $X(u)$, and (ii) the most probable time for the process $\{Y(t) - ct\}$ to reach a large level u . The above-mentioned threshold applies only if both quantities grow linearly with u .

We apply this result to various special cases. In particular, we may take both processes Gaussian. In that case, the process $X(t) + Y(t)$ is also Gaussian, but other properties, like self-similarity, may be lost. Using this procedure, we obtain a modest extension of the exact asymptotics for a class of self-similar Gaussian processes which were obtained in [26].

If the RLE does not hold, then a natural question is what the form of the asymptotics in (1.1) might be. This is the second subject of the paper: In Sections 5–7, we give three examples, which show that X can contribute to the asymptotics through its moderate deviations (Section 5), large deviations (Section 6), or oscillatory behavior (Section 7). In all examples, X is a centered Gaussian process and Y is an On-Off process with peak rate r .

In Section 5, we take $r > c$, but assume that the tail of V_Y^c is not heavy enough for (1.2) to hold. In this case, the asymptotics of V_{X+Y}^c are shown to be rather complicated. This section relies on recent work of Foss & Korshunov [20].

In Sections 6 and 7, we drop the assumption $r > c$, which implies that the right-hand side of (1.2) is 0. In this scenario, the typical way for the process $\{X(t) + Y(t) - ct\}$ to reach a large value is fundamentally different, depending on whether $r = c$ or $r < c$ (needless to say, they both differ from the case $r > c$). We obtain exact asymptotics for (1.1) in both

cases.

The remainder of the paper is organized as follows. In Section 2 we introduce some notation and review some background results for the classes of processes we consider. The main contributions of the paper may be found in Sections 3–7. In Section 3 we formulate the conditions for the RLE to hold. These conditions are applied to the above-mentioned classes of processes in Section 4. The cases indicated above where the RLE does not hold, are examined in Sections 5, 6 and 7. Additional proofs are gathered in Section 8. The paper concludes with Section 9.

2 Model description

In this section we introduce some notation and review some background results for the various processes that we consider. For convenience, we adopt the terminology of queues, although one should keep in mind the connections with risk theory (the overflow probabilities to be studied may be interpreted as ruin probabilities, see for example [5, 6]).

2.1 Notation

We consider a fluid queue with infinite buffer size and constant drain rate c fed by two independent traffic processes X and Y . Denote by $X(t)$ and $Y(t)$ the amount of traffic generated by the two processes during the time interval $[-t, 0]$. If $\mathbb{E}\{X(1) + Y(1)\} < c$, then the random variable

$$V_{X+Y}^c = \sup_{t \geq 0} [X(t) + Y(t) - ct]$$

is finite a.s. and can be identified with the buffer content in steady state.

We will frequently make comparisons with the buffer content for each of the two processes in isolation. For $c > \mathbb{E}\{X(1)\}$ and $D \subseteq [0, \infty)$, define $V_X^c(D) := \sup_{t \in D} [X(t) - ct]$, and let $V_X^c := V_X^c([0, \infty))$ be a random variable representing the stationary workload in a buffer with drain rate c fed by the process $X(t)$ only. Similarly, for $c > \mathbb{E}\{Y(1)\}$ and $D \subseteq [0, \infty)$, define $V_Y^c(D) := \sup_{t \in D} [Y(t) - ct]$, and let $V_Y^c := V_Y^c([0, \infty))$ be a random variable representing the stationary workload in a buffer with drain rate c fed by the process $Y(t)$ only.

For any two real functions $f(\cdot)$ and $g(\cdot)$, we use the notational convention $f(u) \sim g(u)$ to denote that $f(u) = g(u)(1 + o(1))$ as $u \rightarrow \infty$, i.e., $\lim_{u \rightarrow \infty} f(u)/g(u) = 1$. We further write $f(u) \lesssim (\lesssim)g(u)$ to indicate that $\limsup_{u \rightarrow \infty} f(u)/g(u) \leq (<)1$. Analogously, $f(u) \gtrsim (\gtrsim)g(u)$ indicates that $\liminf_{u \rightarrow \infty} f(u)/g(u) \geq (>)1$.

Throughout the paper, we use various classes of distributions. In particular, we consider the class \mathcal{L} of long-tailed distributions, the class \mathcal{S} of subexponential distributions, and the class \mathcal{R} of regularly varying distributions. We also consider the subclass \mathcal{S}^* of \mathcal{S} . For definitions and further background on these classes, we refer to Embrechts *et al.* [19].

2.2 Gaussian processes

Let $X(t)$ be a centered Gaussian process with stationary increments, a.s. continuous sample paths, $X(0) = 0$ a.s., and variance function $\text{Var}\{X(t)\} = \sigma_X^2(t)$. We often impose (a

subset of) the following conditions:

C1 $\sigma_X^2(t) \in C([0, \infty))$ is increasing;

C2 $\sigma_X^2(t)$ is regularly varying at 0 with index $\beta \in (0, 2]$ and $\sigma_X^2(t)$ is regularly varying at ∞ with index $\alpha \in (0, 2)$;

C3 $\sigma_X^2(t)$ is differentiable and its derivative is regularly varying with index $\alpha - 1$.

We now state some known results on the tail behavior of V_X^c . The logarithmic asymptotics for V_X^c follow immediately from Dębicki [13]. Note that the technical condition in [13] can easily be verified by invoking the ergodic theorem.

Proposition 2.1 *If $X(t)$ satisfies conditions C1-C2 and $c > 0$, then*

(i) $\log(\mathbb{P}\{V_X^c > u\}) \sim -M_{X,c}(u)$, where $M_{X,c}(u) := \min_{t \geq 0} \frac{(u+cAt)^2}{2\sigma_X^2(t)}$;

(ii) $M_{X,c}(u) \sim 2 \frac{(2-\alpha)^{\alpha-2}}{\alpha^\alpha} c^\alpha \frac{u^2}{\sigma_X^2(u)}$;

(iii) $M_{X,c}(u) \sim \frac{(u+ct_u)^2}{2\sigma_X^2(t_u)}$ for $t_u = \frac{1}{c} \frac{\alpha}{2-\alpha} u$.

Exact asymptotics for V_X^c are known in some special cases only, namely in the case of fractional Brownian motion (FBM) [26] and Gaussian integrated (GI) input [14].

In particular, suppose that $X(t) = B_H(t)$, where $B_H(t)$ is a fractional Brownian motion (FBM) with Hurst parameter $H \in (0, 1)$, i.e., a centered Gaussian process with stationary increments, a.s. continuous sample paths, and variance function $\sigma_{B_H}^2(t) = t^{2H}$ for $H \in (0, 1)$. The following result is taken from [26].

Theorem 2.1 *As $u \rightarrow \infty$,*

$$\mathbb{P}\{V_{B_H}^c > u\} \sim \frac{\mathcal{H}_{B_H}}{2^{1/2H}} c^{1-2H} \left(\frac{1-H}{H}\right)^{\frac{3}{2}-2H} (1-H)^{1-1/H} u^{\frac{(1-H)(1-2H)}{H}} \times \exp\left(-\frac{c^{2H}}{2} \frac{(1-H)^{2H-2}}{H^{2H}} u^{2-2H}\right).$$

Here, \mathcal{H}_{B_H} is *Pickands constant*. We note that [26] considers a somewhat more general case, allowing for example self-similar processes non-stationary increments.

2.3 On-Off processes

Throughout the paper, we will frequently assume that $Y(t), t \geq 0$ is an (integrated) On-Off process with stationary increments. For future use, we give an explicit construction of such a process, following Heath *et al.* [25]: Let $\{T_{on,m}, m \geq 0\}$ be a sequence of i.i.d. random variables representing the On periods of the source. Similarly, let $\{T_{off,m}, m \geq 1\}$ be the Off periods. Define three additional random variables $T_{on,0}^r, T_{off,0}^r$, and I such that $T_{on,0}^r \stackrel{d}{=} T_{on}^r, T_{off,0}^r \stackrel{d}{=} T_{off}^r$, and

$$p = \mathbb{P}\{I = 1\} = \frac{\mathbb{E}\{T_{on,1}\}}{\mathbb{E}\{T_{on,1}\} + \mathbb{E}\{T_{off,1}\}} = 1 - \mathbb{P}\{I = 0\}.$$

To obtain a stationary alternating renewal process, we define the delay random variable D_0 by

$$D_0 = IT_{on,0}^r + (1-I)(T_{off,0}^r + T_{on,0}^r).$$

Then the delayed renewal sequence

$$\{Z_n, n \geq 0\} = \{D_0, D_0 + \sum_{m=1}^n (T_{off,m} + T_{on,m}), n \geq 1\}$$

is stationary.

Next, we define the process $\{J(t), t \geq 0\}$ as follows. $J(t)$ is the indicator of the event that the source is On at time t . Formally, we have

$$J(t) = I1_{\{t < T_{on,0}^r\}} + (1 - I)1_{\{T_{off,0}^r \leq t < T_{off,0}^r + T_{on,0}\}} + \sum_{n=0}^{\infty} 1_{\{Z_n + T_{off,n+1} \leq t < Z_{n+1}\}}.$$

The On-Off process $\{J(t), t \geq 0\}$ is strictly stationary, see Theorem 2.1 of [25]. The process $\{Y(t), t \geq 0\}$ is then defined by

$$Y(t) := r \int_0^t J(s) ds.$$

Note that the mean rate of $Y(t)$ is given by $\rho = pr$.

The next theorem, due to Jelenković & Lazar [27], provides the tail asymptotics for V_Y^c .

Theorem 2.2 *If $T_{on}^r \in \mathcal{S}$ and $\rho < c < r$, then*

$$\mathbb{P}\{V_Y^c > u\} \sim (1 - p) \frac{\rho}{c - \rho} \mathbb{P}\{T_{on}^r > \frac{u}{r - c}\}.$$

2.4 Instantaneous input

A similar result as in the previous subsection holds for the $GI/G/1$ queue where the input is instantaneous instead of gradual. Let $Y(t)$ be (a stationary version) of the input process of a $GI/G/1$ queue with generic service time B and traffic intensity ρ . Then the following result holds; see e.g. Pakes [38] and Veraverbeke [45].

Theorem 2.3 *If $B^r \in \mathcal{S}$ and $\rho < c$, then*

$$\mathbb{P}\{V_Y^c > u\} \sim \frac{\rho}{c - \rho} \mathbb{P}\{B^r > u\}.$$

3 Reduced-load equivalence I: General results

In this section, we investigate under what conditions a reduced-load equivalence (RLE) holds, as explained in the Introduction. Throughout the section, Y will play the role of the dominant source. This question has been analyzed before in a number of specific cases, in particular when Y is an On-Off source; see e.g. [2] and [28]. In the latter paper, the case is considered where X has a regenerative structure (covering the case of compound Poisson processes and On-Off processes) and Y is some process such that the tail of V_Y^c is *square-root insensitive*, i.e.,

$$\mathbb{P}\{V_Y^c > u - \sqrt{u}\} \sim \mathbb{P}\{V_Y^c > u\}. \quad (3.1)$$

A similar assumption is used in a related problem investigated in [4]. Note that we use terminology from [28]. It is also not uncommon to phrase (3.1) as $\mathbb{P}\{V_Y^c > \cdot\}$ is flat for \sqrt{x} , cf. [8].

It turns out that the condition (3.1) is implicitly based on the following two assumptions:

- The (most probable) time for the process $\{Y(t) - ct\}$ to reach a large level u is linear in u .
- The variance of the process $\{X(u)\}$ is linear in u . Thus, the expected deviation of $X(u)$ from its mean is $O(\sqrt{u})$.

The special cases examined before in [2, 28] show that these assumptions are satisfied if both X and Y are On-Off processes. However, in the more general setting of the present paper, these properties may not hold. If X is Gaussian, the variance function may not be linear in u . Furthermore, the time to overflow of the process $\{Y(t) - ct\}$ may be non-linear in u as well. An example of sublinear time to overflow occurs if Y is compound Poisson; see Asmussen & Klüppelberg [3]. An example of superlinear time to overflow is provided in Section 7 of the present paper. If the most probable time to overflow is of the order $f(u)$, then the extended form of (3.1) is

$$\mathbb{P}\{V_Y^c > u - \sigma_X(f(u))\} \sim \mathbb{P}\{V_Y^c > u\}. \quad (3.2)$$

In words, the tail distribution of V_Y^c should be flat for $\sigma_X(f(u))$.

We now state and prove two theorems. The first theorem gives sufficient conditions for a reduced-load equivalence (RLE) to hold. The second theorem states necessary conditions. In order to demonstrate the importance of the covariance structure of X , we take X to be a Gaussian process. We note however that this assumption is not essential. The only place where this assumption is used is in Lemma 3.1 below. A similar result for a large class of regenerative processes may be found in [28].

Theorem 3.1 (*Sufficient conditions for RLE*)

Let $X(t)$ be a centered Gaussian process with stationary increments satisfying conditions **C1-C2** and let $c > 0$. Assume that there exists an increasing positive function f and some $\epsilon > 0$ such that $\sigma_X(f(u)) = o(u)$, and

$$\lim_{l \rightarrow \infty} \limsup_{u \rightarrow \infty} \frac{\mathbb{P}\{V_Y^{c-\epsilon}[[lf(u), \infty]] > u\}}{\mathbb{P}\{V_Y^c > u\}} = 0. \quad (3.3)$$

Furthermore, assume that

$$\mathbb{P}\{V_X^\epsilon > u\} = o(\mathbb{P}\{V_Y^c > u\}), \quad (3.4)$$

and

$$\mathbb{P}\{V_Y^c > u - \sigma_X(f(u))\} \sim \mathbb{P}\{V_Y^c > u\}. \quad (3.5)$$

Then,

$$\mathbb{P}\{V_{X+Y}^c > u\} \sim \mathbb{P}\{V_Y^c > u\}. \quad (3.6)$$

Our main tool to control $X(t)$ is the following inequality, which is proved in Subsection 8.1.

Lemma 3.1 *If $X(t)$ has stationary increments and satisfies conditions **C1-C2**, then there exist constants $K < \infty$, $\kappa > 0$, and $q > 0$ such that for every x and t ,*

$$\mathbb{P}\{\sup_{s \leq t} X(s) > x\} \leq K e^{-\kappa \left(\frac{x}{\sigma_X(t)} - q\right)^2}.$$

We also need the following lemma, whose proof may be found in Subsection 8.2.

Lemma 3.2 *Let W be some non-negative random variable with $Q(u) = -\log \mathbb{P}\{W > u\}$. Then, for a given $l > 0$, $\mathbb{P}\{W > u - \sigma_X(lu)\} \sim \mathbb{P}\{W > u\}$ implies $Q(u) = o(u/\sigma_X(lu))$. Furthermore, the following statements are equivalent:*

- (i) $\mathbb{P}\{W > u - \sigma_X(f(u))\} \sim \mathbb{P}\{W > u\}$;
- (ii) $\mathbb{P}\{W > u - \sigma_X(lf(u))\} \sim \mathbb{P}\{W > u\}$;
- (iii) $\mathbb{P}\{W > u - k\sigma_X(lf(u))\} \sim \mathbb{P}\{W > u\}$, $k > 0$;
- (iv) $\mathbb{P}\{W > u - Z\sigma_X(lf(u)) \mid Z > k\} \sim \mathbb{P}\{W > u\}$, $k > 0$,

with Z a random variable which has density $\kappa_1 z e^{-\kappa_2 z^2}$.

We now provide a proof of Theorem 3.1. The proof is an extension of Theorem 2 in [28] (with Lemma 3.1 playing the role of Proposition 1 of [28]).

Proof of Theorem 3.1

The proof consists of a lower and an upper bound. We will repeatedly use the equivalence between (i), (ii), and (iii) in Lemma 3.2 without mention.

We start with the upper bound. Write

$$\mathbb{P}\{V_{X+Y}^c > u\} \leq \mathbb{P}\{V_{X+Y}^c[0, lf(u)] > u\} + \mathbb{P}\{V_{X+Y}^c[lf(u), \infty] > u\}.$$

Our first step is to show that the second term can be neglected as $u, l \rightarrow \infty$. Note that the assumption (3.3) implies that for each $\eta > 0$ there exist l_η, u_η such that, if $l \geq l_\eta$ and $u \geq u_\eta$, then

$$\mathbb{P}\{V_Y^{c-\epsilon}[lf(u), \infty] > u\} \leq \eta \mathbb{P}\{V_Y^c > u\}. \quad (3.7)$$

Now, write

$$\begin{aligned} & \mathbb{P}\{V_{X+Y}^c[lf(u), \infty] > u\} \\ & \leq \mathbb{P}\{V_Y^{c-\epsilon}[lf(u), \infty] + V_X^\epsilon > u\} \\ & \leq \mathbb{P}\{V_Y^{c-\epsilon}[lf(u - V_X^\epsilon), \infty] > u - V_X^\epsilon\} \\ & \leq \mathbb{P}\{V_X^\epsilon > u - u_\eta\} + \int_0^{u-u_\eta} \mathbb{P}\{V_Y^{c-\epsilon}[lf(u-z), \infty] > u-z\} d\mathbb{P}\{V_X^\epsilon \leq z\} \\ & \leq \mathbb{P}\{V_X^\epsilon > u - u_\eta\} + \eta \int_0^{u-u_\eta} \mathbb{P}\{V_Y^c > u-z\} d\mathbb{P}\{V_X^\epsilon \leq z\} \\ & \leq \mathbb{P}\{V_X^\epsilon > u - u_\eta\} + \eta \mathbb{P}\{V_Y^c + V_X^\epsilon > u\} \\ & = o(\mathbb{P}\{V_Y^c > u - u_\eta\}) + \eta \mathbb{P}\{V_Y^c + V_X^\epsilon > u\} \\ & \sim \eta \mathbb{P}\{V_Y^c > u\}, \end{aligned}$$

where the last steps follows from Assumption (3.4) and the fact that $V_Y^c \in \mathcal{L}$. This holds for any $\eta > 0$. Hence, we conclude

$$\lim_{l \rightarrow \infty} \limsup_{u \rightarrow \infty} \frac{\mathbb{P}\{V_{X+Y}^c[lf(u), \infty] > u\}}{\mathbb{P}\{V_Y^c > u\}} = 0. \quad (3.8)$$

Thus, we can focus on analyzing $\mathbb{P}\{V_{X+Y}^c[0, lf(u)]\}$ for some large l . Using sample path arguments, we obtain

$$\begin{aligned} & \mathbb{P}\{V_{X+Y}^c[0, lf(u)] > u\} \\ &= \mathbb{P}\left\{\sup_{0 \leq t \leq lf(u)} [X(t) + Y(t) - ct] > u\right\} \\ &\leq \mathbb{P}\{V_Y^c > u - k\sigma_Y(lf(u))\} + \mathbb{P}\{V_Y^c \leq u - k\sigma_Y(lf(u)); V_Y^c + V_X^0[0, lf(u)] > u\}. \end{aligned}$$

We need to show that the second term can be asymptotically neglected. Write

$$\begin{aligned} & \mathbb{P}\{V_Y^c \leq u - k\sigma_X(lf(u)); V_Y^c + V_X^0[0, lf(u)] > u\} \\ &= \int_0^{u - k\sigma_X(lf(u))} \mathbb{P}\{V_X^0[0, lf(u)] > u - y\} d\mathbb{P}\{V_Y^c \leq y\} \\ &\leq K \int_0^{u - k\sigma_X(lf(u))} e^{-\kappa\left(\frac{u-y}{\sigma_X(lf(u))} - q\right)^2} d\mathbb{P}\{V_Y^c \leq y\}. \end{aligned}$$

In the last step, we applied Lemma 3.1. Next, use integration by parts to get the upper bound

$$K e^{-\kappa \frac{u^2}{\sigma_X^2(lf(u))}} + 2K\kappa \int_0^{u - k\sigma_X(lf(u))} \mathbb{P}\{V_Y^c > y\} \frac{u-y}{\sigma_X^2(lf(u))} e^{-\kappa\left(\frac{u-y}{\sigma_X(lf(u))} - q\right)^2} dy.$$

The first term can be neglected in view of the first part of Lemma 3.2 (or alternatively in view of Assumption (3.4) combined with Proposition 2.1). Substituting $z = (u - y)/\sigma_X(lf(u))$, the second term can be rewritten as

$$\begin{aligned} & 2K\kappa \int_k^{\frac{u}{\sigma_X(lf(u))}} z e^{-\kappa z^2} \mathbb{P}\{V_Y^c > u - \sigma_X(lf(u))z\} dz \\ &\leq K_1 \mathbb{P}\{V_Y^c > u - \sigma_X(lf(u))Z; Z > k\}, \end{aligned}$$

where Z is a random variable with density proportional to $ze^{-\kappa z^2}$ and K_1 is some constant. Using Lemma 3.2 (iv), we conclude that

$$\limsup_{u \rightarrow \infty} \frac{\mathbb{P}\{V_Y^c \leq u - k\sigma_X(u); V_Y^c + V_X^0[0, lf(u)] > u\}}{\mathbb{P}\{V_Y^c > u\}} \leq K_1 \mathbb{P}\{Z > k\}.$$

The proof of the upper bound now follows by letting $k \rightarrow \infty$.

We now turn to the lower bound. Using properties of the sup operator, we obtain

$$\mathbb{P}\{V_{X+Y}^c > u\} \geq \mathbb{P}\{V_Y^c[0, lf(u)] - V_{-X}^0[0, lf(u)] > u\}.$$

Hence, for some $k > l$,

$$\mathbb{P}\{V_{X+Y}^c > u\} \geq \mathbb{P}\{V_Y^c[0, lf(u)] > u + k\sigma_X(f(u))\} \mathbb{P}\{V_{-X}^0[0, lf(u)] \leq k\sigma_X(f(u))\}.$$

Now, write

$$\begin{aligned} \frac{\mathbb{P}\{V_{X+Y}^c > u\}}{\mathbb{P}\{V_Y^c > u\}} &\geq \frac{\mathbb{P}\{V_Y^c[0, lf(u)] > u + k\sigma_X(f(u))\}}{\mathbb{P}\{V_Y^c > u + k\sigma_X(f(u))\}} \frac{\mathbb{P}\{V_Y^c > u + k\sigma_X(f(u))\}}{\mathbb{P}\{V_Y^c > u\}} \times \\ &\quad \mathbb{P}\{V_{-X}^0[0, lf(u)] \leq k\sigma_X(f(u))\}, \end{aligned}$$

and take the liminf of each of the three terms as $u \rightarrow \infty$. The first term converges to a limit $U_1(l)$, which, in view of Assumption (3.3) tends to 1 as $l \rightarrow \infty$. Since the first term

(before taking $u \rightarrow \infty$) is non-decreasing in l , and $\sigma_X(f(u)) = o(u)$, $U_1(l)$ is independent of k . The second term tends to 1 in view of Assumption (3.5). The third term converges to a limit $U_2(k, l)$, which has the property that $U_2(k, l) \rightarrow 1$, as $k \rightarrow \infty$ for every l , in view of Lemma 3.1. Thus, the proof of the lower bound is completed by letting first $u \rightarrow \infty$, then $k \rightarrow \infty$, and finally $l \rightarrow \infty$. \square

The next theorem provides corresponding necessary conditions for a RLE to hold.

Theorem 3.2 (*Necessary conditions for RLE*)

Let $X(t)$ be a centered Gaussian process satisfying conditions **C1** – **C2** and let $c > 0$. Assume that there exists a function $g(\cdot)$ such that

$$\liminf_{u \rightarrow \infty} \frac{\mathbb{P}\{V_Y^c[g(u), \infty] > u\}}{\mathbb{P}\{V_Y^c > u\}} > 0. \quad (3.9)$$

Furthermore, assume that

$$\mathbb{P}\{V_{X+Y}^c > u\} \sim \mathbb{P}\{V_Y^c > u\}. \quad (3.10)$$

Then,

$$\limsup_{u \rightarrow \infty} \frac{\mathbb{P}\{V_Y^c > u - \sigma_X(g(u))\}}{\mathbb{P}\{V_Y^c > u\}} < \infty. \quad (3.11)$$

Proof

Write

$$\mathbb{P}\{V_{X+Y}^c > u\} \geq \mathbb{P}\{V_{X+Y}^c > u \mid V_Y^c > u - \sigma_X(g(u))\} \mathbb{P}\{V_Y^c > u - \sigma_X(g(u))\}. \quad (3.12)$$

Denote the right-hand side as $I(u)II(u)$. Define $\tau(u) := \inf\{t : Y(t) - ct \geq u - \sigma_X(g(u))\}$. Then

$$\begin{aligned} I(u) &\geq \mathbb{P}\{V_{X+Y}^c > u; g(u) < \tau(u) < \infty \mid \tau(u) < \infty\} \\ &\geq \mathbb{P}\{X(\tau(u)) > \sigma_X(g(u)); g(u) < \tau(u) < \infty \mid \tau(u) < \infty\} \\ &\geq \mathbb{P}\{X(g(u)) > \sigma_X(g(u)); g(u) < \tau(u) < \infty \mid \tau(u) < \infty\} \\ &= \mathbb{P}\{X(1) > 1\} \mathbb{P}\{g(u) < \tau(u) < \infty\}, \end{aligned}$$

where the third inequality follows from the fact that $1(X(t) > \sigma_X(g(u)))$ is stochastically monotone in t . Hence, Assumption (3.9) implies that $\liminf_{u \rightarrow \infty} I(u) > 0$. Combining this with the fact that reduced-load equivalence is assumed to hold (i.e. (3.10)) and (3.12), the desired statement follows. \square

A similar result in the setting of [2] has been developed by Vincent Dumas (personal communication).

Unfortunately, Theorems 3.1 and 3.2 do not provide a conclusive answer to the question whether Property (3.2) is both necessary and sufficient for a RLE to hold, even if one is allowed to choose $f(u) = g(u)$. We conjecture that if

$$\mathbb{P}\{V_Y^c > u + \sigma(X(f(u)))\} \sim \eta \mathbb{P}\{V_Y^c > u\},$$

with $0 < \eta < 1$, a RLE will *not* hold.

4 Reduced-load equivalence II: Applications

In this section, we apply the results of the previous section to the various processes mentioned in the Introduction.

4.1 Gaussian processes

In case X and Y are both Gaussian, one would hope for simple conditions for (3.6) in terms of the coefficients α_X and α_Y . Unfortunately, it appears that the logarithmic asymptotics of $\mathbb{P}\{V_Y^c > u\}$ do not suffice to obtain such conditions. One must invoke additional regularity assumptions:

Corollary 4.1 *Let Y be Gaussian with stationary increments, and suppose that Y satisfies conditions C1-C2.*

(i) *In addition, assume that V_Y is in the maximum domain of attraction of the Gumbel distribution with auxiliary function $a(x)$. Then (3.6) holds if $\sigma_X(u) = o(a(u))$. If $a(u) = o(\sigma_X(u))$, then (3.6) does not hold.*

(ii) *Assume in addition that $\mathbb{P}\{V_Y^c > u\} \sim e^{-Q(u)}$, with $Q'(u)$ ultimately monotone. Then (3.6) holds if $\alpha_Y > 1 + \alpha_X/2$. If $\alpha_Y < 1 + \alpha_X/2$, then (3.6) does not hold.*

Proof

First of all, note that (see e.g. Corollary 6.1 below) one can take $f(u) = g(u) = u$. Assertion (i) follows immediately from standard results in extreme-value theory, see e.g. Balkema & De Haan [7], Embrechts *et al.* [19], or Goldie & Resnick [22].

To establish the second assertion, note that, according to Proposition 2.1, $Q(u)$ is regularly varying of index $2 - \alpha_Y$. In view of the monotone density theorem (see [9]), the derivative of Q is regularly varying of index $1 - \alpha_Y$. Now, Assumption (3.5) can be written as

$$\int_{u-\sigma_X(u)}^u Q'(y)dy \rightarrow 0.$$

Since $Q'(u)$ is both ultimately monotone and regularly varying, this integral behaves like $Q'(u)\sigma_X(u)$, which is regularly varying of index $\alpha_X/2 + 1 - \alpha_Y$. This easily yields Assertion (ii). □

In particular, we have the following corollary.

Corollary 4.2 *If Y is FBM with $\alpha_Y > 1$ and $X(t)$ satisfies conditions C1-C2, then (3.6) holds if $\alpha_Y > 1 + \alpha_X/2$. If $\alpha_Y < 1 + \alpha_X/2$, then (3.6) does not hold.*

Proof

By Theorem 2.1, we can write

$$\mathbb{P}\{V_Y^c > u\} \sim p(u)e^{-\gamma u^{2-\alpha_Y}},$$

with γ some constant and $p(u)$ a power function. Hence, $Q'(y)$ is ultimately monotone. □

This result shows that the exact asymptotics of [26] can be extended to a class of non-self-similar Gaussian processes with covariance function $t^\alpha(1 + F(t))$, with $F(t) = o(t^{2(H-1)})$.

4.2 Processes with subexponential jumps

In the next example we assume that Y is an input process of a $GI/G/1$ queue. This is an important situation where $f(u)$ and $g(u)$ may be sublinear, as follows from a result of Asmussen & Klüppelberg [3], which has -again- strong connections with extreme-value theory.

Corollary 4.3 *Let V_Y^c be the stationary workload in a $GI/G/1$ queue such that the service time B is in the maximum domain of the Gumbel distribution. Then RLE holds if Assumption (3.4) is satisfied.*

Proof

Under the extreme-value assumption, Asmussen & Klüppelberg [3] have shown that $f(u)$ and $g(u)$ can be chosen such that $f(u) = g(u) = o(u)$. Thus, Assumption (3.5) is always satisfied, and the desired statement follows from Theorem 3.1. \square

In the risk setting, this result has been proven by Schlegel [42].

4.3 On-Off processes

In the final example, we take Y to be an On-Off process.

Corollary 4.4 *Let Y be an On-Off process with peak rate $r > c$; assume that $u^\delta \mathbb{P}\{T_{on}^r > u\}$ is ultimately decreasing for some $\delta > 0$ and that $T_{on}^r \in \mathcal{S}$. Then, the conditions of Theorems 3.1 and 3.2 are satisfied with $f(u) = g(u) = u$. In particular, in case $\mathbb{P}\{T_{on}^r > u\} \sim \beta_1 u^{\beta_2} e^{-\beta_3 u^{\beta_4}}$, Assumption (3.6) is satisfied if $u^{\beta_4-1} \sigma_Y(u)$ converges to 0, and does not hold if this quantity tends to ∞ .*

Proof

The condition that $u^\delta \mathbb{P}\{T_{on}^r > u\}$ is ultimately decreasing, together with Theorem 2.2, implies that Assumption (3.3) is satisfied with $f(u) = u$, see [28]. Furthermore, the time to overflow is also at least linear, since $Y(u) \leq ru$. The desired statement now easily follows from Theorems 2.2 and 3.1. \square

Note that the above corollary relied on the assumption $r > c$. If this inequality does not hold, then $V_Y^c \equiv 0$, implying that the asymptotics for V_{X+Y}^c must be entirely different. This is the subject of Sections 6 and 7. The next section considers the case that a RLE does not hold when $r > c$.

5 Moderately heavy tails and moderate deviations

The necessary and sufficient conditions in the previous section show that the tail of V_Y^c must be heavy enough for a RLE to apply. In the present section we consider a case where the tail of V_Y^c is still subexponential, but not heavy enough for a RLE to apply. Following [4], we then call V_Y^c *moderately heavy tailed*. In this case, the tail asymptotics

of V_{X+Y}^c differ from those of V_Y^c . This leaves the question of what the tail asymptotics might be and how the process X contributes to these asymptotics. In the present section, we focus on a specific case: we assume Y to be an On-Off source, and X to be a Brownian motion. The independent increments of X allow us to treat this problem within the regenerative framework of Asmussen *et al.* [6]: The increment process of $\{X(t)+Y(t)-ct\}$ is regenerative, with regeneration points being the ends of On periods.

In particular, the analysis consists of two steps. First, we investigate the tail behavior of $X(T) + (r - c)T$, with T a subexponential random variable. After that, we apply the results of the first step to obtain the tail behavior of V_{X+Y}^c .

We expect that the results and techniques in this section hold under more general conditions on X ; the main purpose of this section is to show how the process X may contribute to large values of V_{X+Y}^c .

We will often impose the following condition. A similar condition has been used by Borovkov [10] to attack a related problem.

T1 The tail of the random variable T has the form $\mathbb{P}\{T > u\} = e^{-L(u)u^\beta}$, with $0 < \beta < 1$, and $L(u)$ slowly varying and twice differentiable. Moreover, $L'(t) = o(L(t)/t)$ and $L''(t) = o(L(t)/t^2)$.

The next result shows that T is indeed subexponential. In fact, one can show a slightly stronger result:

Lemma 5.1 *If T satisfies T1, then $T \in \mathcal{S}^*$. In particular, $T, T^r \in \mathcal{S}$.*

Proof

The hazard function Q and hazard rate q of T are given by $Q(u) = L(u)u^\beta$ and

$$q(u) = \beta L(u)u^{\beta-1} + u^\beta L'(u).$$

Hence, we have $q(u) \rightarrow 0$, $uq(u) \rightarrow \infty$, and $uq(u)/Q(u) \rightarrow \beta \in (0, 1)$. According to Corollary 3.9 of Goldie & Klüppelberg [24], this implies that $T \in \mathcal{S}^*$, which in turn implies $T, T^r \in \mathcal{S}$. □

5.1 Sampling a Brownian motion at a subexponential time

Let $B(t), t \geq 0$, be a standard Brownian motion, and define $B_\mu(t) = B(t) + \mu t, t \geq 0$. Suppose that T is a random variable which is long-tailed, and independent of $\{B_\mu(t)\}$. Define also the running maximum $M_\mu(t) = \max_{0 < s < t} B_\mu(s)$. The goal of this subsection is to determine the tail behavior of $B_\mu(T)$, when $\mu > 0$.

The first step of our analysis is to show that $B_\mu(T)$ and $M_\mu(T)$ are tail equivalent. The following lemma establishes this tail equivalence under minimal assumptions.

Lemma 5.2 *If $T \in \mathcal{L}$, then $B_\mu(T), M_\mu(T) \in \mathcal{L}$, and*

$$\mathbb{P}\{B_\mu(T) > u\} \sim \mathbb{P}\{M_\mu(T) > u\}.$$

Proof

Let $\tau(x) = \inf\{t : B_\mu(t) = x\}$ and fix y . Note that $\tau(x+y) \stackrel{d}{=} \bar{\tau}(x) + \bar{\tau}(y)$, with the latter two random variables distributed as $\tau(x)$ and $\tau(y)$, but mutually independent. Write, for some M and K ,

$$\begin{aligned} \frac{\mathbb{P}\{M_\mu(T) > x+y\}}{\mathbb{P}\{M_\mu(T) > x\}} &= \frac{\mathbb{P}\{T > \bar{\tau}(x) + \bar{\tau}(y)\}}{\mathbb{P}\{T > \tau(x)\}} \\ &\geq \mathbb{P}\{\tau(y) < M\} \frac{\mathbb{P}\{T > \tau(x) + M\}}{\mathbb{P}\{T > \tau(x)\}} \\ &\geq \mathbb{P}\{\tau(y) < M\} \int_K^\infty \frac{\mathbb{P}\{T > z+y\}}{\mathbb{P}\{T > z\}} d\mathbb{P}\{\tau(x) \leq z\}. \end{aligned}$$

Now, use the fact that $T \in \mathcal{L}$ and $\tau(x) \rightarrow \infty$ a.s. Thus, for each $\epsilon > 0$, there exist appropriately chosen M and K such that

$$\frac{\mathbb{P}\{M_\mu(T) > x+y\}}{\mathbb{P}\{M_\mu(T) > x\}} \geq 1 - \epsilon.$$

This shows that $M_\mu(T) \in \mathcal{L}$.

Next, observe that

$$\begin{aligned} \mathbb{P}\{M_\mu(T) > x\} &\leq \mathbb{P}\{B_\mu(T) \geq x-y\} + \mathbb{P}\{B_\mu(T) < x-y; M_\mu(T) > x\} \\ &\leq \mathbb{P}\{B_\mu(T) \geq x-y\} + \mathbb{P}\{\tau(x) < T; B_\mu(T) - B_\mu(\tau(x)) < -y\} \\ &\leq \mathbb{P}\{B_\mu(T) \geq x-y\} + \mathbb{P}\{\tau(x) < T; \inf_{t>\tau(x)} [B_\mu(t) - B_\mu(\tau(x))] < -y\} \\ &= \mathbb{P}\{B_\mu(T) > x-y\} + \mathbb{P}\{M_\mu(T) > x\} \mathbb{P}\{V_{B_0}^\mu > y\}, \end{aligned}$$

where the last inequality follows from the strong Markov property of $B_\mu(t)$. We conclude that

$$\mathbb{P}\{B_\mu(T) > x-y\} \geq \mathbb{P}\{M_\mu(T) > x\} \mathbb{P}\{V_{B_0}^\mu \leq y\}.$$

From this inequality, the obvious property $\mathbb{P}\{B_\mu(T) > x-y\} \leq \mathbb{P}\{M_\mu(T) > x-y\}$, and the fact that $M_\mu(T) \in \mathcal{L}$, one obtains the tail equivalence of $B_\mu(T)$ and $M_\mu(T)$, and in particular the property $B_\mu(T) \in \mathcal{L}$. □

We now investigate the asymptotic behavior of $\mathbb{P}\{B_\mu(T) > u\}$ in the moderately-heavy tailed regime. A related problem has been investigated by Foss & Korhunov [20]: They consider the random variable $N(T)$, with $N(\cdot)$ a renewal process.

As their analysis shows, the computations in the moderately-heavy tailed regime are very technical. We could apply a similar approach here (using explicit formulas for Brownian motion and the Laplace method), but we will follow a different approach: we construct a renewal process $N_\mu(t)$ with the property

$$M_\mu(t) - 1 \leq N_\mu(t) \leq M_\mu(t), \tag{5.13}$$

which, in view of Lemma 5.2, reduces the problem to the one studied in [20]. This approach avoids a lot of tedious computations and may be of independent interest.

We construct $N_\mu(t)$ as follows. Define a sequence of stopping times $\tau_i, i \geq 1$, by

$$\tau_i = \inf\{t : B_\mu(t) = i\}.$$

Then, define

$$N_\mu(t) = \max\{n : \tau_n \leq t\}.$$

It is obvious that (5.13) holds. Moreover, $N_\mu(t)$ is a renewal process, since $\tau_i - \tau_{i-1}, i \geq 1$, is an i.i.d. sequence. Define

$$\Lambda(x) = \sup_y \{xy - \log \mathbb{E}\{e^{y\tau_1}\}\}.$$

Let $\lambda(x)$ be the optimizing point in the above supremum. Since

$$\mathbb{E}\{e^{y\tau_1}\} = e^{\mu - \sqrt{\mu^2 - 2y}},$$

we have

$$\Lambda(x) = \frac{\mu^2}{2}x - \mu + \frac{1}{2x},$$

and

$$\lambda(x) = \frac{\mu^2}{2} - \frac{1}{2x^2}.$$

We now state the main result of this subsection.

Proposition 5.1 *If T satisfies T1, then*

$$\mathbb{P}\{B_\mu(T) > u\} \sim \mathbb{P}\{M_\mu(T) > u\} \sim \mathbb{P}\{N_\mu(T) > u\} \sim e^{-H(t(u), u)},$$

with

$$H(t, u) = Q(t) + u\Lambda(t/u),$$

and $t(u)$ a solution of

$$Q'(t) = -\lambda(t/u).$$

Proof

Assumption T1 implies that $Q(u) = -\log \mathbb{P}\{T > u\}$ is twice differentiable and that $uQ''(u) \rightarrow 0$. This allows us to apply Theorem 5.1 of [20] to obtain the tail behavior of $N_\mu(T)$. The remaining assertions follow from Lemma 5.2 and (5.13). □

If T has a Weibullian tail, i.e., $Q(u) = u^\beta, 0 < \beta < 1$, then Lemma 6.3 of [20] implies

$$u/\mu - t(u) \sim uQ'(u/\mu) = \beta u^\beta \mu^{1-\beta}. \quad (5.14)$$

This indicates that a large value of $B_\mu(T)$ is caused by a realization of T which is about $u/\mu - \beta u^\beta \mu^{1-\beta}$. This implies that $B_\mu(u) - \mu u$ must be of the order u^β .

Hence, if $1/2 < \beta < 1$ (in which case the asymptotic equivalence $\mathbb{P}\{B_\mu(T) > u\} \sim \mathbb{P}\{\mu T > u\}$ does not hold), $B_\mu(u) - \mu u$ contributes to the asymptotics by means of its *moderate deviations*.

We finish this subsection with another question, namely whether or not the tail distribution of $B_\mu(T)$ is subexponential. This is of crucial importance in the next subsection.

Proposition 5.2 *If T satisfies **T1**, then $B_\mu(T) \in \mathcal{S}^*$.*

Proof

Define an auxiliary random variable X such that

$$\mathbb{P}\{X > u\} = e^{-H(t(u), u)}.$$

First, we show that $X \in \mathcal{S}^*$. Since \mathcal{S}^* is closed under tail equivalence, this implies that $B_\mu(T)$ and $M_\mu(T)$ are in \mathcal{S}^* as well. According to Corollary 3.9 in [24], it suffices to show that the hazard rate of $q_X(u)$ of X is regularly varying of index $\in (-1, 0)$. In fact, we will prove that $q_X(u) = \frac{d}{du}H(t(u), u)$ is regularly varying of index $\beta - 1$. From the expression for $H(t(u), u)$ we obtain

$$q_X(u) = t'(u)Q'(t(u)) + \frac{\mu^2}{2}t'(u) - \mu + \frac{u}{t(u)} - \frac{1}{2} \frac{u^2}{t(u)^2}t'(u),$$

where $t'(u)$ satisfies

$$Q''(t(u)t'(u)) = \frac{u}{t(u)^2} - \frac{u^2}{t(u)^3}t'(u).$$

From this equation, one can show using **T1**, that there exists a constant κ such that

$$t'(u) = \frac{1}{\mu} + (\kappa + o(1))L(u)u^{\beta-1}.$$

From a similar computation (see also [20]), one can show that

$$t(u) = \frac{u}{\mu} - \beta\mu^{1-\beta}u^\beta L(u)(1 + o(1)).$$

Combining the above equations, one obtains after a tedious but straightforward computation that $q_X(u)$ is indeed regularly varying of index $\beta - 1$.

Thus, we conclude that $X \in \mathcal{S}^*$. By Proposition 5.1, $B_\mu(T)$ and X are tail equivalent. Since \mathcal{S}^* is closed under tail equivalence, we conclude that $B_\mu(T) \in \mathcal{S}^*$. □

5.2 Workload asymptotics

In this subsection, we apply the results of the previous subsection to obtain tail asymptotics of the workload distribution. As mentioned before, we will follow the framework of Asmussen *et al.* [6]; See also Foss & Zachary [21] for more recent work in this direction. Recall that the increment process associated with $\{X(t) + Y(t) - ct\}$ (with $X(t) = B(t)$) is regenerative w.r.t. the delayed renewal process $\{Z_n, n \geq 0\}$ defined in Subsection 2.3. Thus, we consider the embedded process

$$S_n = X(Z_n) + Y(Z_n) - cZ_n =: U_0 + U_1 + \dots + U_n.$$

Note that $S_n - S_0, n \geq 1$, is a random walk. Furthermore, define

$$M_0 = \sup_{0 < t < Z_0} [X(t) + Y(t) - ct],$$

$$M_n = \sup_{Z_{n-1} < t < Z_n} [X(t) + Y(t) - ct - S_{n-1}].$$

In order to obtain the asymptotics of V_{X+Y}^c , we will apply the results of Section 3.2 of [6]. To check the assumptions stated there, we need the asymptotic behavior of the random variables U_0, U_1, M_0 , and M_1 . This is covered by the following lemma.

Lemma 5.3 (i) If T_{on}^r satisfies **T1**, then $U_0, M_0 \in \mathcal{S}$, and

$$\mathbb{P}\{U_0 > u\} \sim \mathbb{P}\{M_0 > u\} \sim p\mathbb{P}\{B_{r-c}(T_{on}^r) > u\}.$$

(ii) If T_{on} satisfies **T1**, then $U_1, M_1 \in \mathcal{S}$, and

$$\mathbb{P}\{U_1 > u\} \sim \mathbb{P}\{M_1 > u\} \sim \mathbb{P}\{B_{r-c}(T_{on}) > u\}.$$

Proof

We only prove the statement for U_0 and M_0 (the proof for U_1 and M_1 is similar, but easier). Recall the construction of the On-Off process given in Subsection 2.3. With a slight abuse of notation we can write

$$U_0 \stackrel{d}{=} IB_{r-c}(T_{on}^r) + (1 - I)(B_{-c}(T_{off}) + B_{r-c}(T_{on})).$$

In this expression, all components are independent. Since $B_{-c}(T_{off}) \leq \sup_{t>0} B_{-c}(t)$, this random variable is light-tailed. Secondly, since $T_{on} \in \mathcal{L}$, we have $\mathbb{P}\{T_{on} > x\} = o(\mathbb{P}\{T_{on}^r > x\})$. This implies, using Lemma 5.2,

$$\begin{aligned} \mathbb{P}\{B_{r-c}(T_{on}) > u\} &\sim \mathbb{P}\{M_{r-c}(T_{on}) > u\} \\ &= \mathbb{P}\{T_{on} > \tau(u)\} \\ &= o(\mathbb{P}\{T_{on}^r > \tau(u)\}) \\ &= o(\mathbb{P}\{M_{r-c}(T_{on}^r) > u\}) \\ &= o(\mathbb{P}\{B_{r-c}(T_{on}^r) > u\}). \end{aligned}$$

Thus, using standard properties of subexponential distributions, we conclude that

$$\begin{aligned} \mathbb{P}\{U_0 > u\} &= p\mathbb{P}\{B_{r-c}(T_{on}^r) > u\} + (1 - p)\mathbb{P}\{B_{-c}(T_{off}) + B_{r-c}(T_{on}) > u\} \\ &\sim p\mathbb{P}\{B_{r-c}(T_{on}^r) > u\}. \end{aligned}$$

To show the tail behavior of M_0 , note that (with a slight abuse of notation)

$$M_0 \leq IM_{r-c}(T_{on}^r) + (1 - I)(\sup_{t>0} B_{-c}(t) + M_{r-c}(T_{on})).$$

Hence, using a similar argument as above, we obtain

$$\mathbb{P}\{M_0 > u\} \lesssim p\mathbb{P}\{M_{r-c}(T_{on}^r) > u\} \sim p\mathbb{P}\{B_{r-c}(T_{on}^r) > u\}.$$

The asymptotic lower bound is trivial, since $M_0 \geq U_0$. □

Informally, Lemma 5.3 states that U_i is not much smaller than M_i . Thus, it is no surprise that $\sup_{n \geq 0} S_n$ is not much smaller than V_{X+Y}^c . In fact, we have

Theorem 5.1 Suppose that Y is an On-Off process with peak rate $r > c$, that T_{on}, T_{on}^r satisfy **T1**, and that X is a standard Brownian motion. Then

$$\mathbb{P}\{V_{X+Y}^c > u\} \sim p\mathbb{P}\{M_{r-c}(T_{on}^r) > u\} + \frac{p}{c - \rho} \frac{\mathbb{E}\{M_{r-c}(T_{on})\}}{\mathbb{E}\{T_{on}\}} \mathbb{P}\{M_{r-c}(T_{on})^r > u\}.$$

We prefer to present the asymptotics in terms of $M_{r-c}(\cdot)$, rather than $B_{r-c}(\cdot)$, the reason being that the random variable $B_{r-c}(T_{on})^r$ is not well-defined. The asymptotics for $M_{r-c}(T_{on}^r)$ and $M_{r-c}(T_{on})^r$ are given in Proposition 5.1. Note that if $\mathbb{P}\{T > x\} = e^{-x^\beta}$, then T and T^r satisfy **T1**.

Proof

Lemma 5.3 allows us to apply Corollary 3.2 (ii) of [6], which yields

$$\mathbb{P}\{V_{X+Y}^c > u\} \sim \mathbb{P}\{U_0 > u\} + \mathbb{P}\{\sup_{n \geq 1} S_n - S_0 > u\}.$$

The first term is covered by Lemma 5.3 and the assumption that T_{on}^r satisfies **T1**. To deal with the second term, note that $[U_1^r]^+ \stackrel{d}{=} [B_{r-c}(T_{on}) + B_{-c}(T_{off})]^+ \in \mathcal{S}$, since T_{on} satisfies **T1**, and Lemmas 5.2 and 5.3. This implies, using Veraverbeke's Theorem [45],

$$\begin{aligned} & \mathbb{P}\{\sup_{n \geq 1} S_n - S_0 > u\} \\ & \sim \frac{1}{-[\mathbb{E}\{B_{r-c}(T_{on})\} + \mathbb{E}\{B_{-c}(T_{off})\}]} \int_u^\infty \mathbb{P}\{B_{r-c}(T_{on}) + B_{-c}(T_{off}) > v\} dv \\ & \sim \frac{1}{-[\mathbb{E}\{B_{r-c}(T_{on})\} + \mathbb{E}\{B_{-c}(T_{off})\}]} \int_u^\infty \mathbb{P}\{M_{r-c}(T_{on}) > v\} dv \\ & = \frac{1}{-[(r-c)\mathbb{E}\{T_{on}\} - c\mathbb{E}\{T_{off}\}]} \mathbb{E}\{M_{r-c}(T_{on})\} \mathbb{P}\{M_{r-c}^r(T_{on}) > u\}. \end{aligned}$$

Finally, using the formulas

$$p = \frac{\mathbb{E}\{T_{on}\}}{\mathbb{E}\{T_{on} + T_{off}\}}, \quad \rho = rp,$$

we conclude that

$$\mathbb{P}\{\sup_{n \geq 1} S_n - S_0 > u\} \sim \frac{p}{c - \rho} \frac{\mathbb{E}\{M_{r-c}(T_{on})\}}{\mathbb{E}\{T_{on}\}} \mathbb{P}\{M_{r-c}^r(T_{on}) > u\},$$

which completes the proof of the theorem. □

There are several questions that remain. First of all, it is not clear whether $B_\mu(T)^r$ and $B_\mu(T^r)$ are tail equivalent. (This would simplify the theorem.) One could try to verify this using the sequential approximation of $H((t(u), u)$ given in Section 7 of [20], but it may be better to look for a more direct proof. Another question is whether the class of distributions that satisfy **T1** is closed under the operation $T \rightarrow T^r$. We conjecture that this is the case (for Weibull the proof is tedious, but straightforward). Finally, we expect that similar results will hold if we replace Brownian motion by a Levy process or more general processes, satisfying a moderate deviations principle.

We leave all these questions as a topic for future research.

6 Large deviations: Reduced-peak equivalence

In this section we consider the case that X is Gaussian and Y is an On-Off process with peak rate $r < c$. We assume that the tail of V_Y^d , $\rho < d < r$, is heavier than that of V_X^c .

Under these conditions, it is clear that a reduced-load equivalence (which is covered by Corollary 4.4) cannot hold. Informally, one can observe that $X(t)$ cannot be replaced by its mean (0), since $V_X^c \equiv 0$, nor can $Y(t)$ be replaced by its mean, since it has heavier tails than $X(t)$.

In fact, the next theorem shows that *both* $X(t)$ and $Y(t)$ need to behave atypically in order for the process $\{X(t) + Y(t) - ct\}$ to reach a large value.

Theorem 6.1 *Suppose that $X(t)$ has stationary increments, and that $X(t)$ satisfies conditions C1-C2. Furthermore, let $Y(t)$ be an On-Off process with T_{on}^r regularly varying, and $r < c$. Then*

$$\mathbb{P}\{V_{X+Y}^c > u\} \sim p\mathbb{P}\{V_X^{c-r} > u\}\mathbb{P}\{T_{on}^r > \frac{1}{c-r} \frac{\alpha}{2-\alpha} u\}.$$

The above theorem may be combined with the results in [26] or [14] to obtain an explicit expression for the asymptotic behavior of $\mathbb{P}\{V_{X+Y}^c > u\}$.

Remark

We expect the result to extend to a larger class of subexponential On periods. A proof would require different methods than the ones used here, and one might possibly need to impose additional assumptions on $X(t)$. Note that, if $X(t)$ is long-range dependent, then the asymptotics may have the form of the product of two Weibullian tails. We leave this as a subject for future research.

Before giving a proof of Theorem 6.1, we first provide a heuristic explanation of the result. We refer to [11] for similar results and a more detailed discussion for the case where $X(t)$ has light tails.

Recall that $V_{X+Y}^c = \sup_{t \geq 0} [X(t) + Y(t) - ct]$. The most probable way for the process $\{X(t) + Y(t) - ct\}$ to reach a large level u may be described as follows.

- The Gaussian process $X(t)$ shows similar abnormal behavior as is the typical cause of overflow in an isolated system with drain rate $c - r$.
- During that period, of length $t_u = \frac{1}{c-r} \frac{\alpha}{2-\alpha} u$, the On-Off process $Y(t)$ constantly produces traffic at the maximum rate r , leaving a rate $c - r$ available for the process $X(t)$.

Thus, roughly speaking, V_{X+Y}^c behaves like V_X^{c-r} , i.e., the drain rate c is reduced by the peak rate r of $Y(t)$, hence the term ‘reduced-peak equivalence’.

We now state some auxiliary results whose proofs may be found in Subsections 8.3 and 8.4.

Lemma 6.1 *Let $t_u = \frac{1}{A} \frac{\alpha}{2-\alpha} u$. If $X(t)$ satisfies conditions C1-C3, then for every $\epsilon \in (0, 1)$ and $A > 0$,*

(i)

$$M_{X,A}(u) \lesssim \min_{t \geq (1+\epsilon)t_u} \frac{(u + cAt)^2}{2\sigma_X^2(t)},$$

(ii)

$$M_{X,A}(u) \lesssim \min_{t \leq (1-\varepsilon)t_u} \frac{(u + cAt)^2}{2\sigma_X^2(t)}.$$

As a consequence of Lemma 6.1, we have the following corollary.

Corollary 6.1 *Let $t_u = \frac{1}{A} \frac{\alpha}{2-\alpha} u$. If $X(t)$ satisfies conditions **C1-C3**, then for every $\varepsilon \in (0, 1)$ and $A > 0$,*

(i)

$$\lim_{u \rightarrow \infty} \frac{\mathbb{P}\{V_X^A([0, (1-\varepsilon)t_u]) > u\}}{\mathbb{P}\{V_X^A > u\}} = 0, \text{ or equivalently, } \mathbb{P}\{V_X^A([0, (1-\varepsilon)t_u]) > u\} \sim \mathbb{P}\{V_X^A > u\},$$

(ii)

$$\lim_{u \rightarrow \infty} \frac{\mathbb{P}\{V_X^A([(1+\varepsilon)t_u, \infty)) > u\}}{\mathbb{P}\{V_X^A > u\}} = 0, \text{ or equivalently, } \mathbb{P}\{V_X^A([(1+\varepsilon)t_u, \infty)) > u\} \sim \mathbb{P}\{V_X^A > u\}.$$

We are now ready to provide a proof of Theorem 6.1.

Proof of Theorem 6.1

Let $t_u = \frac{1}{c-r} \frac{\alpha}{2-\alpha} u$ and $\varepsilon \in (0, \infty)$ be given. The proof consists of a lower and an upper bound. To obtain a lower bound, note that

$$\begin{aligned} & \mathbb{P}\{V_{X+Y}^c > u\} \\ &= \mathbb{P}\{\sup_{t \geq 0} X(t) + Y(t) - ct > u\} \\ &\geq \mathbb{P}\{\sup_{t \leq (1+\varepsilon)t_u} X(t) + Y(t) - ct > u\} \\ &\geq \mathbb{P}\{\sup_{t \leq (1+\varepsilon)t_u} X(t) + Y(t) - ct > u \mid Y((1+\varepsilon)t_u) = r(1+\varepsilon)t_u\} \\ &\quad \times \mathbb{P}\{Y((1+\varepsilon)t_u) = r(1+\varepsilon)t_u\} \\ &= \mathbb{P}\{\sup_{t \leq (1+\varepsilon)t_u} X(t) + Y(t) - ct > u \mid Y(t) = rt \text{ for all } t \leq (1+\varepsilon)t_u\} \\ &\quad \times p \mathbb{P}\{T_{on}^r > (1+\varepsilon)t_u\} \\ &= \mathbb{P}\{\sup_{t \leq (1+\varepsilon)t_u} X(t) - (c-r)t > u\} p \mathbb{P}\{T_{on}^r > (1+\varepsilon)t_u\} \\ &= \mathbb{P}\{V_X^{c-r}([0, (1+\varepsilon)t_u]) > u\} p \mathbb{P}\{T_{on}^r > (1+\varepsilon)t_u\}. \end{aligned}$$

Using Corollary 6.1, we have

$$\begin{aligned} \frac{\mathbb{P}\{V_{X+Y}^c > u\}}{p \mathbb{P}\{V_X^{c-r} > u\} \mathbb{P}\{T_{on}^r > t_u\}} &\geq \frac{\mathbb{P}\{V_X^{c-r}([0, (1+\varepsilon)t_u]) > u\} \mathbb{P}\{T_{on}^r > (1+\varepsilon)t_u\}}{\mathbb{P}\{V_X^{c-r} > u\} \mathbb{P}\{T_{on}^r > t_u\}} \\ &\gtrsim \frac{\mathbb{P}\{T_{on}^r > (1+\varepsilon)t_u\}}{\mathbb{P}\{T_{on}^r > t_u\}}. \end{aligned}$$

Letting $\varepsilon \downarrow 0$ and using the fact that T_{on}^r is regularly varying then completes the proof of the lower bound.

To obtain a matching upper bound we proceed as follows. For every $\delta \in (0, \infty)$ and $\zeta \in (0, \infty)$ we have

$$\begin{aligned}
& \mathbb{P}\{V_{X+Y}^c > u\} \\
&= \mathbb{P}\{\sup_{t \geq 0} X(t) + Y(t) - ct > u\} \\
&\leq \mathbb{P}\{\sup_{t \leq (1-\varepsilon)t_u} X(t) + Y(t) - ct > u\} + \mathbb{P}\{\sup_{t \geq (1-\varepsilon)t_u} X(t) + Y(t) - ct > u\} \\
&\leq \mathbb{P}\{\sup_{t \leq (1-\varepsilon)t_u} X(t) - (c-r)t > u\} \\
&+ \mathbb{P}\{\sup_{t \geq (1-\varepsilon)t_u} X(t) + Y(t) - ct > u | V_Y^{r-\delta}([(1-\varepsilon)t_u, \infty)) \leq 0\} \mathbb{P}\{V_Y^{r-\delta}([(1-\varepsilon)t_u, \infty)) \leq 0\} \\
&+ \mathbb{P}\{\sup_{t \geq (1-\varepsilon)t_u} X(t) + Y(t) - ct > u | V_Y^{r-\delta}([(1-\varepsilon)t_u, \infty)) > 0\} \mathbb{P}\{V_Y^{r-\delta}([(1-\varepsilon)t_u, \infty)) > 0\} \\
&\leq \mathbb{P}\{V_X^{c-r}([0, (1-\varepsilon)t_u]) > u\} + \mathbb{P}\{\sup_{t \geq (1-\varepsilon)t_u} X(t) - (c-r+\delta)t > u\} \\
&+ \mathbb{P}\{\sup_{t \geq (1-\varepsilon)t_u} X(t) - (c-r)t > u\} \mathbb{P}\{\sup_{t \geq (1-\varepsilon)t_u} Y(t) - (r-\delta-\zeta)t > \zeta(1-\varepsilon)t_u\} \\
&\leq \mathbb{P}\{V_X^{c-r}([0, (1-\varepsilon)t_u]) > u\} + \mathbb{P}\{V_X^{c-r+\delta} > u\} + \mathbb{P}\{V_X^c > u\} \mathbb{P}\{V_Y^{r-\delta-\zeta} > \zeta(1-\varepsilon)t_u\}.
\end{aligned}$$

According to Theorem 2.2,

$$\mathbb{P}\{V_Y^{r-\delta-\zeta} > \zeta(1-\varepsilon)t_u\} \sim p \frac{r-\rho}{r-\delta-\zeta-\rho} \mathbb{P}\{T_{on}^r > \frac{\zeta}{\delta+\zeta}(1-\varepsilon)t_u\}.$$

Using Corollary 6.1, we thus obtain

$$\begin{aligned}
& \frac{\mathbb{P}\{V_{X+Y}^c > u\}}{\mathbb{P}\{V_X^{c-r} > u\} \mathbb{P}\{T_{on}^r > t_u\}} \\
&\leq \frac{1}{\mathbb{P}\{T_{on}^r > t_u\}} \frac{\mathbb{P}\{V_X^{c-r}([0, (1-\varepsilon)t_u]) > u\}}{\mathbb{P}\{V_X^{c-r} > u\}} + \frac{1}{\mathbb{P}\{T_{on}^r > t_u\}} \frac{\mathbb{P}\{V_X^{c-r+\delta} > u\}}{\mathbb{P}\{V_X^{c-r} > u\}} \\
&+ \frac{r-\rho}{r-\delta-\zeta-\rho} \frac{\mathbb{P}\{T_{on}^r > \frac{\zeta}{\delta+\zeta}(1-\varepsilon)t_u\}}{\mathbb{P}\{T_{on}^r > t_u\}} \\
&\lesssim \frac{r-\rho}{r-\delta-\zeta-\rho} \frac{\mathbb{P}\{T_{on}^r > \frac{\zeta}{\delta+\zeta}(1-\varepsilon)t_u\}}{\mathbb{P}\{T_{on}^r > t_u\}}.
\end{aligned}$$

Letting $\delta \downarrow 0$ and then $\zeta \downarrow 0$, $\varepsilon \downarrow 0$, and using the fact that T_{on}^r is regularly varying completes the proof of the upper bound. \square

7 Oscillatory behavior

As in the previous section, we consider the case that X is Gaussian and Y is an On-Off process with peak rate r . The central assumption of this section is that $r = c$. Under this critical condition, the process $S(t) = X(t) + Y(t) - ct$ will oscillate during the On periods of $Y(t)$.

The next theorem presents the main result of this section.

Theorem 7.1 *If $X(t)$ has stationary increments and satisfies conditions C1-C2, and T_{on} is regularly varying of index $-\nu < -1$ such that $\mathbb{P}\{T_{on}^r > x\} = L(x)x^{1-\nu}$, then*

$$\mathbb{P}\{V_{X+Y}^r > u\} \sim p \mathbb{E}\{\bar{B}_H^{H(\nu-1)}\} \mathbb{P}\{\sigma_X(T_{on}^r) > u\},$$

with $H = \alpha/2$ and $\bar{B}_H = \sup_{0 \leq s \leq 1} B_H(s)$. In particular, V_{X+Y}^r is regularly varying of index $(1 - \nu)/H$.

The above theorem shows that the heaviness of the tail of V_{X+Y}^r is a combined effect of the heaviness of T_{on}^r and the degree of dependence in X .

Informally, if $r = c$, then a large value of V_{X+Y}^c is most likely caused by a single long On period which started at time 0. During this long On period the net input process has zero drift. This implies that the net input process at time t is $O(\sigma_X(t))$. Hence, in order to reach level u , we need an On period of length $O(\sigma_X^{-1}(u))$.

In the proof of Theorem 7.1, we make these heuristics precise. We use the following auxiliary lemma, which is the main result of [17].

Lemma 7.1 *Under the conditions of Theorem 7.1, we have*

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T_{on}^r} X(t) > u\right\} \sim \mathbb{E}\{\bar{B}_H^{H(\nu-1)}\} \mathbb{P}\{\sigma_X(T_{on}^r) > u\}.$$

The main idea of the proof of Theorem 7.1 is to separate the processes X and Y by adding and subtracting *non-linear* perturbations. To handle such perturbations, we need an auxiliary lemma, whose proof may be found in Subsection 8.5.

Lemma 7.2 *Let $X(t)$ be a centered Gaussian process satisfying conditions C1-C2. If $\eta > \alpha/2$, then*

$$\log \mathbb{P}\left\{\sup_{t \geq 0} X(t) - dt^\eta > u\right\} \sim \min_{t \geq 0} \frac{(u + dt^\eta)^2}{2\sigma_X^2(t)}.$$

Proof of Theorem 7.1

The lower bound is trivial, in view of Lemma 7.1 and the construction of the process $Y(t)$ given in Section 2.

For the upper bound, write for some $\gamma \in (0, 1)$,

$$\mathbb{P}\left\{\sup_{t \geq 0} S(t) > u\right\} \leq \mathbb{P}\left\{\sup_{t \leq Z_0} S(t) > (1 - \gamma)u\right\} + \mathbb{P}\left\{\sup_{t > Z_0} S(t) - S(Z_0) > \gamma u\right\}.$$

We need to show that the second term can be asymptotically neglected. Using sample path arguments, we have

$$\begin{aligned} & \mathbb{P}\left\{\sup_{t > Z_0} S(t) - S(Z_0) > \gamma u\right\} \\ &= \mathbb{P}\left\{\sup_{t > Z_0} [X(t) - X(Z_0) + Y(t) - Y(Z_0) - r(t - Z_0)] > \gamma u\right\} \\ &\leq \mathbb{P}\left\{\sup_{t > Z_0} [Y(t) - Y(Z_0) - r(t - Z_0) + d(t - Z_0)^\eta] > \gamma u/2\right\} + \\ & \quad \mathbb{P}\left\{\sup_{t > Z_0} [X(t) - X(Z_0) - d(t - Z_0)^\eta] > \gamma u/2\right\} \\ &= \text{I} + \text{II}, \end{aligned}$$

where we take $1 > \eta > \alpha/2$ and d small.

We first deal with term I. Observe that

$$d(Z_n - Z_0)^\eta \leq d \sum_{i=1}^n (Z_i - Z_{i-1})^\eta,$$

from which it follows that

$$I \leq \mathbb{P}\{\sup_{n \geq 1} S_n > \gamma u/2\},$$

where S_n is a random walk with generic step size $U = dT_{on}^\eta + dT_{off}^\eta - rT_{off}$.

We can choose d small enough such that U has negative mean. Noting that $dT_{off}^\eta - rT_{off}$ is bounded from above, we conclude that the right tail of U is regularly varying. This allows us to apply Veraverbeke's Theorem [45], yielding

$$I \leq \mathbb{P}\{\sup_{n \geq 1} S_n > \gamma u/2\} \sim \frac{1}{-\mathbb{E}\{U\}} \int_{\gamma u/2}^{\infty} \mathbb{P}\{U > y\} dy,$$

which is regularly varying of index $1 - \nu\eta$. We can choose η such that $1 - \nu\eta > (1 - \nu)H$ (i.e. $\eta < H + \frac{1-H}{\nu}$).

We now turn to term II. This term is somewhat easier: since $X(t)$ has stationary increments, we have

$$II = \mathbb{P}\{\sup_{t \geq 0} [X(t) - dt^\eta] > x\}.$$

This probability is decreasing faster than any polynomial, in view of Lemma 7.2.

Thus, we can conclude that, for any $\gamma > 0$

$$\mathbb{P}\{\sup_{t \geq 0} S(t) > u\} \lesssim \mathbb{P}\{\sup_{t \leq Z_0} S(t) > (1 - \gamma)u\}. \quad (7.15)$$

We determine the probability on the right-hand side by conditioning upon the state of the On-Off source at time 0.

$$\begin{aligned} \mathbb{P}\{\sup_{t \leq Z_0} S(t) > (1 - \gamma)u\} &= p \mathbb{P}\{\sup_{t \leq T_{on}^r} X(t) > (1 - \gamma)u\} \\ &+ (1 - p) \mathbb{P}\{\sup_{t < T_{off}^r + T_{on}} [Y(t) + X(t) - rt] > (1 - \gamma)u\}. \end{aligned}$$

Using similar methods as above, it is straightforward to show that the second term is regularly varying of index $-\nu/H$. From the proof of the lower bound, we already know that the first term is regularly varying with index $(1 - \nu)/H$. Hence, we conclude from (7.15) and Lemma 7.1,

$$\limsup_{u \rightarrow \infty} \frac{\mathbb{P}\{\sup_{t \geq 0} S(t) > u\}}{p \mathbb{E}\{\bar{B}_H^{H(\nu-1)}\} \mathbb{P}\{T_{on}^r > \sigma_X(u)\}} \leq (1 - \gamma)^{-(1-\nu)/H}$$

for all $\gamma > 0$. □

Corollary 7.1 *In addition to the assumptions of Theorem 7.1, assume that $X(t), t \geq 0$ has stationary increments, and satisfies conditions C1-C2 with $\alpha = 1$. Then,*

$$\mathbb{P}\{V_{X+Y}^r > u\} \sim p \frac{1}{\sqrt{\pi}} 2^{1+\nu} \Gamma(\nu + \frac{1}{2}) \mathbb{P}\{\sigma_X(T_{on}^r) > u\},$$

Proof

Follows straightforwardly from Theorem 7.1, combined with Proposition 2.1 in [17]. □

8 Additional proofs

8.1 Proof of Lemma 3.1

Using the Borell inequality (Theorem D.1 in [40]), we obtain

$$\mathbb{P}\left\{\sup_{0 \leq s \leq t} X(s) > u\right\} \leq 2e^{-\left(\frac{u}{\sigma_X(t)} - q\right)^2},$$

with q chosen such that

$$\mathbb{P}\left\{\sup_{0 \leq s \leq t} X(s) > q\sigma_X(t)\right\} \leq \frac{1}{2},$$

for every t ; we must show that such a choice of q is possible.

Define $X_t(s) = X(st)/\sigma_X(s)$. From Lemma 4.2 in [17], we have that the process $\{X_t(s), 0 \leq s \leq 1\}$ converges to $\{B_H(s), 0 \leq s \leq 1\}$ in $C([0, 1])$ as $t \rightarrow \infty$, with $H = \alpha/2$. Since the sup operator is continuous in $C([0, 1])$, we conclude that

$$\mathbb{P}\left\{\sup_{0 \leq s \leq t} X(s) > q\sigma_X(t)\right\} \leq \mathbb{P}\left\{\sup_{0 \leq s \leq 1} X_t(s) > q\right\} \rightarrow \mathbb{P}\left\{\sup_{0 \leq s \leq 1} B_H(s) > q\right\},$$

as $t \rightarrow \infty$. This makes a proper choice of q obvious. □

8.2 Proof of Lemma 3.2

Note that $\mathbb{P}\{X > x - \sigma_Y(lx)\} \sim \mathbb{P}\{X > x - \sigma_Y(lx)\}$ implies that, for every $\epsilon > 0$, there exists an x_ϵ such that $Q(x + \sigma_Y(lx)) \leq Q(x) + \epsilon$. Using the monotonicity of $Q(\cdot)$ and iterating this bound n times we obtain, for $x_0 > x_\epsilon$,

$$Q(x_0 + n\sigma_Y(lx)) \leq Q(x_0) + n\epsilon.$$

Taking $n = (x - x_0)/\epsilon$, we have

$$Q(x) \leq Q(x_0) + \epsilon \frac{x - x_0}{\sigma_Y(lx)}.$$

Hence,

$$\limsup_{x \rightarrow \infty} \sigma_Y(lx)Q(x)/x \leq \epsilon$$

for any $\epsilon > 0$. This gives the first assertion.

We proceed to prove that statements (i)-(iv) are all equivalent. The equivalence between (ii) and (iii) is trivial. The equivalence between (i) and (ii) follows from the equivalence between (i) and (iii), combined with the bounds

$$l^{\beta_1}\sigma(x) \leq \sigma(lx) \leq l^{\beta_2}\sigma(x),$$

(for suitable choices of β_1, β_2), which follow from Potter's theorem.

To prove that (i) implies (iv), we write, for some large M ,

$$\begin{aligned} & \mathbb{P}\{W > x - Z\sigma_Y(lx) \mid Z > k\} \\ & \leq \mathbb{P}\{X > x - Mk\sigma_Y(lx)\} + \int_{Mk}^{\infty} z^\gamma e^{-z^2/2} e^{\kappa k^2} \mathbb{P}\{W > x - z\sigma_Y(lx)\} dz. \end{aligned}$$

To bound the integral, we note that, by a similar argument as above, we have the bound

$$\mathbb{P}\{W > x - z\sigma_Y(lx)\} \leq e^{\epsilon z} \mathbb{P}\{W > x\}.$$

Substituting this bound in the integral and invoking (i) then easily yields (iv). The reverse implication is trivial. \square

8.3 Proof of Lemma 6.1

We only present the proof of (i). The proof of (ii) is analogous.

Let $\varepsilon \in (0, 1)$ be given. First, we note that Theorem 1.5.3(ii) in [9] implies that

$$\min_{t \geq (1+\varepsilon)t_u} \frac{(u + cAt)^2}{2\sigma_X^2(t)} \sim \frac{(u + A(1+\varepsilon)t_u)^2}{2\sigma_X^2((1+\varepsilon)t_u)}$$

since $\sigma_X^2(t)$ is regularly varying of index $\alpha_X < 2$.

Hence, it suffices to show that

$$\lim_{u \rightarrow \infty} \frac{M_{X,A}(u)}{\frac{(u + A(1+\varepsilon)t_u)^2}{2\sigma_X^2((1+\varepsilon)t_u)}} < 1.$$

Using the fact that $\sigma_X^2(t)$ is regularly varying, we have

$$\lim_{u \rightarrow \infty} \frac{M_{X,A}(u)}{\frac{(u + A(1+\varepsilon)t_u)^2}{2\sigma_X^2((1+\varepsilon)t_u)}} = \frac{4(1+\varepsilon)^\alpha}{(2+\varepsilon\alpha)^2} =: g_\varepsilon(\alpha) < 1,$$

since $g_\varepsilon(\alpha)$ is strictly increasing in α and equals 1 if $\alpha = 2$. \square

8.4 Proof of Corollary 6.1

By Proposition 2.1,

$$\log(\mathbb{P}\{V_X^A > u\}) \sim -M_{X,A}(u).$$

By Lemma 6.1,

$$M_{X,A}(u) \lesssim \min_{t \geq (1-\varepsilon)t_u} \frac{(u + cAt)^2}{2\sigma_X^2(t)}.$$

Let $X_u(t) = \frac{X(t)}{u+At}$. Using the Borell inequality (Adler [1], p. 43), we obtain, for all $u > 0$

$$\begin{aligned} \mathbb{P}\{V_X^A([0, (1-\varepsilon)t_u]) > u\} &= \mathbb{P}\left\{\sup_{t \in [0, (1-\varepsilon)t_u]} X_u(t) > 1\right\} \\ &\leq 2 \exp\left(-\left(1 - \mathbb{E}\left\{\sup_{t \geq 0} X_u(t)\right\}\right)^2 \min_{t \leq (1-\varepsilon)t_u} \frac{(u + cAt)^2}{2\sigma_X^2(t)}\right). \end{aligned}$$

Since $\lim_{u \rightarrow \infty} \mathbb{E}\left\{\sup_{t \geq 0} X_u(t)\right\} = 0$ by Lemma 2.2 in Dębicki [13], we have

$$\log(\mathbb{P}\{V_X^A([0, (1-\varepsilon)t_u]) > u\}) \lesssim - \min_{t \leq (1-\varepsilon)t_u} \frac{(u + cAt)^2}{2\sigma_X^2(t)}.$$

Hence, $\log(\mathbb{P}\{V_X^A([0, (1-\varepsilon)t_u]) > u\}) \lesssim \log(\mathbb{P}\{V_X^A > u\})$, which completes the proof of (i).

The proof of (ii) is analogous. \square

8.5 Proof of Lemma 7.2

The proof relies on Theorem 3.1 in Deĭbicki [13], which states that it suffices to check that $\mathbb{P}\{\sup_{t \geq 0} [X(t) - dt^\eta] > u\} \rightarrow 0$.

Since $\sigma_X^2(t)$ is regularly varying at 0 with index β , there exists a $T > 0$ such that

$$\sigma_X^2(t) \leq \sigma_X^2(1)t^{\frac{\beta}{2}},$$

and

$$\exp\left(-t^{\frac{\beta}{2}}\right) \leq 1 - \frac{t^{\frac{\beta}{2}}}{2}$$

for every $t \in [0, T]$. Moreover, let $k_0 \in \mathbb{N}$ be such that $\sigma_X^2(t) \geq \sigma_X^2(1)$ for every $t \geq k_0 T$. Now, for every $\delta > 0$ and $u > 0$,

$$\begin{aligned} \mathbb{P}\{V_X^\delta > u\} &\leq \mathbb{P}\left\{\sup_{t \in [0, k_0 T]} X(t) - \delta t > u\right\} + \sum_{k=k_0}^{\infty} \mathbb{P}\left\{\sup_{t \in [kT, (k+1)T]} X(t) - dt^\eta > u\right\} \\ &\leq \mathbb{P}\left\{\sup_{t \in [0, k_0 T]} X(t) > u\right\} \\ &\quad + \sum_{k=k_0}^{\infty} \mathbb{P}\left\{\sup_{t \in [kT, (k+1)T]} \frac{X(t)}{\sigma_X(t)} > \frac{u + d(kT)^\eta}{\sigma_X((k+1)T)}\right\}. \end{aligned} \quad (8.16)$$

Since $\lim_{u \rightarrow \infty} \mathbb{P}\left(\sup_{t \in [0, k_0 T]} X(t) > u\right) = 0$, it suffices to bound the sum in (8.16).

Let $Z(t)$ be a centered stationary Gaussian process with covariance function $\text{Cov}\{Z(s) + t, Z(s)\} = \exp\left(-t^{\frac{\beta}{2}}\right)$. Note that for every $s, t \in [kT, (k+1)T]$ ($k \geq k_0$),

$$\begin{aligned} \text{Cov}\left\{\frac{X(t)}{\sigma_X(t)}, \frac{X(s)}{\sigma_X(s)}\right\} &= \frac{\sigma_X^2(t) + \sigma_X^2(s) - \sigma_X^2(|t-s|)}{2\sigma_X(t)\sigma_X(s)} \\ &\geq 1 - \frac{\sigma_X^2(|t-s|)}{2\sigma_X(t)\sigma_X(s)} \\ &\geq 1 - \frac{\sigma_X^2(|t-s|)}{2\sigma_X^2(1)} \\ &\geq 1 - \frac{|t-s|^{\frac{\beta}{2}}}{2} \\ &\geq \exp\left(-|t-s|^{\frac{\beta}{2}}\right) \\ &= \text{Cov}\{Z(s), Z(t)\}. \end{aligned}$$

Thus, for every $k \geq k_0$ and $u \geq C_T := \mathbb{E}\left\{\sup_{t \in [0, T]} Z(t)\right\}$

$$\mathbb{P}\left\{\sup_{t \in [kT, (k+1)T]} \frac{X(t)}{\sigma_X(t)} > \frac{u + d(kT)^\eta}{\sigma_X((k+1)T)}\right\} \leq \mathbb{P}\left(\sup_{t \in [0, T]} Z(t) > \frac{u + d(kT)^\eta}{\sigma_X((k+1)T)}\right) \quad (8.17)$$

$$\leq 2 \exp\left(-\frac{1}{2} \left(\frac{u + d(kT)^\eta}{\sigma_X((k+1)T)} - C_T\right)^2\right) \quad (8.18)$$

where (8.17) follows from the Slepian inequality (Theorem C.1 in Piterbarg [40]), and (8.18) is due to the Borell inequality (Theorem 2.1 in Adler [1]).

Combining (8.16) with (8.18), we obtain

$$\sum_{k=k_0}^{\infty} \mathbb{P} \left(\sup_{t \in [kT, (k+1)T]} \frac{X(t)}{\sigma_X(t)} > \frac{u + d(kT)^\eta}{\sigma_X((k+1)T)} \right) \leq \sum_{k=k_0}^{\infty} 2 \exp \left(- \frac{\left(\frac{u + d(kT)^\eta}{\sigma_X((k+1)T)} - C_T \right)^2}{2} \right) \rightarrow 0$$

as $u \rightarrow \infty$, since $\eta > \alpha/2$. This completes the proof.

9 Conclusion

We have analyzed the tail asymptotics of a fluid model fed by two stochastic processes, of which at least one has subexponential characteristics. The results show that (i) the question whether or not a RLE holds is determined by a number of structural properties (ii) a wide variety of different asymptotics may arise when a RLE does not hold.

Several interesting questions remain to be explored. In particular, a restrictive assumption that we made is that, in all cases, the tail of V_X^c is heavier than that of V_X^ε . For example, the case of two *identical* On-Off processes has only been treated for the case of regularly varying On periods, see [46]. In case the On periods are Weibullian, the results are expected to be fundamentally different; we refer to Likhanov *et al.* [34] for some interesting asymptotic lower and upper bounds.

An exception is when X and Y are identical Gaussian processes, in particular, when both X and Y are fractional Brownian motions (FBM). In that case, the process $X(t) + Y(t)$ is a fractional Brownian motion as well, and we have, due to the scaling property,

$$\mathbb{P}\{V_{X+Y}^c > u\} = \mathbb{P}\{V_X^{\bar{c}} > u\},$$

with $\bar{c} = c(1/2)^{1/2H}$.

Thus, $\mathbb{P}\{V_{X+Y}^c > u\}/\mathbb{P}\{V_X^c > u\} \rightarrow \infty$ for any value of c . We expect that the corresponding case of identical On-Off processes with Weibullian On periods will lead to fundamentally different results, depending on the value of the peak rate r .

Nevertheless, we believe that the similarities between Gaussian and On-Off processes treated in this paper hold more generally. For example, we conjecture that the asymptotics given in Section 5 remain the same when the On-Off process is replaced by a Gaussian process, for example a FBM with Hurst parameter $H \in (\frac{1}{2}, \frac{3}{4})$.

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