Solution to problem 90-12 : Two integrals arising from a cloud model

Point of Minimum Temperature

Problem 91-15*, *by* M. S. KLAMKIN (University of Alberta).

A homogeneous convex centrosymmetric body with constant thermal properties is initially at temperature zero and its boundary is maintained at a temperature $T_b > 0$. Prove or disprove that at any time $t > 0$, the point of minimum temperature is the center. Also, prove or disprove that the isothermal surfaces are convex and centrosymmetric. Note that the convexity of the isothermal surfaces will imply that the center is the point of minimum temperature.

SOLUTIONS

Two Integrals Arising from a Cloud Model

Problem 90-12, *by* JOHANNES VERLINDE (Colorado State University).

Determine a closed-form solution or a good approximation to the following integrals, where d is a rational number:

$$I_\Gamma(\alpha,\nu,b,c,d)=\int_0^\infty t^{(\alpha-1)}e^{-bt}\Gamma(\nu,ct^d)\,dt, \tag{1.1}$$

$$I_\gamma(\alpha,\nu,b,c,d)=\int_0^\infty t^{(\alpha-1)}e^{-bt}\gamma(\nu,ct^d)\,dt. \tag{1.2}$$

We can get a general form for the closed form of this integral for the special case $d = 1$ from Prudnikov, Brychkov, and Marichev [1, form. 2.10.3.2]:

$$I_\Gamma(d=1)=-\frac{c^\nu\Gamma(\alpha+\nu)}{\nu b^{(\alpha+\nu)}}{}_2F_1\left(\nu,\alpha+\nu;\nu+1;-\frac{c}{b}\right)+\frac{\Gamma(\nu)\Gamma(\alpha)}{b^\alpha}, \tag{1.3}$$

$$I_\gamma(d=1)=\frac{c^\nu\Gamma(\alpha+\nu)}{\nu b^{(\alpha+\nu)}}{}_2F_1\left(\nu,\alpha+\nu;\nu+1;-\frac{c}{b}\right). \tag{1.4}$$

This problem arises in the mean collection growth equation as used in large mesoscale numerical cloud models.

REFERENCE

[1] A. P. PRUDNIKOV, Y. A. BRYCHKOV, AND O. I. MARICHEV, *Integrals and Series*, Vol. 2, Gordon and Breach, New York, 1986.

Solution by J. BOERSMA *and* P. J. DE DOELDER (Eindhoven University of Technology, Eindhoven, the Netherlands).

For simplicity, it is assumed that all parameters α, ν, b, c, d are real and positive, to ensure convergence of the integrals. Since

$$I_\Gamma+I_\gamma=\Gamma(\nu)\int_0^\infty t^{\alpha-1}e^{-bt}\,dt=\frac{\Gamma(\nu)\Gamma(\alpha)}{b^\alpha}, \tag{1}$$

it is sufficient to consider the integral I_γ only.

By inversion of the Mellin transform

$$\int_0^\infty \gamma(\nu,x)x^{s-1}\,dx=-\frac{\Gamma(\nu+s)}{s},\qquad -\nu<\operatorname{Re} s<0,$$

we obtain the representation

$$\gamma(\nu,x)=-\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma(\nu+s)}{s}x^{-s}\,ds,\qquad -\nu<\delta<0,$$

which is inserted into the integral I_γ. By interchanging the order of integration, we are led to a representation of I_γ by a Mellin–Barnes integral:

$$\begin{aligned}(2)\qquad I_\gamma&=-\frac{b^{-\alpha}}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma(\nu+s)\Gamma(\alpha-ds)}{s}\left(\frac{c}{b^d}\right)^{-s}ds\\&=\frac{b^{-\alpha}}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma(\nu+s)\Gamma(\alpha-ds)\Gamma(-s)}{\Gamma(1-s)}\left(\frac{c}{b^d}\right)^{-s}ds,\qquad -\nu<\delta<0.\end{aligned}$$

The latter integral can be expressed in terms of Fox's H-function. Using the definition and notation from [2, § 8.3], we find

$$(3)\qquad I_\gamma=b^{-\alpha}H_{22}^{12}\left[\frac{c}{b^d}\middle|\begin{matrix}(1-\alpha,d),(1,1)\\(\nu,1),(0,1)\end{matrix}\right]=b^{-\alpha}H_{22}^{21}\left[\frac{b^d}{c}\middle|\begin{matrix}(1-\nu,1),(1,1)\\(\alpha,d),(0,1)\end{matrix}\right].$$

The first integral in (2) can also be evaluated as a series of residues at the poles $s=-\nu-k$, $k=0,1,2,\cdots$ to the left, or at the poles $s=0$ and $s=(\alpha+k)/d$, $k=0,1,2,\cdots$ to the right of the integration contour. Closing the contour to the left (right) can be shown to be permissible if $d<1$ ($d>1$) and if $d=1$, $c<b$ ($d=1$, $c>b$). Thus we obtain the series representations

$$(4)\qquad I_\gamma=\frac{c^\nu}{b^{\alpha+d\nu}}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+d\nu+dk)}{(\nu+k)k!}\left(-\frac{c}{b^d}\right)^k,\qquad d<1\quad\text{or}\quad d=1,\quad c<b,$$

$$(5)\qquad I_\gamma=\frac{\Gamma(\nu)\Gamma(\alpha)}{b^\alpha}-\frac{1}{c^{\alpha/d}}\sum_{k=0}^{\infty}\frac{\Gamma(\nu+d^{-1}\alpha+d^{-1}k)}{(\alpha+k)k!}\left(-\frac{b}{c^{1/d}}\right)^k,\ d>1\text{ or }d=1,\ c>b.$$

These results also follow from [1, form. 2.10.1.5]. From (1), (4), and (5) we deduce the symmetry relation

$$I_\gamma(\alpha,\nu,b,c,d)=\frac{c^\nu}{b^\alpha}I_\Gamma(\nu,\alpha,c,b,d^{-1}),$$

which can also be verified in a direct manner.

In the special case $d=1$, the representations (4) and (5) simplify to

$$(6)\qquad I_\gamma(d=1)=\frac{c^\nu\Gamma(\alpha+\nu)}{\nu b^{\alpha+\nu}}{}_2F_1\left(\alpha+\nu,\nu;\nu+1;-\frac{c}{b}\right),\qquad c<b,$$

$$(7)\qquad I_\gamma(d=1)=\frac{\Gamma(\nu)\Gamma(\alpha)}{b^\alpha}-\frac{\Gamma(\alpha+\nu)}{\alpha c^\alpha}{}_2F_1\left(\alpha+\nu,\alpha;\alpha+1;-\frac{b}{c}\right),\qquad c>b.$$

These results are related by analytic continuation and can be combined into the single expression

$$\begin{aligned}I_\gamma(d=1)&=\frac{c^\nu\Gamma(\alpha+\nu)}{\nu(b+c)^{\alpha+\nu}}{}_2F_1\left(\alpha+\nu,1;\nu+1;\frac{c}{b+c}\right)\\&=\frac{c^\nu\Gamma(\alpha+\nu)}{\nu b^\alpha(b+c)^\nu}{}_2F_1\left(1-\alpha,\nu;\nu+1;\frac{c}{b+c}\right);\end{aligned}$$

see [3, form. 2.10 (2), (6)].

Consider next the case where d is rational, say, $d = p/q$, where p and q are positive integers that are mutually prime. It is known [4], [2, form. 8.3.2.22] that for rational d Fox's H-function in (3) can be reduced to a Meijer's G-function. To that end we start from the first integral in (2). By setting $d = p/q$ and by replacing s by qs, we obtain

$$I_\gamma = -\frac{b^{-\alpha}}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{\Gamma(\nu+qs)\Gamma(\alpha-ps)}{s}\left(\frac{c^q}{b^p}\right)^{-s}ds, \qquad -\nu/q<\delta<0.$$

Using the Gauss–Legendre multiplication formula for the Γ-function (cf. [3, form. 1.2 (11)]), we write

$$\Gamma(\nu+qs) = (2\pi)^{(1-q)/2}q^{\nu+qs-1/2}\prod_{k=0}^{q-1}\Gamma\left(\frac{\nu+k}{q}+s\right)$$

and similarly for $\Gamma(\alpha-ps)$. Thus we are led to the Mellin–Barnes integral representation

$$(8)\quad I_\gamma = \frac{M}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\prod_{k=0}^{q-1}\Gamma\left(\frac{\nu+k}{q}+s\right)\prod_{k=0}^{p-1}\Gamma\left(\frac{\alpha+k}{p}-s\right)\frac{\Gamma(-s)}{\Gamma(1-s)}\left(\frac{p^pc^q}{q^qb^p}\right)^{-s}ds,$$

$$-\nu/q<\delta<0,$$

in which

$$(9)\qquad M = (2\pi)^{1-(p+q)/2}p^{\alpha-1/2}q^{\nu-1/2}b^{-\alpha}.$$

The integral (8) is readily expressed in terms of Meijer's G-function. Using the definition and notation from [2, § 8.2], we find

$$I_\gamma = MG^{q,p+1}_{p+1,q+1}\left(\frac{p^pc^q}{q^qb^p}\middle|\begin{matrix}1-\alpha_1, 1-\alpha_2, \cdots, 1-\alpha_p, 1\\ \nu_1, \nu_2, \cdots, \nu_q, 0\end{matrix}\right)$$

$$(10)\qquad = MG^{p+1,q}_{q+1,p+1}\left(\frac{q^qb^p}{p^pc^q}\middle|\begin{matrix}1-\nu_1, 1-\nu_2, \cdots, 1-\nu_q, 1\\ \alpha_1, \alpha_2, \cdots, \alpha_p, 0\end{matrix}\right),$$

with the short notation $\alpha_k = (\alpha + k - 1)/p$, $\nu_k = (\nu + k - 1)/q$.

Finally, the G-function can be expressed as a finite sum of generalized hypergeometric series. Omitting further details, we present the following series representations for I_γ in the two cases of rational $d < 1$ and $d > 1$:

$$(11)\quad I_\gamma = \frac{c^\nu}{b^{\alpha+d\nu}}\sum_{k=1}^{q}\frac{\Gamma(\alpha+d(\nu+k-1))}{(\nu+k-1)(k-1)!}\left(-\frac{c}{b^d}\right)^{k-1}$$

$$\cdot{}_{p+1}F_q\left[\begin{matrix}\nu_k+\alpha_1, \nu_k+\alpha_2, \cdots, \nu_k+\alpha_p, \nu_k;\\ \frac{k}{q}, \frac{k+1}{q}, \cdots, *, \cdots, \frac{k+q-1}{q}, \nu_k+1;\end{matrix}(-1)^q\frac{p^pc^q}{q^qb^p}\right],$$

$$p<q \quad \text{or}$$

$$p=q, \quad c<b;$$

$$(12)\quad I_\gamma = \frac{\Gamma(\nu)\Gamma(\alpha)}{b^\alpha} - \frac{1}{c^{\alpha/d}} \sum_{k=1}^{p} \frac{\Gamma(\nu + d^{-1}(\alpha+k-1))}{(\alpha+k-1)(k-1)!} \left(-\frac{b}{c^{1/d}}\right)^{k-1} \cdot {}_{q+1}F_p \left[\begin{matrix} \alpha_k+\nu_1, \alpha_k+\nu_2, \cdots, \alpha_k+\nu_q, \alpha_k; \\ \frac{k}{p}, \frac{k+1}{p}, \cdots, *, \cdots, \frac{k+p-1}{p}, \alpha_k+1; \end{matrix} (-1)^p \frac{q^q b^p}{p^p c^q} \right],$$

$$p > q \quad \text{or}$$

$$p = q, \quad c > b.$$

Here the asterisk denotes that the value 1 should be omitted in the sequence of parameters. In the special case $d = 1$, we have $p = q = 1$, and the representations (11) and (12) simplify to those in (6) and (7).

REFERENCES

[1] A. P. PRUDNIKOV, Y. A. BRYCHKOV, AND O. I. MARICHEV, *Integrals and Series*, Vol. 2, Gordon and Breach, New York, 1986.

[2] ———, *Integrals and Series*, Vol. 3: *More special functions*, Gordon and Breach, New York, 1990.

[3] A. ERDÉLYI, W. MAGNUS, F. OBERHETTINGER, AND F. G. TRICOMI, *Higher Transcendental Functions*, Vol. I, McGraw–Hill, New York, 1953.

[4] J. BOERSMA, *On a function, which is a special case of Meijer's G-function*, Compositio Math., 15 (1962), pp. 34–63.

Also solved by the proposer.

A Two-Point Boundary Problem for Airy Functions

Problem 90-13*, *by* RICHARD B. EVANS (Science Application International Corporation, New London, Connecticut).

Airy functions satisfy the differential equation

$$(1)\qquad w''(z) - zw(z) = 0,$$

where z may be complex. Find two distinct complex numbers z_1 and z_2 and a nonzero solution to (1) such that

$$(2)\qquad w(z_1) = 0,$$

$$(3)\qquad w(z_2) = 0,$$

$$(4)\qquad \int_{z_1}^{z_2} w^2(z)\,dz = 0.$$

The value of the integral in (4) is independent of the path of integration since

$$w^2\,dz = d[zw^2 - (w')^2],$$

and (4) may be replaced by the condition

$$w'(z_1) = \pm w'(z_2).$$

This problem arises from a study of underwater acoustic propagation in a waveguide where the complex index of refraction is a linear function of depth.