

# On the number of positive integers $\leq x$ and free of prime factors $> y$

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ON THE NUMBER OF POSITIVE INTEGERS  
 $\leq x$  AND FREE OF PRIME FACTORS  $> y$

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1. Introduction

Let  $\Psi(x, y)$  denote the number of integers specified in the title. A number of estimates and asymptotic formulae for this function have been given (cf. [1] and the literature mentioned there). Recently DE BRUIJN ([2]) proved an asymptotic formula for  $\log \Psi(x, y)$  which holds uniformly for  $2 < y \leq x$ . Part of the proof consisted of showing that

$$\Psi(x, y) \geq \binom{\pi(y) + u}{u} \text{ where } u = [(\log x)/(\log y)].$$

It is the purpose of this note to extend this inequality to an asymptotic formula (which is weaker than DE BRUIJN's result). In fact we shall prove:

*Theorem 1:* For  $2 < y \leq x$  we have for  $x \rightarrow \infty$ , uniformly in  $y$ ,

$$\log \Psi(x, y) \sim \log \binom{\pi(y) + u}{u} \quad (1)$$

where

$$u = [(\log x)/(\log y)].$$

We remark that this of course follows from DE BRUIJN's theorem. Our interest lies mainly in giving a short fairly straightforward proof. For some ranges of values of  $y$  (1) is nearly trivial and the most interesting part of the proof concerns the range  $(\log x)^{\varepsilon} < y < (\log x)^{1+\varepsilon}$ .

## 2. Proof of Theorem 1

We shall prove (1) by showing that for every  $\varepsilon > 0$  we have

$$\binom{\pi(y)+u}{u} < \Psi(x,y) < \binom{\pi(y)+u}{u}^{1+\varepsilon} \quad \text{for } x > x_0(\varepsilon). \quad (2)$$

- a. The first inequality immediately follows from the fact that  $\binom{\pi(y)+u}{u}$  represents the number of solutions of

$$\sum_{p \leq y} \alpha_p \leq u = [(\log x)/(\log y)]$$

in nonnegative integers  $\alpha_p$  and this number is less than the number of solutions of

$$\sum_{p \leq y} \alpha_p \log p \leq \log x$$

which is  $\Psi(x,y)$  by definition.

In sections **b**, **c** and **d** we prove the second inequality of (2).

- b. We now consider  $y < (\log x)^{1+\varepsilon}$ . We first remark that (2) is trivial for very small values of  $y$ , for example  $y < (\log x)^{\varepsilon/2}$ , because

$$\Psi(x,y) \leq \left(\frac{\log x}{\log 2} + 1\right)^{\pi(y)} < \binom{\pi(y)+u}{u}^{1+\varepsilon} \quad \text{if } y < (\log x)^{\varepsilon/2}.$$

Hence we can assume that  $y > (\log x)^{\varepsilon/2}$ .

Let  $N_1$  denote the number of integers  $\leq x$ , free of prime factors  $> y^{1-\varepsilon}$  and let  $N_2$  denote the number of integers  $\leq x$  all of whose prime factors satisfy  $y^{1-\varepsilon} < p \leq y$ . Then  $\Psi(x,y) \leq N_1 N_2$ .

Trivially  $N_1 < \left(\frac{\log x}{\log 2} + 1\right)^{\pi(y^{1-\varepsilon})}$ . Furthermore

$\binom{\pi(y)+u}{u} > \left(\frac{\pi(y)+u}{\pi(y)}\right)^{\pi(y)}$ . From this it follows that

$$\frac{\log N_1}{\log \binom{\pi(y)+u}{u}} = O\left(\frac{\log \log x}{(\log x)^{\frac{1}{2}\varepsilon^2}}\right)$$

i.e.  $N_1 < \binom{\pi(y)+u}{u}^\varepsilon$  for  $x > x_1(\varepsilon)$ .

Now  $N_2$  is less than the number of solutions of

$$\sum_{y^{1-\varepsilon} < p \leq y} \alpha_p \leq \frac{\log x}{(1-\varepsilon)\log y}$$

in nonnegative integers  $\alpha_p$  and this number does not exceed

$$\binom{\pi(y)+u'}{u'} \text{ where } u' = \left\lfloor \frac{\log x}{(1-\varepsilon)\log y} \right\rfloor.$$

We now use the fact that if  $a, b$  and  $b(1+\varepsilon)$  are positive integers then

$$\binom{a+b}{a}^{1+\varepsilon} = \prod_{i=0}^{a-1} \left(1 + \frac{b}{a-i}\right)^{1+\varepsilon} > \prod_{i=0}^{a-1} \left(1 + \frac{b(1+\varepsilon)}{a-i}\right) = \binom{a+b(1+\varepsilon)}{a}.$$

It follows that  $N_2 < \binom{\pi(y)+u}{u}^{1+o(\varepsilon)}$ .

Combining the estimates for  $N_1$  and  $N_2$  we find

$$\Psi(x, y) < \binom{\pi(y)+u}{u}^{1+o(\varepsilon)}$$

proving (2) for  $y < (\log x)^{1+\varepsilon}$ .

c. For  $y > (\log x)^{n(\varepsilon)}$  where for instance  $n(\varepsilon) = 2/\varepsilon$  the right-hand side of (2) is trivial because

$$\binom{\pi(y)+u}{u} > \left(\frac{\pi(y)}{u}\right)^u \text{ which implies } \binom{\pi(y)+u}{u}^{1+\varepsilon} > x.$$

d. The case  $(\log x)^{1+\varepsilon} < y < (\log x)^{2/\varepsilon}$  will be treated by writing  $y = (\log x)^\alpha$  and proving

$$\Psi(x, y) = x^{1-1/\alpha+o(1)} \tag{3}$$

which implies (2).

We first remark that  $\binom{\pi(y)+u}{u} = x^{1-1/\alpha+o(1)}$  and the same holds for  $\binom{\pi(y)}{u}$ . So it remains to show that

$$\Psi(x, y) < x^{1-1/\alpha+o(1)}.$$

To do this we split the integers counted by  $\Psi(x, y)$  into two classes. First those with at least  $u$  distinct prime factors. Their number is less than

$$\frac{x}{u!} \left( \sum_{p < x} \frac{1}{p} \right)^u < x \left( \frac{2e \log \log x}{u} \right)^u = x^{1-1/u+o(1)}.$$

The number of integers  $\leq x$  with less than  $u$  distinct prime factors  $\leq y$  is less than

$$\binom{\pi(y)}{u} \binom{\left( \frac{\log x}{\log 2} + u \right)}{u}$$

because there are  $\binom{\pi(y)}{u}$  different  $u$ -tuples of primes  $\leq y$  and for each of these the sum of the exponents is less than  $\frac{\log x}{\log 2}$ . We have

$$\binom{\left( \frac{\log x}{\log 2} + u \right)}{u} = x^{o(1)} \left( \text{this follows from } \binom{n}{k} < \frac{n^k}{k!} < \left( \frac{ne}{k} \right)^k \right).$$

So the number of integers in the second class is also  $x^{1-1/u+o(1)}$ .

This completes the proof of Theorem 1.

#### REFERENCES

- [1] N.G. DE BRUIJN, On the number of positive integers  $\leq x$  and free of prime factors  $> y$ , *Proc. Kon. Ned. Akad. v. Wetensch.*, 54, 50-59 (1951).
- [2] N.G. DE BRUIJN, On the number of positive integers  $\leq x$  and free of prime factors  $> y$ , II, *Proc. Kon. Ned. Akad. v. Wetensch.*, 69, 335-348 (1966).