Taylor-series method for solving two-dimensional differential equations

TAYLOR–SERIES METHOD FOR SOLVING TWO–DIMENSIONAL DIFFERENTIAL EQUATIONS.

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SUMMARY

The variable order Taylor–series method is used to calculate temperature fields in 2D solid complex shaped objects, exposed to a homogeneous external heatsource. A coordinate transformation is used to project the geometry on a rectangle on which the transformed heat equation is solved additionaly. In this paper first 2D calculations are presented.

1. INTRODUCTION

Within the field of mechanical engineering, especially within the area of heat– and mass transfer processes in combustion and heating equipment, contact phenomena and the interaction between fluids and solids is the main issue of research. Many different aspects, such as fluid flow, conduction, chemical reactions, phasechanges and material properties are involved here. However, most of the effort is spent on the treatment of phenomena appearing in the fluid flow. Nevertheless, in order to develop a complete picture of the operation of the apparatus, it is of crucial importance that the heat transport within the often complex–shaped material of the equipment itself and interaction with the fluid part is taken into account correctly. For example, the efficiency of heating equipment depends on the heat transfer and the temperature

distribution in the fins of the heat–exchanger. The need for numerical tools to handle a wide variety of complex geometries and to solve transport problems thereon efficiently, becomes clear. For these reasons, we developed a numerical program based on a variable order method, which gives us the possibility to study the temperature distribution in complex–shaped 2D solid bodies, exposed to external heat sources.

In this paper we present first 2D calculated results. We investigate temperature fields in 2D solid objects shown Fig.1, exposed to a homogeneous external heatsource. First, we study the projection of a complex–shaped object on a rectangle by means of a coordinate transformation. Then, we solve the transformed heat equation on the rectangular object. These two distinct operations may be described by analogous differential equations, as we will show. In section 2 we explain the method we use for the projection and for the heat problem. The numerical method is explained in section 3. Resulting grids and temperature fields are evaluated in section 4. In the future we plan to evaluate the method more intensively and to improve the algorithm in order to investigate the interaction phenomena between the solid and fluid parts in combustion apparatus.

2. PROBLEM DEFINITION.

The physical problem of heat conduction in a solid surface Ω with boundary Γ is described by the two–dimensional Poisson equation

$$\lambda \nabla^2 T + S_{[T]} = 0 \qquad \text{on } \Omega. \tag{1}$$

Here λ denotes the thermal conductivity and $S_{[T]}$ the heat source term. The boundary conditions depend on the physical situation. Here, as a start we take Dirichlet conditions $T=T_0$ on Γ. To handle this heat–conduction problem on an arbitrarily shaped (physical) domain, the geometrical part of the problem is first removed by generating a boundary conforming grid, i.e. transforming the physical domain (x,y) to a rectangular calculation domain (ξ,η). The correspondence between (x,y) and (ξ,η) on the boundaries is specified by the user, while the

transformation $x(\xi,\eta)$ and $y(\xi,\eta)$ in the interior will be determined by the well–known [2] technique of solving an elliptic system of partial differential equations (PDE), consisting of two Laplace equations

$$\nabla^2\xi = 0, \qquad \text{on } \Omega, \tag{2a}$$

$$\nabla^2\eta = 0, \qquad \text{on } \Omega, \tag{2b}$$

Note that both problems, i.e. grid generation and heat–conduction, are described by the same type of PDE and the same kind of boundary conditions. For computational reasons we reformulate both steady problems as transient problems on the calculation domain

$$\left(\frac{\partial z}{\partial t}\right)_{\underline{\xi}} - \dot{\underline{x}}\cdot\underline{\nabla} z = \lambda\nabla^2 z + S_{[z]}, \tag{3}$$

in which the scalar z represents the appropriate scalar unknown (T or ξ,η). The subscript $\underline{\xi}$ indicates a fixed position in the calculation domain. All calculations will be done in the rectangular calculation domain. The velocity vector $\dot{\underline{x}}$ indicates the gridspeed in the physical domain. The appropriate Dirichlet boundary conditions are now understood to be given on the new coordinate boundaries, which coincide with straight line segments. For each problem separately, i.e. grid generation and heat–conduction, Eq.(3) will be examinated further. Without loss of generality we take λ=1 from now on.

2.1 Grid generation.

For the purpose of grid generation the scalar z in Eq.(3) will be replaced by the new coordinates ξ and η, successively. The grid remains fixed (i.e. steady) in the calculation domain implying that the first term in the left–hand side of Eq.(3) equals zero. The grid still has all its moving
abilities in physical space represented by $\dot{\underline{x}}$. We take $S_{[\xi]}$= 0 and $S_{[\eta]}$= 0, which can be interpreted as absence of additional requirements to

redistribute the grid spacing. The result is denoted by the set

$$\dot{\underline{x}}\cdot\underline{\nabla}\xi = -\nabla^2\xi, \tag{4a}$$
$$\dot{\underline{x}}\cdot\underline{\nabla}\eta = -\nabla^2\eta, \tag{4b}$$

which after evalutation of the gradient and Laplacian can be written as

$$\dot{x} = \mathcal{D}x, \tag{5a}$$
$$\dot{y} = \mathcal{D}y. \tag{5b}$$

with $\mathcal{D}$ the second-order differential operator

$$\mathcal{D}\phi = \frac{1}{g}[(x_\eta^2+y_\eta^2)\phi_{\xi\xi} - 2(x_\xi x_\eta + y_\xi y_\eta)\phi_{\xi\eta} + (x_\xi^2+y_\xi^2)\phi_{\eta\eta}] =$$
$$\equiv \alpha\phi_{\xi\xi} - 2\beta\phi_{\xi\eta} + \gamma\phi_{\eta\eta} \tag{6}$$

Note that these two PDE's are non–linear and coupled, although Eq.(5) would suggest otherwise.

2.2 Heat conduction.

Since the medium of heat–transport is a solid, the grid will remain fixed ($\dot{\underline{x}}=\underline{0}$) in physical space:

$$\left(\frac{\partial T}{\partial t}\right)_{\underline{\xi}} = \nabla^2 T + S_{[T]}. \tag{7}$$

Evaluating the Laplacian in the new coordinates, the above equation is written as

$$\dot{T} = \mathcal{D}T + (\nabla^2\xi)T_\xi + (\nabla^2\eta)T_\eta + S_{[T]}, \tag{8}$$

representing the heat–conduction problem expressed in the new

coordinates. For calculations $S_{[T]}=0.1$. Note that Eq.(8) leads to

$$\dot{T} = \mathcal{D}T + S_{[T]}, \tag{9}$$

when the grid reaches the final (i.e. steady) solution of Eq.(4). The (non)linear behavior of Eq.(9) is completely depending on the source term. We will use the resemblance of Eqs.(5) and (9) for solving both problems numerically with the Taylor–series method. From Eqs.(5) and (9) we may conclude that both problems, i.e. grid generation and heat–conduction, are described by the steady part of the non–linear equation ($z=T,\xi,\eta$):

$$\dot{z} = \mathcal{D}z + S_{[z]}, \tag{10}$$

with $S_{[\xi]}=S_{[\eta]}=0$.

3. APPLICATION OF THE TAYLOR–SERIES METHOD.

In this section we will present a method to find the steady solution of Eq.(10), by means of the Taylor–series method (TSM). Solving this (set of) equation(s) must be done by some means of successive linearization. Time has been added from a physical background as relaxation tool in order to control the numerical marching process from an initial guess to the final steady solution by taking small time steps. This approach is known as a pseudo–transient technique [3]. Using explicit time discretization

$$z^{n+1} = z^n + \Delta t\{\mathcal{D}z + S_{[z]}\}, \tag{11}$$

is the recurrence, with superscript n indicating the time step. The steady part will be discretized with the TSM.

Applying the TSM, the domain is divided into elements, which we chose, without loss of generality, to be equidistant and of size

$2\Delta\xi \times 2\Delta\eta = 2 \times 2$. The solution of the PDE is locally (elementwise) expanded in a two–dimensional Taylor–series around the middle of each element

$$z = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} z_{i,j}\, d\xi^i d\eta^j, \tag{12}$$

where $d\xi$ and $d\eta$ are local coordinates. These series and their derivatives are substituted in the PDE and in the corresponding boundary conditions. Collecting same powers of $d\xi$ and $d\eta$, and equating them to zero (because the equations must hold for every $d\xi$ and $d\eta$ within the element) give the necessary relations for the variables $z_{i,j}$. As demonstration, suppose that Eq.(10) represents the heat–conduction problem in general coordinates (ξ,η), according to Eq.(9). Substitution and recollection of the Taylor–series expansion in the PDE yields

$$z^{n+1}_{i,j} = z^n_{i,j} + \Delta t\{[\alpha\frac{(i+2)!}{i!}]z^n_{i+2,j} - [2\beta\frac{(i+1)!\,(j+1)!}{i!\quad j!}]z^n_{i+1,j+1} + [\gamma\frac{(j+2)!}{j!}]z^n_{i,j+2} + S_{i,j}\}. \tag{13}$$
$$i,j=0,1,..,\infty$$

The explicit 'time' discretization is used, because Eqs.(13) are always linear in the variables z^{n+1}, although they are nonlinear in the unknowns z^n, in general. The same procedure of substitution and recollection is used for the boundary conditions which must be given as Taylor–series expansions around the middle of each elementboundary. E.g. for a Dirichlet condition $z(-\Delta\xi,d\eta)=W(d\eta)=\Sigma\, W_j d\eta^j$ on the west boundary ($d\xi = -\Delta\xi$) we find

$$\sum_{i=0}^{\infty} z^{n+1}_{i,j}(-\Delta\xi)^i = W_j, \qquad j=0,1,..,\infty. \tag{14}$$

where W_j is the appropriate Taylor–coefficient of the Dirichlet condition. Note that these equations are always linear in the unknowns. The continuity conditions between each pair of elements form a similar

type of equation. Dealing with the Poisson equation, second–order in ξ, demands continuity in this direction of z and its first derivatives i.e. z_ξ. For similar reasons continuity in η–direction has to be satisfied in z and z_η. The result of Eqs.(13) and (14) is an inifinite set of linear equations. To make the set finite, the number of equations will be bounded, lets say to i=0,1,..,k1 and j=0,1,..,k2. In fact this truncation is the approximating step of the method. Some bookkeeping will show that for each element we have 4 dependent equations, which can clearly be understood from the fact that the continuity conditions on boundaries in ξ–direction have to meet the corresponding continuity conditions on boundaries in η–direction in the 4 elementcorners. Removing 2 equations for each continuity condition in either the ξ–direction or the η–direction will result in a proper set of (k1+3)(k2+3) linear equations per element. Storing the unknowns in a vector, the complete set is denoted by

$$A\underline{z}^{n+1} = b(\underline{z}^n), \tag{15}$$

in which the superscript n refers to the actual 'time'step which are of magnitude Δt. The vektor z^n is the solutionvector containing the unknowns of time step n. Solving this linear set repetitively for sufficient small Δt will result in an approximation of order (k1,k2). A change in one or both of these parameters is easily made and will change the order of accuracy accordingly. Because the set is linear, LU–factorisation of A is performed only once. Although a considerable reduction of computing time is possible by using more economic solvers (e.g. splitting techniques), LU factorisation is used for the time being. The TSM is an easily variable order method. As a matter of fact a certain kind of 'optimal choice' between the number of elements and the order of accuracy becomes available similar as was shown for one–dimensional problems in Sonnemans, et.al. [4].

4. RESULTS AND CONCLUSIONS.

The heat conduction problem for the three systems, given in

Fig.(1), is solved with the method presented in the previous sections.

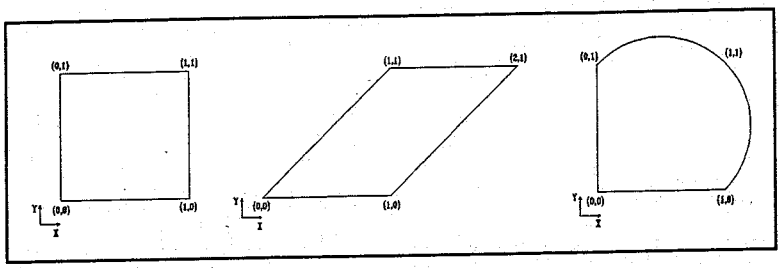

Fig.1. Three geometrical systems on which the heat equation is solved.

The grids are solved according to Eq.(5) and shown for the third example in Fig(2a).

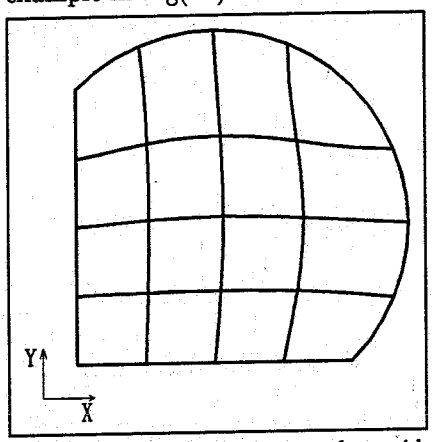

Fig.2a. The generated grid (k=2,N=4) of the third system.

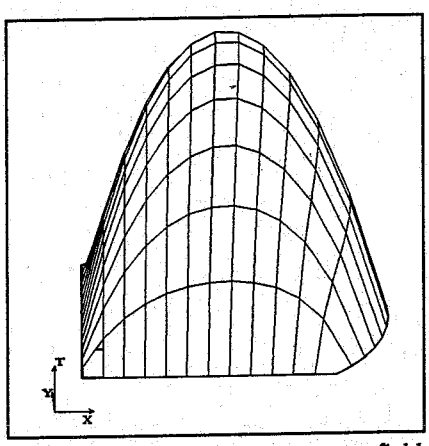

Fig.2b. The temperature field (k=2,N=4) on the third system.

For different numbers of k and for the same number of elements the grid changes slightly. The boundary conditions, described by cubic splines, are well satisfied, although projection of Fig.(2a) onto a rectangle displays a problem in the corner $(\xi,\eta)=(1,1)$, where the Jacobian of the transformation vanishes. The Taylor–series, however, will never be expanded in this point. Nevertheless, the series $x(\xi,\eta)$ and $y(\xi,\eta)$ should be expanded in a point at a sufficiently large distance from this corner to avoid bad convergence of the series. We may conclude that the proposed method works well for generating grids.

The temperature field is calculated on the generated grids as is shown for the third example in Fig.(2b) with a finer mesh for more detail. The characteristic 'temperature bulge' can clearly be seen, besides teh satisfaction of the Dirichlet boundary conditions.

Results of the temperature at the centres, as indicated in Fig.(1), of the three objects, are given in Tables I,II and III, for calculations with different N and k values.

	N=1	N=2	N=3	N=4
k=1	0.6250	0.6667	0.7073	0.7187
k=2	0.7292	0.7411	0.7374	0.7370
k=3	0.7292	0.7363	0.7365	–
k=4	0.7366	0.7365	0.7367	–
Analytical solution : 0.7367135				

Tabel I. Temperature * 10^2.

The calculations for k=3,4 and N=4 are omitted due to memory capacity limits. The accuracy of the numerical coordinate transformation is only of importance when the heat–equation is solved either by Eq.(8) or Eq.(9). Comparisons of the temperature in the physical domain are made, since it is not meaningful to study the accuracy in the computational domain. In Table I the analytical result [3] is included for comparison. The results of Table I denote that the accuracy increases for increasing k and N as one might expect and agrees with analogous results in 1D situations as was shown in [4]. Higher accuracy demands can be satisfied by increasing k and/or N. Both parameters also have their influence on the required computing time which itself depends on the solving procedure. For complete LU–factorisation both parameters have equal influence. From an 'economic' point of view higher accuracy demands can be satisfied with a preference for higher–order approximations as can be seen from Table I, which is a simalar tendency as shown in [4] for 1D problems. This preference, the basic idea of using the Taylor–series method optimally, will in general depend on the

problem being solved, i.e. the order of the PDE and the behavior of the solution quantified by the radius of convergence. To quantify this dependency for 2D problems a more profound research in this direction is necessary.

The results from tables II and III are not as clear as one might expect.

	N=1	N=2	N=3	N=4
k=1	0.4891	0.5600	0.5411	0.5389
k=2	0.5194	0.5440	0.5389	0.5381
k=3	0.5374	0.5641	0.5460	–
k=4	0.5363	0.5613	0.5430	–

Tabel II. Temperature * 10^2.

	N=1	N=2	N=3	N=4
k=1	0.8045	0.8162	0.8916	0.9175
k=2	0.9359	1.0347	0.9772	0.9642
k=3	0.8947	0.9230	0.9506	–
k=4	0.9405	0.9780	0.7241	–

Tabel III. Temperature * 10^2.

The reason can be seen if the extension of the Taylor–series of the solution are studied. It appears that the truncated part of these series is not negligible. The elements are taken too large relative to the radius of convergence of the series. The approximations remains of comparable magnitude, therefore no important improvement of accuracy can be observed. More (smaller) elements may be used to make the truncation error sufficiently small as we are doing now.

In future we plan to make use of the special structure of the set of equations to develop a more sophisticated algorithm in order to solve the system more economically, for example by applying direction splitting techniques.

Usage of larger number of elements will become possible, which makes practical problems more accessible.

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