Solution to Problem 73-8: A polynomial diophantine equation

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Here $\varepsilon = +1$ with $\delta = \pm 1$ and $\varepsilon = -1$ with $\delta = -1$ covers all possibilities for

$$t_1 = \frac{n}{4} - \frac{4 + \varepsilon + \delta}{2}, \qquad t_2 = \frac{n}{4} - \frac{1 - \delta}{2},$$

$$t_3 = \frac{n}{4} - \frac{1 - \varepsilon}{2}, \qquad t_4 = \frac{n}{4}.$$

M is the matrix of a 'third' minor. The evaluation of

$$\det(M^T \cdot M) = \left|\begin{array}{cccc} M_1 & & +1 & \\ & M_2 & & \\ +1 & & M_3 & \\ & & & M_4 \end{array}\right|, \quad \text{where}$$

$$M_v = \begin{bmatrix} n-3 & & & \\ & \cdot & & -3 \\ -3 & & \cdot & \\ & & & n-3 \end{bmatrix}$$

of order t_v, gives

$$\det M = 0, \qquad \text{for } \varepsilon = +1, \delta = +1, n > 8;$$

$$\det M = 0, \qquad \text{for } \varepsilon = +1, \delta = -1, n \geqq 8;$$

$$\det M = \pm 4n^{n/2-3} = \pm 4g_n/n^3, \quad \text{for } \varepsilon = -1, \delta = -1, n \geqq 8.$$

The problem also shows that the $(n - 2)$- and $(n - 3)$-order nonsingular submatrices of an n-order Hadamard all have inverses whose nonzero entries can be only $+2/n$ or $-2/n$.

Problem 73-8. *A Polynomial Diophantine Equation*, by M. S. KLAMKIN (Ford Motor Company).

Determine all real solutions of the polynomial Diophantine equation

$$P(x)^2 - P(x^2) = x\{Q(x)^2 - Q(x^2)\}. \tag{1}$$

Solution by O. P. LOSSERS (Technological University, Eindhoven, the Netherlands).

From the given equation, it follows that

$$\begin{aligned} P(x^4) - x^2Q(x^4) &= P^2(x^2) - x^2Q^2(x^2) \\ &= \{P(x^2) - xQ(x^2)\}\{P(x^2) + xQ(x^2)\}. \end{aligned}$$

Letting $F(x) = P(x^2) - xQ(x^2)$, we have

$$F(x^2) = F(x)F(-x). \tag{2}$$

Conversely, any solution of (1) may be obtained from a solution of (2) by taking

$$P(x) = \frac{1}{2}\{F(\sqrt{x}) + F(-\sqrt{x})\},$$

$$Q(x) = \frac{1}{2x}\{-F(\sqrt{x}) + F(-\sqrt{x})\}.$$

Polynomial solutions of (2) may be written in the form

$$F(x) = C(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) \qquad (C \text{ is a constant}).$$

Then

$$F(-x) = (-1)^n C(x + \alpha_1)(x + \alpha_2) \cdots (x + \alpha_n),$$

so that

$$F(x)F(-x) = (-1)^n C^2 (x - \alpha_1)(x + \alpha_1)(x - \alpha_2)(x + \alpha_2) \cdots (x - \alpha_n)(x + \alpha_n).$$

On the other hand, taking β_i such that $\beta_i^2 = \alpha_i (i = 1, \cdots, n)$, we find

$$F(x^2) = C(x - \beta_1)(x + \beta_1)(x - \beta_2)(x - \beta_2) \cdots (x - \beta_n)(x + \beta_n).$$

Therefore, in view of (2), excluding the trivial case $C = 0$, we obtain $C = (-1)^n$ and $(\alpha_i)_{i=1}^n$ is a permutation of $(\beta_i)_{i=1}^n$.

Finite, squaring-invariant subsets of the complex plane can only contain 0 and roots of unity of odd order. The irreducible polynomials corresponding to these roots are

$$\lambda_0(x) = x, \qquad \lambda_k(x) = \prod_{(2k-1,l)=1} [x - \exp[2\pi i l/(2k - 1)]], \qquad k = 1, 2, 3, \cdots,$$

(the cyclotomic polynomials). Since for all $k = 1, 2, 3, \cdots$, the set $\{\exp[2\pi i l/(2k - 1)]\}_{(l,2k-1)=1}$ is squaring-invariant and the set of solutions of (2) is closed under multiplication, the general polynomial solution of (2) is

$$F(x) = (-1)^{\deg F} \prod_{k=0}^{\infty} (\lambda_k(x))^{n_k},$$

the n_k being nonnegative integers, $n_k \neq 0$, for a finite number of indices k. These polynomials all have integral coefficients.

Also solved by the proposer, who notes that one can give extensions by considering higher order roots of unity. For example, letting $\omega^3 = 1$, consider $F(x^3) = F(x)F(\omega x)F(\omega^2 x)$, where $F(x) = P(x^3) + \omega x Q(x^3) + \omega^2 x^2 R(x^3)$.