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Abstract

This paper deals with a mixture of H_2 and H_∞. We have two inputs and one output. One input signal is a white noise stochastic process, and represents errors e.g. resulting from measurement noise. The other input has a more deterministic character. If one has a reference signal (e.g. a step) as input one can not model this as white noise, but it fits nicely into this new class of inputs. The objective is to minimize the effect of these exogenuous signals on the output of the system. We define a cost function which enables us to combine the structural difference between these two exogenuous inputs. The analysis of this function leads to a standard H_∞ Riccati equation. We will motivate this cost function by looking at two theoretical applications: the derivation of robust performance bounds and a tracking problem.

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1 Introduction

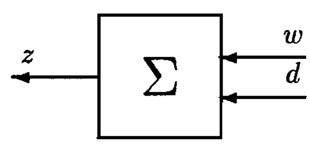
Since the seventies the progress made in H_2 and H_∞ theory is enormous. H_2 theory deals with an exogenuous, white noise, disturbance signal entering the system. It gives a measure of the influence that this signal has on the output. On the other hand H_∞ theory deals with square integrable, unknown, disturbance signals, and discusses the worst-case influence of such signals on the output. The H_∞ norm can be used to derive an indication for the sensitivity of the system due to perturbations of the model. It is a good measure for the robustness problem, where we want to guarantee stability for all systems in some neigbourhood of the model. The model uncertainty in this setup is incorporated via some disturbance system in the model, which is bounded in an H_∞ sense. The H_2 norm is better suited to treat errors resulting from e.g. measurement noise. The H_2 and H_∞ control problems, which minimize these norms, are already solved (see e.g. [2, 4, 9, 10]).
The growing interest for so-called mixed H_2/H_∞ is a logical consequence, and it has already been investigated extensively ([1, 3, 5, 6, 7, 8, 11]). In general we are interested in the H_∞ norm from one input to one output and, simultanuously the H_2 norm from another input to another output. We can try to minimize the nominal H_2 norm (the influence of exogenuous signals on the output), while keeping the H_∞ norm small (we still want stability if the plant is not equal to our model, but close to it). This is done for example in [1, 5], where they minimize some upperbound (called auxilary cost) for the H_2 norm under an H_∞ constraint. From a practical point of view it seems more appropriate to consider some kind of worst-case H_2 norm, such that for all systems close to the model the H_2 norm has a guaranteed upper bound. For example, assume that the plant can be represented by incorporating a disturbance system Δ in the model, where Δ has an H_∞ bound. Then we would like to minimize the H_2 norm for the worst disturbance system Δ. This disturbance system includes modelling errors as well as discarded non-linearities, time-variations and dynamics. The latter arises because of the desire to get a model of sufficiently simple structure for the design of controllers. Instead of maximizing the H_2 norm over all possible disturbance systems Δ, in [8] the approach is taken to maximize instead over all signals, constrained such that they can be connected via a disturbance system Δ. This approach has the disadvantage that it is difficult to incorporate causality of Δ. Moreover, except for SISO systems, the results only yield bounds for worst-case performance.
In this paper we want to combine the H_2 and H_∞ norm in one cost function where we have two inputs and one output. The objective is to study a system with two inputs. One input is a white noise input, like in H_2, resulting from disturbances which have a clear random character. The other input, resulting e.g. from command signals or from modelling uncertainty, has a more deterministic character. The latter is for instance expressed in the fact that this signal need not have zero expectation. It is not clear what measure we should take on the output space. THe H_∞ norm is normally defined via L_2-signals and hence is defined as a measure on the transient behaviour. On the other hand, the H_2 norm is defined via the steady-state behaviour. However, there is a natural way of defining the H_∞ norm via "power signals" and this shows that the H_∞ norm can also be defined as a measure on the steady state behaviour. Because of the above reasoning we decided to define our cost function as a measure for the steady state behaviour. Our cost function expresses the worst-case effect of the inputs on the steady state behaviour of the output.
In section 2 this will be formulated more explicitly. In section 3 the cost function is evaluated for the finite horizon case, which avoids a number of technicalities. In section 4 this will be

extended to the infinite horizon case, which leads to an expression in terms of a standard H_∞ Riccati equation. In section 5 this cost function is minimized with state feedback using well-known H_∞ techniques. In section 6 an interpretation of the cost function is given, and the conclusions are stated in section 7. Since some of the proofs are rather lenghty and technical, they are given in two appendices.

The notation used is fairly standard. We will use that $< x, y >_T := \int_0^T x'(t)y(t)dt$ and $\|x\|_{2,T}^2 :=< x, x >_T$. Moreover, by $z_{w,\beta}$ we denote the output z for initial condition $x(0) = 0$ and inputs w and v, where v is a Brownian motion with $\mathcal{E}v(t) = 0$ and $\mathcal{E}v(t)v'(s) = \beta I \min(t, s)$.

2 Problem formulation

To get a somehow natural mixture of the H_2 norm and the H_∞ norm, it is important to decide which definitions of these norms to use, how to combine these in some mixed cost function, which class of signals to take, and so on. In this paper we consider the case of 2 inputs and 1 output, as is shown in the following picture:



The objective we have for this paper was already pursued in [11]. In [11] the authors give a cost function which mixes H_2 and H_∞. However their work is not very satisfactory since it lacks a solid mathematical foundation. For instance, one of their inputs is a deterministic function which behaves like white noise; a function, the authors admit, that does not exist. We will pursue a cost function which is intuitive, and also fits into a solid mathematical framework. At first we assume Σ to be a linear, time-invariant, finite-dimensional system which has the following structure:

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Gw(t) + Ed(t) \quad , \quad x(0) = 0 \\ z(t) &= Cx(t) \end{cases}$$

with A stable. Here d is a white noise process, which in fact does not exists in continuous time. We can overcome this by writing $d(t)dt$ as $dv(t)$, where v is a Brownian motion. In this way we can get a solid framework by writing Σ in terms of a stochastical differential equation:

$$\Sigma : \begin{cases} dx(t) &= Ax(t)dt + Gw(t)dt + Edv(t) \\ z(t) &= Cx(t) \end{cases} \tag{1}$$

Here we assume v to be a Brownian motion with $\mathcal{E}v(t) = 0$ and $\mathcal{E}v(t)v'(s) = \beta I \min(t, s)$. Obviously the direct feedthrough matrix from v to z has to be zero, since otherwise the H_2 norm from v to z is infinite. For simplicity we assume the direct feedthrough matrix from w to z to be zero.

For the infinite horizon case we have, as mentioned in the introduction, the problem that the standard definition for the H_2 norm investigates the transient behaviour while the standard definition for the H_2 norm investigates the steady state behaviour. This makes it difficult to combine the two. Hence we first investigate the finite horizon case where this problem does not arise.

As in H_∞ we investigate worst-case behaviour. Because we allow the disturbance to depend on the current state we see that in general w will be a stochastic process, albeit not necessarily Gaussian or with zero expectation. In section 6, as we investigate some applications, we will see why it is natural to allow w to be stochastic.

We define Ω_T as the set of all stochastic processes on $[0,T]$ such that

$$\mathcal{E}\|w\|_{2,T}^2 := \mathcal{E}\int_0^T w'(t)w(t)dt$$

is well-defined and bounded. Then it is possible to define a finite horizon cost function in terms of an induced norm:

$$\mathcal{J}(\Sigma,T) \;=\; \sup_{w,\beta} \frac{\mathcal{E}\|z_{w,\beta}\|_{2,T}^2}{\mathcal{E}\|w\|_{2,T}^2+\beta T} \tag{2}$$

where $w \in \Omega_T$ is constrained to be such that $w(t)$ is $\mathcal{F}_v^t$-measurable. Here $\mathcal{F}_v^t$ is the σ-algebra generated by $\{v(s),\ 0 \le s \le t\}$. The latter says nothing more than the only stochastics contained in $w(t)$ is a dependence on past values of v. Note that the term βT is a little surprising. As we already stated we have one input d which is white noise with $\mathcal{E}d(t)d'(t) = \beta I$ for all t. Hence $\mathcal{E}\|d\|_{2,T}^2 = \beta T$, and we get a cost function

$$\bar{\mathcal{J}}(\Sigma,T) \;=\; \sup_{w,\beta} \frac{\mathcal{E}\|z_{w,\beta}\|_{2,T}^2}{\mathcal{E}(\|w\|_{2,T}^2+\|d\|_{2,T}^2)} \;=\; \sup_{w,\beta} \frac{\mathcal{E}\|z_{w,\beta}\|_{2,T}^2}{\mathcal{E}\|w\|_{2,T}^2+\beta T}$$

In discrete time this would be the formal definition. It is in some way the operator norm of Σ on $[0,T]$ induced by the norm $\mathcal{E}\|\cdot\|_{2,T}$, except that, but for the magnitude, the distribution of the random signal d is fixed. However in continuous time white noise is not well-defined. This forces us to work with v instead of d. But the above reasoning motivates the definition of $\mathcal{J}(\Sigma,T)$. We will derive a test, whether $\mathcal{J}(\Sigma,T)$ is smaller than a given bound, in terms of a Riccati differential equation.

We would like to extend this definition to the infinite horizon case. The definition of the H_2 norm is in terms of the steady state covariance of the output, i.e.

$$\lim_{T\to\infty} \mathcal{E}\frac{1}{T}\|z\|_{2,T}^2 \tag{3}$$

It seems appropriate to define $\mathcal{J}(\Sigma)$ in terms of the steady state behaviour of the signals. However, we would like to work with input and output spaces that are linear vector spaces. The class of stochastic processes for which the limit in (3) exists is easily seen not to be linear. Furthermore we could not show that for w in this class and $\beta > 0$ the output z is in this class. Hence it is better to take input and output functions in Ω which is defined as the set of stochastic processes w for which

$$\|w\|_\Omega^2 \;=\; \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}\|w\|_{2,T}^2$$

is well-defined and finite. It is easily checked that Ω is a linear space. On the other hand note that $\|.\|_\Omega$ is not a norm, since it equals zero for all signals in L_2 (it is however a semi-norm). With this class we are able to extend $\mathcal{J}(\Sigma,T)$ to the infinite horizon case:

$$\mathcal{J}(\Sigma) \;=\; \sup_{w,\beta} \frac{\|z_{w,\beta}\|_\Omega^2}{\|w\|_\Omega^2+\beta} \tag{4}$$

where $w \in \Omega$ is constrained to be such that $w(t)$ is $\mathcal{F}_v^t$-measurable. It will be shown that this cost function is well-defined and finite for stable systems. In the appendix we show that $\mathcal{J}(\Sigma)$ equals the square of the H_∞ norm if $\beta = 0$. It is easily seen that $\mathcal{J}(\Sigma)$ equals the square of the H_2 norm for $w = 0$, which indicates the natural mixture of these norms. We will derive a test, whether $\mathcal{J}(\Sigma)$ is smaller than a given bound. This test will be in terms of an algebraic Riccati equation. However we will need quite a bit of work to derive this test. In section 6 we will show that $\mathcal{J}(\Sigma)$ has nice interpretations, e.g. for robust performance and for tracking problems.

3 The finite horizon case

First note that for the finite horizon problem, we do not have to deal with stability properties. So we can drop the assumption that A is stable. For $T > 0$ we define the cost function $\mathcal{J}(\Sigma, T)$ by (2) where the supremum is taken over all w such that $\mathcal{E}\|w\|_{2,T}^2$ is well-defined and finite, $\beta \geq 0$ and $(w, \beta) \neq (0, 0)$. If we only consider deterministic signals w (so w does not depend on v), it is easy to compute $\mathcal{J}(\Sigma, T)$:

Lemma 3.1 : *Let $\gamma_1(T)$ be the finite horizon H_∞ norm from w to z, and $\gamma_2(T)$ the finite horizon H_2 norm from v to z:*

$$\gamma_1(T) = \sup_{w \in L_{2,loc}} \frac{\|z_{w,0}\|_{2,T}}{\|w\|_{2,T}} \quad \textit{and} \quad \gamma_2(T) = \sqrt{\mathcal{E}\|z_{0,1}\|_{2,T}^2}$$

If we require in definition (2) that w is a deterministic function and hence independent of v, then $\mathcal{J}(\Sigma, T) = max\{\gamma_1^2(T), \gamma_2^2(T)\}$. □

Proof : Note that w is deterministic and hence it is not difficult to see that

$$\mathcal{E} < z_{w,0}, z_{0,\beta} >_T = 0.$$

Therefore, we get

$$\begin{aligned} \frac{\mathcal{E}\|z_{w,\beta}\|_{2,T}^2}{\|w\|_{2,T}^2 + \beta T} &= \frac{\|z_{w,0}\|_{2,T}^2 + \mathcal{E}\|z_{0,\beta}\|_{2,T}^2}{\|w\|_{2,T}^2 + \beta T} \\ &\leq \frac{\gamma_1^2(T)\|w\|_{2,T}^2 + \gamma_2^2(T)\beta T}{\|w\|_{2,T}^2 + \beta T} \\ &\leq \max\{\gamma_1^2(T), \gamma_2^2(T)\} \end{aligned}$$

If $\gamma_2(T) \geq \gamma_1(T)$, then this upper bound is attained by $w = 0$. This is seen by noting that

$$\mathcal{E}\|z_{0,\beta}\|_{2,T}^2 = \beta\mathcal{E}\|z_{0,1}\|_{2,T}^2.$$

On the other hand, if $\gamma_1(T) > \gamma_2(T)$, then $\mathcal{J}(\Sigma, T) = \gamma_1^2$ which is attained for $\beta = 0$. ■

Now consider the general case, where w is allowed to depend on v, i.e. w is stochastic. It is easy to find an upper and lower bound for $\mathcal{J}(\Sigma, T)$:

Lemma 3.2 : *Let γ_1 and γ_2 be as defined in lemma 3.1. For $\mathcal{J}(\Sigma, T)$, defined by (2), we have the following bounds*

$$max\ \{\gamma_1^2(T), \gamma_2^2(T)\} \leq \mathcal{J}(\Sigma, T) \leq 2max\ \{\gamma_1^2(T), \gamma_2^2(T)\} \qquad \square$$

Proof: By taking $w = 0$ we see that $\mathcal{J}(\Sigma, T) \geq \gamma_2^2(T)$, and for $\beta = 0$ we get $\mathcal{J}(\Sigma, T) \geq \gamma_1^2(T)$. This proves the first inequality. The second one folows by writing

$$\begin{aligned}
\mathcal{E}\|z_{w,\beta}\|_{2,T}^2 &= \mathcal{E}\|z_{w,0} + z_{0,\beta}\|_{2,T}^2 \\
&\leq 2\mathcal{E}(\|z_{w,0}\|_{2,T}^2 + \|z_{0,\beta}\|_{2,T}^2) \\
&\leq 2(\gamma_1^2(T)\mathcal{E}\|w\|_{2,T}^2 + \gamma_2^2(T)\beta T) \\
&\leq 2\max\ \{\gamma_1^2(T), \gamma_2^2(T)\}(\mathcal{E}\|w\|_{2,T}^2 + \beta T)
\end{aligned}$$

■

To evaluate this cost function, we relate it to another problem. Just as in H_∞ theory it is useful to work with a quadratic cost criterion. For $\alpha > 0$ we define

$$J(T, \alpha) = \sup_{w,\beta} \mathcal{E}(\|z_{w,\beta}\|_{2,T}^2 - \alpha\|w\|_{2,T}^2 - \alpha\beta T)$$

Obviously $\mathcal{J}(\Sigma, T) \leq \alpha$ if and only if $J(T, \alpha) \leq 0$. We have seen that $\mathcal{J}(\Sigma, T) \geq \gamma_1^2(T)$. We know that for $\alpha > \gamma_1^2(T)$ the Riccati differential equation

$$\dot{P}(t) = A'P(t) + P(t)A + \frac{1}{\alpha}P(t)GG'P(t) + C'C \quad , \quad P(0) = 0$$

has a solution P_α with $P_\alpha(t) \geq 0$ for $t \geq 0$. We can prove the following (the proof is given in appendix A):

Theorem 3.3 : *Let $\gamma_1(T)$ be the finite horizon H_∞ norm from w to z. Then $J(T, \alpha) < \infty$ if and only if $\alpha > \gamma_1^2(T)$ and $\int_0^T$ Trace $E'P_\alpha(t)Edt \leq \alpha T$. Moreover, in that case, $J(T, \alpha) = 0$, which is attained by $w^*(t) = \alpha^{-1}G'P_\alpha(T - t)x(t)$ and $\beta^* = 0$.* □

Since P_α decreases if α increases (while αT of course increases), it is easy to compute $\mathcal{J}(\Sigma)$. If $\int_0^T$ Trace $E'P_\alpha Edt \leq \alpha T$, then $\mathcal{J}(\Sigma, T) \leq \alpha$, and if $\int_0^T$ Trace $E'P_\alpha Edt > \alpha T$, then $\mathcal{J}(\Sigma, T) > \alpha$. Hence is possible to give an explicit formula for our cost function:

Corollary 3.4 : *We have* $\mathcal{J}(\Sigma, T) = \inf_{\alpha > \gamma_1^2(T)} \{\alpha | \int_0^T \text{Trace } E'P_\alpha Edt \leq \alpha T\}$ □

and a binary search will lead to $\mathcal{J}(\Sigma, T)$.

4 The infinite horizon case

In the infinite horizon case the cost function will in general be unbounded if A is unstable. Hence a standing assumption in this section will be that A is stable. The results found in the previous section can be extended to the infinite horizon case, although the mathematical details become more complicated. Consider the cost function $\mathcal{J}(\Sigma)$, defined by (4), where the supremum is taken over all $w \in \Omega$ such that $w(t)$ is $\mathcal{F}_v^t$-measurable and over all $\beta \geq 0$ where $||w||_\Omega$ and β are not both zero. We defined the class Ω at the end of section 2. Notice that $||.||_\Omega$ is a semi-norm, not a norm. As in the finite horizon case it is possible to give an upper and lower bound for $\mathcal{J}(\Sigma)$:

Lemma 4.1 : *Let γ_1 be the H_∞ norm from w to z, and γ_2 be the H_2 norm from v to z:*

$$\gamma_1 = \sup_{w \in L_2} \frac{||z_{w,0}||_2}{||w||_2} \quad \textit{and} \quad \gamma_2 = \sqrt{\lim_{T\to\infty} \mathcal{E}\frac{1}{T}||z_{0,1}||_{2,T}^2}$$

Then we have max $\{\gamma_1^2, \gamma_2^2\} \leq \mathcal{J}(\Sigma) \leq 2$max $\{\gamma_1^2, \gamma_2^2\}$. □

Proof : That $\mathcal{J}(\Sigma) \geq \gamma_2^2$ follows immediately by taking $w = 0$. In the appendix we show that $\mathcal{J}(\Sigma)$ equals γ_1^2 for $\beta = 0$, which proves the first inequality. The second one follows by writing

$$\begin{aligned}
\limsup_{T\to\infty} \mathcal{E}\frac{1}{T}||z_{w,\beta}||_{2,T}^2 &= \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}||z_{w,0} + z_{0,\beta}||_{2,T}^2 \\
&\leq 2\limsup_{T\to\infty} \mathcal{E}\frac{1}{T}||z_{w,0}||_{2,T}^2 + 2\limsup_{T\to\infty} \mathcal{E}\frac{1}{T}||z_{0,\beta}||_{2,T}^2 \\
&\leq 2\gamma_1^2 \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}||w||_{2,T}^2 + 2\gamma_2^2\beta
\end{aligned}$$

where in the last step we used that $\mathcal{J}(\Sigma)$ equals γ_1^2 for $\beta = 0$. ■

In the same way as in lemma 3.1, it is easily seen that $\mathcal{J}(\Sigma) = \max\{\gamma_1^2, \gamma_2^2\}$ if we only consider deterministic functions w which are hence independent of v. The evaluation of $\mathcal{J}(\Sigma)$ goes in an analogous way as for the finite horizon case. For $\alpha > 0$ we define

$$K(\alpha) = \sup_{w,\beta} \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}||z_{w,\beta}||_{2,T}^2 - \alpha \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}||w||_{2,T}^2 - \alpha\beta$$

Obviously $\mathcal{J}(\Sigma) \leq \alpha$ if and only if $K(\alpha) \leq 0$. Since $K(\alpha)$ involves limsup twice, this is not so attractive for computations. We would rather relate it to

$$J(\alpha) = \sup_{w,\beta} \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}(||z_{w,\beta}||_{2,T}^2 - \alpha||w||_{2,T}^2) - \alpha\beta \tag{5}$$

It is easily seen that $K(\alpha) \leq J(\alpha)$. To show that they are equal we will first evaluate $J(\alpha)$. We know that $\mathcal{J}(\Sigma) \geq \gamma_1^2$, and that for $\alpha > \gamma_1^2$ the Riccati equation

$$A'P + PA + \alpha^{-1}PGG'P + C'C = 0 \tag{6}$$

has a solution $P_\alpha \geq 0$ such that

$$A + \alpha^{-1} G G' P_\alpha \qquad (7)$$

is stable.

Theorem 4.2 : *Let γ_1 be the H_∞ norm from w to z. Then $J(\alpha) < \infty$ if and only if $\alpha > \gamma_1^2$ and Trace $E'P_\alpha E \leq \alpha$. In that case $J(\alpha) = 0$, which is (not uniquely) attained by $w^* = \alpha^{-1} G' P_\alpha x$ and $\beta^* = 0$.* □

The proof is given in appendix B. Using the optimal w^* and β^* from this theorem we find the following lemma (also proven in the appendix):

Lemma 4.3 : *For $\alpha > \mathcal{J}(\Sigma)$ there holds $J(\alpha) = K(\alpha)$.* □

This lemma shows that $\mathcal{J}(\Sigma) \leq \alpha$ if and only if $J(\alpha) \leq 0$. In combination with the previous theorems the following is an immediate result:

Corollary 4.4 : *We have $\mathcal{J}(\Sigma) = \inf_{\alpha > \gamma_1^2} \{\alpha \mid \text{Trace } E'P_\alpha E \leq \alpha\}$* □

Proof : We have that

$$\begin{aligned} \mathcal{J}(\Sigma) < \alpha &\implies K(\alpha) = J(\alpha) \\ \mathcal{J}(\Sigma) \leq \alpha &\implies K(\alpha) \leq 0 \end{aligned}$$

Combined this yields

$$\mathcal{J}(\Sigma) < \alpha \implies J(\alpha) \leq 0$$

On the other hand, we have

$$J(\alpha) \leq 0 \implies K(\alpha) \leq 0 \implies \mathcal{J}(\Sigma) \leq \alpha$$

The above immediately yields a proof of the corollary ■

Note that this cost-function equals γ_1^2 if $E = 0$, and γ_2^2 if $G = 0$, exactly what is expected. The term Trace $E'P_\alpha E$ is the so called auxilary cost, which has been used in [5, 8, 11]. In these papers the auxilary cost is used as an upperbound for the nominal H_2 norm, which is minimized under an H_∞ constraint.
Since P_α increases if α decreases, it only requires a binary search to compute $\mathcal{J}(\Sigma)$.

5 Minimization by state feedback

In the previous section we analysed a cost function $\mathcal{J}(\Sigma)$, and we showed that it can be related to a standard H_∞ Riccati equation. Using well-known techniques from H_∞ optimization (see e.g. [9]), the minimization of $\mathcal{J}(\Sigma)$ using state feedback is relatively easy. First consider the open-loop system

$$\Sigma : \begin{cases} dx(t) = Ax(t)dt + Bu(t)dt + Gw(t)dt + Edv(t) \\ z(t) = Cx(t) \quad + Du(t) \end{cases}$$

We make the following assumptions:

Assumption 5.1

(*i*) *The pair* (A, B) *is stabilizable,*

(*ii*) *The pair* (C, A) *is detectable,*

(*iii*) $D'[C\ D] = [0\ I]$. □

These assumptions are standard for H_∞ theory. For the third assumption it is sufficient that D is injective. In that case assumption (*iii*) can be achieved via a preliminary feedback.
For an arbitrary, possibly non-linear, time-varying or dynamic but causal compensator: $u = F(x)$ we denote by $\Sigma \times F$ the closed loop interconnection. The interconnection is well-posed if the state and output signals, for given input and zero inititial conditions, are unique and well-defined stochastic processes. We call a compensator stabilizing if the interconnection is well-posed and stable. From lemma 4.1 we know that we can never achieve $\mathcal{J}(\Sigma \times F) < \gamma_{1,*}^2$, where $\gamma_{1,*}$ is the infimum over all $\gamma > 0$ for which there exists a stabilizing compensator for which the closed loop H_∞ norm from w to z is less than γ. We know the following ([9]):

Corollary 5.2 : *Let* $\gamma > 0$. *There exists an internally stabilizing feedback such that the* H_∞ *norm from* w *to* z *is less than* γ *if and only if there exists a positive semi-definite solution* Q *of*

$$A'Q + QA + \gamma^{-2}QGG'Q - QBB'Q + C'C = 0$$

such that $A + \gamma^{-2}GG'Q - BB'Q$ *is stable.* □

This enables us to solve the problem of minimizing $\mathcal{J}(\Sigma)$ using state feedback:

Theorem 5.3 : *There exists an internally stabilizing feedback* $u = F(x)$ *such that* $\mathcal{J}(\Sigma \times F) \leq \alpha$ *if and only if there exists a positive semi-definite solution* Q *of*

$$A'Q + QA + \alpha^{-1}QGG'Q - QBB'Q + C'C = 0$$

such that $A + \alpha^{-1}GG'Q - BB'Q$ *is stable and such that Trace* $E'QE \leq \alpha$. *In that case the linear, static and time-invariant compensator* $u = -B'Qx$ *is internally stabilizing and assures* $\mathcal{J}(\Sigma \times F) \leq \alpha$. □

Proof : We denote by $R(\Sigma \times F, \alpha)$ the cost function $J(\alpha)$ as defined in 5 for the system $\Sigma \times F$. Obviously (see the previous section) $\mathcal{J}(\Sigma \times F) \leq \alpha$ if and only if $R(\Sigma \times F, \alpha) \leq 0$. For any stabilizing feedback $u = F(x)$ we can write using assumption (iii):

$$
\begin{aligned}
& R(\Sigma \times F, \alpha) \\
= & \sup_{w,\beta} \|z_{w,\beta}\|_\Omega^2 - \alpha\|w\|_\Omega^2 - \alpha\beta \\
= & \sup_{w,\beta} \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}(\|Cx + Du\|_{2,T}^2 - \alpha\|w\|_{2,T}^2) - \alpha\beta \\
= & \sup_{w,\beta} \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}\left(\int_0^T |Cx(t)|^2 + |u(t)|^2 - \alpha|w(t)|^2 + \frac{d}{dt}x'(t)Qx(t)dt - x'(T)Qx(T)\right) - \alpha\beta \\
= & \sup_{w,\beta} \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}(\int_0^T |Cx(t)|^2 + |u(t)|^2 - \alpha|w(t)|^2 + 2x'(t)QAx(t) + 2u'(t)B'Qx(t) \\
& \qquad +2w'(t)G'Qx(t)dt - x'(T)Qx(T)) + \beta(\text{Trace } E'QE - \alpha) \\
= & \sup_{w,\beta} \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}(\int_0^T |u(t) + B'Qx(t)|^2 - \alpha|w(t) - \alpha^{-1}G'Qx(t)|^2 dt \\
& \qquad -x'(T)Qx(T)) + \beta(\text{Trace } E'QE - \alpha)
\end{aligned}
$$

Here we used that $\int_0^T x'(t)QEdv(t) = 0$ since v is a Brownian motion process and since $x(t)$ is $\mathcal{F}_v^t$-measurable. Obviously (for any fixed stabilizing compensator) this supremum is bounded if and only if Trace $E'QE \leq \alpha$, which gives

$$R(\Sigma \times F, \alpha) = \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}(\int_0^T |u(t) + B'Qx(t)|^2 - x'(T)Qx(t)dt)$$

which is (not uniquely) attained by $w^* = \alpha^{-1}G'Qx$ and $\beta^* = 0$. By taking $u^* = -B'Qx$, it is easy to check (using that u^* and w^* result in the stable closed-loop matrix $A + \alpha^{-1}GG'Q - BB'Q$) that

$$\lim_{T\to\infty} \mathcal{E}\frac{1}{T}x'(T)Qx(t) = 0$$

and hence $R(\Sigma \times F, \alpha) = 0$. This completes the proof. ■

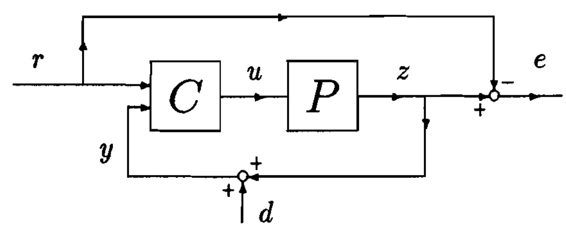
We see that the minimization of $\mathcal{J}(\Sigma)$ using state feedback is not so difficult to solve. The minimization using dynamic feedback will be much harder, as in H_∞ theory. This is the subject of future research.

6 Norminterpretation

In this section we will give some intuition why one might be interested in the cost function $\mathcal{J}(\Sigma)$ introduced in this paper. One application is related to robust performance while the other is basically the tracking problem.

6.1 Tracking problem

In this subsection we show how a standard tracking problem leads naturally to the performance measure $\mathcal{J}(\Sigma)$ as introduced in this paper. Consider the following setup:

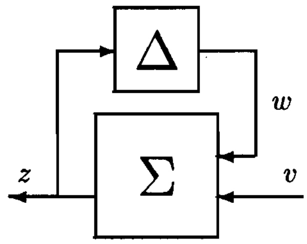


Here r is an, a priori unknown, reference signal. The objective is to make the output z resemble this reference signal, i.e. to minimize the tracking error e. We can affect this tracking signal by designing a suitable compensator C. However, our sensors measuring the output z of the system, will always induce some measurement noise.
Clearly, it is natural to assume that the measurement noise is a white noise stochastic process (in case of coloured noise we can easily incorporate this via a shaping filter). However, for the reference signal r it is much better to model this as an unknown signal in the set Ω. After all, the set Ω contains standard reference signals like steps, sinusoidal functions and bounded piecewise-constant functions. Information with respect to the frequency content of the reference signal can again be incorporated via a suitable weighting.
We would like to minimize the steady state tracking error. We know that if the inputs are in the above classes then the resulting output will be in Ω and $\|\cdot\|_\Omega$ is indeed a measure for the steady state tracking error.
One might argue that the measurement noise and the reference signal are independent. However, there are two reasons why some dependency can occur. First of all if the reference signal is produced by a human operator, the operator might make the reference signal depending upon observations he personally makes of the closed loop system. Secondly the size of the measurement error at a certain time might depend on the size of the signal to be measured. After all, if the signal to be measured is large, then the measurement error might increase: either because the measurement error is relative to the size of the signal or because of saturation effects.
After the above reasoning it should be clear why the measure $\mathcal{J}(\Sigma)$ is a natural measure because it expresses the size of the tracking error relative to the size of the reference signal and the measurement noise.
Concluding it is our feeling that the effect of measurement noise should be measured in an H_2 sense while the effect of reference signals on the tracking error is better measured in an H_∞ sense with suitable weighting. The measure introduced in this paper combines these two.

6.2 Robust performance

In this paper we defined a cost function $\mathcal{J}(\Sigma)$ which is a natural mixture of the H_2 and H_∞ norm of a system with two inputs and one output. We have seen a very attractive equivalent expression for $\mathcal{J}(\Sigma)$, which is given in terms of a Riccati equation. In this section we want to give some results which can be derived using this cost function.

We consider the problem of robust performance. Assume that $w = \Delta z$ for some Δ, as is shown in the following picture:



This system described by this interconnection will be denoted by Σ_Δ. The disturbance system Δ contains modelling errors, non-linearities, time-delays, and so on. We assume Δ to be causal, and bounded (say by 1) in the sense of Ω induced semi-norms (which, for most systems, is equal to being bounded by 1 in L_2 induced norm). Moreover, we require Δ to yield a well-posed interconnection, i.e. given v, the disturbance system Δ and the model Σ with zero initial conditions, thesignals w and z are unique and well-defined stochastic processes. This unknown structure of Δ requires an investigation of the definition of the H_2 norm. The stochastic definition of the H_2 norm can be extended to non-linear, time-varying systems: we define the H_2 norm for a (possibly) non-linear, time-varying system Σ_n as:

$$\|\Sigma_n\|_2^2 := \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}\|z_n\|_{2,T}^2,$$

where z_n is the output of Σ_n with a Brownian motion v_n as input where $\mathcal{E}v_n(t)v_n'(s) = I\min(t,s)$. We take *limsup* instead of *lim*, because we can only guarantee that z_n is bounded in this sense, and not that the limit exists (especially for time-varying perturbations). From $\mathcal{J}(\Sigma)$ we can derive an upperbound for the worst-case H_2 norm:

Lemma 6.1 : *If* $\mathcal{J}(\Sigma) \leq \alpha < 1$, *then* $\sup_{\|\Delta\|_\infty \leq 1} \|\Sigma_\Delta\|_2^2 \leq \frac{\alpha}{1-\alpha}$. □

Proof : Notice that $\|\Sigma_\Delta\|_2^2 = \|z_{\Delta z,1}\|_\Omega^2$ ($\beta = 1$). Say $w = \Delta z$ with $\|\Delta\|_\infty \leq 1$. Then

$$\frac{\|z_{\Delta z,1}\|_\Omega^2}{\|z_{\Delta z,1}\|_\Omega^2 + 1} \leq \frac{\|z_{\Delta z,1}\|_\Omega^2}{\|\Delta z_{\Delta z,1}\|_\Omega^2 + 1} = \frac{\|z_{w,1}\|_\Omega^2}{\|w\|_\Omega^2 + 1} \leq \alpha$$

Since $\alpha < 1$, the result follows immediately. ■

It is not difficult to check that this bound is not tight. For example if $G = 0$ the disturbed system will always have the same H_2 norm, namely γ_2 (notice that $\mathcal{J}(\Sigma) = \gamma_2^2$). And if $E = 0$ it should be zero (here $\mathcal{J}(\Sigma) = \gamma_1^2$). It is possible to give a better bound:

Lemma 6.2 : *If* $\gamma_1^2 < 1$, *then*

$$\sup_{\|\Delta\|_\infty \leq 1} \|\Sigma_\Delta\|_2^2 \leq \inf_{1>\alpha>\gamma_1^2} \frac{\mathit{Trace}\ E'P_\alpha E}{1-\alpha}$$

where P_α *for* $\alpha > \gamma_1^2$ *is equal to the unique solution* $P \geq 0$ *satisfying (6) such that (7) is stable.* □

Proof : This bound can be derived by a careful reexamination of the sequence of equalities used in the proof of B.3. For $1 > \alpha > \gamma_1^2$ (take $\beta = 1$) we find

$$\sup_w \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}(\|z_{w,1}\|_{2,T}^2 - \alpha\|w\|_{2,T}^2) - \alpha$$

$$= \sup_w \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}\left(-\|w - \alpha^{-1}G'P_\alpha x\|_{2,T}^2 - x'(T)P_\alpha x(T)\right) + \text{Trace } E'P_\alpha E - \alpha$$

which is equal to Trace $E'P_\alpha E - \alpha$ and this supremum is (not uniquely) attained by $w^* = \alpha^{-1}G'P_\alpha x$. Hence

$$\sup_w \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}(\|z_{w,1}\|_{2,T}^2 - \alpha\|w\|_{2,T}^2) = \text{Trace } E'P_\alpha E$$

So for $w = \Delta z$ with $\|\Delta\|_\infty \leq 1$ we get

$$\begin{aligned}(1-\alpha)\limsup_{T\to\infty}\mathcal{E}\frac{1}{T}\|z_{\Delta z,1}\|_{2,T}^2 &\leq \limsup_{T\to\infty}\mathcal{E}\frac{1}{T}(\|z_{\Delta z,1}\|_{2,T}^2 - \alpha\|\Delta z_{\Delta z,1}\|_{2,T}^2)\\ &= \limsup_{T\to\infty}\mathcal{E}\frac{1}{T}(\|z_{w,1}\|_{2,T}^2 - \alpha\|w\|_{2,T}^2)\\ &\leq \text{Trace } E'P_\alpha E\end{aligned}$$

From this the result follows immediately. ■

Note that P_α decreases if α increases ($\alpha < 1$), while $\frac{1}{1-\alpha}$ increases. This makes this bound not so easy to compute. This bound seems to be a tight one, for example if $E = 0$ or $G = 0$ it gives exactly the results given above (notice that $\gamma_1 = 0$ if $G = 0$, and in that case $P_\alpha = L$ where L is the observability grammian).

7 Conclusions

In this paper we investigate a mixed cost function which combines the H_2 and the H_∞ norm in a stochastic framework. We consider the case of two inputs and one output. First we chose a suitable class of functions for the input space from which we can define the H_2 norm and the H_∞ norm in terms of the same kind of measure on the output. The cost function $\mathcal{J}(\Sigma)$ is then defined in a natural way as an induced semi-norm. This paper mainly deals with the analysis of this mixed cost function. Our objective was to show how the H_2 and H_∞ norm can be combined in a logical and elegant way. We find a very attractive expression for $\mathcal{J}(\Sigma)$ in terms of a standard H_∞ Riccati equation, which is known in literature as the auxilary cost. In section 5 we show that using this expression the minimization of the cost function for the state feedback case is relative easy. We also give some connections with problems like robust performance and tracking problems, for which we can easily derive results using $\mathcal{J}(\Sigma)$.
For the combination of H_2 and H_∞ in one mixed cost function there are many open problems which seem rather tractable. We can look at the dual version of the problem stated here, i.e. the case where we have one input and two outputs. Then we can try to consider the general case of two inputs and two outputs. Especially this last set-up is very useful for the problem of robust performance but hard to handle. Furthermore it is interesting to investigate the use of the mixed cost functions for certain applications in more detail.

Appendices

A Proofs for the finite horizon case

The proof of theorem 3.3 will only be given for $\alpha = 1$. The general result can be obtained via scaling.

Theorem A.1 : *Let $\gamma_1(T)$ be the finite horizon H_∞ norm from w to z. Then $J(T,1) < \infty$ if and only if $\gamma_1^2(T) < 1$ and $\int_0^T$ Trace $E'P(t)Edt \leq T$. In that case $J(T,1) = 0$ and the supremum is attained for $w^*(t) = G'P(T-t)x(t)$ and $\beta^* = 0$. Here $P(t)$ is defined by*

$$\dot{P}(t) = A'P(t) + P(t)A + P(t)GG'P(t) + C'C \ ; \ P(0) = 0 \qquad \square$$

Proof : $J(T,1)$

$$= \sup_{w,\beta} \mathcal{E}(\|z_{w,\beta}\|_{2,T}^2 - \|w\|_{2,T}^2 - \beta T + \int_0^T \frac{d}{dt} x'(t)P(T-t)x(t)dt)$$

$$= \sup_{w,\beta} \mathcal{E}\left(\int_0^T [x'(t)C'Cx(t) - w'(t)w(t)]\, dt - \beta T \right.$$
$$+ \int_0^T 2x'(t)P(T-t)Edv(t) - \int_0^T x'(t)\dot{P}(T-t)x(t)dt + \int_0^T \beta \text{ Trace } E'P(t)Edt$$
$$\left. + \int_0^T 2x'(t)P(T-t)\,[Ax(t) + Gw(t)]\, dt\right)$$

$$= \sup_{w,\beta} \mathcal{E} \int_0^T [2x'(t)P(T-t)Gw(t) - w'(t)w(t) - x'(t)P(T-t)GG'P(T-t)x(t)]\, dt$$
$$+\beta(\int_0^T \text{ Trace } E'P(t)Edt - T)$$

Here we used Ito's differential rule and the fact that $\mathcal{E} \int_0^T f(t)dv(t) = 0$ for any $f(t)$ that is $\mathcal{F}_v^t$-measurable. Hence we get

$$J(T,1) = \sup_{w,\beta} \beta(\int_0^T \text{ Trace } E'P(t)Edt - T) - \mathcal{E}\|w(t) - G'P(T-t)x(t)\|_{2,T}^2$$

which is finite if and only if $\int_0^T$ Trace $E'P(t)Edt \leq$ T, and in that case $J(T,1) = 0$ and the supremum is attained for $w^*(t) = G'P(T-t)x(t)$ and $\beta^* = 0$. ■

B Proofs for the infinite horizon case

The results found for the finite horizon case can be extended to the infinite horizon case, although it needs some more extensive calculations. At first we will give an equivalent definition for the H_∞ norm, which is used in lemma 4.1. This result is similar result given in [11]. However, because we work with *limsup* instead of *limits* and because we do not make assumptions on the input signals to allow a frequency domain analysis, we need to do more work.

Lemma B.1 : *Let the system (1) be given. We have*

$$\sup_w \frac{\|z_{w,0}\|_\Omega}{\|w\|_\Omega} = \gamma_1$$

where γ_1 is the H_∞ norm from w to z. □

Proof : Note that since $\beta = 0$, we only deal with deterministic signals. We first show that the above supremum is larger than or equal to γ_1. Let $\epsilon > 0$ be arbitrary small and choose $w_* \in L_2$ such that

$$\frac{\|z_{w_*,0}\|_2^2}{\|w_*\|_2^2} > \gamma_1^2 - \frac{\epsilon}{4}$$

This clearly implies that there exists T_1 such that for all $t > T_1$:

$$\frac{\|z_{w_*,0}\|_{2,t}^2}{\|w_*\|_{2,t}^2} > \gamma_1^2 - \frac{\epsilon}{2}$$

Define

$$\kappa = \sqrt{\int_0^\infty \|Ce^{At}\|^2 dt} \ , \quad \delta = \frac{\|w_*\|_{2,T_1}^2}{4\kappa\|z_{w_*,0}\|_2}\epsilon$$

where $\|.\|$ denotes the largest singular value of a matrix. Since the input w_* is in L_2 and A is stable, it is well-known that then

$$x(t) = \int_0^t e^{A(t-\tau)} E w_*(\tau) d\tau \to 0$$

as $t \to \infty$. Using this it can be shown that there exists $T_2 > T_1$ such that for all $t > T_2$:

$$\left| \int_0^t e^{A(t-\tau)} E w_*(\tau) d\tau \right| < \frac{1}{2}\delta$$

Since A is stable, it is obvious that there exists $s > T_2$ such that for all $t > s$ we have

$$\|e^{At}\| < \frac{1}{2}$$

Next, we will define a input function with period s, and compute a lower bound for the corresponding output. Define $w \in \Omega$ by

$$w(ks + t) = w_*(t)$$

for $k \in I\!N$ and $0 \leq t < s$. Because w is periodic, it is easy to see that this function is indeed in Ω. We will show that $|x(ks)| < \delta$ for all $k \in I\!N$. Obviously this is true for $k = 0$ ($x(0) = 0$). Suppose it is true for some $k \in I\!N$. Then

$$\begin{aligned}
|x((k+1)s)| &= |e^{As}x(ks) + \int_{ks}^{(k+1)s} e^{A((k+1)s-t)}Ew(t)dt| \\
&\leq \|e^{As}\||x(ks)| + |\int_0^s e^{A(s-t)}Ew_*(t)dt| \\
&< \frac{1}{2}\delta + \frac{1}{2}\delta
\end{aligned}$$

The next step is to give a lower bound for $\|z_{w,0}\|^2_{2,ks}$ for $k \in I\!N$:

$$\begin{aligned}
&\|z_{w,0}\|^2_{2,(k+1)s} \\
&= \|z_{w,0}\|^2_{2,ks} + \int_{ks}^{(k+1)s} |z_{w,0}(t)|^2 dt \\
&= \|z_{w,0}\|^2_{2,ks} + \int_{ks}^{(k+1)s} |Ce^{A((k+1)s-t)}x(ks) + z_{w_*,0}(t-ks)|^2 dt \\
&\geq \|z_{w,0}\|^2_{2,ks} + \|z_{w_*,0}\|^2_{2,s} - 2\delta\kappa\|z_{w_*,0}\|_2
\end{aligned}$$

By applying the same step to this expression, we find in a recursive way

$$\|z_{w,0}\|^2_{2,ks} \geq k\|z_{w_*,0}\|^2_{2,s} - 2k\delta\kappa\|z_{w_*,0}\|_2$$

This enables us to prove the result

$$\begin{aligned}
\frac{\|z_{w,0}\|^2_\Omega}{\|w\|^2_\Omega} &\geq \frac{\limsup\limits_{k\to\infty} \frac{1}{ks}(k\|z_{w_*,0}\|^2_{2,s} - 2k\delta\kappa\|z_{w_*,0}\|_2)}{\frac{1}{s}\|w_*\|^2_{2,s}} \\
&= \frac{\|z_{w_*,0}\|^2_{2,s}}{\|w_*\|^2_{2,s}} - \frac{1}{2}\frac{\|w_*\|^2_{2,T_1}}{\|w_*\|^2_{2,s}}\epsilon \\
&\geq \gamma_1^2 - \frac{1}{2}\epsilon - \frac{1}{2}\epsilon \;=\; \gamma_1^2 - \epsilon
\end{aligned}$$

We used that $\|w_*\|^2_{2,T_1} \leq \|w_*\|^2_{2,s}$ since $T_1 < s$. Since $\epsilon > 0$ was arbitrary, this proves that the supremum is larger than or equal to γ_1.

We will now show that we actually have equality. Suppose there exists w_* such that

$$\frac{\|z_{w_*,0}\|^2_\Omega}{\|w_*\|^2_\Omega} \geq \gamma_1^2 + \epsilon$$

for some $\epsilon > 0$. Let $\delta > 0$ be such that $\delta < \frac{\epsilon}{\epsilon + 2\gamma_1^2}$. By definition of the limsup there exists T_1 such that for all $t > T_1$:

$$\frac{1}{t}\|w_*\|^2_{2,t} \leq (1+\delta)\|w_*\|^2_\Omega$$

Moreover, there exists $s > T_1$ such that

$$\frac{1}{s}||z_{w_*,0}||_{2,s}^2 \geq (1-\delta)||z_{w_*,0}||_\Omega^2$$

Using this we find

$$\begin{aligned}
\sup_{w \in L_2[0,s]} \frac{||z_{w,0}||_{2,s}^2}{||w||_{2,s}^2} &\geq \frac{||z_{w_*,0}||_{2,s}^2}{||w_*||_{2,s}^2} \\
&\geq \frac{(1-\delta)||z_{w_*,0}||_\Omega^2}{(1+\delta)||w_*||_\Omega^2} \\
&= \frac{(1-\delta)}{(1+\delta)}(\gamma_1^2 + \epsilon) \\
&> \gamma_1^2 \\
&= \sup_{w \in L_2} \frac{||z_{w,0}||_2^2}{||w||_2^2}
\end{aligned}$$

Since the infinite horizon H_∞ norm is larger than or equal to the finite horizon H_∞ norm, this is a contradiction. This completes the proof. ■

To prove that $J(\alpha) = 0$ if $J(\alpha) < \infty$ (theorem 4.2), we need the following lemma:

Lemma B.2 : *If $w \in \Omega$ then $\limsup_{T\to\infty} \mathcal{E}\frac{1}{T}|x(T)|^2 < \infty$ for all $\beta \geq 0$.* □

Proof : We have

$$\begin{aligned}
\limsup_{T\to\infty} \mathcal{E}\frac{1}{T}|x(T)|^2 &= \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}\left|\int_0^T e^{A(T-t)}Gw(t)dt + \int_0^T e^{A(T-t)}Edv(t)\right|^2 \\
&\leq 2\limsup_{T\to\infty} \mathcal{E}\frac{1}{T}|\int_0^T e^{A(T-t)}Gw(t)dt|^2 \\
&\qquad +2\limsup_{T\to\infty} \mathcal{E}\frac{1}{T}|\int_0^T e^{A(T-t)}Edv(t)|^2
\end{aligned}$$

The first term can be bounded as follows:

$$\limsup_{T\to\infty} \mathcal{E}\frac{1}{T}|\int_0^T e^{A(T-t)}Gw(t)dt|^2 \leq \left(\int_0^\infty ||e^{At}G||^2 dt\right)||w||_\Omega^2$$

using lemma B.1 and the fact that A is stable. Using standard properties of stochastic integrals we also find

$$\limsup_{T\to\infty} \mathcal{E}\frac{1}{T}\left|\int_0^T e^{A(T-t)}Edv(t)\right|^2 = \limsup_{T\to\infty} \frac{\beta}{T}\int_0^T \text{Trace } E'e^{A'(T-t)}e^{A(T-t)}Edt$$

which tends to zero since A is stable. ■

This lemma enables us to prove theorem 4.2 (for $\alpha = 1$):

Theorem B.3 : *Let $\gamma_1^2 < 1$, where γ_1 is the H_∞ norm from w to z. Then $J(1) < \infty$ if and only if Trace $E'PE \leq 1$, and in that case $J(1) = 0$, which is (not uniquely) attained by $w^* = G'Px$ and $\beta^* = 0$. Here $P \geq 0$ satisfies $A'P + PA + PGG'P + C'C = 0$ with $A + GG'P$ stable.* □

Proof : We can write using lemma B.2 :

$$\begin{aligned} & J(1) \\ = & \sup_{w,\beta} \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}\left(\|z_{w,\beta}\|_{2,T}^2 - \|w\|_{2,T}^2 - \beta + \int_0^T \left[\frac{d}{dt}x'(t)Px(t)\right]dt - x'(T)Px(T)\right) \\ = & \sup_{w,\beta} \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}\left(\int_0^T (2x'(t)PGw(t) - w'(t)w(t) - x'(t)PGG'Px(t))\,dt\right. \\ & \left. - x'(T)Px(T)\right) + \beta(\text{Trace } E'PE - 1) \end{aligned}$$

Hence we get

$$J(1) = \sup_{w,\beta} \; \beta(\text{Trace } E'PE - 1) + \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}\left(-\|w - G'Px\|_{2,T}^2 - x'(T)Px(T)\right)$$

which is finite if and only if Trace $E'PE \leq 1$, and in that case we see that $J(1) = 0$, which is (not uniquely!) attained by $w^*(t) = G'Px(t)$ and $\beta^* = 0$. Note that if we have inputs w^* and β^* (i.e. $v = 0$) then it is easy to show that

$$\lim_{T\to\infty} \mathcal{E}\frac{1}{T}x'(T)Px(T) = 0$$

Therefore $J(1) = 0$. Any w and β which attain this supremum are of the form $\beta^* = 0$ and $w^* = G'Px + w_1$ for some function w_1 with $\|w_1\|_\Omega = 0$. ■

Now we will prove lemma 4.3, which claims that $J(\alpha) = K(\alpha)$. Again we take $\alpha = 1$.

Lemma B.4 : *If $\mathcal{J}(\Sigma) < 1$, then*

$$\begin{aligned} J(1) &= \sup_{w,\beta} \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}(\|z_{w,\beta}\|_{2,T}^2 - \|w\|_{2,T}^2) - \beta \\ &= \sup_{w,\beta} \|z_{w,\beta}\|_\Omega^2 - \|w\|_\Omega^2 - \beta \;=\; K(1) \end{aligned}$$

Proof : The left-hand side equals $J(1)$, for which theorem B.3 says that this supremum is (not uniquely) attained by a state feedback $w^* = G'Px$. This feedback results in the system

$$\Sigma^* : \begin{cases} dx(t) &= (A + GG'P)x(t)dt + Edv(t) \\ z(t) &= Cx(t) \\ w^*(t) &= G'Px(t) \end{cases}$$

where $A+GG'P$ is stable. It is not very hard to show that for every $\beta \geq 0$ the following limit is well-defined and bounded:

$$\lim_{T\to\infty} \mathcal{E}\frac{1}{T}\|w^*\|_{2,T}^2$$

and hence

$$\begin{aligned} J(1) &= \limsup_{T\to\infty} \mathcal{E}\frac{1}{T}(\|z_{w^*,\beta}\|_{2,T}^2 - \|w^*\|_{2,T}^2) - \beta \\ &= \|z_{w^*,\beta}\|_\Omega^2 - \|w^*\|_\Omega^2 - \beta. \end{aligned}$$

From this it easily follows that $J(\alpha) \leq K(\alpha)$. That $J(\alpha) \geq K(\alpha)$ is trivial. ■

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List of COSOR-memoranda - 1992

Number	Month	Author	Title
92-01	January	F.W. Steutel	On the addition of log-convex functions and sequences
92-02	January	P. v.d. Laan	Selection constants for Uniform populations
92-03	February	E.E.M. v. Berkum H.N. Linssen D.A. Overdijk	Data reduction in statistical inference
92-04	February	H.J.C. Huijberts H. Nijmeijer	Strong dynamic input-output decoupling: from linearity to nonlinearity
92-05	March	S.J.L. v. Eijndhoven J.M. Soethoudt	Introduction to a behavioral approach of continuous-time systems
92-06	April	P.J. Zwietering E.H.L. Aarts J. Wessels	The minimal number of layers of a perceptron that sorts
92-07	April	F.P.A. Coolen	Maximum Imprecision Related to Intervals of Measures and Bayesian Inference with Conjugate Imprecise Prior Densities
92-08	May	I.J.B.F. Adan J. Wessels W.H.M. Zijm	A Note on "The effect of varying routing probability in two parallel queues with dynamic routing under a threshold-type scheduling"
92-09	May	I.J.B.F. Adan G.J.J.A.N. v. Houtum J. v.d. Wal	Upper and lower bounds for the waiting time in the symmetric shortest queue system
92-10	May	P. v.d. Laan	Subset Selection: Robustness and Imprecise Selection
92-11	May	R.J.M. Vaessens E.H.L. Aarts J.K. Lenstra	A Local Search Template (Extended Abstract)
92-12	May	F.P.A. Coolen	Elicitation of Expert Knowledge and Assessment of Imprecise Prior Densities for Lifetime Distributions
92-13	May	M.A. Peters A.A. Stoorvogel	Mixed H_2/H_∞ Control in a Stochastic Framework