Riesz bases of special polynomials in weighted Sobolev spaces of analytic functions

Eindhoven University of Technology
Department of Mathematics and Computing Science

RANA 90-14
December 1990

RIESZ BASES OF SPECIAL POLYNOMIALS IN WEIGHTED SOBOLEV SPACES OF ANALYTIC FUNCTIONS

by
J. de Graaf

Reports on Applied and Numerical Analysis
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven
The Netherlands

RIESZ BASES OF SPECIAL POLYNOMIALS IN WEIGHTED SOBOLEV SPACES OF ANALYTIC FUNCTIONS

by

J. de Graaf

Abstract

The main subject of this paper is the construction of Riesz bases in Weighted Sobolev Spaces of Analytic Functions on open sets in C. For this two main tools are introduced.
First, starting from weighted Sobolev spaces on a disc or an annulus and supplied with the Taylor basis, we move to an 'arbitrary' open set by means of a conformal mapping. In many cases our weighted Sobolev spaces behave naturally under an analytic pull-back. The analytic functions in our spaces are characterized by the asymptotic behaviour of expansion coefficients and by boundary conditions in ordinary Sobolev spaces.
The second tool is of a completely different nature: By means of upper triangular transition matrices S we construct new Riesz bases out of a given one. Our transition matrices are diagonalizers of a class of given upper triangular matrices. Thus we are able to make estimations in both S and S^{-1} at once. In one of the applications we describe the domains of exponentiated square roots of Jacobi operators in ordinary Sobolev spaces on $[-1,1]$. This case was left in [GE]. We also relate this to a refinement of Szegö's result on series of Jacobi polynomials on an ellips. (Thm 4.1).

Contents:

1. Weighted Sobolev spaces of analytic functions on the annulus.
2. Weighted Sobolev spaces of analytic functions on open domains.
3. The general functional analytic classification problem.
4. Application to the Jacobi operators.

1. Weighted Sobolev spaces of analytic functions on the annulus.

For an arbitrary subset Ω of the complex plane $\mathbb{C}$ we denote by $\mathcal{A}(\Omega)$ the set of complex valued functions on Ω which are analytic at each point $z = x + iy \in \Omega$.
Now let Ω be open. For a fixed weight function $\mu \in L_1(\Omega)$, $\mu \geq 0$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ we define

$$\|\cdot\|_{\Omega,\mu,m} \quad : \quad \mathcal{A}(\Omega) \to [0,\infty] ,$$

by

$$f \mapsto \Big(\int\limits_{\Omega} \{|f(z)|^2 + |f^{(m)}(z)|^2\} \ \mu(z) \ dxdy \Big)^{\frac{1}{2}} .$$

We introduce the complex vectorspace

$$\mathcal{A}_2(\Omega;\mu,m) = \{ f \mid \|f\|_{\Omega,\mu,m} < \infty \}$$

provided with the inner product

$$(f,g)_{\Omega,\mu,m} = \int\limits_{\Omega} \{ f(z) \ \overline{g(z)} + f^{(m)}(z) \ \overline{g^{(m)}(z)} \} \ \mu(z) \ dxdy .$$

Instead of the triple $(\Omega,\mu,0)$ we denote (Ω,μ). The first special open set Ω that we meet is the *annulus* $A = \{z \mid 0 < a < |z| < b < \infty\}$ for fixed a and b. The *special annulus* A_T is defined by $A_T = \{z \mid e^{-T} < |z| < e^{T}\}$, for some fixed $T > 0$.
The subset $A^{\varepsilon} \subset A$, $0 < \varepsilon < \frac{1}{2}(b-a)$, is defined by $A^{\varepsilon} = \{z \mid 0 < a < |z| < a + \varepsilon \ \text{ or } \ b - \varepsilon < |z| < b\}$.
For any given weight function μ on A the new weight function μ_{ε} on A is defined to be zero on the annulus $A \backslash A^{\varepsilon}$ and equal to μ on A^{ε}. So, μ_{ε} equals μ near the boundary of A.

THEOREM 1.1.
Let the weight function μ on the annulus A be such that

- $\forall z \in A \quad \mu(z) = \mu(|z|)$
- $\exists \varepsilon > 0 \ \ \forall z \in A^{\varepsilon} \ \ \mu(z) > 0$ and μ is continuous at z.

Let $m \in \mathbb{N}_0$ be fixed.

(A) $\forall f \in \mathcal{A}(A) \ \ \forall \varepsilon > 0 \ \ \forall j \in \mathbb{N}_0 \ \ \exists M_{\varepsilon,j} > 0$

$$\sup \{ |f^{(j)}(z)| \ |z \in A \backslash A^{\varepsilon} \} \leq M_{\varepsilon,j} \Big(\int\limits_{A^{\varepsilon}} |f(z)|^2 \ \mu(z) \ dxdy \Big)^{\frac{1}{2}} .$$

(B) The space $\mathcal{A}_2(A;\mu,m)$ is a Hilbert space.
The functions $\{z \mapsto z^n \mid n \in \mathbb{Z}\}$ establish an *orthogonal* basis in $\mathcal{A}_2(A;\mu,m)$.

(C) For $f \in \mathcal{A}(A)$ expressed as a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n\, z^n$ we have

$$f \in \mathcal{A}_2(A;\mu,m) \Longleftrightarrow \sum_{n=-\infty}^{\infty} |a_n|^2\, \|z^n\|^2_{A,\mu,m} = \sum_{n=-\infty}^{\infty} 2\pi\, \alpha_n^{-2}\, |a_n|^2 < \infty\ .$$

Here $(\alpha_n)^{-2} = \int_a^b r^{2n+1}\, \mu(r)dr + |(n-m+1)_m|^2 \int_a^b r^{2(n-m)+1}\, \mu(r)dr$, with Pochhammer's symbol

$$(n-m+1)_m = n\cdot(n-1)\cdot\ldots\cdot(n-m+1)\ .$$

So $(\alpha_n z^n)_{n=-\infty}^{\infty}$ is a Riesz basis in $\mathcal{A}_2(A;\mu,m)$. (That means: each $g \in \mathcal{A}_2$ has a unique expansion $g = \sum_{n=-\infty}^{\infty} a_n(\alpha_n z^n)$, with $(a_n)_{n=-\infty}^{\infty} \in l_2$, which converges in $\mathcal{A}_2$.)

(D) If μ is replaced by ν (not necessarily $\nu(z) = \nu(|z|)$ such that

$$\exists\varepsilon > 0\ \exists\delta > 0\ \exists M > 0\ \forall z \in A^\varepsilon\quad 0 < \delta < \frac{\nu(z)}{\mu(z)} < M < \infty\ ,$$

then the spaces $\mathcal{A}_2(A;\mu,m)$ and $\mathcal{A}_2(A;\nu,m)$ have equivalent norms, i.e. they are the same as topological vector spaces. Note that in both these Hilbert spaces the functions $\{z \mapsto \alpha_n\, z^n \mid n \in \mathbb{Z}\}$ establish a Riesz basis.

(E) Let $j \in \mathbb{N},\ 0 < j < m$. Then

$$\forall\varepsilon > 0\ \exists M^\varepsilon > 0\ \forall f \in \mathcal{A}_2(A;\mu,m)$$

$$\|f^{(j)}\|^2_{A,\mu} \le M_\varepsilon \|f\|^2_{A,\mu} + \varepsilon \|f^{(m)}\|^2_{A,\mu}\ .$$

(F) Let m distinct point $\underline{z} = (z_0,\ldots,z_{m-1}) \subset A$ be given. The norm $\|\cdot\|_{A,\underline{z},\mu,m}$, defined by

$$\|f\|^2_{A,\underline{z},\mu,m} = \sum_{k=0}^{m-1} |f(z_k)|^2 + \|f^{(m)}\|^2_{A,\mu}\ ,$$

is equivalent to the norm $\|\cdot\|_{A,\mu,m}$.

Proof

(A) We use the "mean value property" for analytic functions

$$f^{(j)}(w) = c_j\, \rho^{-(2j+2)} \int_{|w-z|<\rho} f(z)\, (\bar{z} - \bar{w})^j\, dxdy\ ,\quad j \in \mathbb{N}_0\ ,$$

with $c_j = (2\pi)^{-1}\,(2j+2)j!$. Take w on one of the circles $|z| = b - \frac{1}{2}\varepsilon$, $|z| = a + \frac{1}{2}\varepsilon$. Take $\rho = \frac{1}{4}\varepsilon$.
Let $\theta > 0$ be such that on $A^{\frac{3}{4}\varepsilon} \backslash A^{\frac{1}{4}\varepsilon}$ one has $\mu(z) > \theta$. Now application of the Cauchy Schwarz inequality to

$$f^{(j)}(w) = c_j \left(\frac{\varepsilon}{4}\right)^{-(2j+2)} \int_{|w-z|<\frac{3}{4}} f(z) \frac{(\bar{z}-\bar{w})^j}{\mu} \mu \, dxdy$$

leads to

$$|f^{(j)}(w)| \le M_{\varepsilon,j} \|f\|_{A^\varepsilon,\mu}$$

with $M_{\varepsilon,j} = (\pi\theta)^{-\frac{1}{2}} (j+1)^{\frac{1}{2}} \left(\frac{4}{3}\right)^{j+1} j!$.
Because of the maximum principle this result extends to the annular domain between the two circles.

(B) Because of part (A) a Cauchy sequence in $\mathcal{A}_2(A;\mu,m)$ converges uniformly on compact sets in A. So, the limit exists as an analytic function. Further, since $\mu(z) = \mu(|z|)$ it is clear that the functions $\{z^n \mid n \in \mathbb{Z}\}$ are orthogonal. We show that the span of these functions is dense: Let $g \in \mathcal{A}_2(A;\mu,m)$ and suppose $g \perp z^n$ for all $n \in \mathbb{Z}$. Write $g = \sum_{n=-\infty}^{\infty} b_n z^n$, evaluate $(g, z^n)_{A\backslash A^\varepsilon,\mu,m}$ and take the limit $\varepsilon \downarrow 0$. This leads to

$$b_n\{\|z^n\|^2_{A,\mu,m} + |n\cdot(n-1)\cdot\ldots\cdot(n-m+1)|^2 \, \|z^{n-m}\|^2_{A,\mu,m}\} = 0 \; .$$

Hence the result that all $b_n = 0$.

(C) Let $f = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathcal{A}(A)$. Since the functions $\{z^n \mid n \in \mathbb{Z}\}$ are mutually orthogonal in all spaces $\mathcal{A}_2(A\backslash A^\varepsilon;\mu,m)$ we have

$$\|f\|^2_{A\backslash A^\varepsilon,\mu,m} = \sum_{n=-\infty}^{\infty} |a_n|^2 \, \|z^n\|^2_{A\backslash A^\varepsilon,\mu,m} \; .$$

Now let $\varepsilon \downarrow 0$ and observe that the sum is finite iff the integral is finite.

(D) First we show that we arrive at equivalent norms of μ is replaced by μ_ε. On one hand there is the trivial inequality

$$\|f\|^2_{A,\mu_\varepsilon,m} \le \|f\|^2_{A,\mu,m} \; .$$

On the other hand

$$\|f\|^2_{A,\mu,m} = \|f\|^2_{A^\varepsilon,\mu,m} + \|f\|^2_{A\backslash A^\varepsilon,\mu,m} \; .$$

The first term is equal to $\|f\|^2_{A,\mu_\varepsilon,m}$. The second term can be estimated by

$$(M^2_{\varepsilon,0} + M^2_{\varepsilon,m}) \left(\int_A \mu(z)\, dxy\right) \|f\|^2_{A,\mu_\varepsilon} \, ,$$

because of (A). Next we compare ν with μ_ε. On one hand we have

$$\|f\|^2_{A,\mu^\varepsilon,m} = \|f\|^2_{A^\varepsilon,\mu,m} \le \frac{1}{\delta} \|f\|^2_{A^\varepsilon,\nu,m} \le \frac{1}{\delta} \|f\|^2_{A,\nu,m} \; .$$

On the other hand

$$\|f\|^2_{A,\nu,m} = \|f\|^2_{A^\epsilon,\nu,m} + \|f\|^2_{A\setminus A^\epsilon,\nu,m} \leq$$

$$\leq M\ \|f\|^2_{A^\epsilon,\mu,m} + (M^2_{\epsilon,0} + M^2_{\epsilon,m})\ (\int\limits_A \nu(z)dxdy)\ \|f\|^2_{A,\mu_\epsilon}$$

$$\leq \{M + (M^2_{\epsilon,0} + M^2_{\epsilon,m}) \int\limits_A \nu(z)dxdy\}\ \|f\|^2_{A^\epsilon,\mu,m}\ .$$

(E) Write

$$\|f^{(j)}\|^2_{A,\mu} = \sum_{k=-\infty}^{\infty} |a_k|^2\ |k\cdot(k-1)\cdot\ldots\cdot(k-j+1)|^2\ \|z^{k-j}\|^2_{A,\mu}\ .$$

Because of $\|z^{k+l}\|^2_{A,\mu} \leq b^{2l}\ \|z^k\|^2_{A,\mu}$, if $l \in \mathbb{N}_0$, we estimate, with $N \in \mathbb{N}$

$$\|f^{(j)}\|^2_{A,\mu} \leq \sum_{|k|\leq N} |a_k|^2\ |k\cdot(k-1)\cdot\ldots\cdot(k-j+1)|^2\ \|z^{k-j}\|^2_{A,\mu}+$$

$$+b^{2(m-j)} \sum_{|k|>N} \frac{|k\cdot(k-1)\cdot\ldots\cdot(k-m+1)|^2}{|(k-j)\cdot\ldots\cdot(k-m+1)|^2}\ |a_k|^2\ \|z^{k-m}\|^2_{A,\mu}\ .$$

Let $\varepsilon > 0$ be given. Take N so large that for all $k \in \mathbb{Z}$, $|k| > N$, $|(k-j)\cdot\ldots\cdot(k-m+1)|^{-2}\ b^{2(m-j)} < \varepsilon$.
Then the second sum can be estimated by $\varepsilon\|f^{(m)}\|^2_{A,\mu}$.
The first sum can be estimated by $M_\epsilon\|f\|^2_{A,\mu}$ if we take

$$M_\epsilon = \max_{-N\leq k\leq N}\ \{|k\cdot\ldots\cdot(k-j+1)|^2\ \|z^{k-j}\|^2_{A,\mu}\ \|z^k\|^{-2}_{A,\mu}\}\ .$$

(F) Because of part (A) of the theorem there is a constant c_1, which depends only the choice of $z_0, \ldots, z_{m-1}$, such that

$$\sum_{k=0}^{m-1} |f(z_k)|^2 \leq c_1\|f\|^2_{A,\mu}\ .$$

Therefore

$$\|f\|^2_{A,\underline{z},\mu,m} \leq (c_1+1)\ \|f\|^2_{A,\mu,m}\ .$$

For the converse inequality, split

$$f = p + f_1 \quad \text{with} \quad p(z) = \sum_{k=0}^{m-1} a_k\ z^k\ .$$

Since p is a polynomial of, at most, degree $m-1$, there is a constant $c_2 > 0$ such that

$$\|p\|_{A,\mu}^2 \le c_2 \sum_{k=0}^{m-1} |p(z_k)|^2 \le$$

$$\le 2\ c_2 \sum_{k=0}^{m-1} |f(z_k)|^2 + 2\ c_2 \sum_{k=0}^{m-1} |f_1(z_k)|^2 .$$

Again, part (A) of the theorem, there is $c_3 > 0$,

$$\|f\|_{A,\mu}^2 \le 2\ c_2 \sum_{k=0}^{m-1} |f(z_k)|^2 + (2c_2c_3 + 1)\ \|f\|_{A,\mu}^2 .$$

Finally,

$$\|f_1\|_{A,\mu}^2 = \sum_{k \neq 0,\dots,m} |a_k|^2\ \|z^k\|_{A,\mu}^2 \le$$

$$\le b^{2m} \sum_{k=-\infty}^{\infty} |a_k|^2\ |k \cdot (k-1) \cdot \ldots \cdot (k-m+1)|^2\ \|z^{k-m}\|_{A,\mu}^2 =$$

$$= b^{2m}\ \|f^{(m)}\|_{A,\mu}^2 .$$

(Remind that b is the outer radius of A.) □

Note on part (D) of the theorem. In the estimate $\|\cdot\|_{A,\mu,m}^2 \le C\|\cdot\|_{A,\mu_\epsilon,m}^2$ the best possible constant C is given by

$$C = 1 + \max\{(\int_a^b \mu(r)dr)\ (\int_{b-\epsilon}^b \mu(r)dr)^{-1},\ (\int_a^b \mu(r)dr)\ (\int_a^{a+\epsilon} \mu(r)dr)^{-1}\}$$

as a simple monotonicity argument shows.

In the next theorem we consider important closed linear subspaces of $\mathcal{A}_2(A;\mu,m)$ and Riesz bases for them. We define four linear subspaces $\mathcal{A}(A;+)$, $\mathcal{A}(A;-)$, $\mathcal{A}(A;e)$, $\mathcal{A}(A;o)$ of $\mathcal{A}(A)$ in the following way: $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ belongs to those spaces if, respectively, $\forall n \in \mathbb{N}\ :\ a_{-n} = 0$, $\forall n \in \mathbb{N}\ :\ a_n = 0$, $\forall n \in \mathbb{Z}\ :\ a_n = a_{-n}$, $\forall n \in \mathbb{Z}\ :\ a_n = -a_{-n}$.

THEOREM 1.2
Suppose μ satisfies the conditions of Theorem 1.1.

(A) The sets

$$W_+ = \mathcal{A}_2(A;+;\mu,m) = \mathcal{A}_2(A;\mu,m) \cap \mathcal{A}(A;+)$$

$$W_- = \mathcal{A}_2(A;-;\mu,m) = \mathcal{A}_2(A;\mu,m) \cap \mathcal{A}(A;-)$$

are closed linear subspaces in $\mathcal{A}_2(A;\mu,m)$.
The functions in those subspaces can be continued analytically into, respectively, the whole disc $\{z \mid |z| < b\}$, and the set $\{z \mid |z| > a\} \cup \{\infty\}$.

(B) Consider the case $A = A_T$. Define the mapping $Z : \mathcal{A}(A_T) \to \mathcal{A}(A_T)$ by $(Zf)(z) = f(\frac{1}{z})$.
Suppose $\exists C > 0\ \forall z \in A_T\ \mu(\frac{1}{z}) \le C|z|^4\ \mu(z)$.
Then the operator Z is a continuous bijection on $\mathcal{A}_2(A^T;\mu,m)$. Further, the sets

$$W_e = \mathcal{A}_2(A_T;e;\mu,m) = \mathcal{A}_2(A_T;\mu,m) \cap \mathcal{A}(A;e)$$

$$W_o = \mathcal{A}_2(A_T;o;\mu,m) = \mathcal{A}_2(A_T;\mu,m) \cap \mathcal{A}(A;o)$$

are closed linear subspaces of $\mathcal{A}_2(A_T;\mu,m)$.

(C) With the same additional conditions on μ, the functions

$$\{\beta_0, \beta_n(z^n + z^{-n}),\ \ \beta_n(z^n - z^{-n}) \mid n \in I\!N\}$$

with

$$\beta_n = \alpha_n\ \alpha_{-n}(\alpha_n^2 + \alpha_{-n}^2)^{-\frac{1}{2}}$$

establish a Riesz basis in $\mathcal{A}_2(A_T;\mu,m)$.

Proof.

(A) Elementary.

(B) First, we show that Z is well defined.
Suppose $f \in \mathcal{A}_2(A_T;\mu,m)$. If both $\|Z f\|^2_{A_T,\mu} < \infty$ and $\|(Z f)^{(m)}\|^2_{A_T,\mu} < \infty$ then $Z f \in \mathcal{A}_2(A_T;\mu,m)$. We only prove the second inequality. By induction one shows

$$\frac{d^m}{dz^m} f\left(\frac{1}{z}\right) = \sum_{j=1}^{m} f^{(j)}\left(\frac{1}{z}\right)\ p_{jm}\left(\frac{1}{z}\right)\ ,$$

where the p_{jm} are polynomials of degree at most $2m$. These p_{jm} do not depend on f. (Cf. Faà di Bruno's formula [AS].)
Put $M = \sup\{|p_{jm}(\frac{1}{z})|\ |z \in A_T,\ 1 \le j \le m\}$. Now

$$\int_{A_T} |(Z f)^{(m)}(z)|^2\ \mu(z)dxdy \le m\,M^2 \sum_{j=1}^{m} \int_{A_T} |f^{(j)}\left(\frac{1}{z}\right)|^2\ \mu(z)dxdy\ .$$

Transform the latter integral $z \mapsto \frac{1}{z}$ and use the condition on μ. Then

$$\|(Z\,f)^{(m)}\|^2_{A,\mu} \le m\,M^2 \sum_{j=1}^{m} \int_{A_T} |f^{(j)}(z)|^2\,|z|^{-4}\,\mu\left(\frac{1}{z}\right)dxdy$$

$$\le m\,M^2 C \sum_{j=1}^{m} \int_{A_T} |f^{(j)}(z)|^2\,\mu(z)dxdy\ .$$

After removing the intermediate derivatives with Theorem 1.1 (E), we arrive at

$$\|Z\,f\|^2_{A_T,\mu,m} \le C_1\ \|f\|^2_{A_T,\mu,m}$$

for some $C_1 > 0$ not dependend on f.
Hence Z is a continuous bijection with $Z^2 = I$. The mentioned subspaces are merely the ranges of the respective projection operators $\frac{1}{2}(I+Z)$ and $\frac{1}{2}(I-Z)$.

(C) First we show that from our assumptions follows the existence of constants A_1 and A_2 such that for all $n \in I\!N_0$

$$(*) \qquad 0 < A_1 < \frac{\alpha_{-n}}{\alpha_n} < B_1\ .$$

Note that, with $b = e^T$,

$$(\alpha_{-n})^{-2} = 2\pi \int_{b^{-1}}^{b} r^{-2n+1}\ \mu(r)dr+$$

$$+2\pi|(-n)\cdot\ldots\cdot(-n-m+1)|^2 \int_{b^{-1}}^{b} r^{-2(n+m)+1}\ \mu(r)dr\ .$$

Change the variable $r \mapsto \frac{1}{r}$. Take $n > 2(m-1)$

$$(\alpha_{-n})^2 = 2\pi \int_{b^{-1}}^{b} r^{2n-3}\ \mu\left(\frac{1}{r}\right)\ dr+$$

$$+2\pi[n(n+1)\cdot\ldots\cdot(n+m-1)]^2 \int_{b^{-1}}^{b} r^{2(n+m)-3}\ \mu\left(\frac{1}{r}\right)\ dr \le$$

$$\le 2\pi\,C \int_{b^{-1}}^{b} r^{2n+1}\ \mu(r)\ dr + 2\pi\,C\ b^{4m}[n\cdot\ldots\cdot(n+m-1)]^2 \int_{b^{-1}}^{b} r^{2(n-m)+1}\ \mu(r)\ dr$$

$$\le 2\pi\,C(1 + b^{4m}\,2^{4m})\ (\alpha_n)^{-2}\ .$$

An inequality of type $(\alpha_n)^2 \le C_m(\alpha_{-n})^2$ is derived in a similar way.

Now consider $f \in \mathcal{A}_2(A_T; m, \mu)$

$$f(z) = \sum_{n=-\infty}^{\infty} a_n \alpha_n z^n = a_0 \alpha_0 + \sum_{n=1}^{\infty} \frac{1}{2}(a_n \alpha_n + a_{-n} \alpha_{-n}) (z^n + z^{-n}) +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{2}(a_n \alpha_n - a_n \alpha_{-n}) (z^n - z^{-n}) .$$

We write

$$f(z) = a_0 \alpha_0 + \sum_{n=1}^{\infty} b_n \beta_n (z^n + z^{-n}) + \sum_{n=1}^{\infty} c_n \beta_n (z^n - z^{-n})$$

with

$$\begin{matrix} b_n \\ c_n \end{matrix} = \frac{1}{2} a_n \left(1 + \left(\frac{\alpha_n}{\alpha_{-n}}\right)^2\right)^{\frac{1}{2}} \begin{matrix} + \\ - \end{matrix} \frac{1}{2} a_{-n} \left(1 + \left(\frac{\alpha_{-n}}{\alpha_n}\right)^2\right)^{\frac{1}{2}} .$$

Because of (*) the sequences (b_n) and (c_n) are l_2 iff (a_n) is l_2. □

Note that in case μ is of the form

$$\mu(z) = \mu(|z|) = |z|^{-2} \chi(|z|) \chi(|z|^{-1}) ,$$

with χ arbitrary, and $m = 0$, the basis in Theorem 1.2 (C) is orthonormal.

DEFINITION 1.3.
Let ∂A denote the boundary of A. We have $\partial A = \partial_a A \cup \partial_b A$ with $\partial_a A$ and $\partial_b A$ the circles $|z| = a$ and $|z| = b$ respectively.
For $f \in \mathcal{A}(A)$ and $a < r < b$ consider the function

$$f_r : [-\pi, \pi] \to C , \quad t \mapsto f(r e^{it}) .$$

We say that f satisfies a boundary condition in the Sobolev space $H^\nu(\partial_b A)$, $\nu \in \mathbb{R}$, if $f_b = \lim_{r \uparrow b} f_r$ exists in $H^\nu_{\text{per}}([-\pi, \pi])$.
Similarly we say that $f_a \in H^\theta(\partial_a A)$, $\theta \in \mathbb{R}$, if $f_a = \lim_{r \uparrow a} f_r \in H^\theta_{\text{per}}([-\pi, \pi])$.

THEOREM 1.4.
Consider the Hilbert space $\mathcal{A}_2(A; \mu_{\theta\nu}, m)$ with $\mu_{\theta\nu}(|z|) = (b - |z|)^{2\nu-1} (|z| - a)^{2\theta-1}$.
Let $f \in \mathcal{A}(A)$, $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$.
The following three conditions are equivalent

(i) $f \in \mathcal{A}_2(A; \mu_{\theta\nu}, m)$

(ii) $(b^n n^{m-\nu} a_n)_n^\infty \in l_2$ and $(a^n |n|^{m-\theta} a_n)_{n=-\infty}^0 \in l_2$

(iii) f satisfies the boundary conditions

$$f_a \in H^{m-\theta}(\partial_a A) \quad \text{and} \quad f_b \in H^{m-\nu}(\partial_b A)\ .$$

So the functions $\{1,\ b^{-n}\, n^{-(m-\nu)}\, z^n,\ a^n\, n^{-(m-\theta)}\, z^{-n} \mid n \in I\!N\}$ establish a Riesz basis in $\mathcal{A}_2(A; \mu_{\theta\nu}, m)$.

Proof.
We apply Theorem 1.1 (C), so we have to calculate the asymptotic behaviour for $k \to \pm\infty$ of the integral

$$\gamma_{2k+1} = \int_a^b r^{2k+1}(b-r)^{2\nu-1}\,(r-a)^{2\theta-1}\,dr =$$

$$b^{2\nu-1}\,a^{2\theta-1}\int_{1/b}^{1/a} r^{-(2k+1+2\nu+2\theta)}\left(\frac{1}{a}-r\right)^{2\theta-1}\left(r-\frac{1}{b}\right)^{2\nu-1}\,dr =$$

$$= b^{2k+1}(b-a)^{2\nu+2\theta-1}\int_0^1 (1-s)^{2\theta-1}\,s^{2\nu-1}\Big(1-s\left(1-\frac{a}{b}\right)\Big)^{2k+1}\,ds =$$

$$= \left(\frac{1}{a}\right)^{-(2k+1)}\left(\frac{a}{b}\right)^{2\theta}(b-a)^{2\nu+2\theta-1}\ .$$

$$\cdot\int_0^1 (1-s)^{2\nu-1}\,s^{2\theta-1}\Big(1-s\left(1-\frac{a}{b}\right)\Big)^{-(2k+1+2\nu+2\theta)}\,ds\ .$$

Note that, for $0 < B < 1\ ,\ \ l \in I\!N$

$$(l\,B)^{2\nu}\int_0^1 (1-s)^{2\theta-1}\,s^{2\nu-1}(1-s\,B)^l\,ds =$$

$$\int_0^1 (1-s)^{2\theta-1}\,(l\,B\,s)^{2\nu-1}\left(1-\frac{1}{l}\,l\,B\,s\right)^l l\,B\,ds =$$

$$= \int_0^{lB}\left(1-\frac{\sigma}{l\,B}\right)^{2\theta-1}\sigma^{2\nu-1}\left(1-\frac{\sigma}{l}\right)^l\,d\sigma \to$$

$$\rightarrow \int_0^\infty \sigma^{2\nu-1}\, e^{-\sigma}\, d\sigma = \Gamma(2\nu)\ , \quad \text{as } l \rightarrow \infty\ .$$

Since

$$(\alpha_n)^{-2} = \gamma_{2n+1} + |(n-m+1)_m|^2\, \gamma_{2(n-m)+1}$$

we find that the following two limits exist

$$\lim_{n\rightarrow\infty} \alpha_n[b^n\, n^{m-\nu}] = L_1 > 0\ ,$$

$$\lim_{n\rightarrow -\infty} \alpha_n\Big[\Big(\frac{1}{a}\Big)^{-n}\ |n|^{m-\nu}\Big] = L_2 > 0\ .$$

(i) $\Longleftrightarrow$ (ii) We apply Theorem 1.1 (C). Since $b > a$ we have $(\alpha_n^{-1}\, a_n)_{n=-\infty}^{\infty} \in l_2$ iff

$$(b^n\, |n|^{m-\nu}\, a_n)_{n=-\infty}^{\infty} \in l_2 \quad \underline{\text{and}} \quad (a^n\, |n|^{m-\theta}\, a_n)_{n=-\infty}^{\infty} \in l_2$$

(ii) $\Longleftrightarrow$ (iii) Note that

$$f_b = \sum_{n=-\infty}^{\infty} a_n\, b^n\, e^{in\phi} \in H^{m-\nu}(\partial_b A)$$

iff $(a_n\, b^n\, n^{m-\nu})_{n=-\infty}^{\infty} \in l_2$.

For f_a the argument is similar. □

As an application of Theorem 1.2 (B) we find

COROLLARY 1.5.
In the Hilbert space $\mathcal{A}_2(A_T; \mu_\nu, m)$, with $\mu_\nu(|z|) = (e^T - |z|)^{2\nu-1}\, (|z| - e^{-T})^{2\nu-1}$, $T > 0$, $\nu > 0$, the functions

$$\{1,\ e^{-nT}\, n^{-(m-\nu)}(z^n + z^{-n}),\ \ e^{-nT}\, n^{-(m-\nu)}(z^n - z^{-n})\ |\ n \in I\!N\}$$

establish a Riesz basis.

2. Weighted Sobolev spaces of analytic functions on $\Omega \subset C$.

Consider an open set $\Omega \subset C$ and a function $\psi \in \mathcal{A}(\Omega)$ which maps Ω bijectively onto an open subset $\psi(\Omega) \subset A$, the annulus of section 1. The linear subspace $\mathcal{A}(\Omega;\psi) \subset \mathcal{A}(\Omega)$ is defined by a pull-back

$$\mathcal{A}(\Omega;\psi) = \{F \mid F(\zeta) = f(\psi(\xi)),\ \ \zeta = \xi + i\eta \in \Omega,\ \ f \in \mathcal{A}(A)\}\ .$$

There is a bijective linear correspondence

$$J_\psi\ :\ \mathcal{A}(A) \to \mathcal{A}(\Omega;\psi)\ ,\ \ (J_\psi f)\ (\zeta) = f(\psi(\zeta))\ ,$$

so $J_\psi(\mathcal{A}(A)) = \mathcal{A}(\Omega;\psi)$.
Note that $\psi(\Omega) = A$ implies $\mathcal{A}(\Omega;\psi) = \mathcal{A}(\Omega)$. The Hilbert subspace $J_\psi(\mathcal{A}_2(A;\mu,m))$ carries the inner product

$$(F,G)_\psi = (J_\psi^{-1} F,\ J_\psi^{-1} G)_{A,\mu,m}\ .$$

In this section we investigate the relation between the mentioned "pull back inner product" and an ordinary Sobolev inner product in $\mathcal{A}(\Omega)$. In the next Lemma an auxilliary result concerning the chain rule is presented. The proof is omitted. An explicit expression for the R_{mj} is given by Faà di Bruno's formula [AS].

LEMMA 2.1.
Let f and ψ be analytic functions. We have

(i) $(f \circ \psi)^{(m)}(\zeta) = \sum\limits_{j=1}^{m} (f^{(j)} \circ \psi)\ (\zeta) \cdot R_{mj}(\zeta)$.

- The R_{mj}, $m \in I\!N$, $1 \leq j \leq m$, are homogeneous polynomials of degree j in ψ', $\psi^{(2)}, \ldots, \psi^{(m-j+1)}$. Put $R_{m,j} = 0$ if $j < 1$ or $j > m$.
- $R_{m+1,j} = R'_{m,j} + \psi' \cdot R_{m,j-1}$, $R_{11} = \psi'$.
- $R_{mm} = (\psi')^m\ ,\quad R_{m1} = \psi^{(m)}$.

(ii) $f^{(m)}(z) = \left(\dfrac{d\psi^{\leftarrow}}{dz}\right)^m \left\{(f \circ \psi)^{(m)}(\psi^{\leftarrow}(z)) - \sum\limits_{j=1}^{m-1} f^{(j)}(z)\ R_{mj}(\psi^{\leftarrow}(z))\right\}$

THEOREM 2.2.
Consider an open set $\Omega \subset C$. Let $\psi \in \mathcal{A}(\Omega)$, $\psi\ :\ \Omega \to A$ be injectief. Denote $\Omega^\epsilon = \psi^{\leftarrow}(A^\epsilon)$. Let $\omega \in L_1(\Omega)$ be a weight function and let $m \in I\!N_0$ be fixed. Finally, let $\mu \in L_1(A)$ be a weight function on A which satisfies the conditions of Theorem 1.1.
Suppose

(a) $\sup_{0\leq q\leq m,\ \zeta\in\text{supp}\,\omega} |\psi^{(q)}(\zeta)| = M_1 < \infty$

(b) $\exists\varepsilon > 0\ \exists M_2 > 0\ \exists M_3 > 0\ \forall\zeta \in \Omega^\varepsilon$
$M_2\ \mu(\psi(\zeta))\ |\psi'(\zeta)|^2 \leq \omega(\zeta) \leq M_3\ \mu(\psi(\zeta)\ |\ |\psi'(\zeta)|^2$

(c) If $m > 0$ then
$\exists\varepsilon > 0\ \exists M_4 > 0\ \forall\zeta \in \Omega^\varepsilon\ |\psi'(\zeta)|^{-1} \leq M_4.$

Then

(A) If $\psi(\Omega)$ is dense in A, then on $\mathcal{A}(\Omega;\psi)$ the norms $F \mapsto \|J_\psi^{-1} F\|_{A,\mu,m}$ and $F \mapsto \|F\|_{\Omega,\omega,m}$ are equivalent. So the functions $(\alpha_n(\psi(\zeta))^n)_{n=-\infty}^{\infty}$ establish a Riesz basis in $\mathcal{A}_2(\Omega;\psi;\omega,m)$.

(B) Let W be a closed subspace of $\mathcal{A}_2(A;\mu,m)$ and suppose that the norms $\|\cdot\|_{A,\mu,m}$ and $\|\cdot\|_{\psi(\Omega),\mu,m}$, when restricted to W, are equivalent. Then on the Hilbert space $J_\psi(W) \subset \mathcal{A}(\Omega;\psi;W)$ the norms $F \mapsto \|J_\psi^{-1} F\|_{\psi(\Omega);\mu,m}$ and $F \mapsto \|F\|_{\Omega,\omega,m}$ are equivalent. So $(\sigma_n(z))_{n\in I}$ is a Riesz basis in W iff $(\sigma_n(\psi(\zeta))_{n\in I}$ is a Riesz basis in $\mathcal{A}_2(\Omega;\psi;W;\omega,n)$.

Proof.

(A1) Let $F \in \mathcal{A}(\Omega;\psi)$. There exists a unique $f \in \mathcal{A}(A)$ such that $F = f\circ\psi$. For $\varepsilon > 0$ sufficiently small we estimate the two terms of

$$\|F^{(m)}\|^2_{\Omega,\mu,m} = \|F^{(m)}\|^2_{\Omega^\varepsilon,\mu,m} + \|F^{(m)}\|^2_{\Omega\backslash\Omega^\varepsilon,\mu,m}$$

separately. With Lemma 2.1.(i) and the condition on ω we estimate

$$\int_{\Omega^\varepsilon} |(f\circ\psi)^{(m)}(\zeta)|^2\ \omega(\zeta)\ d\xi d\eta \leq$$

$$\leq c_1 \sum_{j=1}^{m} \int_{\Omega^\varepsilon} |(f^{(j)}\circ\psi)\ (\zeta)|^2\ \mu(\psi(\zeta))\ |\psi'(\zeta)|^2\ d\xi d\eta\ .$$

The constant c_1 depends on m, M_1, M_3 and the shape of the R_{mj}. The latter expression is equal to

$$c_1 \sum_{j=1}^{m} \int_{A^\varepsilon} |f^{(j)}(z)|^2\ \mu(z)\ dxdy \leq c_2\ \|f\|^2_{A,\mu,m}\ .$$

The constant c_2 depends on c_1 and on the constants mentioned in Theorem 1.1, parts D and E.
Next we show that

$$\exists c_3 > 0\ \forall\zeta \in \Omega\backslash\Omega^\varepsilon\ |F^{(m)}(\zeta)| \leq c_3\ \|f\|_{A,\mu}\ .$$

Indeed, for $\zeta \in \text{supp}\ \omega \cap (\Omega\backslash\Omega^\varepsilon)$ write

$$F^{(m)}(\zeta) = (f \circ \psi)^{(m)}(\zeta) =$$

$$= \frac{1}{2\pi i} \sum_{j=1}^{m} (-1)^j \, j! \, R_{mj}(\zeta) \int_K f(z) \, (\psi(\zeta) - z)^{-1-j} \, dz \, .$$

Here K consists of two suitably oriented circles in the interior of $A^{\frac{2}{3}\epsilon} \backslash A^{\frac{1}{3}\epsilon}$. On K we have, with the constants of Theorem 1.1.(A), $|f(z)| \leq M_{\frac{1}{3}\epsilon,0} \, \|f\|_{A^{\frac{1}{3}},\mu}$. From this we infer the existence of c_3. We gather that

$$\|F^{(m)}\|^2_{\Omega,\mu} \leq (c_2 + c_3\|\omega\|_1) \, \|f\|^2_{A,\mu,m} \, .$$

Since such an estimate also holds for $m = 0$ we arrive at the existence of $c_4 > 0$ such that

$$\|F\|^2_{\Omega,\mu,m} \leq c_4 \, \|f\|^2_{A,\mu,m} \, .$$

(A2) Now let $f \in \mathcal{A}(A)$. With Lemma 2.1.(ii) and conditions (b), (c)

$$\|f^{(m)}\|^2_{A^\epsilon,\mu} \leq c_4 \int_{A^\epsilon} |(f \circ \psi)^{(m)} \, (\psi^{\leftarrow}(z))|^2 \, \mu(z) \, dxdy +$$

$$+ c_5 \sum_{j=1}^{m-1} \int_{A^\epsilon} |f^{(j)}(z) \, R_{mj}(\psi^{\leftarrow}(z))|^2 \, \mu(z) \, dxdy \, .$$

The constant c_5 depends only on M_4 and m.
From the boundedness of the R_{mj} and application of Theorem 1.1.(E) it follows that

$$\|f^{(m)}\|^2_{A^\epsilon,\mu} \leq c_5 \int_{A^\epsilon} |(f \circ \psi)^{(m)} \, (\psi^{\leftarrow}(z)|^2 \, \mu(z) \, dxdy + c_6 \, \|f\|^2_{A^\epsilon,\mu} \, .$$

The constant c_6 depends only on c_5, M_1 and the shape of the R_{mj}. Transform the integral over A^ϵ to an integral over Ω^ϵ and estimate again

$$\|f^{(m)}\|^2_{A^\epsilon,\mu} \leq c_6 \, M^{-1} \, \|F^{(m)}\|^2_{\Omega^\epsilon,\omega} + c_6 \, \|f\|^2_{A^\epsilon,\mu} \, .$$

Add to this expression the inequality

$$2\, c_6 \, \|f\|^2_{A^\epsilon,\mu} \leq 2\, c_6 \, M^{-1} \, \|F\|^2_{\Omega^\epsilon,\mu} \, .$$

Finally, with Theorem 1.1.D, we are led to

$$\|f\|^2_{A,\mu,m} \leq c_7 \, \|f\|^2_{A^\epsilon,\mu,m} \leq c_8 \, \|F\|^2_{\Omega^\epsilon,\omega,m} \leq c_8 \, \|F\|^2_{\Omega,\omega,m} \, .$$

(B) The proof of part (B) is merely a simple adaptation of the proof of (A). □

REMARK 2.3.
The conditions (a) and (c) of Theorem 2.2 are automatically satisfied if $\bar{\Omega}$ is compact and $\psi \in \mathcal{A}(\bar{\Omega})$.

A simple modification of Theorem 2.2 is the following

THEOREM 2.4.
Consider an open set $\Omega \subset C$. Let $\psi \in \mathcal{A}(\Omega)$, $\psi : \Omega \to A$ be injective. Let $\omega \in L_1(\Omega)$ be a weight function such that for some $M_1 > 0$, $M_2 > 0$

$$M_1\, \mu(\psi(\zeta)) \leq \omega(\zeta) \leq M_2\, \mu(\psi(\zeta))\,.$$

Let $W \subset \mathcal{A}_2(A;\mu,1)$ be a closed subspace as in Theorem 2.2.(B). Fix $p \in \Omega$. Then on the Hilbert space $J_\psi(W)$ the norms

$$F \mapsto \|J_\psi^{-1} F\|_{\psi(\Omega);\mu,1} \quad \text{and} \quad F \mapsto (|F(p)|^2 + \|F'\|^2_{\Omega,\omega})^{\frac{1}{2}}$$

are equivalent.
The mapping J_ψ becomes unitary if $\omega(\zeta) = \mu(\psi(\zeta))$ and $\mathcal{A}_2(A;\mu,1)$ carries the norm

$$f \mapsto (|f(\psi(p))|^2 + \|f'\|^2_{A,\mu})^{\frac{1}{2}}\ .$$

□

In the next theorem we study a class of weight functions on A and Ω which can easily be compared. We need

CONDITION 2.5.
Suppose that $\psi : \Omega \to A$ extends continuously to $\psi_e : \bar{\Omega} \to \bar{A}$. Fix a subset $\Delta \subset \partial\bar{\Omega}$. Consider the distance functions $\operatorname{dis}(\zeta,\Delta)$ on Ω and $\operatorname{dis}(z,\psi(\Delta))$ on $\psi(\Omega) \subset A$. Let $\varepsilon > 0$ be fixed. Assume

(i) Each $\zeta \in \Omega^\varepsilon$ can be connected with Δ by means of a differentiable curve within Ω^ε and which has length $\operatorname{dis}(\zeta,\Delta)$.

(ii) Each $z \in \psi(\Omega^\varepsilon)$ can be connected with $\psi_e(\Delta)$ by means of a differentiable curve within $\psi(\Omega^\varepsilon)$ and which has length $\operatorname{dis}(z,\psi(\Delta))$.

THEOREM 2.6.
Assume Conditions 2.5. Suppose in addition

$$\exists M > 0\ \forall \zeta \in \Omega^\epsilon\ |\psi'(\zeta)| \le M \quad \text{and} \quad |\psi'(\zeta)|^{-1} \le M\ .$$

Let $\theta > -1$. On Ω^ϵ define $\omega(\zeta) = (\mathrm{dis}(\zeta, \Delta))^\theta$.
On $A^\epsilon \cap \psi(\Omega)$ define $\mu(z) = (\mathrm{dis}(z, \psi(\Delta)))^\theta$.
Then

$$\forall \zeta \in \Omega^\epsilon \quad M^{-|\theta|}\ \mu(\psi(\zeta)) \le \omega(\zeta) \le M^{|\theta|}\ \mu(\psi(\zeta))$$

and condition (b) of Theorem 2.2 is satisfied.

Proof.
Let $\zeta \in \Omega^\epsilon$. Take a curve $s \mapsto \chi(s)$ within $\psi(\Omega^\epsilon)$ which connects $\psi(\zeta)$ and $\psi(\Delta)$ and which has length $L = \mathrm{dis}(\psi(\zeta),\ \psi(\Delta))$. Estimate

$$\mathrm{dis}(\zeta, \Delta) \le \int_0^L \left| \frac{d\psi^{\leftarrow}(\chi(s))}{ds} \right| ds \le ML = M\ \mathrm{dis}(\psi(\zeta),\ \psi(\Delta))\ .$$

Similarly we estimate for $z \in \psi(\Omega^\epsilon)$

$$\mathrm{dis}(z, \psi(\Delta)) \le M\ \mathrm{dis}(\psi^{\leftarrow}(z), \Delta)\ .$$

Taking the θ-th power leads to the desired inequalities. $\square$

APPLICATION I.
Consider the rectangle

$$\Omega = S_T = \{\zeta = \xi + i\,\eta \mid \xi \in (-\pi,\pi)\ ,\ \ \eta \in (-T,T)\}\ .$$

Take $\psi(\zeta) = e^{i\zeta}$, $\psi(S_T) = A_T \setminus (-\infty, 0)$. Note that $\psi(S_T)$ is an open dense set in A_T. We find that $\mathcal{A}(S_T;\psi) = \mathcal{A}(S_T; 2\pi\text{-per})$, i.e. the space of functions on S_T which extend to 2π-periodic analytic functions.
For the weight functions on S_T and A_T take

$$\omega_\theta(\zeta) = \omega_\theta(\xi + i\eta) = (T-\eta)^{2\theta-1}(\eta+T)^{2\theta-1}\ ,$$

$$\mu_\theta(|z|) = \mu_\theta(|z|) = (e^T - |z|)^{2\theta-1}(|z| - e^{-T})^{2\theta-1}\ .$$

Note that the conditions (a), (b), (c) of Theorem 2.2 are satisfied. Combining Theorem 1.4 and Theorem 2.2.A we find that for all $\theta > 0$ and all $m \in \mathbb{N}_0$

$$J_\psi\ :\ \mathcal{A}_2(A_T;\mu_\theta, m) \to \mathcal{A}_2(S_T; 2\pi\text{−per}; \omega_\theta, m)$$

is a continuous bijection between both Hilbert spaces. As a corollary thereof, the functions

$$(\zeta \mapsto |n|^{-(m-\theta)}\, e^{-|n|T}\, e^{in\zeta})_{n\in\mathbb{Z}}$$

establish a Riesz basis in $\mathcal{A}_2(S_T; 2\pi\text{-per};\omega_\theta, m)$. Note also that $f \in \mathcal{A}(S_T; 2\pi\text{-per})$ belongs to $\mathcal{A}_2(S_T; 2\pi\text{-per};\omega_\theta, m)$ iff the limits of $\xi \mapsto f(\xi + i\eta)$ as $\eta \uparrow T$ or $\eta \downarrow -T$ exist in the Sobolev space $H^{m-\theta}_{\text{per}}(-\pi,\pi)$.
A combination of Theorem 1.2.B and Theorem 2.2.B leads to the Riesz bases

$$(\zeta \mapsto n^{-(m-\theta)}\, e^{-nT}\ \cos\, n\,\zeta)_{n\in\mathbb{N}_0}$$

$$(\zeta \mapsto n^{-(m-\theta)}\, e^{-nT}\ \sin\, n\,\zeta)_{n\in\mathbb{N}}$$

for the even and odd subspaces of $\mathcal{A}_2(S_T; 2\pi\text{-per};\omega_\theta, m)$ respectively.

APPLICATION II.
Consider the ellips $E_T = \{\zeta = \xi + i\eta \mid 1 - \left(\frac{\xi^2}{\cosh^2 T} + \frac{\eta^2}{\sinh^2 T}\right) > 0\}$.
Take $\Omega = E_T' = E_T \setminus (-\infty, -1] \cup [1, \infty)$.
Take $\psi(\zeta) = e^{i\arccos\zeta}$. Then ψ maps E_T' bijectively onto A_T^+, i.e. the part of A_T which lies in the open upper half plane.
Note that $J_\psi(\mathcal{A}(A_T; e)) = \mathcal{A}(E_T)$ because of its symmetry $f(e^{i\arccos\zeta})$ does not jump at the real axis and extends to an analytic function on the whole of E_T.
Further, since on $\mathcal{A}(A_T; e)$ the norms $\|\cdot\|_{A_T,\mu,m}$ and $\|\cdot\|_{A_T^+,\mu,m}$ are equivalent for any choice of μ which satisfies the conditions of Theorem 1.1, we can apply Theorem 2.2.B.
In particular we apply Theorem 2.6. Take $\Delta = \partial E_T$. On E_T introduce the weight function $\omega_\theta(\zeta) = (\text{dis}(\zeta, \partial E_T))^{2\theta-1}$, $\theta > 0$. On A_T^+ introduce μ_θ as in Application I. We find that for all $\theta > 0$ and all $m \in \mathbb{N}_0$

$$J_\psi \ : \ \mathcal{A}_2(A_T; e; \mu_\theta, m) \to \mathcal{A}_2(E_T; \omega_\theta, m)$$

is a continuous bijection between both Hilbert spaces. As a consequence the Chebyshev polynomials

$$(n^{-(m-\theta)}\, e^{-nT}\, T_n(\zeta))_{n\in\mathbb{N}_0}$$

with $T_n(\zeta) = \cos(n \ \arccos \ \zeta)$, $n \in \mathbb{N}$ and $T_0(\zeta) = \pi^{-\frac{1}{2}}$ establish a Riesz basis in $\mathcal{A}_2(E_T;\omega_\theta, m)$.

As a corollary of the general Theorems in sections 1 and 2 we mention the following

Characterization Result:
Let $f \in \mathcal{A}([-1,1])$, $f(x) = \sum_{n=0}^{\infty} a_n T_n(x)$, let $\theta > 0$, $m \in \mathbb{N}_0$, then the following three conditions are equivalent:

- $f \in \mathcal{A}_2(E_T; \omega_\theta, m)$
- $(n^{(m-\theta)}\, e^{nT}\, a_n)_{n=0}^{\infty} \in l_2$
- $f \in \mathcal{A}(E_T)$ and $f|_{\partial E_T} \in H^{m-\theta}(\partial E_T)$

In the next section we extend this expansion characterization to a much wider class of suitably normalized polynomials which encompasses all (normalized) Jacobi polynomials. In this way Szegö's result [Sz] is refined in several directions at once.

APPLICATION III.
Consider the set $\Omega = B_T = \mathbb{R}\times i\,(-\frac{\pi}{2},\ \frac{\pi}{2}) \setminus \bar{\mathcal{U}}_T$ with $\mathcal{U}_T = \{\xi + i\eta \mid |\sin\xi| > \tanh T \ \cosh\eta\}$. Note that $\bar{\mathcal{U}}_T$ is an infinite set of "ovals" centered at the points $\xi + i\eta = i(k+\frac{1}{2})\pi$, $k \in \mathbb{Z}$. Take $\psi(\zeta) = e^{2i \operatorname{arctan} \sinh \zeta} = \exp 2i \int_0^\zeta \frac{1}{\cosh w}\, dw$, $\psi(B_T) = A_T \setminus (-\infty, 0)$, which is open and dense in A_T. We find that $\mathcal{A}(B_T;\psi) = \mathcal{A}(B_T; 2\pi i\text{-per}, \partial e\infty)$. A function g belongs to this class iff it extends to $\tilde{g} \in \mathcal{A}(\mathbb{C}\setminus\bar{\mathcal{U}}_T)$ such that

(i) $\tilde{g}$ is $2\pi i$-periodic

(ii) $\forall\zeta \in \mathbb{C}\ \ \tilde{g}(i\frac{\pi}{2}+\zeta) = \tilde{g}(i\frac{\pi}{2}-\zeta)$

(iii) $\lim_{|\Re\zeta|\to\infty} \tilde{g}(\zeta)$ exists.

There are natural subspaces of $\mathcal{A}(B_T;\psi)$ which exhibit even more symmetry:

- $J_\psi(\mathcal{A}(A_T; e)) = \mathcal{A}(B_T; \pi i\text{-per, even, } \partial e\infty)$ which is the subspace of even πi-periodic functions.
- $J_\psi(\mathcal{A}(A_T; o)) = \mathcal{A}(B_T; 2\pi i\text{-per, odd, } \partial e\infty)$ which is the subspace of odd $2\pi i$-periodic functions.

Now, let $\Delta = \partial\mathcal{U}_T \cap \bar{B}_T$. Denote the function

$$\delta^\theta(\zeta) = (\min\{\text{dis}(\zeta,\Delta),1\})^{2\theta-1}, \quad \theta > 0 \ .$$

For the weight function on B_T we take

$$\omega_\theta(\zeta) = \delta^\theta(\zeta)\ |\cosh\zeta|^{-2} \ .$$

On A_T introduce the weight function μ_θ, as in Application I. For $\varepsilon < (1-e^{-T})$ the conditions of Theorem 2.2 are satisfied (use Theorem 2.6). So we find that for all $\theta > 0$ and all $m \in I\!N_0$

$$J_\psi \ : \ \mathcal{A}_2(A_T;\mu_\theta,m) \to \mathcal{A}_2(B_T;2\pi i\text{-per},\partial e\infty;\omega_\theta,m)$$

is a continuous bijection between both Hilbert spaces. As a consequence we find for the subspace $\mathcal{A}_2(B_T;\pi i\text{-per, even, } \partial e\infty;\omega_\theta,m)$ a Riesz basis

$$\left(n^{-(m-\theta)}\, e^{-nT}\, T_{2n}\left(\frac{1}{\cosh\zeta}\right)\right)_{n=0}^\infty$$

and for the subspace $\mathcal{A}_2(B_T;2\pi i\text{-per, odd, } \partial e\infty;\omega_\theta,m)$ we have the Riesz basis

$$\left(n^{-(m-\theta)}\, e^{-nT}\, (\tanh\zeta)\, U_{2n-1}\left(\frac{1}{\cosh\zeta}\right)\right)_{n=1}^\infty .$$

Here the polynomials U en T are defined by $T_n(\cos\theta) = \cos\, n\,\theta$ and $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$. As a corollary of the general theorems in sections 1 and 2 we mention the following

Characterization Result:
Let $f \in \mathcal{A}(I\!R)$, $f(x) = \sum_{n=0}^\infty a_n\, T_{2n}\left(\frac{1}{\cosh x}\right) + \tanh x\ \sum_{n=1}^\infty b_n\, U_{2n-1}\left(\frac{1}{\cosh x}\right)$. Let $\theta > 0$, $m \in I\!N_0$, then the following three conditions are equivalent:

- $f \in \mathcal{A}_2(B_T;2\pi i\text{-per},\partial e\infty;\omega_\theta,m)$
- $(n^{(m-\theta)}\, e^{nT}\, a_n)_{n=0}^\infty \in l_2$ and $(n^{(m-\theta)}\, e^{nT}\, b_n)_{n=1}^\infty \in l_2$
- $f \in \mathcal{A}(\mathbb{C}\setminus\bar{\mathcal{U}}_T)$, f is $2\pi i$-periodic,

$$\lim_{|\Re\zeta|\to\infty} f(\zeta) \text{ exists} \quad \text{and} \quad f|_{\partial\mathcal{U}_T} \in H_{loc}^{m-\theta}(\partial\mathcal{U}_T) \ .$$

Finally it follows that

$$\left(\pi^{-1}(\cosh x)^{-\frac{1}{2}}\, T_{2n-2}\left(\frac{1}{\cosh x}\right),\ \pi^{-1}(\cosh x)^{-\frac{1}{2}}\ \tanh x\, U_{2n-1}\left(\frac{1}{\cosh x}\right)\right)_{n=1}^\infty$$

is a complete orthonormal system in $L_2(I\!R)$.
In [BG1] and [BG2] we have characterized the Fourier and Fourier-Jacobi images of some of the spaces mentioned here.

APPLICATION IV.
Consider the set $\Omega = R_{ab} = \mathbb{C} \setminus (D_a \cup D_b)$ with $0 < a < 1 < b$. Here D_c, $c > 0$, denotes the closed disc $\{\zeta \mid |\zeta + i\,\frac{c^2+1}{c^2-1}| \leq \frac{2c}{|c^2-1|}\}$.
Take $\psi(\zeta) = \frac{i-\zeta}{i+\zeta}$, then $\psi(R_{ab}) = A \setminus \{-1\}$.
We find that $\mathcal{A}(\Omega;\psi) = \mathcal{A}(R_{ab};\infty)$, i.e. the space of functions which are analytic outside the discs D_a and D_b and at infinity.
Note that if we take $W_+ = \mathcal{A}(A;+)$ then

$$\mathcal{A}(\Omega;\psi;W_+) = \mathcal{A}(\mathbb{C} \setminus D_b;\infty)$$

and, similarly

$$\mathcal{A}(\Omega;\psi;W_-) = \mathcal{A}(\mathbb{C} \setminus D_a;\infty)\ .$$

Let $\Delta = \partial D_a \cup \partial D_b$. Denote the function

$$\delta^\theta(\zeta) = (\min\{\mathrm{dis}(\zeta,\Delta),1\})^{2\theta-1},\ \ \theta > 0\ .$$

For the weight functions on R_{ab} and A we take

$$\omega_\theta(\zeta) = \delta^\theta(\zeta)\ |i+\zeta|^{-4}$$

$$\mu_\theta(\zeta) = (b-|z|)^{2\theta-1}\ (|z|-a)^{2\theta-1}\ .$$

For $\varepsilon < \frac{1}{2}(b-a)$ the conditions of Theorem 2.2 are satisfied (via Theorem 2.6).
So we find that for all $\theta > 0$ an all $m \in \mathbb{N}_0$

$$J_\psi\ :\ \mathcal{A}_2(A;\mu_\theta,m) \to \mathcal{A}_2(R_{ab};\infty;\omega_\theta,m)$$

is a continuous bijection between both Hilbert spaces.
As a consequence, for $\mathcal{A}_2(R_{ab};\infty;\omega_\theta,m)$ we have the Riesz basis

$$\Big\{1,\ b^{-n}\,n^{-(m-\theta)}\ \left(\frac{i-\zeta}{i+\zeta}\right)^n,\ \ a^n\,n^{-(m-\theta)}\ \left(\frac{i-\zeta}{i+\zeta}\right)^{-n}\Big\}_{n\in\mathbb{N}}\ .$$

Finally, it follows that

$$\left((\pi(1+x^2))^{-\frac{1}{2}}\ \frac{i-x}{i+x}\right)_{n\in\mathbb{Z}}$$

is an orthonormal basis in $L_2(\mathbb{R})$.

Spaces of this example play a key role in [D].

3. The general functional analytic classification problem.

Let $K = (K_{mn})_{m,n=0}^{\infty}$ be an infinite matrix with $K_{mn} = 0$ if $m > n$.
We denote $K \in UTM$, i.e. the class of *u*pper *t*riangular *m*atrices. If we suppose that $K_{ii} \neq K_{jj}$ if $i \neq j$ there exists a unique $S \in UTM$ with all $S_{ii} = 1$ such that $S^{-1} K S = \Delta_k = \text{diag}(K_{00}, K_{11}, ...)$. If K has suitable growth properties we construct classes of diagonal matrices $M = \text{diag}(\mu_0, \mu_1, ...)$ such that MSM^{-1} and $MS^{-1}M^{-1}$ are l_2-bounded matrices. In general the growth properties of M are related to those of K.

This general result will enable us to construct a great variety of (polynomial) Riesz bases out of a given one, in the weighted Sobolev spaces of the preceding sections. One of the consequences will be a refinement and an extension of Szegö's classical theorem of Jacobi expansions of analytic functions on an ellips.

We start with a separable Hilbert space X and a selected orthonormal basis $(T_n)_{n=0}^{\infty} \subset X$. Associated with a sequence $\mu = (\mu_n)_{n=0}^{\infty}$, $\mu_n > 0$, $\mu_n \to \infty$ as $n \to \infty$, we introduce a Hilbert space X_μ which is dense in X,

$$X_\mu = \{f \mid f \in X, \sum_{n=0}^{\infty} \mu_n^2 \, |(f, T_n)|^2 < \infty\} \, .$$

Note that $f = \sum_{n=0}^{\infty} a_n T_n \in X_\mu$ iff $\sum_{n=0}^{\infty} \mu_n^2 \, |a_n|^2 < \infty$. Note also that $(\mu_n^{-1} T_n)_{n=0}^{\infty}$ is an orthonormal basis in X_μ. By means of a transition matrix $S \in UTM$ we define a sequence of vectors $(R_n)_{n=0}^{\infty} \subset X$ by $R_n = \sum_{j=0}^{n} S_{jn} T_j$. We take all $S_{nn} = 1$ so that S has an inverse $S^{-1} \in UTM$ of the same type and $T_n = \sum_{j=0}^{n} S_{jn}^{-1} R_j$. Our first question is: When is the sequence $(\mu_n^{-1} R_n)_{n=0}^{\infty}$ a Riesz basis in X_μ?

THEOREM 3.1.
Let $M \in UTM$ be defined by $M = \text{diag}(\mu_0, \mu_1, ...)$. The sequence $(\mu_n^{-1} R_n)_{n=0}^{\infty}$ is Riesz basis in X_μ iff both matrices MSM^{-1} and $MS^{-1}M^{-1}$ are l_2-bounded. This means that they can be regarded as bounded operators in l_2.

Proof.
On $span(T_n)$ define the operators M and S by $\mathrm{M}\, T_n = \mu_n T_n$ and $\mathrm{S}\, T_n = R_n$, followed by linear extension. On $span(T_n)$ the inverse S^{-1} exists and $\mathrm{S}^{-1} R_n = T_n$. Observe that $(\mu_n^{-1} R_n)_{n=0}^{\infty}$ is a Riesz basis in X_μ iff S extends to a continuous bijection on X_μ iff both S and S^{-1} extend to continuous mappings on X_μ iff both matrices MSM^{-1} and MS^{-1} are l_2-bounded. The latter equivalence follows from $\|\mathrm{S}\, f\|_\mu = \|\mathrm{M}\, f\| = \|(\mathrm{MSM}^{-1})\, \mathrm{M}\, f\|$. □

Let K be a densely defined operator in X which acts invariantly on $span(T_n)$. Suppose that, with respect to the basis $(T_n)_{n=0}^{\infty}$, the operator K is represented by a matrix $K \in UTM$ with mutually distinct entries on its diagonal. Then there exists a unique $S \in UTM$, with all $S_{ii} = 1$, such that $S^{-1} K S = \Delta_K = \text{diag}(K_{00}, K_{11}, ...)$. Consequently, there exists a unique sequence $(T_n^K)_{n=0}^{\infty} \subset span(T_n)$ of eigenvectors of K.
Note that $T_n^K = \sum_{j=0}^{n} S_{jn} T_j$ and $\mathrm{K}\, T_n^K = K_{nn} T_n^K$. Now we turn $span(T_n)$ into a pre-Hilbert

space by declaring the sequence $(\mu_n^{-1} T_n^K)_{n=0}^\infty$ to be an *orthonormal* sequence: The Hilbert space completion is then denoted by $X_\mu^{\mathbf{K}}$. Note in particular that $(\mu_n^{-1} T_n^K)_{n=0}^\infty$ is a Riesz basis in X_μ iff $X_\mu = X_\mu^{\mathbf{K}}$ as topological vector spaces. Note that $X_\mu = X_\mu^{\mathbf{M}}$. Two problems can now be posed:

- **The classification problem.**
  For which pairs $(\mathbf{K}, \mu)$, $(\mathbf{K}', \mu')$ do we have $X_\mu^{\mathbf{K}} = X_\mu^{\mathbf{K}'}$ as topological vector spaces. Or, more generally, when a second Hilbert space Y with a similar construction is involved. When do we have $X_\mu^{\mathbf{K}} = Y_\nu^{\mathbf{L}}$?

- **The characterization problem.**
  If X is a function space, describe the elements of $X_\mu^{\mathbf{K}}$ in classical analytic terms.

The characterization problem has been, in fact, the subject of the preceding sections. When dealing with the classification problem we want to apply Theorem 3.1. We solve a problem of the following type: Let there be given an infinite matrix $K \in UTM$ with distinct diagonal elements. Let $S \in UTM$ with all $S_{ii} = 1$ be the diagonalizer of K, so $S^{-1} K S = \Delta_K = \mathrm{diag}(K_{00}, K_{11}, \ldots)$. Find diagonal matrices $M = \mathrm{diag}(\mu_0, \mu_1, \ldots)$ with the property MSM^{-1} and $MS^{-1}M^{-1}$ are l_2-bounded. We consider matrices M of the form $M = e^{t\Lambda}$ with $\mathrm{Re}\, t > 0$ and $\Lambda = \mathrm{diag}(\lambda_0, \lambda_1, \ldots)$. The conditions we impose on K and the conditions on Λ that we look for are growth conditions.

THEOREM 3.2.
Let $K \in UTM$. Put $C_n = \max\limits_{1 \le i < j \le n} |K_{ij}|$. Suppose

(i) $\quad \exists D > 0 \;\forall n > m \quad |K_{mm} - K_{nn}|^{-1} \le \dfrac{D}{n(n-m)}$

(ii) $\quad \sup\limits_{n \ge 1} \left(\dfrac{1}{n} C_n\right) = \delta < \infty$.

Let $\mu \ge 1$ be fixed. Let $\Lambda = \mathrm{diag}(\lambda_0, \lambda_1, \ldots)$ with

(iii) $\quad \mathrm{Re}\, \lambda_n = n^\mu (1 + \varepsilon_1(n)),\; \varepsilon_1(n) \to 0 \;\text{ as }\; n \to \infty$

$$\mathrm{Im}\, \lambda_n = \varepsilon_2(n)\, \mathrm{Re}\, \lambda_n,\; \varepsilon_2(n) \to 0 \;\text{ as }\; n \to \infty$$

$$\mathrm{Re}(\lambda_n - \lambda_m) \ge n^{\mu-1}(n-m)\;(1 + \varepsilon_3(n,m)),\; n > m$$

$$\text{with }\; \varepsilon_3(n,m) \to 0 \;\text{ as }\; \min(n,m) \to \infty$$

$$\mathrm{Im}(\lambda_n - \lambda_m) = \varepsilon_4(n,m)\; \mathrm{Re}(\lambda_n - \lambda_m)$$

$$\text{with }\; \varepsilon_4(n,m) \to 0 \;\text{ as }\; \min(n,m) \to \infty \;.$$

Let $S \in UTM$ be the diagonalizer of K with $S_{ii} = 1,\ 0 \le i < \infty$. The for all $t \in \mathbb{C}$, $\mathrm{Re}\, t > 0$, the matrices

$$e^{t\Lambda}\, S\, e^{-t\Lambda} \quad \text{and} \quad e^{t\Lambda}\, S^{-1}\, e^{-t\Lambda} \text{ are } l_2\text{-bounded.}$$

Proof.

The proof consists of parts (a)-(g).

(a) Since $K\,S = S\,\Delta_K$ and $S^{-1}\,K = \Delta_K\,S^{-1}$ we have the following recurrence relations for the entries of S and S^{-1}:

- $$S_{n-q,n} = (K_{n-q,n} - K_{nn})^{-1} \sum_{k=0}^{q-1} K_{n-q,n}\, S_{n-k,n}\ ,$$

  $$S_{nn} = 1, \quad 1 \le q \le n-1$$

- $$S^{-1}_{m,m+q} = (K_{mm} - K_{m+q,m+q})^{-1} \sum_{k=0}^{q-1} K_{m+k,m+q}\, S^{-1}_{m,m+k}\ ,$$

  $$S^{-1}_{mm} = 1, \quad q \ge 1\ .$$

(b) We show that

$$\left.\begin{matrix} |S_{m,n}| \\ |S^{-1}_{m,n}| \end{matrix}\right\} \le \exp 2\,\{D\,\delta(n-m)\}^{\frac{1}{2}}\ .$$

From the recurrence relations for the entries of S we estimate

$$|S_{n-q,n}| \le \frac{D\,C_n}{n\,q} \sum_{k=0}^{q-1} |S_{n-k,n}|\ , \quad 1 \le q \le n,\ S_{nn} = 1\ .$$

By induction, for $1 \le q \le n$

$$|S_{n-q,n}| \le \frac{D\,C_n}{n\,q} \prod_{l=1}^{q-1} \left(1 + \frac{1}{l}\,\frac{D\,C_n}{n}\right) \le \prod_{l=1}^{q} \left(1 + \frac{1}{l}\,D\,\delta\right) \le e^{2\sqrt{2D\delta}}\ .$$

For the second one

$$|S^{-1}_{m,m+q}| \le \frac{D\,C_{m+q}}{2(m+q)} \sum_{k=0}^{q-1} |S^{-1}_{m,m+k}|\ , \quad q = 1,2,\ldots, S^{-1}_{mm} = 1\ .$$

By induction, $q = 1,2,\ldots,$

$$|S^{-1}_{m,m+q}| \le \frac{D}{q}\,\frac{C_{m+q}}{m+q} \prod_{l=1}^{q-1} \left(1 + \frac{D}{l}\,\frac{C_{m+1}}{m+l}\right) \le \prod_{l=1}^{q} \left(1 + \frac{D}{l}\,\frac{C_{m+l}}{m+l}\right) \le e^{2\sqrt{qD\delta}}\ .$$

(c) Put $t = t_1 + i\,t_2$ and $\sigma^t_{mn} = (e^{t\Lambda}\,S\,e^{-t\Lambda})_{mn} = S_{mn}\,e^{-t(\lambda_n-\lambda_m)}$. So with (b)

$$|\sigma^t_{mn}| \leq \exp\{2[D\,\delta(n-m)]^{\frac{1}{2}} - t_1\,\mathrm{Re}(\lambda_n-\lambda_m) + t_2\,\mathrm{Im}(\lambda_n-\lambda_m)\}\ .$$

(d) We now show that each row in σ^t in an l_2-sequence. So let m be fixed

$$|\sigma^t_{mn}| \leq \exp\{t_1\,\mathrm{Re}\,\lambda_m - t_2\,\mathrm{Im}\,\lambda_m\} \cdot \exp\{2[D\,\delta\,n]^{\frac{1}{2}} - t_1\,n^{\mu}(1+\varepsilon_1(n))\left(1 - \frac{t_2}{t_1}\,\varepsilon_1(n)\right)\ .$$

Take N so large that for $n > N$ we have

$$2\,n^{-\frac{1}{2}}(D\,\delta)^{\frac{1}{2}} \leq \frac{1}{4}t_1 \quad \text{and} \quad (1+\varepsilon_1(n))\left(1 - \frac{t_2}{t_1}\,\varepsilon_2(n)\right) \geq \frac{1}{2}\ .$$

For $n > N$ the second factor is smaller than $e^{-\frac{1}{4}t_n n}$. Hence the result.

(e) Starting from the inequality in (c) we find with conditions (iii)

$$|\sigma^t_{mn}| \leq \exp\{2(D\,\delta(n-m))^{\frac{1}{2}} - t_1\,n^{\mu-1}(n-m)\,(1+\varepsilon_3)\left(1 - \frac{t_2}{t_1}\,\varepsilon_4\right)\}\ .$$

Below the M^{th} row, M sufficiently large, we have

$$|\sigma^t_{mn}| \leq \exp\{2(D\,\delta(n-m))^{\frac{1}{2}} - \frac{1}{2}\,t_1(n-m)\}\ .$$

Therefore each diagonal is a bounded sequence. Finally for $m \geq M$ and $(n-m) \geq \frac{1}{t_1}\,8(D\,\delta)^{\frac{1}{2}}$

$$|\sigma^t_{mn}| \leq e^{-\frac{1}{4}t_1(n-m)}\ .$$

(f) We now split the matrix σ^t in 2 parts:

- The first M rows. They represent a bounded l_2-operator which is Hilbert Schmidt, cf. (d).
- The part below the M^{th} row also represents a bounded l_2-operator because of the "codiagonal estimate", use

$$\sum_{k=0}^{\infty}(\sup_{j-i=k}\ |K_{ij}| < \infty\ .$$

(g) For $e^{t\Lambda}\,S^{-1}\,e^{-t\Lambda}$ the parts (c)-(g) of the proof apply in exactly the same way. □

REMARK.
If $\mu > 1$ the conditions (i) and (ii) can be relaxed considerably, see [GE] and [G]. However in this paper we need the case $\mu = 1$.

THEOREM 3.3.

(a) The conditions on Λ in Theorem 3.2 are satisfied if $\lambda_n = n^\mu(1+\rho(n))$, $\mu \geq 1$, and

$$\lim_{n\to\infty} \rho(n) = 0 \quad \lim_{m\to\infty} \sup_{n\geq m+1} \left|\frac{\rho(n)\,n - \rho(m)\,m}{n-m}\right| = 0\,.$$

(b) The conditions of (a) are satisfied if ρ is a (complex valued) differentiable function on $(0,\infty)$ with $\lim_{\xi\to\infty} \rho(\xi) = 0$ and $\lim_{\xi\to\infty} \xi\,\rho'(\xi) = 0$.

(c) If $\Lambda = N_a^\mu = \operatorname{diag}(\dots,(n(n+a))^{\frac{1}{2}\mu},\dots)$, $a \in C$, then condition (b) is satisfied.

Proof.

(a) We compute

$$\lambda_n - \lambda_m = (n-m)\,n^{\mu-1}\left\{1 + \left(\frac{1-(\frac{m}{n})^\mu}{1-\frac{m}{n}} - 1\right) + \left(\frac{n\,\rho(n) - m\,\rho(m)}{n-m}\right) + \left(\frac{m}{n}\,\frac{1-(\frac{m}{n})^{\mu-1}}{1-\frac{m}{n}}\,\rho(m)\right)\right\}.$$

On the interval $[0,1)$ one has

$$\mu \geq \frac{1-x^\mu}{1-x} \geq 1 \quad \text{and} \quad \frac{1-x^{\mu-1}}{1-x} \leq \max(1,\mu-1) \leq \mu\,.$$

For $\varepsilon_3(n,m)$ take the real parts of the 3rd and 4th terms between $\{\ \}$. We omit the simple verification of the other conditions.

(b) Follows by application of the mean value estimation on $(n-m)^{-1}(\rho(n)\,n - \rho(m)\,m)$.

(c) We have $\rho(x) = \left(1+\frac{a}{x}\right)^{\frac{\mu}{2}} - 1$ which satisfies (b). □

Note that sequences like $((\log n)^A\, n^B \exp(t\,n^\mu + Cn + D\sqrt{n}))_{n=1}^\infty$ with $A, B, C, D \in \mathbb{R}$, $t > 0$, $\mu \geq 1$ are of type $(\exp(t\,n^\mu(1+\rho(n))))$.
Now we formulate the main result of this section which is a consequence of Theorems 3.1 and 3.2.

THEOREM 3.3. (Classification)

- Consider a Hilbert space X and fix an orthonormal basis $(T_n)_{n=0}^\infty \subset X$.
- Let $\mathcal{K}$ denote the class of all upper triangular matrices $K \in UTM$ which satisfy the conditions (i) and (ii) of Theorem 3.2.
- Let $\mathcal{Y}$ denote the class of sequences $(\lambda_n)_{n=0}^\infty$ which satisfy condition (iii) of Theorem 3.2.

- Let the sequence $\mu = (\mu_n)_{n=0}^{\infty}$, $\mu_n = |e^{t\lambda_n}|$ with $(\lambda_n)_{n=0}^{\infty} \in \mathcal{Y}$, be fixed.

Then

(A) For all $K \in \mathcal{K}$ the sequence $(e^{-t\lambda_n} T_n^K)_{n=0}^{\infty}$ is a Riesz basis in X_μ. In other words, the Hilbert spaces X_μ and X_μ^K are the same as topological vector spaces.

(B) If for some positive sequence $\theta = (\theta_n)_{n=0}^{\infty}$ and some $K \in \mathcal{K}$ the sequence $(\theta_n T_n^K)_{n=0}^{\infty}$ happens to be a Riesz basis in a Hilbert space Y, then for all $K \in \mathcal{K}$ and all $L \in \mathcal{K}$ we have the equality

$$Y_\alpha^K = X_\mu^L , \quad \text{with } \alpha = \mu\theta^{-1} ,$$

as Hilbertizable topological vector spaces.

4. Application to the Jacobi operators.

The results of the preceding section can be applied in the following way. If the, suitably normalized, eigenvectors of some operator establish a Riesz basis in a "scale" of smooth Hilbert spaces then certain perturbed operators will do the same. To put it differently: All perturbed operators when "substituted" into a suitable fixed unbounded function lead to closed operators which all have the same domain.
Such a procedure leads in a straight forward way to long lists of Riesz bases for e.g. the spaces of analytic functions that we discussed in the Applications I-IV in Section 2.
In this final section we carry out such a program for the Jacobi operators which can be considered as perturbations of the Chebyshev operator.

Consider the general Jacobi differentiation operator

$$\mathcal{A}_{\alpha\beta} = -(1-x^2)\,\frac{d^2}{dx^2} + (\alpha+\beta+2)\,x\,\frac{d}{dx} - (\beta-\alpha)\,\frac{d}{dx}\ , \quad \alpha,\beta \in \boldsymbol{C},\ \text{fixed}\ .$$

In the special case $\alpha, \beta \in I\!R$, $\alpha > -1$, $\beta > -1$, the differentiation operator $\mathcal{A}_{\alpha\beta}$ can be regarded as a self-adjoint operator in $X_{\alpha\beta} = L_2([-1,1],\ (1-x)^\alpha\,(1+x)^\beta\,dx)$. In this special case for the eigenfunctions we take the normalized (in $X_{\alpha\beta}$) Jacobi polynomials

$$R_n^{\alpha\beta} = \kappa_n^{\alpha\beta}\, P_n^{\alpha\beta}\ .$$

The explicit expression for the normalizing coefficient $\mathcal{K}_n^{\alpha\beta}$ is mentioned in [GE], one has

$$\lim_{n\to\infty}\ (2^{\alpha+\beta}\,n^{-1})^{\frac{1}{2}}\ \kappa_n^{\alpha\beta} = 1\ .$$

Special cases are the normalized Chebyshev polynomials of the first and second kind: In a local convention we have

$$R_0^{-\frac{1}{2},-\frac{1}{2}}(\cos\theta) = T_0(\cos\theta) = \pi^{-\frac{1}{2}}\ ,\quad R_n^{-\frac{1}{2},-\frac{1}{2}}(\cos\theta) = T_n(\cos\theta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\ \cos n\theta\ ,\quad n \in I\!N$$

$$R_n^{\frac{1}{2},\frac{1}{2}}(\cos\theta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\ \frac{\sin(n+1)\,\theta}{\sin\theta}\ ,\quad n \in I\!N_0\ .$$

Next, let $K = A_{\alpha\beta}$ denote the matrix of $\mathcal{A}_{\alpha\beta}$, $\alpha \in \boldsymbol{C}$, $\beta \in \boldsymbol{C}$, with respect to the basis $(T_n)_{n=0}^{\infty}$ in the space of all polynomials.
In [GE] we showed that $K_{mn} = (\mathcal{A}_{\alpha\beta}T_m, T_n)_{-\frac{1}{2},-\frac{1}{2}}$, $K_{nn} = n(n+\alpha+\beta+1)$ and $|K_{mn}| \le (2\,|\beta-\alpha| + |\alpha+\beta+1|)\,n$. Note that this matrix satisfies the conditions (i) and (ii) of Theorem 3.2.
The matrix $S_{\alpha\beta} \in UTM$ is defined by

$$S_{\alpha\beta}^{-1}\, A_{\alpha\beta}\, S_{\alpha\beta} = \operatorname{diag}(\ldots, n(n+\alpha+\beta+1), \ldots)\ .$$

Via the transition matrix $S_{\alpha\beta}$ we construct the polynomial basis $(T_n^{\alpha\beta})_{n=0}^{\infty}$ from the Chebyshev basis $(T_n)_{n=0}^{\infty}$. If $a > -1$, $\beta > -1$, we have $T_n^{\alpha\beta} = r_n^{\alpha\beta} R_n^{\alpha\beta}$ where the sequences $(r_n^{\alpha\beta})_{n=0}^{\infty}$ and $((r_n^{\alpha\beta})^{-1})_{n=0}^{\infty}$ are bounded.
Now we are in a position to extend the results of Application II in Section 2.

APPLICATION V. (sequel to Applications II)
First we fix some notations.
For all $\alpha > -1$, $\beta > -1$, $m \in \mathbb{N}_0$, $\theta > 0$, $t > 0$ we consider the domain $D(\mathcal{A}_{\alpha\beta}^{\frac{1}{2}(m-\theta)} \exp t\,\mathcal{A}_{\alpha\beta}^{\frac{1}{2}})$ of the selfadjoint operator $\mathcal{A}_{\alpha\beta}^{\frac{1}{2}(m-\theta)} \exp t\,\mathcal{A}_{\alpha\beta}^{\frac{1}{2}}$ in $X_{\alpha\beta} = L_2([0,1]\,(1-x)^{\alpha}\,(1+x)^{\beta}\,dx)$. Note that the sequence $\mu = (\mu_n)_{n=0}^{\infty}$ of eigenvalues of this operator is given by

$$([n(n+\alpha+\beta+1)]^{\frac{1}{2}(m-\theta)} \exp t\,[n(n+\alpha+\beta+1)]^{\frac{1}{2}})_{n=0}^{\infty} .$$

Note also that for fixed m, θ, t the quotient of two such sequences (for different pairs (α, β)) is always bounded. As a corollary of the general theory in Section 3 we obtain

THEOREM 4.1 (Cf. [Sz] Thm 9.1.1)
Let $f \in \mathcal{A}([-1,1])$. For $\alpha, \beta \in \mathbb{C}$, $K \in \mathcal{K}$, expand

$$f(x) = \sum_{n=0}^{\infty} a_n^{\alpha\beta}\, T_n^{\alpha\beta}(x) = \sum_{n=0}^{\infty} a_n^{K}\, T_n^{K}(x) .$$

Denote $a_n^{-\frac{1}{2},-\frac{1}{2}} = a_n$. Let $m \in \mathbb{N}_0$, $\theta > 0$, $T > 0$. The following nine conditions are equivalent.

- $f \in \mathcal{A}_2(E_T; \omega_\theta, m)$
- $(n^{(m-\theta)}\, e^{nT}\, a_n)_{n=0}^{\infty} \in l_2$
- $f \in \mathcal{A}(E_T)$ and $f|_{\partial E_T} \in H^{m-\theta}(\partial E_T)$.
- $\exists_{\alpha \in \mathbb{C}}\, \exists_{\beta \in \mathbb{C}} \quad (n^{(m-\theta)}\, e^{nT}\, a_n^{\alpha\beta})_{n=0}^{\infty} \in l_2$
- $\forall_{\alpha \in \mathbb{C}}\, \forall_{\beta \in \mathbb{C}} \quad (n^{(m-\theta)}\, e^{nT}\, a_n^{\alpha\beta})_{n=0}^{\infty} \in l_2$
- $\exists_{\alpha > -1}\, \exists_{\beta > -1} \quad f \in D(\mathcal{A}_{\alpha\beta}^{\frac{1}{2}(m-\theta)} \exp T\,\mathcal{A}_{\alpha\beta}^{\frac{1}{2}})$
- $\forall_{\alpha > -1}\, \forall_{\beta > -1} \quad f \in D(\mathcal{A}_{\alpha\beta}^{\frac{1}{2}(m-\theta)} \exp T\,\mathcal{A}_{\alpha\beta}^{\frac{1}{2}})$
- $\exists K \in \mathcal{K} \quad (n^{(m-\theta)}\, e^{nT}\, a_n^{K})_{n=0}^{\infty} \in l_2$
- $\forall K \in \mathcal{K} \quad (n^{(m-\theta)}\, e^{nT}\, a_n^{K})_{n=0}^{\infty} \in l_2$. □

Note that it is an interesting problem to look for other "known" polynomials which correspond to eigenvectors of a matrix $K \in \mathcal{K}$.

References

[AS] Abramowitz, M. and I.A. Stegun. Handbook of mathematical functions. National Bureau of Standards. Washington 1968.

[BG1] Berkel, C.A.M. van, and J. de Graaf. On a property of the Fourier-cosine transform. Appl. Math. Lett. 1 (3), 307-310, 1988.

[BG2] Berkel, C.A.M. van, and J. de Graaf. The Fourier-Jacobi transform of analytic functions which are (almost) periodic in the imaginary direction. Preprint. Submitted to SIAM J. Math. An.

[D] Duran, A.J. The analytic functionals in the lower half plane as a Gel'fand-Shilov space. To appear.

[G] Graaf, J. de. On approximation of (slowly growing) analytic functions by special polynomials. To appear in "Orthogonal polynomials" Conference Erice 1990.

[GE] Graaf, J. de, and S.J.L. van Eijndhoven. Domains of exponentiated fractional Jacobi Operators: Characterizations, Classifications, Expansion Result. Constr. Approx. **5**, 271-296, 1989.

[Sz] Szegö, G. Orthogonal polynomials, AMS Coll. Publ. XXIII, 1959.

PREVIOUS PUBLICATIONS IN THIS SERIES:

Number	Author(s)	Title	Month
90-07	M. Kuipers A.A.F. van de Ven	Unilateral contact of a springboard and a fulcrum	July '90
90-08	P.H. van Lieshout A.A.F. van de Ven	A variational approach to the magneto-elastic buckling problem of an arbitrary number of superconducting beams	July '90
90-09	A. Reusken	A multigrid method for mixed finite element discretizations of current continuity equations	August '90
90-10	G.A.L. van de Vorst R.M.M. Mattheij H.K. Kuiken	A boundary element solution for 2-dimensional viscous sintering	October '90
90-11	A.A.F. van de Ven	A note on 'A nonequilibrium theory of thermoelastic superconductors' by S-A. Zhou and K. Miya	October '90
90-12	M.E. Kramer R.M.M. Mattheij	Timestepping for solving nonlinear equations	October '90
90-13	A. Reusken	Multigrid applied to mixed finite element schemes for current continuity equations	November '90
90-14	J. de Graaf	Riesz bases of special polynomials in weighted Sobolev spaces of analytic functions	December '90