Generalized Fischer-Fock spaces

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Summary

This paper is on functional Hilbert spaces of entire analytic functions which extend the class of Fischer-Fock spaces. They are related with Bargmann's description of Schwarz' test space of rapidly decreasing C^{∞}-functions and its dual the space of tempered distributions.

Preliminaries

Let $I\!P$ denote the collection of all entire analytic functions f for which all derivatives $f^{(n)}(0)$, $n = 0,1,2, \cdots$, in $z = 0$, are strictly positive. For each $f \in I\!P$ the function K_f on $\mathbb{C} \times \mathbb{C}$ defined by

$$(0.1) \qquad K_f(z,w) = f(z\bar{w}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} (z\bar{w})^n, \quad z,w \in \mathbb{C}$$

is of positive type. To K_f there is associated precisely one functional Hilbert space $\mathbf{H}[f]$, cf. [Ar]. The Hilbert space $\mathbf{H}[f]$ consists of all entire analytic functions ϕ with the property that

$$(0.2) \qquad \|\phi\|_f^2 := \sum_{n=0}^{\infty} \frac{|\phi^{(n)}(0)|^2}{n!\, f^{(n)}(0)} < \infty.$$

The functions $\phi \in \mathbf{H}[f]$ satisfy the estimation

$$(0.3) \qquad |\phi(z)|^2 \le f(|z|^2)\, \|\phi\|_f^2, \quad z \in \mathbb{C}.$$

The normalized monomials $\left[\dfrac{f^{(n)}(0)}{n!}\right]^{1/2} z^n$ establish an orthonormal basis in $\mathbf{H}[f]$.

In $I\!P$ we introduce an order relation by

$$(0.4) \qquad f_1 \lesssim f_2 :\Leftrightarrow \exists_{\lambda>0} : \lambda f_2 - f_1 \in I\!P.$$

As one can readily check, $f_1 \lesssim f_2$ implies that $\mathbf{H}[f_1]$ can be continuously injected into $\mathbf{H}[f_2]$. Further, the class $I\!P$ is closed with respect to addition, $f_1 + f_2$, and joint multiplication, $f_1 f_2$. In this connection we mention the following interesting result of Burbea, cf. [Bu]:

Let $\phi_j \in \mathbf{H}[f_j]$, $j = 1,2$. Then $\phi_1 \phi_2 \in \mathbf{H}[f_1 f_2]$ and

$$\|\phi_1 \phi_2\|_{f_1 f_2} \le \|\phi_1\|_{f_1} \|\phi_2\|_{f_2}.$$

In this paper we concentrate on the confluent hypergeometric functions $f_{a,b,c} \in I\!P$, $a,b,c > 0$, defined by

$$(0.5) \qquad f_{a,b,c}(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{(cz)^n}{n!}, \quad z \in \mathbb{C}.$$

(We use Pochhammer's symbol $(r)_n = \dfrac{\Gamma(r+n)}{\Gamma(r)}$, $r \in I\!R$.)

The functions $f_{a,b,c}$ satisfy the order relation

$$(0.6) \qquad f_{a,b,c} \lesssim f_{\tilde{a},\tilde{b},\tilde{c}}$$

in case

* $c < \tilde{c}$ and $a, b, \tilde{a}$ and $\tilde{b}$ arbitrary,
* $c = \tilde{c}$ and $\tilde{a} - a \geq \tilde{b} - b$.

The space $H[f_{1,1,1}]$ is the classical Fischer-Fock space or Bargmann space, cf. [NeSh] and [Ba1]. The functional Hilbert space $H[f_{1,b,c}]$ are introduced in [Bu], where they are called generalized (b,c)-Fischer spaces. So $H[f_{a,b,c}]$ may be called the generalized (a,b,c)-Fischer space.

1. Generalized (a,b,c)-Fischer spaces

For $\kappa, \mu \in \mathbb{R}$, let $W_{\kappa,\mu}$ denote the Whittaker function of the second kind which for $\mu - \kappa > -\frac{1}{2}$ satisfies

$$W_{\kappa,\mu}(t) = \frac{1}{\Gamma(\frac{1}{2} + \mu - \kappa)}\, t^{\mu+\frac{1}{2}} \exp(-\tfrac{1}{2} t) \int_0^\infty e^{-st} s^{\mu-\kappa-\frac{1}{2}} (1+s)^{\mu+\kappa-\frac{1}{2}}\, ds\,,$$

cf. [MOS], p.313. So for each $\kappa, \mu \in \mathbb{R}$ with $\mu - \kappa > -\frac{1}{2}$ the function $W_{\kappa,\mu}$ is positive on $(0,\infty)$. Consider the following integral relations, cf. [MOS], p. 316,

$$(1.1) \qquad \int_0^\infty t^{n+\nu-1} \exp(-\tfrac{1}{2} ct)\, W_{\kappa,\mu}(ct)\, dt =$$

$$= \frac{\Gamma(\frac{1}{2} + \mu + \nu + n)\, \Gamma(\frac{1}{2} - \mu + \nu + n)}{\Gamma(1 - \kappa + na + n)} \left[\frac{1}{c}\right]^{n+\nu}.$$

We set

$$(1.2) \qquad G_{a,b,c}(t) = \frac{c^\nu\, \Gamma(b)}{\pi\, \Gamma(a)}\, t^{\nu-1} \exp(-\tfrac{1}{2} ct)\, W_{\kappa,\mu}(ct)$$

with

$$\kappa = \frac{b - 2a + 2}{2}, \quad \mu = \frac{b-1}{2}, \quad \nu = \frac{b}{2}.$$

Then from (1.1) we deduce

$$(1.3) \qquad \int_0^\infty t^n\, G_{a,b,c}(t)\, dt = \frac{1}{\pi} \frac{(b)_n}{(a)_n} \frac{n!}{c^n}, \quad n \in \mathbb{N}_0.$$

Next we introduce the space $\mathbf{F}_{a,b,c}$ of all entire analytic functions ϕ for which the integral

$$\int_{\mathbb{R}^2} |\, \phi(x+iy)\, |^2\, G_{a,b,c}(x^2+y^2)\, dx\, dy$$

is finite. With the natural inner product $(\,,\,)_{a,b,c}$,

$$(\phi,\psi)_{a,b,c} = \int_{\mathbb{R}^2} \phi(x+iy)\,\overline{\psi(x+iy)}\,G_{a,b,c}(x^2+y^2)\,dx\,dy\ ,$$

$\mathbf{F}_{a,b,c}$ is a Hilbert space.

(1.4) *Theorem.*
The Hilbert space $\mathbf{F}_{a,b,c}$ equals the functional Hilbert space $\mathbf{H}[f_{a,b,c}]$, i.e.

$$* \qquad \forall_{w \in \mathbb{C}} : \phi(w) = \int_{\mathbb{R}^2} \phi(z)\, \overline{{}_1F_1(a,b,c\,\overline{w}\,z)}\, G_{a,b,c}\,(|z|)^2\, dx\,dy\ ,\quad z = x+iy\ ,$$

$$* \qquad (\phi,\psi)_{a,b,c} = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)\,\overline{\psi^{(n)}(0)}}{n!}\,\frac{(b)_n}{(a)_n}\,\frac{1}{c^n}.$$

Proof.

From relation (1.3) we deduce that the normalized monomials $u_n(a,b,c\,;z) = \left\{\frac{(a)_n\, c^n}{(b)_n\, n!}\right\}^{1/2} z^n$, $z \in \mathbb{C}$, establish an orthonormal set in $\mathbf{F}_{a,b,c}$. Already we know that the $u_n(a,b,c)$ establish an orthonormal basis in $\mathbf{H}[f_{a,b,c}]$.
Now for $\phi \in \mathbf{F}_{a,b,c}$ the series

$$\sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{n!}\, z^n$$

converges to ϕ uniformly on each disc $D_r = \{z \in \mathbb{C} \mid |z| \le r\}$. So we have

$$\int_{D_r} \Big[\sum_{n,m=0} \frac{\phi^{(n)}(0)\,\overline{\phi^{(m)}(0)}}{n!\,m!}\,(x+iy)^n\,(x-iy)^m\Big]\, G_{a,b,c}(x^2+y^2)\,dx\,dy =$$

$$= \sum_{n,m=0} \frac{\phi^{(n)}(0)\,\overline{\phi^{(m)}(0)}}{n!\,m!} \int_{D_r} (x+iy)^n\,(x-iy)^m\, G_{a,b,c}(x^2+y^2)\,dx\,dy =$$

$$= \sum_{n=0}^{\infty} \left[\frac{|\phi^{(n)}(0)|}{n!}\right]^2 \Big[\pi \int_0^r t^n\, G_{a,b,c}(t)\,dt\Big].$$

Letting $r \to \infty$ we obtain the identity

$$\|\phi\|^2_{a,b,c} = \sum_{n=0}^{\infty} \frac{|\phi^{(n)}(0)|^2}{n!}\,\frac{(b)_n}{(a)_n}\,\frac{1}{c^n}.$$

Thus the result follows. □

(1.5) *Special cases.*

* $a=b=1,\ c>0$

$$G_{1,1,c}\,(x^2+y^2)=\exp\,[-c(x^2+y^2)\,].$$

The space $\mathbf{F}_{1,1,c}$ equals the Bargmann-Fock space with reproducing kernel

$$f_{1,1,c}\,(z\,\overline{w})=\exp[-c\,z\,\overline{w}\,]$$

* $a=1,\ b,c>0$

$$G_{1,b,c}\,(x^2+y^2)=c^{b/2}\,\Gamma(b)\,t^{b/2-1}\,\exp[-\tfrac{1}{2}\,ct\,]\,W_{\frac{b}{2},\frac{b}{2}-\frac{1}{2}}\,(ct)$$

$$=c^{b/2}\,\Gamma(b)\,(x^2+y^2)^{b-1}\,\exp[-c(x^2+y^2)\,].$$

The space $\mathbf{F}_{1,b,c}$ equals the generalized (b,c)-Fischer space with reproducing kernel

$$f_{1,b,c}(z\,\overline{w})={}_1F_1(1,b\,;c\,z\,\overline{w})$$

* $b=1,\ a,c>0$

$$G_{a,\,1,c}(x^2+y^2)=\frac{c^{1/2}}{\Gamma(a)}\,(x^2+y^2)^{-1/2}\,\exp[-\tfrac{1}{2}\,c(x^2+y^2)]\,W_{\frac{3-2a}{2},0}(c(x^2+y^2))$$

$$=\frac{c^{1/2}}{\Gamma(a)}\,\exp[-c(x^2+y^2)]\,L_{1-a}\,(c(x^2+y^2))$$

where

$$L_{1-a}(t)=\sum_{n=0}^{\infty}\frac{(a-1)_n}{n!}\,\frac{t^n}{n!}\,,\quad t\in\mathbb{R}.$$

(1.6) **Corollary.**
Let $a,b,c>0$. The functions $\phi\in\mathbf{F}_{a,b,c}$ satisfy the following growth estimate

$$|\,\phi(z)\,|=O(|\,z\,|^{a-b}\,\exp(\tfrac{1}{2}\,c\,|z|^2)),\quad |\,z\,|>1.$$

Proof.
By (0.3) for each $z\in\mathbb{C}$ we have

$$|\,\phi(z)\,|\leq\|\,\phi\|_{a,b,c}\,({}_1F_1(a,b\,;c\,|z|^2))^{1/2}.$$

So the result follows from the asymptotics of the confluent hypergeometric functions for large values of the argument. □

(1.7) **Corollary.**
Let $a,b,c>0$. Then $\phi\in\mathbf{F}_{a,b,c}$ iff ϕ is entire analytic and

$$\sum_{n=0}^{\infty} \frac{|\phi^{(n)}(0)|^2}{n!} \frac{(n+1)^{b-a}}{c^n} < \infty.$$

Proof.

Since the limit $\lim_{n \to \infty} (n+1)^{a-b} \frac{(b)_n}{(a)_n}$ exists, the assertion is a consequence of the previous theorem. □

(1.8) **Corollary.**

Let $c > 0$ and let $a, b, \tilde{a}, \tilde{b} > 0$ with $a - b = \tilde{a} - \tilde{b}$. Then $\mathbf{F}_{a,b,c} = \mathbf{F}_{\tilde{a},\tilde{b},c}$ as function spaces with equivalent inner products. □

On $F_{a,b,c} \times F_{b,a,\frac{1}{c}}$ we introduce the sesquilinear form

(1.9) $$<\phi, \psi>_{a,b,c} = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)\,\overline{\psi^{(n)}(0)}}{n!}$$

which is well-defined because

$$\sum_{n=0}^{\infty} \left| \frac{\phi^{(n)}(0)\,\psi^{(n)}(0)}{n!} \right| \leq \|\phi\|_{a,b,c}\, \|\psi\|_{b,a,\frac{1}{c}}.$$

Since for each $r > 0$

$$\frac{1}{\pi} \int_{|x+iy| \leq r} \psi(x+iy)\, \overline{\phi(x+iy)} \exp[-(x^2+y^2)]\, dx\, dy =$$

$$= \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0)\,\overline{\phi^{(n)}(0)}}{n!} \left[\frac{1}{n!} \int_0^r t^n e^{-t}\, dt\right]$$

it follows letting $r \to \infty$ that

(1.10) $$<\psi, \phi> = \frac{1}{\pi} \int_{\mathbb{R}^2} \psi(x+iy)\, \overline{\phi(x+iy)} \exp(-x^2-y^2)\, dx\, dy.$$

2. Projective and inductive limits

Let **A** denote the vector space of all entire analytic functions endowed with the Frechet topology generated by the norms

$$\|\phi\|_k = \sup_{n \in \mathbb{N}_0} |\phi^{(n)}(0)| \frac{k^n}{n!}.$$

Dual to **A** is the vector space **E** of all entire analytic functions of exponential type. So $\psi \in \mathbf{E}$ if there are $K > 0$ and $c > 0$ such that

$$|\psi(z)| \leq K \exp(c\,|z|).$$

The space **E** is a countable inductive limit of Banach spaces. To be more specific,

$$\mathbf{E} = \bigcup_{k=1}^{\infty} \mathbf{E}_k$$

where $\mathbf{E}_k$ is the subspace of **E** consisting of all $\psi \in \mathbf{E}$ with the property that

$$\sup_{n \in \mathbb{N}} |\psi^{(n)}(0)|\, k^{-n} < \infty.$$

The spaces **E** and **A** are each other's strong duals where the duality is established by the sesquilinear form

$$<\psi,\phi> = \sum_{n=0}^{\infty} \frac{\psi^{(n)}(0)\,\overline{\phi^{(n)}(0)}}{n!}\ , \quad \psi \in \mathbf{E}\,,\ \phi \in \mathbf{A}.$$

As in (1.10) we have

$$<\psi,\phi> = \frac{1}{\pi} \int_{\mathbb{R}^2} \psi(x+iy)\,\overline{\phi(x+iy)}\exp(-x^2-y^2)\,dx\,dy.$$

The spaces $\mathbf{E}$, $\mathbf{F}_{1,1,1}$ and **A** constitute a Gelfand triple,

(2.1) $\qquad \mathbf{E} \hookrightarrow \mathbf{F}_{1,1,1} \hookrightarrow \mathbf{A}$

where $<\psi,\phi> = (\psi,\phi)_{1,1,1}$ for all $\psi \in \mathbf{E}$ and $\phi \in \mathbf{F}_{1,1,1}$, cf. [AnVa]. The space **E** is about the smallest space that contains the coherent states e_w, $e_w(z) = \exp(\overline{w}\,z)$. So for all $\phi \in \mathbf{A}$

$$\phi(w) = \overline{<e_w,\phi>}\ , \quad w \in \mathbb{C}.$$

The following lemma indicates that the $\mathbf{F}_{a,b,c}$ give rise to continuous scales of Hilbert spaces.

(2.2) **Lemma.**
The continuous and dense inclusion $\mathbf{F}_{a,b,c} \hookrightarrow \mathbf{F}_{\tilde{a},\tilde{b},\tilde{c}}$ holds true in the following cases

* $c < \tilde{c}$ and $a,b,\tilde{a}$ and $\tilde{b}$ arbitrary,
* $c = \tilde{c}$ and $\tilde{a} - a \geq \tilde{b} - b$.

Proof.
Cf. assertion (0.6) of the preliminaries. □

Clearly all functional Hilbert spaces $\mathbf{F}_{a,b,c}$ are contained in **A** and contain **E** as a dense subspace. The triple (2.1) extends in the following obvious way

$$\mathbf{E} \hookrightarrow \mathbf{F}_{a,b,c} \hookrightarrow \mathbf{F}_{\tilde{a},\tilde{b},\tilde{c}} \hookrightarrow \mathbf{F}_{1,1,1} \hookrightarrow \mathbf{F}_{\tilde{b},\tilde{a},\frac{1}{\tilde{c}}} \hookrightarrow \mathbf{F}_{b,a,\frac{1}{c}} \hookrightarrow \mathbf{A}.$$

Here each space on the left hand side is in duality with a space on the right hand side, where the duality is established by the form

$$\frac{1}{\pi} \int_{\mathbb{R}^2} \psi(x+iy)\, \overline{\phi(x+iy)} \exp(-x^2-y^2)\, dx\, dy.$$

The monomials $u_n(z) = \dfrac{z^n}{\sqrt{n!}}$ form an orthonormal basis in $\mathbf{F}_{1,1,1}$, consisting of eigenfunctions of the differential operator $R = z\dfrac{d}{dz} + 1$ with eigenvalues $n+1$, $n \in \mathbb{N}_0$. In the next lemma we describe the relation between the spaces $\mathbf{F}_{a,b,c}$ and the self-adjoint operator R.

(2.3) **Lemma.**

* Let $0 < c < 1$ and let $a,b > 0$

$$\mathbf{F}_{a,b,c} = R^{\frac{1}{2}(a-b)} \exp[\tfrac{1}{2}(\log c) R]\,(\mathbf{F}_{1,1,1}).$$

* Let $b \geq a > 0$

$$\mathbf{F}_{a,b,1} = R^{\frac{1}{2}(a-b)}\,(\mathbf{F}_{1,1,1}).$$

* Let $a > b > 0$
  $\mathbf{F}_{a,b,1}$ is the completion of $\mathbf{F}_{1,1,1}$ with respect to the norm $\phi \mapsto \| R^{\frac{1}{2}(b-a)} \phi \|_{1,1,1}$, $\phi \in \mathbf{F}_{a,b,1}$.

* Let $c > 1$ and let $a,b > 0$
  $\mathbf{F}_{a,b,c}$ is the completion of $\mathbf{F}_{1,1,1}$ with respect to the norm $\phi \mapsto \| R^{\frac{1}{2}(b-a)} \exp[-\frac{1}{2}(\log c) R]\, \phi \|_{1,1,1}$, $\phi \in \mathbf{F}_{a,b,c}$.

Proof.
For each $\phi \in \mathbf{F}_{1,1,1}$ we have

$$\phi = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(0)}{\sqrt{n!}}\, u_n.$$

So the assertions are consequences of Corollary (1.7) and the spectral theorem for self-adjoint operators. □

We consider the following chains of Hilbert spaces

$$\{\mathbf{F}_{a,1,1} \mid a > 0\},\ \{\mathbf{F}_{1,b,1} \mid b > 0\},\ \{\mathbf{F}_{1,1,c} \mid c > 0\}.$$

They yield the following inductive / projective limits, which fit in the general set up of the paper [EGK].

* The projective limit $\bigcap_{b>0} \mathbf{F}_{1,b,1}$ which is in strong duality with the inductive limit $\bigcup_{a>0} \mathbf{F}_{a,1,1}$.

* The inductive limit $\bigcup_{0<c<1} \mathbf{F}_{1,1,c}$ which is in strong duality with the projective limit $\bigcup_{c>1} \mathbf{F}_{1,1,c}$.

(2.4) **Lemma.**

* The projective limit $\bigcap_{b>0} \mathbf{F}_{1,b,1}$ equals the space of all C^∞-vectors of the operator R, i.e.

$$D^\infty(R) := \bigcap_{n=1}^{\infty} D(R^n) = \bigcap_{b>0} \mathbf{F}_{1,b,1}.$$

* The inductive limit $\bigcup_{0<c<1} \mathbf{F}_{1,1,c}$ equals the space of all analytic vectors of the operator R, i.e.

$$D^\omega(R) := \bigcup_{t>0} e^{-tR}(\mathbf{F}_{1,1,1}) = \bigcup_{0<c<1} \mathbf{F}_{1,1,c}.$$

The operator R is unitarily equivalent to the positive self-adjoint operator H in $L_2(\mathbb{R})$ defined by $H = \frac{1}{2}(-\frac{d^2}{dx^2} + x^2 + 1)$. Indeed, let ψ_n, $n = 0,1,2,\cdots$, denote the n-th Hermite function defined by the formula

$$\psi_n(x) = \frac{(-1)^n}{(\sqrt{\pi}\; n!\; 2^{2n})^{1/2}}\, e^{\frac{1}{2}x^2} \left[\frac{d}{dx}\right]^n (e^{-x^2}).$$

The functions ψ_n establish an orthonormal basis in $L_2(\mathbb{R})$. They are eigenfunctions of the self-adjoint operator H with

$$H\psi_n = (n+1)\psi_n.$$

Now the linear operator A on $L_2(\mathbb{R})$ defined by

$$(Af)(z) = \int_{\mathbb{R}} A(z,x)\, f(x)\, dx$$

where

$$A(z,x) = \pi^{-1/4} \exp[-\tfrac{1}{2}(z^2 + x^2) + \sqrt{2}\, z\, x],$$

maps $L_2(\mathbb{R})$ unitarily onto $\mathbf{F}_{1,1,1}$. In particular,

$$A\,\psi_n = u_n\,,\quad n = 0,1,2,\cdots$$

and

$$A\,H\,A^* = R.$$

The Hermite functions are also eigenfunctions of the Fourier transformation $\mathbb{F}$ on $L_2(\mathbb{R})$, viz. $\mathbb{F}\,\psi_n = (i)^n\,\psi_n$. Thus in a natural way Fourier invariant test- and distribution spaces arise from series expansions with respect to the Hermite functions. We mention Schwarz' test space S of C^∞-functions of rapid decrease and the Gelfand-Shilov spaces S_α^α, $\alpha \geq ½$. Namely, the following characterizations are valid.

(2.5) **Lemma.**

* The space S consists of precisely all square integrable functions ϕ for which $(\phi,\psi_n)_{L_2} = O(n^{-k})$ for all $k \in \mathbb{N}$.
* For each $\alpha \geq ½$, the space S_α^α consists of precisely all square integrable functions ϕ for which $(\phi,\psi_n) = O(\exp(-n^{½\alpha}\,t))$ for some $t > 0$. In particular,

$$S_{½}^{½} = D^\omega(H) = \bigcup_{t>0} e^{-tH}(L_2(\mathbb{R})).$$

Proof.
Cf. [Si] and [Go]. □

Consequently, we have the following results.

(2.6) **Corollary.**

* For each $b > 0$ the image of $H^{-b}(L_2(\mathbb{R}))$ under A equals $\mathbf{F}_{1,b+1,1}$. In particular, $A(S) = \bigcap_{b>0} \mathbf{F}_{1,b+1,1}$.
* For each $t > 0$ the image of $e^{-tH}(L_2(\mathbb{R}))$ under A equals $\mathbf{F}_{1,1,e^{-t}}$. In particular $A(S_{½}^{½}) = \bigcup_{0<c<1} \mathbf{F}_{1,1,c}$.
* For each $a > 0$, let $H^a(L_2(\mathbb{R}))$ denote the completion of $L_2(\mathbb{R})$ with respect to the norm $f \mapsto \|H^{-a}f\|_{L_2(\mathbb{R})}$. Then A extends to $H^a(L_2(\mathbb{R}))$ with $A(H^a(L_2(\mathbb{R}))) = \mathbf{F}_{a+1,1,1}$. In particular, $A(S') = \bigcup_{a>0} \mathbf{F}_{a,1,1}$.

Remark. These results are in correspondence with the results stated in [Ba].

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