Recurrent and dissipative sets for the Markov shift

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by

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§ 1. Introduction

Let (X,Σ,m,P) be a Markov process with $P1 = 1$ and $m(X) = 1$, i.e. (X,Σ,m) is a probability space and P a positive linear σ-additive operator on $\mathcal{L}_\infty(m)$, with $P1 = 1$. We consider X as the state space of the process, and form the realization space

$$(\Omega,\mathcal{R}) = \prod_{i=0}^{\infty} (X,\Sigma)_i \quad \text{where } (X,\Sigma)_i = (X,\Sigma) \quad \text{for all } i .$$

Hence a point $\omega \in \Omega$ is a sequence $\omega = (\omega_0,\omega_1,\omega_2,\ldots)$ with $\omega_n \in X$ for all n. We denote by X_n the projection of Ω on the n-th coordinate, i.e. $X_n(\omega) = \omega_n$. In $(\Omega,\mathcal{R})$ we consider the shift transformation S defined by

$$S(\omega_0,\omega_1,\omega_2,\ldots) = (\omega_1,\omega_2,\omega_3,\ldots) .$$

There may exist a probability M on $(\Omega,\mathcal{R})$ such that

$$M(X_0 \in A_0,\ldots,X_n \in A_n) = mI_{A_0}PI_{A_1}P \ldots P1_{A_n} .$$

(This terminology can e.g. be found in Foguel [2], $mf = \int f\,dm$, $I_Af = 1_Af$.) It is well known that we can decompose the state space X into a conservative part C and a dissipative part D for the operator P. A similar decomposition theorem holds for measurable transformations on measure spaces, hence in particular for S on $(\Omega,\mathcal{R},M)$.

The relationship between these decompositions is, under some conditions, given by Harris and Robbins [5] and slightly extended by Simons [10]. In this note we want to give a faster deduction of this relationship, making use of a generalization of embedded Markov processes. This deduction will be given in the third section; in the second section some facts on Markov measures on $(\Omega,\mathcal{R})$ are collected.

To avoid misunderstandings, we remark that all equalities and inequalities in this note on sets and functions are valid modulo null sets.

§ 2. Markov measures on $(\Omega,\mathcal{R})$

Let m_0 be a (not necessarily finite or σ-finite) measure on (X,Σ) with $m_0 \ll m$. A measure M_0 on $(\Omega,\mathcal{R})$ is said to be a Markov measure with initial measure m_0 if for all $A_0,A_1,\dots,A_n \in \Sigma$ we have

(2.1) $$M_0\{X_0 \in A_0,\dots,X_n \in A_n\} = m_0 I_{A_0} P \dots P 1_{A_n} .$$

It follows that for all nonnegative functions $f_0,\dots,f_n$ we have

(2.2) $$\int f_0(\omega_0) f_1(\omega_1) \dots f_n(\omega_n) M_0(d\omega) = m_0 F_0 P F_1 \dots P f_n$$

where F stands for multiplication by the function f.

Let $\mathcal{R}_{0,n}$ be the sub-σ-algebra of $\mathcal{R}$ generated by the sets $\{X_0 \in A_0,\dots,X_n \in A_n\}$. Application of (2.2) yields

(2.3) $$\int f_0(\omega_0) \dots f_n(\omega_n) f_{n+1}(\omega_{n+1}) \dots f_{n+m}(\omega_{n+m}) M_0(d\omega) =$$
$$= \int f_0(\omega_0) \dots f_n(\omega_n)(PF_{n+1}P \dots Pf_{n+m})(\omega_n) M_0(d\omega) .$$

Let $E_{\mathcal{R}_{0,n}}$ be the conditional expectation operator in $(\Omega,\mathcal{R},M_0)$ with respect to $\mathcal{R}_{0,n}$. Then from (2.3) we conclude

(2.4) $$E_{\mathcal{R}_{0,n}} f_{n+1}(\omega_{n+1}) \dots f_{n+m}(\omega_{n+m}) = (PF_{n+1}P \dots Pf_{n+m})(\omega_n) .$$

Note that the conditional expectation is independent of the measure M_0.

In general a Markov measure M_0 with initial measure $m_0 \ll m$ need not exist. However, if the process P is given by a transition probability such that $m(A) = 0$ implies $P(\cdot,A) = 0$ m-almost everywhere, then it follows from the theorem of Ionesco-Tulcea that a Markov probability M with initial probability m exists (cf. [8], V.1). In this case for any initial measure $m_0 \ll m$ there exists a Markov measure M_0 on $(\Omega,\mathcal{R})$. In fact, let $\frac{dm_0}{dm}$ be the Radon-Nikodym derivative of m_0 with respect to m on (X,Σ) and define the measure M_0 on $(\Omega,\mathcal{R})$ by

$$\frac{dM_0}{dM}(\omega) = \frac{dm_0}{dm}(\omega_0) .$$

Then

$$M_0\{X_0 \in A_0,\dots,X_n \in A_n\}$$

$$= \int \frac{dm_0}{dm}(\omega_0) 1_{A_0}(\omega_0) \dots 1_{A_n}(\omega_n) M(d\omega)$$

$$= m \frac{dm_0}{dm} I_{A_0} P \dots P 1_{A_n}$$

$$= m_0 I_{A_0} P \dots P 1_{A_n} .$$

We conclude this section with two technical results which we shall need in the sequel.

Lemma 2.1. Let M_0 be a Markov measure on $(\Omega,\mathcal{R})$ with initial measure $m_0 << m$ such that $\frac{dm_0}{dm} < \infty$ on X. Then M_0 is σ-finite, and M_0 is preserved under the shift if and only if $\frac{dm_0}{dm} P = \frac{dm_0}{dm}$.

Proof. Put

$$A_n = \{\omega \mid \frac{dm_0}{dm}(\omega_0) \in [n-1,n)\} ,$$

then $\Omega = \bigcup_{n=1}^{\infty} A_n$ and $M_0(A_n) < \infty$ for all n, hence M_0 is σ-finite. Suppose M_0 is shift invariant. Then for all $A \in \Sigma$ we have

$$M_0\{X_0 \in A\} = M_0\{X_1 \in A\}$$

$$\int_A \frac{dm_0}{dm} dm = \int \frac{dm_0}{dm} P 1_A dm = \int_A (\frac{dm_0}{dm} P) dm ,$$

$$\frac{dm_0}{dm} = \frac{dm_0}{dm} P .$$

Conversely, suppose $\frac{dm_0}{dm} P = \frac{dm_0}{dm}$. In order to show that M_0 is invariant under S it suffices to prove

$$M_0\{X_0 \in A_0,\dots,X_n \in A_n\} = M_0\{X_1 \in A_0,\dots,X_{n+1} \in A_n\}$$

for all $A_0,\dots,A_n \in \Sigma$.

$$M_0\{X_0 \in A_0,\dots,X_n \in A_n\} = m\,\frac{dm_0}{dm}\, I_{A_0} P I_{A_1} P \dots P 1_{A_n}$$

$$= m\,\frac{dm_0}{dm}\, P I_{A_0} P I_{A_1} P \dots P 1_{A_n}$$

$$= M_0\{X_1 \in A_0,\dots,X_{n+1} \in A_n\} .$$

Lemma 2.2. Let $f \in \mathcal{L}_\infty$ and $A_1,\dots,A_n \in \Sigma$. Then for every k we have

$$E_{\mathcal{B}_{0,k}} 1_{\{(X_{k+1},\dots,X_{k+n}) \notin A_1 \times \dots \times A_n\}}(\omega) f(\omega_{k+n}) =$$

$$= (P^n - P I_{A_1} P \dots P I_{A_n}) f(\omega_k) .$$

Proof. Note that

$$1_{\{(X_{k+1},\dots,X_{k+n}) \in A_1 \times \dots \times A_n\}}(\omega) f(\omega_{k+n}) +$$

$$+ 1_{\{(X_{k+1},\dots,X_{k+n}) \notin A_1 \times \dots \times A_n\}}(\omega) f(\omega_{k+n}) = f(\omega_{k+n}) .$$

Taking on both sides the conditional expectation with respect to $\mathcal{B}_{0,k}$ we obtain by (2.4)

$$P I_{A_1} P \dots P I_{A_n} f(\omega_k) + E_{\mathcal{B}_{0,k}} 1_{\{(X_{k+1},\dots,X_{k+n}) \notin A_1 \times \dots \times A_n\}}(\omega) f(\omega_{k+n}) =$$

$$= P^n f(\omega_k)$$

from which the relation follows.

§ 3. Recurrent and dissipative sets for the Markov shift

It is well known how to decompose the state space X into a conservative part C and a dissipative part D. For a description of this decomposition the reader is referred to [2], chapter 2 or [4]. In this section we mention the properties we shall need in the sequel.

Lemma 3.1. There exists a partition $D_1, D_2, \dots$ of D such that

$$\sum_{n=0}^{\infty} P^n 1_{D_i} \in \mathcal{L}_\infty \qquad \text{for all } i .$$

Proof. See Feldman [1], theorem 2.1 or [4], theorem 1.

Lemma 3.2.

a) For all $g \in \mathcal{L}_\infty^+$ $Pg \geq g (Pg \leq g)$ on C implies $Pg = g$ on C.

b) There exists a function $g \in \mathcal{L}_\infty^+$ with $Pg \leq g$ and $Pg < g$ on D.

Proof.

a) See Foguel [2], chapter 2, theorem B and (2.9).

b) Let $D_1, D_2, \ldots$ be the partition as in lemma 3.1. Put $\alpha_i = \| \sum_{n=0}^{\infty} P^n 1_{D_i} \|_\infty$, and define $f = \sum_{i=1}^{\infty} \frac{1}{2^i \alpha_i} 1_{D_i}$. Then $g = \sum_{n=0}^{\infty} P^n f \leq 1$ and $g - Pg = f > 0$ on D.

Lemma 3.3. The conservative part of X with respect to P^n is the conservative part of X with respect to P.

Proof. Let $D(P^n)$ be the dissipative part of X with respect to P^n. Then there exists a function $g \in \mathcal{L}_\infty^+$ with $P^n g \leq g$ and the $<$ sign holds on $D(P^n)$.
Put $g' = g + Pg + \ldots + P^{n-1}g$, then $Pg' = Pg + \ldots + P^{n-1}g + P^n g$, hence $Pg' \leq g'$, and the $<$ sign holds on $D(P^n)$. It follows that $D(P^n) \subset D$.
Conversely, let $h \in \mathcal{L}_\infty^+$ satisfy $Ph \leq h$, with $<$ on D. Since P is a positive operator, we have $h \geq Ph \geq P^n h$, hence $h > P^n h$ on D and therefore $D \subset D(P^n)$.

In the next lemma we introduce a rather queer type of Markov process, which will turn up in the proof of theorem 3.2. Some special cases of this type of Markov operator however are well known. If $n = 1$ and $H = I_A$, then $H^c = I_{A^c}$ and Q_H is the embedded process; if $n = 1$ and H is the multiplication by a function f with $0 \leq f \leq 1$, then Q_H is the operator T_f as studied by Foguel and Lin [3] and Lin [7].

Lemma 3.4. Let H and H^c be Markov processes on (X, Σ, m) such that $H + H^c = P^{n-1}$. Define for every $g \in \mathcal{L}_\infty^+$

$$Q_H(g) = \sum_{k=0}^{\infty} (PH^c)^k PHg ,$$

then Q_H is a Markov process satisfying $H(I - Q_H)1 = 0$ on C.

Proof. Since P, H and H^c are Markov operators, the operator Q_H is positive, linear and σ-additive. It remains to show that $Q_H 1 \le 1$. This follows from the following relation, which is easily verfified by writing out, by taking $j \to \infty$:

$$(3.1) \qquad \sum_{k=0}^{j} (PH^c)^k PH1 + (PH^c)^{j+1} 1 = 1 .$$

Put $(PH^c)^j 1 = g_j$, then it also follows that (g_j) is a nonincreasing sequence of nonnegative functions, hence $\lim_{j\to\infty} g_j = g$ exists.

$$P^n g_j = PHg_j + PH^c g_j = PHg_j + g_{j+1} .$$

Let $j \to \infty$, then we obtain

$$P^n g = PHg + g ,$$

from which we conclude $P^n g \ge g$, and therefore by lemma 2.3 and lemma 3.2a) $P^n g = g$ on C, PHg = 0 on C. Again by lemma 3.2a) this implies Hg = 0 on C. Since $g = 1 - Q_H 1$, we obtain $H(I - Q_H)1 = 0$ on C.

After these preliminaries, we turn to the main subject of this section. We start with a definition.

Definition 3.1. Let S be a measurable transformation on a (finite or σ-finite) measure space $(\Omega, \mathcal{R}, M)$. A set $W \in \mathcal{R}$ is said to be wandering if $W \cap S^{-n}W = \emptyset$ for n = 1,2,..., or equivalently, if $\{\omega \in W \mid S^n\omega \in W \text{ for some } n \ge 1\} = \emptyset$. A set $A \in \mathcal{R}$ is said to be dissipative if A is a countable union of wandering sets. A set $A \in \mathcal{R}$ is said to be recurrent if $\{\omega \in A \mid S^n\omega \in A \text{ i.o.}\} = A$. ($S^n\omega \in A$ i.o. (infinitely often) means that there exists a sequence $1 \le n_1 < n_2 < \ldots$ such that $S^{n_k}\omega \in A$ for all $k \ge 1$.)

Recall that the conservative part of Ω with respect to S is characterized by the fact that all its subsets are recurrent, while the dissipative part of Ω with respect to S, i.e. the complement of the conservative part, indeed is dissipative (cf. [6], [9]). Obviously, a countable union of dissipative sets again is dissipative, and a countable union of recurrent sets is recurrent. Note, however, that a dissipative set may be recurrent. This is for instance the case if $\Omega = \mathbb{Z}$ and $Sn = n + 1$ for all $n \in \mathbb{Z}$. Then {n} is wandering for all $n \in \mathbb{Z}$, hence Ω is a recurrent dissipative set.

From now on we shall assume that $(\Omega,\mathcal{R},M)$ is the realization space of (X,Σ,m,P), where M is the Markov measure on $(\Omega,\mathcal{R})$ for P with initial measure m.

Theorem 3.1. Let (X,Σ,m,P) be a Markov process with $m(X) = 1$ and $P1 = 1$, let $(\Omega,\mathcal{R},M)$ be the realization space where M is the Markov probability for P with initial measure m, and let D be the dissipative part of X with respect to P. Then $\{X_0 \in D\}$ is a dissipative set in $\mathcal{R}$ for the shift S in $(\Omega,\mathcal{R},M)$.

Proof. Let $D_1,D_2,\ldots$ be the partition of D as in lemma 3.1. Then

$$m\Big(\sum_{n=0}^{\infty} P^n 1_{D_i}\Big) = \sum_{n=0}^{\infty} M\{X_n \in D_i\} < \infty ,$$

and therefore by the Borel-Cantelli lemma

$$M\{X_n \in D_i \text{ i.o.}\} = 0 .$$

It follows that

$$\{X_0 \in D\} = \bigcup_{i=1}^{\infty} \bigcup_{k=0}^{\infty} \{X_0 \in D_i,\ X_m \in D_i \text{ for exactly } k \text{ integer } m > 0\}.$$

Obviously, every set on the right hand side is wandering under S. Hence $\{X_0 \in D\}$ is a dissipative set.

Theorem 3.2. Let (X,Σ,m,P) be a Markov process with $m(X) = 1$ and $P1 = 1$, let $(\Omega,\mathcal{R},M)$ be the realization space where M is the Markov probability for P with initial measure m, and let C be the conservative part of X with respect to P. Let $A_0,\ldots,A_{n-1} \in \Sigma$ be given such that $A_0 \subset C$. Then $\{X_0 \in A_0,\ldots,X_{n-1} \in A_{n-1}\}$ is a recurrent set in $\mathcal{R}$ for the shift S in $(\Omega,\mathcal{R},M)$.

Proof. We consider the following sets in $\mathcal{R}$:

$$B_{k,\ell} = \{(X_k,\ldots,X_{k+n-1}) \in A_0 \times \ldots \times A_{n-1},$$
$$(X_{k+jn},\ldots,X_{k+n-1+jn}) \notin A_0 \times \ldots \times A_{n-1} \quad \text{for } 1 \le j \le \ell\}$$

$$B_{k,\infty} = \{(X_k,\ldots,X_{k+n-1}) \in A_0 \times \ldots \times A_{n-1},$$
$$(X_{k+jn},\ldots,X_{k+n-1+jn}) \notin A_0 \times \ldots \times A_{n-1} \quad \text{for all } j \ge 1\} .$$

Using lemma 2.2 we obtain

$$M(B_{k,\ell}) = mP^k I_{A_0} P \ldots PI_{A_{n-1}} (P^n - PI_{A_0} P \ldots PI_{A_{n-1}})^\ell 1 .$$

Define $Hg = I_{A_0} P \ldots PI_{A_{n-1}} g$ for all $g \in \mathcal{L}_\infty^+$, then H is a Markov process on (X,Σ,m) satisfying $Hg \le P^{n-1}g$ for all $g \in \mathcal{L}_\infty^+$. Hence also $H^c = P^{n-1} - H$ is a Markov process on (X,Σ,m). It follows that

$$M(B_{k,\ell}) = mP^k H(PH^c)^\ell 1 .$$

If $\ell \to \infty$ we get, using (3.1) in the proof of lemma 3.4

$$M(B_{k,\infty}) = mP^k H(I - Q_H) 1 .$$

Since $H(I - Q_H)1 = 0$ outside A_0, and $A_0 \subset C$, it follows from lemma 3.4 that $H(I - Q_H)1 = 0$ on X, and therefore, $M(B_{k,\infty}) = 0$.
Put $B = \{X_0 \in A_0, \ldots, X_{n-1} \in A_{n-1}\}$, then we have $M\{\omega \in B \mid S^{kn} \in B$ for finitely many $k\} = 0$. It follows that

$$B = \{\omega \in B \mid S^{kn} \in B \text{ for infinitely many } k\}$$

$$= \{\omega \in B \mid S^k\omega \in B \text{ i.o.}\} ,$$

hence B is recurrent.

Theorem 3.2 does not exclude that a set $\{X_0 \in A_0, \ldots, X_{n-1} \in A_{n-1}\}$ is dissipative, since dissipative sets can be recurrent. Therefore in general we cannot conclude that $\{X_0 \in C\}$ belongs to the conservative part of Ω with respect to S. However, Harris and Robbins [5] have shown that, under the condition that P admits a finite or σ-finite invariant measure on C, the shift S is conservative on $\{X_0 \in C\}$. Their proof rests on the following lemma.

<u>Lemma 3.5.</u> Let S be a measure preserving transformation in a finite or σ-finite measure space $(\Omega,\mathcal{B},M_0)$. Let $\mathfrak{A}$ be an algebra generating $\mathcal{B}$ such that every $A \in \mathfrak{A}$ is recurrent. Then S is conservative on Ω.

<u>Proof.</u> Let W be a wandering set of finite measure. Choose $\varepsilon > 0$ and $A \in \mathfrak{A}$ such that $M_0(A \Delta W) < \varepsilon$. Since $A \subset \bigcup_{n=1}^{\infty} S^{-n}A$, there exists an integer N such that $M_0(A \setminus \bigcup_{N-1}^{N} S^{-n}A) < \varepsilon$. Then

$$0 = M_0(S^{-N}W \cap \bigcup_{i=0}^{N-1} S^{-i}W) > M_0(S^{-N}A \cap \bigcup_{i=0}^{N-1} S^{-i}W) - \varepsilon$$

$$= \sum_{i=0}^{N-1} M_0(S^{-N}A \cap S^{-i}W) - \varepsilon = \sum_{i=0}^{N-1} M_0(S^{i-N}A \cap W) - \varepsilon$$

$$\geq M_0(\bigcup_{i=1}^{N} S^{-i}A \cap W) - \varepsilon > M_0(A \cap W) - 2\varepsilon .$$

Hence

$$M_0(W) \leq M_0(A\Delta W) + M_0(A \cap W) < 3\varepsilon .$$

It follows that $M_0(W) = 0$ and S is conservative on Ω.

Theorem 3.3. (Harris-Robbins [5]). Let (X,Σ,m,P) be a Markov process with $m(X) = 1$, $P1 = 1$. Let C be the conservative part of X with respect to P. Let $(\Omega,\mathcal{R},M)$ be the realization space of P where M is the Markov probability with initial measure m. Suppose there exists a function u with $0 < u < \infty$ on C, $u = 0$ on D such that $uP = u$. Then $\{X_0 \in C\}$ is the conservative part of Ω for the shift S and $\{X_0 \in D\}$ is the dissipative part of Ω for the shift S.

Proof. Define the measure M_0 on $(\Omega,\mathcal{R})$ by $\frac{dM_0}{dM}(\omega) = u(\omega_0)$, then M_0 is a Markov measure for P with initial measure m_0, where m_0 is determined by $u = \frac{dm_0}{dm}$.
By lemma 2.1 M_0 is σ-finite and invariant under S.
It follows from the definition of M_0 that M_0 is equivalent to M on $\{X_0 \in C\}$ and $M_0 = 0$ on $\{X_0 \in D\}$. Hence the algebra $\mathfrak{A}$ of finite unions of sets $\{X_0 \in A_0,\dots,X_n \in A_n\}$ with $A_0 \subset C$ generates (mod M_0) $\mathcal{R}$. By theorem 3.2 all elements of $\mathfrak{A}$ are recurrent, and therefore by lemma 3.5 S is conservative on $(\Omega,\mathcal{R},M_0)$. Hence, for every wandering set $W \subset \{X_0 \in C\}$ we have $M_0(W) = 0$, and therefore $M(W) = 0$. It follows that $\{X_0 \in C\}$ belongs to the conservative part of $(\Omega,\mathcal{R},M)$. On the other hand, by theorem 3.1 $\{X_0 \in D\}$ belongs to the dissipative part of $(\Omega,\mathcal{R},M)$. This completes the proof of the theorem.

References

[1] Feldman, J.: Subinvariant measures for Markov operators. Duke Math. J. 29, 71-98 (1962).

[2] Foguel, S.R.: The ergodic theory of Markov processes. Van Nostrand Mathematical Studies # 21, Van Nostrand Reinhold Company, New York etc. 1969.

[3] Foguel, S.R. and M. Lin: Some ratio limit theorems for Markov operators. Z. Wahrsch. verw. Geb. 23, 55-66 (1972).

[4] Groenewegen, L.P.J., K.M. van Hee, D.A. Overdijk and F.H. Simons: Embedded Markov processes and recurrence. Internal report, Memorandum COSOR 75-03. Technological University Eindhoven, 1975.

[5] Harris, T.E. and H. Robbins: Ergodic theory of Markov chains admitting an invariant measure. Proc. Nat. Acad. Sci. 39, 860-864 (1953).

[6] Helmberg, G.: Uber die Zerlegung einer messbaren Transformation in konservatieve und dissipatieve Bestandteile. Math. Z. 88, 358-367 (1965).

[7] Lin, M.: On quasi-compact Markov operators. The Annals of Probability, 2, 464-475 (1974).

[8] Neveu, J.: Mathematical foundations of the calculus of probability. Holden-Day, San Francisco etc. 1965.

[9] Simons, F.H.: Ein weiten Beweis eines Zerlegung satzes für messbare Transformationen. Math. Z. 89, 247-249 (1965).

[10] Simons, F.H.: Recurrence properties and periodicity for Markov processes. Ph.D. thesis, Technological University Eindhoven, 1971.