

## Solution to Problem 73-8: A polynomial diophantine equation

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providing details and we will investigate your claim.

Here  $\varepsilon = +1$  with  $\delta = \pm 1$  and  $\varepsilon = -1$  with  $\delta = -1$  covers all possibilities for

$$t_1 = \frac{n}{4} - \frac{4 + \varepsilon + \delta}{2}, \quad t_2 = \frac{n}{4} - \frac{1 - \delta}{2},$$

$$t_3 = \frac{n}{4} - \frac{1 - \varepsilon}{2}, \quad t_4 = \frac{n}{4}.$$

$M$  is the matrix of a ‘third’ minor. The evaluation of

$$\det(M^T \cdot M) = \begin{array}{|c|c|c|c|} \hline M_1 & & & \\ \hline & M_2 & & +1 \\ \hline & +1 & M_3 & \\ \hline & & & M_4 \\ \hline \end{array}, \text{ where}$$

$$M_v = \begin{bmatrix} n-3 & & & \\ & \cdot & & -3 \\ -3 & & \cdot & \\ & & & n-3 \end{bmatrix}$$

of order  $t_v$ , gives

$$\begin{aligned} \det M &= 0, && \text{for } \varepsilon = +1, \delta = +1, n > 8; \\ \det M &= 0, && \text{for } \varepsilon = +1, \delta = -1, n \geq 8; \\ \det M &= \pm 4n^{n/2-3} = \pm 4g_n/n^3, && \text{for } \varepsilon = -1, \delta = -1, n \geq 8. \end{aligned}$$

The problem also shows that the  $(n - 2)$ - and  $(n - 3)$ -order nonsingular submatrices of an  $n$ -order Hadamard all have inverses whose nonzero entries can be only  $+2/n$  or  $-2/n$ .

*Problem 73-8. A Polynomial Diophantine Equation*, by M. S. KLAMKIN (Ford Motor Company).

Determine all real solutions of the polynomial Diophantine equation

$$(1) \quad P(x)^2 - P(x^2) = x\{Q(x)^2 - Q(x^2)\}.$$

Solution by O. P. LOSSERS (Technological University, Eindhoven, the Netherlands).

From the given equation, it follows that

$$\begin{aligned} P(x^4) - x^2Q(x^4) &= P^2(x^2) - x^2Q^2(x^2) \\ &= \{P(x^2) - xQ(x^2)\}\{P(x^2) + xQ(x^2)\}. \end{aligned}$$

Letting  $F(x) = P(x^2) - xQ(x^2)$ , we have

$$(2) \quad F(x^2) = F(x)F(-x).$$

Conversely, any solution of (1) may be obtained from a solution of (2) by taking

$$P(x) = \frac{1}{2}\{F(\sqrt{x}) + F(-\sqrt{x})\},$$

$$Q(x) = \frac{1}{2x}\{-F(\sqrt{x}) + F(-\sqrt{x})\}.$$

Polynomial solutions of (2) may be written in the form

$$F(x) = C(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) \quad (C \text{ is a constant}).$$

Then

$$F(-x) = (-1)^n C(x + \alpha_1)(x + \alpha_2) \cdots (x + \alpha_n),$$

so that

$$F(x)F(-x) = (-1)^n C^2(x - \alpha_1)(x + \alpha_1)(x - \alpha_2)(x + \alpha_2) \cdots (x - \alpha_n)(x + \alpha_n).$$

On the other hand, taking  $\beta_i$  such that  $\beta_i^2 = \alpha_i (i = 1, \dots, n)$ , we find

$$F(x^2) = C(x - \beta_1)(x + \beta_1)(x - \beta_2)(x + \beta_2) \cdots (x - \beta_n)(x + \beta_n).$$

Therefore, in view of (2), excluding the trivial case  $C = 0$ , we obtain  $C = (-1)^n$  and  $(\alpha_i)_{i=1}^n$  is a permutation of  $(\beta_i)_{i=1}^n$ .

Finite, squaring-invariant subsets of the complex plane can only contain 0 and roots of unity of odd order. The irreducible polynomials corresponding to these roots are

$$\lambda_0(x) = x, \quad \lambda_k(x) = \prod_{(2k-1, l)=1} [x - \exp [2\pi il/(2k-1)]], \quad k = 1, 2, 3, \dots,$$

(the cyclotomic polynomials). Since for all  $k = 1, 2, 3, \dots$ , the set  $\{\exp [2\pi il/(2k-1)]\}_{(l, 2k-1)=1}$  is squaring-invariant and the set of solutions of (2) is closed under multiplication, the general polynomial solution of (2) is

$$F(x) = (-1)^{\deg F} \prod_{k=0}^{\infty} (\lambda_k(x))^{n_k},$$

the  $n_k$  being nonnegative integers,  $n_k \neq 0$ , for a finite number of indices  $k$ . These polynomials all have integral coefficients.

Also solved by the proposer, who notes that one can give extensions by considering higher order roots of unity. For example, letting  $\omega^3 = 1$ , consider  $F(x^3) = F(x)F(\omega x)F(\omega^2 x)$ , where  $F(x) = P(x^3) + \omega x Q(x^3) + \omega^2 x^2 R(x^3)$ .