On graphs, geometries, and groups of Lie type

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Ralf Gramlich

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Dit proefschrift is goedgekeurd door de promotoren:

prof.dr. A.M. Cohen
en
prof.dr. S.V. Shpectorov

Copromotor:
dr. F.G.M.T. Cuypers

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Gramlich, Ralf

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Preface

Geometry and group theory

The idea of describing a geometry by a transformation group that leaves that geometry invariant goes back to Felix Klein. In his inaugural thesis [Kle72] at the University of Erlangen, Klein demands that a geometry should be considered independently from an embedding into the ambient space. Then, so he claims, any result one can prove depends exclusively on the transformation group of that geometry. Indeed, if one has a notion of *points* in the geometry, then for instance the fact that the transformation group acts transitively on the set of points only depends on the group and the set of points as a whole, not on single points of that set. Moreover, in this case one can describe the set of all points as follows. Fix one point p and consider its full stabilizer G_p in the transformation group G. Then we can identify the set $\mathcal{P}$ of points of the geometry with the set of cosets of G_p in G via the bijection $G/G_p \to \mathcal{P} : gG_p \mapsto g(p)$. Proposition B.1.2 on the isomorphism between flag-transitive geometries and the corresponding coset geometries is an immediate consequence of Klein's considerations.

The converse idea of describing groups with geometries is also well developed. For example, Jacques Tits used his buildings in order to describe the groups of Lie type as automorphism groups of geometries (e.g., [Tit74]). Other people should be mentioned here. Hans Freudenthal related several exceptional groups (or rather real forms of exceptional groups) to the projective plane over the octonions, cf. [Fre51]. Another interesting example is Bruce Cooperstein's work on long root group geometries, see [Coo79] and [Coo83]. More recent is the search for defining relations of groups by means of simple connectedness of suitable geometries. Tits' lemma B.2.5, which was proved independently by Antonio Pasini [Pas85], Sergey Shpectorov [Shp], and Jacques Tits [Tit86], states that a flag-transitive geometry is *simply connected* (i.e., there does not exist a non-trivial cover of that geometry) if and only if the considered group of automorphisms is the universal completion of the amalgam on its maximal parabolics. Tits' lemma is a powerful tool of proving heavy group-theoretic results by geometric means. Examples are the new proof of the Curtis-Tits theorem by Bernhard Mühlherr ([Müh], also Theorem 3.1.5 of this thesis) and the new proof of one of Phan's theorems by Curt Bennett and Sergey Shpectorov ([BS], also Theorem 3.2.1 and Corollary 3.2.4 of this thesis).

One should notice that Tits' lemma only implies results on defining amalgams of a given group. So, strictly speaking, a new proof of Phan's theorems has to consist of two steps, a simple connectedness proof of a geometry and a uniqueness proof of related amalgams. Bennett and Shpectorov present both steps in their paper [BS].

However, in some cases there is an elegant geometric way of unifying those two steps. Instead of proving simple connectedness of a geometry one rather gives a local characterization. In particular, if there is a unique geometry up to isomorphism with certain local properties, then a local characterization implies simple connectedness as the universal cover of a geometry has the same local properties. If the geometry admits a flag-transitive group of automorphisms, then such a local characterization allows for an immediate identification of this group from internal conditions like centralizers of involutions. For example, Jon Hall's local characterization of the Kneser graphs $K(n, 2)$, for $n \geq 7$ (e.g., [Hal87]), implies Theorem 27.1 of [GLS94] (also Theorem 2.5.5 of this thesis), a local recognition of the symmetric groups of sufficiently large degree from the structure of centralizers of transpositions. In fact, precisely this group-theoretic application was one of the motivations of Jon Hall's to prove his result on Kneser graphs.

The present thesis

The objective of this thesis is to provide several local characterization results for different graphs and geometries as well as group-theoretic consequences of those results. In some cases, we prove simple connectedness and do not obtain a complete local characterization. In Chapter 1 we study graphs on non-incident point-hyperplane pairs of Desarguesian projective spaces where two pairs are adjacent if the point of one pair is contained in the hyperplane of the other and vice versa. The reason why we are interested in these graphs lies in the following group-theoretic description. If the coordinate division ring of the projective space is distinct from the field of two elements, there is a one-to-one correspondence between the reflection tori of the collineation group and the non-incident point-hyperplane pairs of the projective space by assigning a reflection torus to its center and its axis. In the group theoretic setting the adjacency relation is equivalent to the commutation relation. We obtain a local recognition theorem for those graphs, which immediately translates to universal completion results of certain amalgams and local recognition results of certain groups. In the final section of Chapter 1 we study graphs on certain non-incident point-hyperplane pairs in projective spaces that admit a polarity. More precisely, we investigate graphs on non-singular points and their polars under some polarity, which arise as induced subgraphs of the above point-hyperplane graphs over division rings with an involutive anti-automorphism. The results of Chapter 1 have been obtained in collaboration with Arjeh Cohen and Hans Cuypers. Chapter 2 is similar to the first chapter. Here we investigate graphs on non-intersecting line-hyperline pairs of Desarguesian projective spaces where two pairs are adjacent if the line of one pair is contained in the hyperline of the other and vice versa. Naturally, the results we obtain for these graphs are similar to the results collected in Chapter

1. From a group-theoretic point of view, one can describe the line-hyperline graph as the graph on the fundamental SL_2's of the group $PSL_{n+1}(\mathbb{F})$ where n is the dimension of the projective space and $\mathbb{F}$ its coordinate division ring. Again, adjacency translates to the commutation relation. Therefore, besides a local recognition result for the graphs and the companion results in group theory, we are able to give a characterization of the geometry on the root subgroups of $PSL_{n+1}(\mathbb{F})$ as points and the fundamental SL_2's as lines from local knowledge about the centralizers of the fundamental SL_2's. However, this characterization will not be obtained before Chapter 4, and Chapter 2 simply paves the way towards this result. In Chapter 3 we investigate geometries related to the famous Curtis-Tits theorem and Phan's theorems. These theorems are important recognition tools in the classification of finite simple groups, and we are working toward a uniform geometric approach to both the Curtis-Tits theorem and Phan's theorems. Bernhard Mühlherr has recently presented a proof of the Curtis-Tits theorem by way of proving the simple connectedness of the so-called opposites geometry of certain twin buildings. On the other hand, Curt Bennett and Sergey Shpectorov managed to improve one of Phan's theorems by exploiting properties of a geometry that can be naturally embedded into one of Mühlherr's opposites geometries. We first give a review of the results of Mühlherr and Bennett and Shpectorov and then explain the technique of how we expect one can also re-prove Phan's other theorems. Going even further, in the second half of that chapter we present the simple connectedness part of a new Phan-type theorem which has been obtained by the author in collaboration with Corneliu Hoffman and Sergey Shpectorov. There is still a lot of work to be done in this area, and we pose several questions which may lead to further research beyond our results. In the final Chapter 4 we turn to the area of geometries on long root subgroups of Chevalley groups. Our approach to the topic is via centralizers of fundamental SL_2's. We use the results from Chapter 2 to obtain a characterization of the geometry on the root subgroups and fundamental SL_2's of the groups $PSL_{n+1}(\mathbb{F})$. Besides we present characterizations of the corresponding geometries of symplectic and unitary groups and Chevalley groups of (twisted) type F_4. For the symplectic and unitary groups we again study the centralizers of the fundamental SL_2's, but we need a different approach to the long root geometries of type F_4. Our findings enable us to locally recognize Chevalley groups of (twisted) type A_n and C_n. We expect that the types B_n and D_n can be handled in a similar way, but we cannot say anything about the exceptional types. It is our understanding that the approach to the area of geometries on long root subgroups by way of centralizers of fundamental SL_2's has already been suggested by Bill Kantor in the 1980's, as the author was told recently.

The appendices serve as very short introductions to relevant notions. Those who are not familiar with the concepts of geometry and group theory may be better served with one of the references given in the appendices. Those who are familiar with geometry or group theory may find the appendices to be welcome reminders of basic definitions and important facts of the area. However, the first two chapters of this thesis are largely self-contained and only require a bit of intuitive understanding of the notion of graphs and projective spaces.

A list of theorems

Graphs: simple connectedness

- Theorem 1.7.5 (p. 28):
  locally $\mathcal{NO}_6^+(\mathbb{F})$.
- Theorem 1.7.9 (p. 29):
  locally $\mathcal{NO}_6^-(\mathbb{F})$.
- Theorem 1.7.10 (p. 29):
  locally $\mathcal{NU}_5(\mathbb{F}^2)$.

Graphs: local recognition

- Theorem 1.3.21 (p. 18):
  point-hyperplane graphs.
- Theorem 2.5.1 (p. 52):
  line-hyperline graphs.
- Theorem 4.4.22 (p. 100):
  symplectic hyperbolic line graphs.
- Theorem 4.5.3 (p. 103):
  unitary hyperbolic line graphs.

Geometries: simple connectedness

- Theorem 3.7.9 (p. 79):
  flipflop geometry of type C_n.

Geometries: local characterization

- Theorem 4.3.6 (p. 92):
  linear hyperbolic root geometry.
- Corollary 4.3.7 (p. 93):
  linear hyperbolic root geometry.
- Theorem 4.4.23 (p. 101):
  symplectic hyp. long root geometry.
- Theorem 4.5.4 (p. 103):
  unitary hyp. long root geometry.

Groups: universal completion

- Theorem 1.6.2 (p. 21):
  $P\Gamma L_n(\mathbb{F})$.
- Corollary 1.7.7 (p. 28):
  $PSO_n^\pm(\mathbb{F})$.
- Theorem 3.8.1 (p. 79):
  $Sp_{2n}(\mathbb{F})$.
- Theorem 3.8.4 (p. 81):
  $Sp_{2n}(\mathbb{F})$.

Groups: local recognition

- Theorem 1.6.3 (p. 23):
  $PGL_n(\mathbb{F})$.
- Theorem 2.5.3 (p. 53):
  $PGL_n(\mathbb{F})$.
- Theorem 4.4.25 (p. 101):
  $PSp_{2n}(\mathbb{F})$.
- Theorem 4.5.5 (p. 104):
  $PGU_n(\mathbb{F}^2)$.

General notation

As a general rule graphs are denoted by greek capitals (Γ) and vertices of graphs by boldface latin letters ($\mathbf{x}$). Geometries are denoted by calligraphic latin capitals ($\mathcal{G}$), elements of geometries by latin letters (p) and subspaces of geometries by latin capitals (X). Sometimes, geometries are considered as graphs and vice versa, vertices of graphs as elements of geometries, subspaces of geometries as elements of geometries or subgeometries. Then we use the notation which seems the most natural to us. For example, if we choose a vertex of a graph, which will later play the role of an element of a geometry, it will still be denoted in a boldface latin letter.

The end of the proof of a lemma, proposition, corollary, or theorem will always be indicated by $\Box$. If the proof has been given before the statement of the result, or if the proof is immediate, then the $\Box$ will occur after the statement of the result. If a significant result is taken from a source other than the present thesis, the source will occur in parentheses next to the header of the result. If nothing more is said about the proof, then no $\Box$ will occur. If we believe not to have given the original reference for a result, then we put an **e.g.** before the source we have given. To indicate that we have omitted a significant part of the proof of a result, we finish the proof or the statement of that result with $\blacksquare$ instead of $\Box$. We do the same for the results that logically depend on such results. Corollaries always immediately depend on a proposition, a theorem, or another corollary. If it is not clear on which result a corollary depends, that respective result will be indicated next to the header of the corollary.

Thanks

I cannot even begin to find the words that would properly express the gratitude I feel towards Arjeh Cohen and Hans Cuypers for all their help, support, and advice, so I won't try. Simply put: thanks. Thanks a lot!!!

Sergey Shpectorov has earned a big share of gratitude as well. I very much enjoyed his warmth and hospitality during my stay at the Bowling Green State University. Doing research with him is quite an impressive and enlightening experience. Furthermore, he has been very helpful during the final stage of bringing the present thesis into reasonable shape.

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The PhD committee

voorzitter	J.K. Lenstra (Technische Universiteit Eindhoven)
1e promotor	A.M. Cohen (Technische Universiteit Eindhoven)
2e promotor	S.V. Shpectorov (Bowling Green State University)
copromotor	F.G.M.T. Cuypers (Technische Universiteit Eindhoven)
lid kerncommissie	A.E. Brouwer (Technische Universiteit Eindhoven)
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opponent	J.I. Hall (Michigan State University)
opponent	D.E. Taylor (University of Sydney)
opponent uit de zaal	P.J. Cameron (University of London)

Contents

Chapter 1

Point-Hyperplane Pairs

Local recognition of graphs is a problem described, for example, in [Coh90]. The general idea is the following. Choose your favorite graph Δ and try to find all connected graphs Γ that are locally Δ, i.e., graphs whose induced subgraph on the set of all neighbors of an arbitrary vertex of Γ is isomorphic to Δ. One restricts the search to connected graphs, because a graph is locally Δ if and only if all of its connected components are also locally Δ. There has already been done a lot of work in this direction, see, e.g., [BC73], [BC75], [BH77], [Hal80], [Ron81], [Vin81], [BBBC85], [Hal87], [BB89], [Ned93], [Wee94a], [Wee94b].

In this chapter we will study graphs on the non-incident point-hyperplane pairs of a Desarguesian projective space where a point-hyperplane pair is adjacent to another point-hyperplane pair if and only if the point of one pair is contained in the hyperplane of the other pair and vice versa. Alternatively, except for the field of two elements, one can describe these graphs as the graphs on the reflection tori of the group $P\Gamma L_n(\mathbb{F})$ with distinct tori being adjacent if and only if they commute, see Proposition 1.5.5. Thus, a local recognition result has applications in group theory. As opposite elements of the projective geometry are involved, one can expect one of these applications to be closely related to the Curtis-Tits theorem on defining relations of subgroups of the group $P\Gamma L_n(\mathbb{F})$. See Section 3.1 for a precise statement of the Curtis-Tits theorem.

The graph on the non-incident point-hyperplane pairs as defined above (see also Definition 1.1.1) is denoted by $\mathbf{H}_n(\mathbb{F})$, where n is the projective dimension of the projective space and $\mathbb{F}$ is its coordinatizing division ring. For $n \geq 2$, it is possible to recover the space $\mathbb{P}_n(\mathbb{F})$ from the graph $\mathbf{H}_n(\mathbb{F})$, as is shown in Section 1.2. Especially note Proposition 1.2.8 and its corollaries. We can use this reconstruction of the projective space to obtain the (full) automorphism group of $\mathbf{H}_n(\mathbb{F})$ in Corollary 1.2.10. In later sections we study graphs that are connected and locally $\mathbf{H}_n(\mathbb{F})$. We present a local recognition result for $n \geq 3$, see Theorem 1.3.21, as well as an example of a connected, locally $\mathbf{H}_2(2)$ graph that is not isomorphic to $\mathbf{H}_3(2)$ in Section 1.4.

Section 1.6 is devoted to the study of group-theoretic consequences of the results gathered in the preceding sections. In the beginning of the section a result on universal completions of an amalgam is given. However, this result does not require the full strength of the local recognition result Theorem 1.3.21. Instead, simple connectedness of the point-hyperplane graphs is sufficient, a result that is obtained at a very early stage in Lemmas 1.3.3 and 1.3.6. Later in Section 1.6 we turn to the problem of embedding related amalgams into $P\Gamma L_{n+2}(\mathbb{F})$, see Theorem 1.6.2, and the classical problem of recognizing a group from local information on the centralizers of involutions, see Theorem 1.6.3. Here we fully exploit the results of Theorem 1.3.21.

In Section 1.7 we study graphs on the reflection tori of certain subgroups of $P\Gamma L_{n+1}(\mathbb{F})$, namely $P\Gamma O_n(\mathbb{F})$ and $P\Gamma U_n(\mathbb{F})$. These graphs are induced subgraphs of the graph $\mathbf{H}_n(\mathbb{F})$. This section is not self-contained and heavily relies on [CP92] and [Cuyb]. It is a report on local recognition results obtained by Hans Cuypers plus a new idea for improvement of the dimension bound in some cases. More precisely, we prove the simple connectedness of all graphs belonging to those cases and conjecture that this allows for improvement of the local recognition result. Furthermore we include several group-theoretic consequences of our result on simple connectedness and point out corollaries of Cuypers' local recognition result as well as our conjecture.

The contents of this chapter have been obtained in collaboration with Arjeh Cohen and Hans Cuypers and are taken from [CCG].

1.1 The point-hyperplane graph

Definition 1.1.1 Let $n \in \mathbb{N} \cup \{0\}$ and let $\mathbb{F}$ be a division ring. Consider the projective space $\mathbb{P}_n(\mathbb{F}) = \mathbb{P}(V)$ of projective dimension n over $\mathbb{F}$, i.e., the projective space of an $(n+1)$-dimensional vector space V over $\mathbb{F}$. The **point-hyperplane graph** $\mathbf{H}_n(\mathbb{F})$ of $\mathbb{P}_n(\mathbb{F})$ is the graph whose vertices are the non-incident point-hyperplane pairs of $\mathbb{P}_n(\mathbb{F})$ in which a vertex (a, A) is adjacent to another vertex (b, B) (in symbols, $(a, A) \perp (b, B)$) if and only if $a \in B$ and $b \in A$.

By definition, we have $\mathbf{x} \not\perp \mathbf{x}$, so the perp $\mathbf{x}^\perp$ of $\mathbf{x}$ of all vertices of $\mathbf{H}_n(\mathbb{F})$ in $\perp$ relation to $\mathbf{x}$ is the set of vertices in $\mathbf{H}_n(\mathbb{F})$ at distance one from $\mathbf{x}$. Moreover, for a set X of vertices, we define the **perp of** X as $X^\perp := \bigcap_{\mathbf{x}\in X} \mathbf{x}^\perp$ with the understanding that $\emptyset^\perp = \mathbf{H}_n(\mathbb{F})$ and the **double perp of** X as $X^{\perp\perp} := (X^\perp)^\perp$.

If the division ring $\mathbb{F}$ is clear from the context or irrelevant, we sometimes write $\mathbb{P}_n$ and $\mathbf{H}_n$ instead of $\mathbb{P}_n(\mathbb{F})$, respectively $\mathbf{H}_n(\mathbb{F})$, and if $\mathbb{F} = \mathbb{F}_q$ is finite of order q, we also write $\mathbb{P}_n(q)$ and $\mathbf{H}_n(q)$. For a projective space $\mathbb{P}$ isomorphic to $\mathbb{P}_n(\mathbb{F})$, denote by $\mathbf{H}(\mathbb{P})$ the graph on the non-incident point-hyperplane pairs with mutual inclusion of the point of one pair in the hyperplane of another as adjacency. Certainly, $\mathbf{H}(\mathbb{P}) \cong \mathbf{H}_n(\mathbb{F})$.

A point p of the projective space $\mathbb{P}_n(\mathbb{F}) = (\mathcal{P}, \mathcal{L})$ determines the set of vertices $v_p = \{(x, X) \in \mathbf{H}_n(\mathbb{F}) \mid x = p\}$ of the graph $\mathbf{H}_n(\mathbb{F})$. A line l of $\mathbb{P}_n(\mathbb{F})$ determines the union v_l of all sets v_p of vertices for $p \in l$. Clearly the map $v : \mathcal{P} \cup \mathcal{L} \to 2^{\mathbf{H}_n(\mathbb{F})} : x \mapsto v_x$ is injective, and $p \in l$ if and only if $v_p \subset v_l$, so we can identify the projective space with its image under v in the collection of all subsets of the vertex set of $\mathbf{H}_n(\mathbb{F})$. We shall refer to this image in $2^{\mathbf{H}_n(\mathbb{F})}$ as the **exterior projective space on $\mathbf{H}_n(\mathbb{F})$**. Similarly, one can map points Π and lines Λ of the dual projective space $\mathbb{P}_n(\mathbb{F})^{\text{dual}}$ onto subsets of vertices of $\mathbf{H}_n(\mathbb{F})$ of the form $w_\Pi = \{(x, X) \in \mathbf{H}_n(\mathbb{F}) \mid X = \Pi\}$ and $w_\Lambda = \bigcup_{\Pi \supset \Lambda} w_\Pi$ for Π running over all points of $\mathbb{P}_n(\mathbb{F})^{\text{dual}}$ containing Λ. This gives rise to the **dual exterior projective space on $\mathbf{H}_n(\mathbb{F})$**. The subsets v_p, v_l, w_Λ and w_Π so obtained are called **exterior points**, **exterior lines**, **exterior hyperlines**, and **exterior hyperplanes** of $\mathbf{H}_n(\mathbb{F})$, respectively. Note that, if $\mathbb{F} \cong \mathbb{F}^{\text{opp}}$, the projective space $\mathbb{P}_n(\mathbb{F})$ is isomorphic to its dual, and so there is an automorphism of $\mathbf{H}_n(\mathbb{F})$ mapping the image under v onto the image under w. (If π is a duality, then $(x, X) \mapsto (\pi(X), \pi(x))$ is such an automorphism of $\mathbf{H}_n(\mathbb{F})$.) Also, note that $\mathbf{H}(\mathbb{P}_n(\mathbb{F})) \cong \mathbf{H}(\mathbb{P}_n(\mathbb{F})^{\text{dual}})$ by the map $(x, X) \mapsto (X, x)$. In particular, it will not be possible to distinguish exterior points from exterior hyperplanes when one tries to reconstruct the projective space from the graph. Another useful observation is that the exterior points partition the vertex set of $\mathbf{H}_n(\mathbb{F})$. In other words, each vertex of $\mathbf{H}_n(\mathbb{F})$ belongs to a unique exterior point. The same holds for exterior hyperplanes.

One of our goals is to characterize the graph $\mathbf{H}_n(\mathbb{F})$ as the unique connected graph (up to isomorphism) that is locally $\mathbf{H}_{n-1}(\mathbb{F})$, for sufficiently large n. In this light the following two observations are important.

Proposition 1.1.2
Let $n \geq 1$. The graph $\mathbf{H}_n(\mathbb{F})$ is locally $\mathbf{H}_{n-1}(\mathbb{F})$.

Proof. Let $\mathbf{x} = (x, X)$ be a vertex of $\mathbf{H}_n(\mathbb{F})$. Then $X \cong \mathbb{P}_{n-1}(\mathbb{F})$. Identifying X with $\mathbb{P}_{n-1}(\mathbb{F})$ by means of this isomorphism, we establish an isomorphism $\mathbf{x}^\perp \to \mathbf{H}_{n-1}(\mathbb{F})$ as follows. For any vertex $\mathbf{y} = (y, Y)$ adjacent to $\mathbf{x}$, we have $x \in Y$, $y \in X \setminus (X \cap Y)$, and $\dim(X \cap Y) = n - 2$, so $(y, X \cap Y)$ belongs to $\mathbf{H}(X)$.

Conversely, for any vertex of $\mathbf{H}(X)$, i.e., for any non-incident pair (z, Z) consisting of a point z and a hyperline Z of $\mathbb{P}_n(\mathbb{F})$ with $z \in X$, $Z \subseteq X$, the pair $(z, \langle Z, x\rangle)$ is a vertex of $\mathbf{x}^\perp$. (Indeed, $z \notin \langle Z, x\rangle$, since $x \notin X$.)

Clearly, the maps $(y, Y) \mapsto (y, X \cap Y)$ and $(z, Z) \mapsto (z, \langle Z, x\rangle)$ are each other's inverses. Moreover, the maps preserve adjacency, and the proposition follows. □

Let $\mathbb{P}$ be isomorphic to $\mathbb{P}_n(\mathbb{F})$, and let X be a hyperplane of $\mathbb{P}$. By the proof of the preceding proposition, we can consider the graph $\mathbf{H}(X)$ as an induced subgraph of $\mathbf{H}(\mathbb{P})$. Indeed, choose a point x of $\mathbb{P}$ that is not contained in X. This defines a vertex (x, X) of $\mathbf{H}(\mathbb{P})$, and we can embed the vertices of $\mathbf{H}(X)$ in $\mathbf{H}(\mathbb{P})$ by the map $(y, Y) \mapsto (y, \langle Y, x\rangle)$, which are precisely the neighbors of the vertex (x, X). If X is an arbitrary proper subspace of $\mathbb{P}$, we can describe X as the intersection of a number of hyperplanes and therefore we can embed $\mathbf{H}(X)$ into $\mathbf{H}(\mathbb{P})$.

For points x, y the symbol xy denotes the unique line through x and y if the points are distinct and the point $x = y$ if the points are equal.

Proposition 1.1.3
$\mathbf{H}_0$ consists of precisely one point; $\mathbf{H}_1$ is the disjoint union of cliques of size two; the diameter of $\mathbf{H}_2$ equals three; the diameter of $\mathbf{H}_n$, $n \geq 3$, equals two. In particular, $\mathbf{H}_n$ is connected for $n \geq 2$.

Proof. The statements about $\mathbf{H}_0$ and $\mathbf{H}_1$ are obvious. Let $\mathbf{x} = (x, X)$, $\mathbf{y} = (y, Y)$ be two non-adjacent vertices of $\mathbf{H}_2$. The intersection $X \cap Y$ is a point or a line, and xy is a point or a line. The vertices $\mathbf{x}$ and $\mathbf{y}$ have a common neighbor, i.e., they are at distance two, if and only if $X \cap Y \not\subseteq xy$. If $X \cap Y \subseteq xy$, however, it is easily seen, that they are at distance three. Indeed, choose $a \in X \setminus \{y\}$ and $b \in Y \setminus \{x\}$ with $ay \not\ni b$ and $bx \not\ni a$. Then (x, X), (a, bx), (b, ay), (y, Y) establishes a path of length three.

Now let $\mathbf{x} = (x, X)$, $\mathbf{y} = (y, Y)$ be two non-adjacent vertices of $\mathbf{H}_n$, $n \geq 3$. The intersection $X \cap Y$ contains a line. Since $x \notin X$ and $y \notin Y$, we find a point $z \in X \cap Y$ and a hyperplane $Z \supseteq xy$ with $z \notin Z$ and, thus, a vertex (z, Z) adjacent to both $\mathbf{x}$ and $\mathbf{y}$. □

The first main result of this chapter will be a reconstruction theorem of the projective space from graphs isomorphic to the point-hyperplane graph without making use of the coordinates, see the next section. This goal will be achieved by the study of double perps of two vertices, i.e., subsets of $\mathbf{H}_n$ of the form $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$.

Lemma 1.1.4
Let $\mathbf{x} = (x, X)$, $\mathbf{y} = (y, Y)$ be distinct vertices of $\mathbf{H}_n$ with $\{\mathbf{x}, \mathbf{y}\}^{\perp} \neq \emptyset$. Then the double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ equals the set of vertices $\mathbf{z} = (z, Z)$ of $\mathbf{H}_n$ with $z \in xy$ and $Z \supseteq X \cap Y$.

Proof. Distinct vertices with non-empty perp only exist for $n \geq 2$. The vertices of $\{\mathbf{x}, \mathbf{y}\}^{\perp}$ are precisely the non-incident point-hyperplane pairs (p, H) with $p \in X \cap Y$ and $H \supset xy$. Let $\left\{(p_i, H_i) \in \{\mathbf{x}, \mathbf{y}\}^{\perp} \mid i \in I\right\}$ be the set of all these vertices, indexed by some set I. Now $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp} = (\{\mathbf{x}, \mathbf{y}\}^{\perp})^{\perp}$ consists of precisely those vertices $(z, Z) \in \mathbf{H}_n$ with $z \in \bigcap_{i \in I} H_i$ and $Z \supset \langle (p_i)_{i \in I} \rangle$. But since $\{\mathbf{x}, \mathbf{y}\}^{\perp} \neq \emptyset$, we have $\bigcap_{i \in I} H_i = xy$ and $\langle (p_i)_{i \in I} \rangle = X \cap Y$, thus proving the claim. □

In order to recover the projective spaces $\mathbb{P}_n$ and ${\mathbb{P}_n}^{\mathrm{dual}}$ from the information contained in a graph $\Gamma \overset{\phi}{\cong} \mathbf{H}_n$, we have to recognize vertices $\mathbf{x}$, $\mathbf{y}$ of Γ with $x = y$ or, dually, $X = Y$, if $\phi(\mathbf{x}) = (x, X)$, $\phi(\mathbf{y}) = (y, Y)$. Clearly, $x = y$ and $X = Y$ if and only if the vertices $\mathbf{x}$, $\mathbf{y}$ are equal. To recognize the other cases, we make use of the following definition and lemma.

Recall that the **projective codimension** of a subspace X of a projective space $\mathbb{P}$ is the number of elements in a maximal chain of proper inclusions of subspaces properly containing X and properly contained in $\mathbb{P}$. For example, the projective codimension of a hyperplane of $\mathbb{P}$ equals 0.

Definition 1.1.5 Let $n \geq 2$. Vertices $\mathbf{x} = (x, X)$, $\mathbf{y} = (y, Y)$ of $\mathbf{H}_n(\mathbb{F})$ are **in relative position** (i, j) if

$$i = \dim \langle x, y \rangle \text{ and } j = \operatorname{codim}(\mathrm{X} \cap \mathrm{Y})$$

where dim denotes the projective dimension and codim the projective codimension. Note that $i, j \in \{0, 1\}$.

Lemma 1.1.6
Let $n \geq 2$, and let $\mathbf{x}$, $\mathbf{y} \in \mathbf{H}_n$. Then the following assertions hold.

(i) *The vertices $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(0, 0)$ if and only if they are equal.*

(ii) *The vertices $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(0, 1)$ or $(1, 0)$ if and only if they are distinct and the double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is minimal with respect to containment, i.e., it does not contain two vertices with a strictly smaller double perp.*

(iii) *The vertices $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1, 1)$ if and only if they are distinct and the double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is not minimal.*

Proof. Statement (i) is obvious. Suppose $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(0, 1)$. Then $\{\mathbf{x}, \mathbf{y}\}^{\perp} \neq \emptyset$ (since $n \geq 2$), and we can apply Lemma 1.1.4. We obtain $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp} = \{(z, Z) \in \mathbf{H}_n \mid z = x = y, Z \supseteq X \cap Y\}$, whence any pair of distinct vertices contained in $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is in relative position $(0, 1)$ and gives rise to the same double perp. Symmetry handles the case $(1, 0)$. If $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1, 1)$ and $\{\mathbf{x}, \mathbf{y}\}^{\perp} = \emptyset$, then $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp} = \mathbf{H}_n$, which is clearly not minimal. So let us assume $\{\mathbf{x}, \mathbf{y}\}^{\perp} \neq \emptyset$. Again by Lemma 1.1.4, we have $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp} = \{(z, Z) \in \mathbf{H}_n \mid z \in xy, Z \supseteq X \cap Y\}$. This double perp contains a vertex that is at relative position $(0, 1)$ to $\mathbf{x}$, and we obtain a double perp strictly contained in $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$. Statements (ii) and (iii) now follow from the fact that distinct vertices $\mathbf{x} = (x, X)$ and $\mathbf{y} = (y, Y)$ are in relative position $(0, 1)$, $(1, 0)$, or $(1, 1)$. $\square$

We conclude this section with a lemma that will be needed later.

Lemma 1.1.7
Let $\mathbf{x} = (x, X)$ and $\mathbf{y} = (y, Y)$ be two adjacent vertices in $\mathbf{H}_n$. If $\mathbf{x}$ is adjacent to a vertex (z, Z_1) and $\mathbf{y}$ adjacent to a vertex (z, Z_2), then there exists a vertex (z, Z_3) adjacent to both $\mathbf{x}$ and $\mathbf{y}$.

Proof. The statement of the lemma is empty for $n < 2$, and we can assume $n \geq 2$. We have $z \in X \cap Y$. Since $\mathbf{x}$ and $\mathbf{y}$ are adjacent, $x \in Y$ and $y \in X$ are distinct and the line xy does not contain z. Hence the choice of a hyperplane Z_3 that contains xy and does not contain z is possible, and we have found a vertex (z, Z_3) adjacent to both $\mathbf{x}$ and $\mathbf{y}$. $\square$

1.2 Reconstruction of the projective space

This section will concentrate on the unique reconstruction (up to duality) of the projective space from a graph Γ isomorphic to $\mathbf{H}_n(\mathbb{F})$. Abusing notation to some extent, we will sometimes speak of relative positions on Γ, but *only if we have fixed a particular isomorphism* $\Gamma \cong \mathbf{H}_n(\mathbb{F})$. Throughout the whole section, let $n \geq 2$. Furthermore, let $\mathbb{F}$ be a division ring and $\Gamma \cong \mathbf{H}_n(\mathbb{F})$.

Definition 1.2.1 Let $\mathbf{x}$, $\mathbf{y}$ be vertices of Γ. Write $\mathbf{x} \approx \mathbf{y}$ to denote that either $\mathbf{x}$, $\mathbf{y}$ are equal or the double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is minimal with respect to inclusion (in the class of double perps $\{\mathbf{u}, \mathbf{v}\}^{\perp\perp}$ for vertices $\mathbf{u}$, $\mathbf{v}$ with $\mathbf{u} \neq \mathbf{v}$).

For a fixed isomorphism $\Gamma \cong \mathbf{H}_n$ inducing coordinates on Γ the relation $\approx$ coincides with the relation 'being equal or in relative position $(1,0)$ or $(0,1)$' by Lemma 1.1.6(ii). What remains is the problem of distinguishing the dual cases $(0,1)$ and $(1,0)$:

Lemma 1.2.2
On the vertex set of Γ*, there are unique equivalence relations* $\approx^p$ *and* $\approx^h$ *such that* $\approx$ *equals* $\approx^p \cup \approx^h$ *and* $\approx^p \cap \approx^h$ *is the identity relation. Moreover, for a fixed isomorphism* $\Gamma \cong \mathbf{H}_n$*, we either have*

- $\approx^p$ *is the relation 'being equal or in relative position* $(0,1)$*', and* $\approx^h$ *is the relation 'being equal or in relative position* $(1,0)$*', or*
- $\approx^p$ *is the relation 'being equal or in relative position* $(1,0)$*', and* $\approx^h$ *is the relation 'being equal or in relative position* $(0,1)$*'.*

In other words, for a fixed isomorphism $\Gamma \cong \mathbf{H}_n(\mathbb{F})$ and up to interchanging $\approx^p$ and $\approx^h$, we may assume that $\approx^p$ stands for being equal or in relative position $(0,1)$ and $\approx^h$ stands for being equal or in relative position $(1,0)$.

Proof. As we have noticed after Definition 1.2.1, two vertices $\mathbf{x}$, $\mathbf{y}$ of Γ are in relation $\approx$ if and only if their images (x, X) and (y, Y) in $\mathbf{H}_n(\mathbb{F})$ are equal or in relative positions $(0,1)$ or $(1,0)$. Let us consider equivalence relations that are subrelations of $\approx$. Obviously, the identity relation is an equivalence relation. Moreover, the relation 'equal or in relative position $(0,1)$' and the relation 'equal or in relative position $(1,0)$' are equivalence relations. Now let us assume we have vertices $\mathbf{x} = (x, X)$, $\mathbf{y} = (y, Y)$, $\mathbf{z} = (z, Z)$ of $\Gamma \cong \mathbf{H}_n(\mathbb{F})$ such that $\mathbf{x}$, $\mathbf{y}$ are in relative position $(0,1)$ and $\mathbf{x}$, $\mathbf{z}$ are in relative position $(1,0)$. Then $y \neq z$ and $Y \neq Z$ and $\mathbf{y}$, $\mathbf{z}$ cannot be in relative position $(0,1)$ or $(1,0)$. Consequently, if we want to find two sub-equivalence relations $\approx^p$ and $\approx^h$ of $\approx$ whose union equals $\approx$, then either of $\approx^p$ and $\approx^h$ has to be a subrelation of the relation 'equal or in relative position $(0,1)$' or of the relation 'equal or in relative position $(1,0)$'. The lemma is proved. $\square$

Convention 1.2.3 From now on, we will always assume that, as soon as we fix an isomorphism $\Gamma \cong \mathbf{H}_n(\mathbb{F})$, the relation $\approx^p$ corresponds to 'equal or in relative position $(0, 1)$'.

Definition 1.2.4 Let $\mathbf{x}$ be a vertex of Γ. With $\approx^p$ and $\approx^h$ as in Lemma 1.2.2, we shall write $[\mathbf{x}]^p$ to denote the equivalence class of $\approx^p$ containing $\mathbf{x}$, and similarly we shall write $[\mathbf{x}]^h$ to denote the equivalence class of $\approx^h$ containing $\mathbf{x}$. We shall refer to $[\mathbf{x}]^p$ as the **interior point** on $\mathbf{x}$ and to $[\mathbf{x}]^h$ as the **interior hyperplane** on $\mathbf{x}$.

Lemma 1.2.5
For a fixed isomorphism $\Gamma \cong \mathbf{H}_n(\mathbb{F})$, an interior point of Γ is the image of an exterior point of $\mathbf{H}_n(\mathbb{F})$ under this isomorphism, and vice versa. The same correspondence exists between interior hyperplanes of Γ and exterior hyperplanes of $\mathbf{H}_n(\mathbb{F})$.

Proof. This is direct from the above. □

Note that an exterior point and an exterior hyperplane of $\mathbf{H}_n(\mathbb{F})$ are disjoint if and only if the corresponding point and hyperplane of $\mathbb{P}_n(\mathbb{F})$ are incident. The above lemma motivates us to call a pair (p, H) of an interior point and an interior hyperplane of Γ **incident** if and only if $p \cap H = \emptyset$. This makes it possible to define interior lines.

Definition 1.2.6 Let p and q be distinct interior points of Γ. The **interior line** l of Γ spanned by p and q is the union of all interior points disjoint from every interior hyperplane disjoint from both p and q. In other words, the interior line pq consists of exactly those interior points which are incident with every interior hyperplane incident with both p and q. Dually, define the **interior hyperline** spanned by distinct interior hyperplanes H and I as the union of all interior hyperplanes disjoint from every interior point disjoint from both H and I.

Lemma 1.2.7
For a fixed isomorphism $\Gamma \cong \mathbf{H}_n(\mathbb{F})$, each interior line of Γ is the image of an exterior line of $\mathbf{H}_n(\mathbb{F})$ under this isomorphism, and vice versa. The analogue holds for interior hyperlines.

Proof. The proof is straightforward. □

The geometry $(\mathcal{P}, \mathcal{L}, \subset)$ on Γ where $\mathcal{P}$ is the set of interior points of Γ and $\mathcal{L}$ is the set of interior lines of Γ is called the **interior projective space on** Γ. By Lemma 1.2.5 and Lemma 1.2.7, this interior projective space is isomorphic to the exterior projective space on $\mathbf{H}_n(\mathbb{F})$. Proceeding with $\approx^h$ as we did for $\approx^p$, the same holds for the dual of the interior projective space on Γ. We summarize the findings in the following proposition.

Proposition 1.2.8
Let $n \geq 2$. Up to interchanging $\approx^p$ and $\approx^h$ every isomorphism $\Gamma \cong \mathbf{H}_n(\mathbb{F})$ induces an isomorphism between the interior projective space on Γ and the exterior projective space on $\mathbf{H}_n(\mathbb{F})$. The analogue holds for the dual interior projective space on Γ. □

Corollary 1.2.9
Let $n \geq 2$, and let Γ be isomorphic to $\mathbf{H}_n(\mathbb{F})$. Then the interior projective space on Γ is isomorphic to $\mathbb{P}_n(\mathbb{F})$ or $\mathbb{P}_n(\mathbb{F})^{\mathrm{dual}}$. □

Corollary 1.2.10
Let $n \geq 2$, and let Γ be isomorphic to $\mathbf{H}_n(\mathbb{F})$. If $\mathbb{F}$ admits an anti-automorphism, then the automorphism group of Γ is of the form $P\Gamma L_{n+1}(\mathbb{F}).2$.

Proof. Indeed, every automorphism of $\mathbb{P}_n(\mathbb{F})$ induces an automorphism of Γ. Conversely, every automorphism of Γ that preserves the interior projective space gives rise to a unique automorphism of $\mathbb{P}_n(\mathbb{F})$, by the theorem. Moreover, every automorphism of Γ either preserves the interior projective space or maps it onto the dual interior projective space, again by the theorem. Finally, an outer automorphism is induced on Γ by the map $(p, H) \mapsto (\delta(H), \delta(p))$ for a duality δ of the projective space, and the map $(p, H) \mapsto (\delta^2(p), \delta^2(H))$ preserves the interior projective space on Γ. □

The question whether the automorphism group of Γ actually is of the form $P\Gamma L_{n+1}(\mathbb{F}) : 2$ is equivalent to the question whether the division ring $\mathbb{F}$ admits an involutive anti-automorphism.

Corollary 1.2.11
Let $n \geq 2$, and let Γ be isomorphic to $\mathbf{H}_n(\mathbb{F})$. If $\mathbb{F}$ does not admit an anti-automorphism, the automorphism group of Γ is isomorphic to $P\Gamma L_{n+1}(\mathbb{F})$.

Proof. The proof is the same as the proof of the preceding corollary, only that one cannot map the interior projective space onto the dual interior projective space, as they are non-isomorphic. □

Remark 1.2.12 This might be an appropriate moment to address the problem of duality. Although, by Convention 1.2.3, as soon as we fix an isomorphism $\Gamma \cong \mathbf{H}_n(\mathbb{F})$, we also choose the equivalence relation $\approx^p$ to correspond to the relation 'equal or in relative position $(0, 1)$' of $\mathbf{H}_n(\mathbb{F})$, there is a subtle problem—mainly of notation—coming with this: Suppose $\Gamma \cong \mathbf{H}_n(\mathbb{F})$ with $\mathbb{F} \not\cong \mathbb{F}^{\mathrm{opp}}$. Then, by the convention, the interior projective space on Γ will always be isomorphic to $\mathbb{P}_n(\mathbb{F})$. If one wants the interior projective space to be isomorphic to $\mathbb{P}_n(\mathbb{F})^{\mathrm{dual}}$, then one will have to fix an isomorphism $\Gamma \cong \mathbf{H}_n(\mathbb{F}^{\mathrm{opp}})$, although $\mathbf{H}_n(\mathbb{F}) \cong \mathbf{H}_n(\mathbb{F}^{\mathrm{opp}})$ by means of the map $(p, H) \mapsto (H, p)$. The reason for this is that we have defined the graph $\mathbf{H}_n(\mathbb{F})$ as the point-hyperplane graph of the space $\mathbb{P}_n(\mathbb{F})$, which by Convention 1.2.3 determines the isomorphism class of the interior projective space on Γ.

The remainder of this section serves as a collection of results to be used later on. First comes a useful lemma on subspaces of the interior projective space of Γ.

Lemma 1.2.13
Let U be a subspace of the interior projective space on Γ. For any projective basis of U there exists a clique of vertices in Γ such that the interior points containing these vertices are the basis elements.

Proof. Fix an isomorphism $\phi : \Gamma \to \mathbf{H}_n$. By Proposition 1.2.8, we can as well argue with exterior points of $\mathbf{H}_n$. Let x_i, i in some index set I, be exterior points such that $\langle \{x_i \mid i \in I\} \rangle = \phi(U)$. We have $x_i = \{(p_i, H) \in \mathbf{H}_n \mid H$ a hyperplane of $\mathbb{P}_n\}$ for all $i \in I$. In particular, $(p_i, \langle V, \{p_j \mid j \in I \setminus \{i\}\} \rangle) \in x_i$ for all $i \in I$ and a complement V of $\phi(U)$ in $\mathbb{P}_n$, and the claim follows. $\square$

Notation 1.2.14 Let $n \geq 3$. For a vertex $\mathbf{x}$ of $\Gamma \cong \mathbf{H}_n(\mathbb{F})$, we write $\approx_{\mathbf{x}}$ for the relation $\approx$ defined on $\mathbf{x}^\perp$ (bear in mind that the latter is isomorphic to $\mathbf{H}_{n-1}(\mathbb{F})$ by Proposition 1.1.2).

Lemma 1.2.15
Let $n \geq 3$. Let $\mathbf{x}$ be a vertex of Γ. Then $\approx_{\mathbf{x}}$ is the restriction of $\approx$ to $\mathbf{x}^\perp$.

In particular, if p is an interior point of Γ with $p \cap \mathbf{x}^\perp \neq \emptyset$, then $p \cap \mathbf{x}^\perp$ is an interior point or an interior hyperplane of $\mathbf{x}^\perp$, and conversely, if q is an interior point of $\mathbf{x}^\perp$, then there exists an interior point or hyperplane q' of Γ with $q' \cap \mathbf{x}^\perp = q$.

Proof. Fix an isomorphism $\phi : \Gamma \to \mathbf{H}_n$. As above, we argue in $\mathbf{H}_n$ rather than in Γ. Let $\phi(\mathbf{x}) = (x, X)$. Now the lemma follows from the fact that, for $\mathbf{a}, \mathbf{b} \in \mathbf{x}^\perp$ with $\phi(\mathbf{a}) = (a, A)$, $\phi(\mathbf{b}) = (b, B)$, the statements $A \cap X = B \cap X$ and $A = B$ are equivalent. $\square$

Notation 1.2.16 In view of the lemma, we can choose the equivalence relation $\approx_{\mathbf{x}}^p$ on $\mathbf{x}^\perp$ in such a way that $(\approx_{\mathbf{x}})^p = (\approx^p)_{\mathbf{x}}$. In that case, there is no harm in writing $\approx_{\mathbf{x}}^p$ to denote this relation. In particular, there is a one-to-one map from the set of interior points of $\mathbf{x}^\perp$ into the set of interior points of Γ.

Lemma 1.2.17
Let $n \geq 3$ and let $\mathbf{x}$ be a vertex of Γ. Then the interior projective space on $\mathbf{x}^\perp$ is a hyperplane of the interior projective space on Γ.

Proof. Fix an isomorphism $\Gamma \cong \mathbf{H}_n(\mathbb{F})$. By Proposition 1.2.8 this isomorphism of graphs induces an isomorphism between the interior projective space on Γ and the exterior projective space on $\mathbf{H}_n(\mathbb{F})$. The vertex $\mathbf{x} \in \Gamma$ is mapped onto a non-incident point-hyperplane pair of $\mathbf{H}_n(\mathbb{F})$, say (x, X). The neighbors of $\mathbf{x}$ are mapped onto point-hyperplane pairs (y, Y) with $y \in X$, inducing a map of the set of interior points of Γ that meet $\mathbf{x}^\perp$ non-trivially onto the set of exterior points of $\mathbf{H}_n(\mathbb{F})$ that intersect $(x, X)^\perp$ non-trivially. But the latter set of exterior points form a hyperplane of the exterior projective space on $\mathbf{H}_n(\mathbb{F})$, and the lemma is proved. $\square$

1.3 Locally point-hyperplane graphs

Throughout the whole section, let $n \geq 3$, and let Γ be a connected, locally $\mathbf{H}_n(\mathbb{F})$ graph for some division ring $\mathbb{F}$. Recall that, given graphs Γ and Δ, the graph Γ is called **locally** Δ if, for each vertex $\mathbf{x} \in \Gamma$, the induced subgraph $\mathbf{x}^\perp$ on all neighbors of $\mathbf{x}$ in Γ is isomorphic to the graph Δ. Thus, the fact that Γ is locally $\mathbf{H}_n(\mathbb{F})$ means that, for each vertex $\mathbf{x}$ of Γ, there is an isomorphism $\mathbf{x}^\perp \to \mathbf{H}_n(\mathbb{F})$ (as well as an isomorphism $\mathbf{x}^\perp \to \mathbf{H}_n(\mathbb{F}^{\mathrm{opp}})$). Consequently, by Corollary 1.2.9, the interior projective space on $\mathbf{x}^\perp$ is isomorphic to $\mathbb{P}_n(\mathbb{F})$ or its dual. The goal of this section is, by use of these isomorphisms, to show that Γ is isomorphic to the point-hyperplane graph of $\mathbb{P}_{n+1}(\mathbb{F})$.

Notice that the definitions of interior points and lines are only local and may differ on different perps. It is one task of this section to show that there is a well-defined notion of *global points* and *global lines* on the whole graph. To avoid confusion, we will index each interior point p and each interior line l by the vertex $\mathbf{x}$ whose perp it belongs to, so we write $p_\mathbf{x}$ and $l_\mathbf{x}$ instead of p and l. These interior points and lines are called **local points** and **local lines**, respectively. We do the same for the relations $\approx$, $\approx^p$, $\approx^h$ obtaining the local relations $\approx_\mathbf{x}$, $\approx^p_\mathbf{x}$, $\approx^h_\mathbf{x}$.

Lemma 1.3.1
Let $\mathbf{x}$ and $\mathbf{y}$ be two adjacent vertices of Γ. Then there is a choice of local equivalence relations $\approx^p_\mathbf{x}$ and $\approx^p_\mathbf{y}$ such that the restrictions of $\approx^p_\mathbf{x}$ and $\approx^p_\mathbf{y}$ to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ coincide.

Proof. This follows immediately from a repeated application of Lemma 1.2.15 to $\mathbf{x}^\perp \cong \mathbf{H}_n(\mathbb{F})$ and $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ and to $\mathbf{y}^\perp \cong \mathbf{H}_n(\mathbb{F})$ and $\mathbf{x}^\perp \cap \mathbf{y}^\perp$. $\square$

The preceding lemma allows us to transfer points from $\mathbf{x}^\perp$ to $\mathbf{y}^\perp$. Indeed, if there is a local point $p_\mathbf{x}$ in $\mathbf{x}^\perp$ that lies in the hyperplane $Y_\mathbf{x}$ induced by the vertex $\mathbf{y}$ on $\mathbf{x}^\perp$, the point $p_\mathbf{x}$ corresponds to a point $p_\mathbf{y}$ of $\mathbf{y}^\perp$. That point $p_\mathbf{y}$ is simply the $\approx^p_\mathbf{y}$ equivalence class that contains the set $p_\mathbf{x} \cap \mathbf{y}^\perp$. The next lemma will show us that we can transfer interior points at will around a triangle of Γ.

Lemma 1.3.2
Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be three mutually adjacent vertices of Γ. Then there is a choice of local equivalence relations $\approx^p_\mathbf{x}$, $\approx^p_\mathbf{y}$, and $\approx^p_\mathbf{z}$ such that the restrictions of $\approx^p_\mathbf{x}$ and $\approx^p_\mathbf{y}$ to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$, the restrictions of $\approx^p_\mathbf{x}$ and $\approx^p_\mathbf{z}$ to $\mathbf{x}^\perp \cap \mathbf{z}^\perp$, and the restrictions of $\approx^p_\mathbf{y}$ and $\approx^p_\mathbf{z}$ to $\mathbf{y}^\perp \cap \mathbf{z}^\perp$ coincide.

Proof. In view of Lemma 1.3.1, we may assume that $\approx^p_{\mathbf{x}}$ and $\approx^p_{\mathbf{y}}$ have the same restriction to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ and that $\approx^p_{\mathbf{x}}$ and $\approx^p_{\mathbf{z}}$ have the same restriction to $\mathbf{x}^\perp \cap \mathbf{z}^\perp$. Let $p_{\mathbf{x}}$ be an interior point of $\mathbf{x}^\perp$ such that $p_{\mathbf{x}} \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp \neq \emptyset$. By analysis of $\mathbf{x}^\perp$, we can find two vertices, say $\mathbf{u}$ and $\mathbf{v}$ in $p_{\mathbf{x}} \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$. Now the above choices of local equivalence relations imply that $\mathbf{u}$ and $\mathbf{v}$ belong to $\approx^p_{\mathbf{y}} \cap \approx^p_{\mathbf{z}}$ (indeed, $\mathbf{u}$ and $\mathbf{v}$ belong to both $\approx^p_{\mathbf{x}} \cap \approx^p_{\mathbf{y}}$ and $\approx^p_{\mathbf{x}} \cap \approx^p_{\mathbf{z}}$). This forces that $\approx^p_{\mathbf{y}}$ and $\approx^p_{\mathbf{z}}$ have the same restriction to $\mathbf{y}^\perp \cap \mathbf{z}^\perp$ by Lemma 1.2.2. □

Lemma 1.3.3
Let $n \geq 5$ and let $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ be a path of vertices in Γ. Then $\mathbf{w}^\perp \cap \mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp \neq \emptyset$. In particular, the diameter of Γ is two, and Γ, viewed as a two-dimensional simplicial complex whose two-simplices are its triangles, is simply connected.

Proof. Choose local equivalence relations $\approx^p_{\mathbf{w}}$, $\approx^p_{\mathbf{x}}$, $\approx^p_{\mathbf{y}}$, and $\approx^p_{\mathbf{z}}$ such that $\approx^p_{\mathbf{w}}$ and $\approx^p_{\mathbf{x}}$ coincide on $\mathbf{w}^\perp \cap \mathbf{x}^\perp$, such that $\approx^p_{\mathbf{x}}$ and $\approx^p_{\mathbf{y}}$ coincide on $\mathbf{x}^\perp \cap \mathbf{y}^\perp$, and such that $\approx^p_{\mathbf{y}}$ and $\approx^p_{\mathbf{z}}$ coincide on $\mathbf{y}^\perp \cap \mathbf{z}^\perp$ as indicated in Lemma 1.3.1. Suppose we have $\mathbf{x} = (x_{\mathbf{y}}, X_{\mathbf{y}})$ and $\mathbf{z} = (z_{\mathbf{y}}, Z_{\mathbf{y}})$ inside $\mathbf{y}^\perp$. Then the graph $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$ (considered inside $\mathbf{y}^\perp$) consists of the non-incident point-hyperplane pairs whose points are contained in $X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ and whose hyperplanes contain the line $x_{\mathbf{y}} z_{\mathbf{y}}$. Since $n \geq 5$, the space $X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ has at least (projective) dimension three. The line $x_{\mathbf{y}} z_{\mathbf{y}}$ can intersect $X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ in at most a point, as $x_{\mathbf{y}} \notin X_{\mathbf{y}}$ and $z_{\mathbf{y}} \notin Z_{\mathbf{y}}$. Assume it does and let $a_{\mathbf{y}}$ be that intersection point.

By the choice of local equivalence relations and by Lemma 1.2.15, we can consider this configuration also in $\mathbf{x}^\perp$. The vertices $\mathbf{w}$ and $\mathbf{y}$ correspond to point-hyperplane pairs $(w_{\mathbf{x}}, W_{\mathbf{x}})$ and $(y_{\mathbf{x}}, Y_{\mathbf{x}})$, respectively. The space $X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ in $\mathbf{y}^\perp$ corresponds to a space $U_{\mathbf{x}}$ of $\mathbf{x}^\perp$ (of the same dimension as $X_{\mathbf{y}} \cap Z_{\mathbf{y}}$) contained in $Y_{\mathbf{x}}$. The point $a_{\mathbf{y}}$ arises as the point $a_{\mathbf{x}}$ in $\mathbf{x}^\perp$ and is contained in $U_{\mathbf{x}}$. Let $l_{\mathbf{x}}$ be the line in $\mathbf{x}^\perp$ spanned by $a_{\mathbf{x}}$ and $y_{\mathbf{x}}$. In the worst case, we have $l_{\mathbf{x}} \cap U_{\mathbf{x}} \cap W_{\mathbf{x}} \neq \emptyset$. But even then $\langle w_{\mathbf{x}}, l_{\mathbf{x}} \rangle \cap U_{\mathbf{x}} \cap W_{\mathbf{x}}$ is strictly contained in $U_{\mathbf{x}} \cap W_{\mathbf{x}}$. Therefore we can choose a point $p_{\mathbf{x}} \in U_{\mathbf{x}} \cap W_{\mathbf{x}} \setminus \langle w_{\mathbf{x}}, l_{\mathbf{x}} \rangle$ and a hyperplane $H_{\mathbf{x}} \not\ni p_{\mathbf{x}}$ containing $l_{\mathbf{x}}$ and $w_{\mathbf{x}}$. Now $(p_{\mathbf{x}}, H_{\mathbf{x}})$ describes a vertex $\mathbf{p}$ in $\mathbf{x}^\perp$ which is adjacent to $\mathbf{w}$ (as $p_{\mathbf{x}} \in W_{\mathbf{x}}$ and $w_{\mathbf{x}} \in H_{\mathbf{x}}$) and also to $\mathbf{y}$ (as $p_{\mathbf{x}} \in U_{\mathbf{x}} \subseteq Y_{\mathbf{x}}$ and $y_{\mathbf{x}} \in l_{\mathbf{x}} \subset H_{\mathbf{x}}$) and to $\mathbf{z}$ (as $\mathbf{p} = (p_{\mathbf{y}}, H_{\mathbf{y}})$ in $\mathbf{y}^\perp$ with $p_{\mathbf{y}} \in X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ and $x_{\mathbf{y}} a_{\mathbf{y}} \subseteq H$, whence also $z_{\mathbf{y}} \in H_{\mathbf{y}}$), as required. If some of the assumptions made in the proof—like $x_{\mathbf{y}} z_{\mathbf{y}} \cap X_{\mathbf{y}} \cap Z_{\mathbf{y}} \neq \emptyset$—do not hold, then the result is obtained even faster.

The other assertions are direct consequences of the above. Indeed, let $\mathbf{a}$, $\mathbf{b}$ be distinct vertices of Γ. As Γ is connected, there exists a path from $\mathbf{a}$ to $\mathbf{b}$. If the path has length greater than two, we can find a new path of shorter length by the first part of the lemma. Induction proves that there exists a path of length at most two from $\mathbf{a}$ to $\mathbf{b}$. Similarly, one proves that a cycle can be decomposed into triangles. □

The preceding lemma shows that in case $n \geq 5$ the graph Γ is automatically simply connected. The cases $n = 3, 4$ will prove a little bit more difficult, but the series of the following three lemmas will prove simple connectedness of Γ for $n \in \{3, 4\}$ as well.

Lemma 1.3.4
Let $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ *be a path of vertices in* Γ. *Then for* $\mathbf{x} = (x_{\mathbf{y}}, X_{\mathbf{y}})$ *and* $\mathbf{z} = (z_{\mathbf{y}}, Z_{\mathbf{y}})$ *inside* $\mathbf{y}^{\perp}$, *if* $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x_{\mathbf{y}} z_{\mathbf{y}} = \emptyset$ *or if* $X_{\mathbf{y}} = Z_{\mathbf{y}}$, *we have* $\{\mathbf{w}, \mathbf{x}, \mathbf{z}\}^{\perp} \neq \emptyset$.

Notice that, for example, we have $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x_{\mathbf{y}} z_{\mathbf{y}} = \emptyset$, in case $x_{\mathbf{y}} = z_{\mathbf{y}}$.

Proof. Choose local equivalence relations $\approx^p_{\mathbf{w}}$, $\approx^p_{\mathbf{x}}$, $\approx^p_{\mathbf{y}}$, and $\approx^p_{\mathbf{z}}$ such that $\approx^p_{\mathbf{w}}$ and $\approx^p_{\mathbf{x}}$ coincide on $\mathbf{w}^{\perp} \cap \mathbf{x}^{\perp}$, that $\approx^p_{\mathbf{x}}$ and $\approx^p_{\mathbf{y}}$ coincide on $\mathbf{x}^{\perp} \cap \mathbf{y}^{\perp}$, and that $\approx^p_{\mathbf{y}}$ and $\approx^p_{\mathbf{z}}$ coincide on $\mathbf{y}^{\perp} \cap \mathbf{z}^{\perp}$ as indicated in Lemma 1.3.1. Application of Lemma 1.2.17 to the interior projective space of $\mathbf{y}^{\perp} \cong \mathbf{H}_n(\mathbb{F})$ shows that the interior projective spaces on $\mathbf{x}^{\perp} \cap \mathbf{y}^{\perp}$ and on $\mathbf{y}^{\perp} \cap \mathbf{z}^{\perp}$ correspond to hyperplanes of $\mathbf{y}^{\perp} \cong \mathbf{H}_n(\mathbb{F})$. We have to investigate $\mathbf{x}^{\perp} \cap \mathbf{y}^{\perp} \cap \mathbf{z}^{\perp}$. We have $\mathbf{x} = (x_{\mathbf{y}}, X_{\mathbf{y}})$ and $\mathbf{z} = (z_{\mathbf{y}}, Z_{\mathbf{y}})$ inside $\mathbf{y}^{\perp}$. Then the graph $\mathbf{x}^{\perp} \cap \mathbf{y}^{\perp} \cap \mathbf{z}^{\perp}$ (considered inside $\mathbf{y}^{\perp}$) consists of the non-incident point-hyperplane pairs whose points are contained in $X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ and whose hyperplanes contain the line $x_{\mathbf{y}} z_{\mathbf{y}}$.

First, let us assume $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x_{\mathbf{y}} z_{\mathbf{y}} = \emptyset$. Also assume that $x_{\mathbf{y}} \neq z_{\mathbf{y}}$ and denote the intersection $x_{\mathbf{y}} z_{\mathbf{y}} \cap X_{\mathbf{y}}$ by $a_{\mathbf{y}}$. Consider $\mathbf{x}^{\perp}$, in which the point $a_{\mathbf{y}} \in X_{\mathbf{y}}$ arises as $a_{\mathbf{x}}$ inside $Y_{\mathbf{x}}$, and denote $\mathbf{w}$ by $(w_{\mathbf{x}}, W_{\mathbf{x}})$ and $\mathbf{y}$ by $(y_{\mathbf{x}}, Y_{\mathbf{x}})$. We can assume $w_{\mathbf{x}}$ to be contained in $Y_{\mathbf{x}}$. (Indeed, inside $\mathbf{y}^{\perp}$, the intersection $X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ contains a line $l_{\mathbf{y}}$. This line $l_{\mathbf{y}}$ arises as a subspace $l_{\mathbf{x}}$ of $\mathbf{x}^{\perp}$ that is contained in $Y_{\mathbf{x}}$. Choose a hyperplane $H_{\mathbf{x}}$ that contains $a_{\mathbf{x}}$, $w_{\mathbf{x}}$, and $y_{\mathbf{x}}$, and choose a point $p_{\mathbf{x}}$ on $l_{\mathbf{x}}$ off $H_{\mathbf{x}}$. This gives rise to a vertex $\mathbf{y}'$ that is adjacent to $\mathbf{x}$ and $\mathbf{y}$. Local analysis of $\mathbf{y}^{\perp}$ shows that the hyperplane of the vertex $\mathbf{y}'$ contains the point $x_{\mathbf{y}}$ and the point $a_{\mathbf{y}}$, whence also the point $z_{\mathbf{y}}$. Moreover, the point of $\mathbf{y}'$ is contained in $l_{\mathbf{y}}$, whence also in $Z_{\mathbf{y}}$, and $\mathbf{y}'$ is a neighbor of $\mathbf{z}$.) Inside $\mathbf{x}^{\perp}$ we have now the following setting. The hyperplane $Y_{\mathbf{x}}$ contains the points $w_{\mathbf{x}}$ and $a_{\mathbf{x}}$ as well as the line $l_{\mathbf{x}}$. Note that $l_{\mathbf{x}}$ has to intersect the hyperplane $W_{\mathbf{x}}$. If $\langle a_{\mathbf{x}}, w_{\mathbf{x}} \rangle$ does not intersect $l_{\mathbf{x}} \cap W_{\mathbf{x}}$, then we can choose a point inside $l_{\mathbf{x}} \cap W_{\mathbf{x}}$ and a non-incident hyperplane that contains $\langle a_{\mathbf{x}}, w_{\mathbf{x}}, y_{\mathbf{x}} \rangle$, yielding a vertex that is adjacent to $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, and—after local analysis of $\mathbf{y}^{\perp}$—also to $\mathbf{z}$. Therefore assume that $\langle a_{\mathbf{x}}, w_{\mathbf{x}} \rangle$ does intersect $l_{\mathbf{x}} \cap W_{\mathbf{x}}$. Then fix the point $u_{\mathbf{x}} := \langle a_{\mathbf{x}}, w_{\mathbf{x}} \rangle \cap l_{\mathbf{x}} \cap W_{\mathbf{x}}$ and choose a hyperplane $U_{\mathbf{x}}$ of $\mathbf{x}^{\perp}$ that contains $a_{\mathbf{x}}$ and $y_{\mathbf{x}}$ but not $u_{\mathbf{x}}$. The pair $(u_{\mathbf{x}}, U_{\mathbf{x}})$ describes another vertex, $\mathbf{u}$ say, that is adjacent to $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. Locally in $\mathbf{u}^{\perp}$ we have a hyperplane $X_{\mathbf{u}}$ of $\mathbf{x}$, a line $k_{\mathbf{u}}$ in $X_{\mathbf{u}}$ that arises from a line $k_{\mathbf{x}}$ contained in the intersection $U_{\mathbf{x}} \cap W_{\mathbf{x}}$ of the hyperplanes of the vertices $\mathbf{u}$ and $\mathbf{w}$ inside $\mathbf{x}^{\perp}$, and the hyperplane $Z_{\mathbf{u}}$ of $\mathbf{z}$. Choose a point $v_{\mathbf{u}}$ in $k_{\mathbf{u}} \cap Z_{\mathbf{u}}$ and a hyperplane $V_{\mathbf{u}}$ on $x_{\mathbf{u}} z_{\mathbf{u}}$ that does not contain $v_{\mathbf{u}}$. Obviously, this vertex $\mathbf{v} = (v_{\mathbf{u}}, V_{\mathbf{u}})$ is adjacent to $\mathbf{x}$ and $\mathbf{z}$. In $\mathbf{x}^{\perp}$, however, we see $\mathbf{v}$ as $(v_{\mathbf{x}}, V_{\mathbf{x}})$ whose hyperplane $V_{\mathbf{x}}$ contains the points $a_{\mathbf{x}}$ and $u_{\mathbf{x}}$, therefore also $w_{\mathbf{x}}$. Moreover, $v_{\mathbf{x}}$ is contained in $k_{\mathbf{x}}$, whence also in $W_{\mathbf{x}}$, and $\mathbf{v}$ is the required vertex.

The special cases $x_{\mathbf{y}} = z_{\mathbf{y}}$ and $X_{\mathbf{y}} = Z_{\mathbf{y}}$ run along the same lines and are, in fact, easier to prove. $\square$

Lemma 1.3.5
The diameter of Γ *is two.*

Proof. Since for $n \geq 5$, this is a weaker version of Lemma 1.3.3, we will concentrate on the cases $n \in \{3,4\}$. Choose a path $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ of vertices in Γ, and fix local equivalence relations $\approx^p_{\mathbf{w}}$, $\approx^p_{\mathbf{x}}$, $\approx^p_{\mathbf{y}}$, and $\approx^p_{\mathbf{z}}$ as in the proof of the preceding lemma. Inside $\mathbf{y}^\perp$, let $\mathbf{x}$ correspond to $(x_{\mathbf{y}}, X_{\mathbf{y}})$ and $\mathbf{z}$ correspond to $(z_{\mathbf{y}}, Z_{\mathbf{y}})$. Up to a change of $\mathbf{x}$ to a vertex $\mathbf{x}_0 \in \mathbf{w}^\perp \cap \mathbf{x}^\perp \cap \mathbf{y}^\perp$ with $\mathbf{x}_0 = (x^0_{\mathbf{y}}, X^0_{\mathbf{y}})$ inside $\mathbf{y}^\perp$ we can assume that $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x_{\mathbf{y}}z_{\mathbf{y}} = \emptyset$. For, suppose that $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x_{\mathbf{y}}z_{\mathbf{y}} \neq \emptyset$. Then $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x_{\mathbf{y}}z_{\mathbf{y}}$ is a point; $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \backslash x_{\mathbf{y}}z_{\mathbf{y}}$ is at least (the point set of) an affine line, for $n = 3$, and at least (the point set of) a dual affine plane, for $n = 4$; it may be even bigger if $X_{\mathbf{y}} = Z_{\mathbf{y}}$. The set of common neighbors of $\mathbf{x}$ and $\mathbf{z}$ in $\mathbf{y}^\perp$ corresponds to the set of all non-incident point-hyperplane pairs $(p_{\mathbf{y}}, H_{\mathbf{y}})$ with $p_{\mathbf{y}} \in X_{\mathbf{y}} \cap Z_{\mathbf{y}}$ and $H_{\mathbf{y}} \supset x_{\mathbf{y}}z_{\mathbf{y}}$. This implies that for any point $p_{\mathbf{y}} \in X_{\mathbf{y}} \cap Z_{\mathbf{y}} \backslash x_{\mathbf{y}}z_{\mathbf{y}}$ we can find a vertex $(p_{\mathbf{y}}, H_{\mathbf{y}})$ in $\mathbf{y}^\perp$ adjacent to both $\mathbf{x}$ and $\mathbf{z}$. Now consider $\mathbf{x}^\perp$. Let $\mathbf{w} = (w_{\mathbf{x}}, W_{\mathbf{x}})$ and $\mathbf{y} = (y_{\mathbf{x}}, Y_{\mathbf{x}})$. Any vertex $\mathbf{x}_0 = (x^0_{\mathbf{x}}, X^0_{\mathbf{x}})$ adjacent to $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$ consists of a point $x^0_{\mathbf{x}} \in W_{\mathbf{x}} \cap Y_{\mathbf{x}}$ and a non-incident hyperplane $X^0_{\mathbf{x}} \supset w_{\mathbf{x}}y_{\mathbf{x}}$. Hence, as above in $\mathbf{y}^\perp$, we can choose $x^0_{\mathbf{x}}$ freely on an affine line for $n = 3$ or a dual affine plane for $n = 4$. This translates to $\mathbf{y}^\perp$ as follows. The line $w_{\mathbf{x}}y_{\mathbf{x}}$ intersects $Y_{\mathbf{x}}$ in a point, $a_{\mathbf{x}}$ say, which gives rise to a point $a_{\mathbf{y}} \in X_{\mathbf{y}}$ of $\mathbf{y}^\perp$. So all those hyperplanes $X^0_{\mathbf{x}}$ arise as hyperplanes $X^0_{\mathbf{y}}$ in $\mathbf{y}^\perp$ that have to contain the line $x_{\mathbf{y}}a_{\mathbf{y}}$. Notice that this line $x_{\mathbf{y}}a_{\mathbf{y}}$ is the largest subspace of $\mathbf{y}^\perp$ that is contained in all these hyperplanes $X^0_{\mathbf{y}}$. If for some fixed choice of $x^0_{\mathbf{y}}$, there exists a hyperplane $X^0_{\mathbf{y}}$ such that $X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x^0_{\mathbf{y}}z_{\mathbf{y}} = \emptyset$, we are done. Hence, for a fixed $x^0_{\mathbf{y}}$, suppose all choices for $X^0_{\mathbf{y}}$ contain the point $x^0_{\mathbf{y}}z_{\mathbf{y}} \cap Z_{\mathbf{y}}$. Then we can choose another $x^1_{\mathbf{y}}$ instead of $x^0_{\mathbf{y}}$ and find an $X^0_{\mathbf{y}}$ with $X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x^1_{\mathbf{y}}z_{\mathbf{y}} = \emptyset$. For, suppose for a choice $\mathbf{x}^1_{\mathbf{y}}$ distinct from $\mathbf{x}^0_{\mathbf{y}}$ still $X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x^1_{\mathbf{y}}z_{\mathbf{y}} \neq \emptyset$ for all possible $X^0_{\mathbf{y}}$. Then the points $u_{\mathbf{y}} := x^0_{\mathbf{y}}z_{\mathbf{y}} \cap Z_{\mathbf{y}}$ and $v_{\mathbf{y}} := x^1_{\mathbf{y}}z_{\mathbf{y}} \cap Z_{\mathbf{y}}$ span a line as $z_{\mathbf{y}} \notin Z_{\mathbf{y}}$. But this line $u_{\mathbf{y}}v_{\mathbf{y}}$ has to coincide with the line $x_{\mathbf{y}}a_{\mathbf{y}}$. In particular, $x_{\mathbf{y}}$ is contained in $Z_{\mathbf{y}}$. But this contradicts our assumption that $X_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x_{\mathbf{y}}z_{\mathbf{y}} \neq \emptyset$. Hence we can find an $x^1_{\mathbf{y}}$ with $X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} \cap x^1_{\mathbf{y}}z_{\mathbf{y}} = \emptyset$.

We have found a chain of length three from $\mathbf{w}$ to $\mathbf{z}$ that satisfies the hypotheses of Lemma 1.3.4, and the claim follows. □

We would like to credit the following lemma to Andries Brouwer, who observed that the combination of the proofs of the two preceding lemmas yields simple connectedness.

Lemma 1.3.6
The graph Γ, *considered as a two-dimensional simplicial complex whose two-simplices are its triangles, is simply connected.*

Proof. The proof of Lemma 1.3.5 shows that for every path of distinct vertices $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ in Γ we have $x_{\mathbf{y}}z_{\mathbf{y}} \cap X_{\mathbf{y}} \cap Z_{\mathbf{y}} = \emptyset$ or there exists a vertex $\mathbf{x}_0 \in \{\mathbf{w}, \mathbf{x}, \mathbf{y}\}^\perp$ with $x^0_{\mathbf{y}}z_{\mathbf{y}} \cap X^0_{\mathbf{y}} \cap Z_{\mathbf{y}} = \emptyset$. The proof of Lemma 1.3.4, on the other hand, implies that in the former case there exists a path of vertices inside $\mathbf{x}^\perp \cap \mathbf{z}^\perp$ from $\mathbf{y}$ to a vertex $\mathbf{v}$ that is adjacent to $\mathbf{w}$, $\mathbf{x}$, and $\mathbf{z}$ and in the latter case there exists a path of vertices inside $\mathbf{x}_0^\perp \cap \mathbf{z}^\perp$ from $\mathbf{y}$ to a vertex $\mathbf{v}$ that is adjacent to $\mathbf{w}$, $\mathbf{x}_0$, and $\mathbf{z}$. Simple connectedness of Γ follows. □

Lemma 1.3.7
There is a choice of local equivalence relations $\approx_\mathbf{x}^p$ for all $\mathbf{x} \in \Gamma$ such that, for any two adjacent vertices $\mathbf{x}$ and $\mathbf{y}$, the restrictions of $\approx_\mathbf{x}^p$ and $\approx_\mathbf{y}^p$ to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ coincide.

Proof. Since Γ is simply connected (by Lemmas 1.3.3 and 1.3.6), the statement of this lemma follows immediately from the triangle analysis in Lemma 1.3.2. □

Notation 1.3.8 Fix a choice of $\approx_\mathbf{x}^p$ as in Lemma 1.3.7 and set $\approx^p = \bigcup_{\mathbf{x}\in\Gamma} \approx_\mathbf{x}^p$.

Lemma 1.3.9
Let $\mathbf{x}$ and $\mathbf{y}$ be vertices of Γ such that $\mathbf{x} \approx_\mathbf{u}^p \mathbf{y}$ for some vertex $\mathbf{u}$ in $\{\mathbf{x},\mathbf{y}\}^\perp$. Then $\mathbf{x} \approx_\mathbf{v}^p \mathbf{y}$ for every vertex $\mathbf{v}$ in $\{\mathbf{x},\mathbf{y}\}^\perp$.

Proof. Let $\mathbf{u}$, $\mathbf{x}$, $\mathbf{y}$ be as in the hypothesis and let $\mathbf{v} \in \{\mathbf{x},\mathbf{y}\}^\perp$ be an additional vertex. If $\mathbf{u} \perp \mathbf{v}$, then the claim is certainly true. Indeed, in $\mathbf{u}^\perp$ we then have $\mathbf{x} = (x_\mathbf{u}, X_\mathbf{u})$, $\mathbf{y} = (y_\mathbf{u}, Y_\mathbf{u})$, $\mathbf{v} = (v_\mathbf{u}, V_\mathbf{u})$ with $x_\mathbf{u}, y_\mathbf{u} \in V_\mathbf{u}$ and $x_\mathbf{u} = y_\mathbf{u}$ as $\mathbf{x} \approx_\mathbf{u}^p \mathbf{y}$. Thus, it is sufficient to show that the induced subgraph $\{\mathbf{x},\mathbf{y}\}^\perp$ of Γ is connected. In $\mathbf{u}^\perp$ the vertices $\mathbf{x}$ and $\mathbf{y}$ correspond to point-hyperplane pairs $(x_\mathbf{u}, X_\mathbf{u})$, respectively $(y_\mathbf{u}, Y_\mathbf{u})$. Likewise, in $\mathbf{x}^\perp$ we have the correspondence $\mathbf{u} = (u_\mathbf{x}, U_\mathbf{x})$ and $\mathbf{v} = (v_\mathbf{x}, V_\mathbf{x})$. Moreover, the intersection $X_\mathbf{u} \cap Y_\mathbf{u}$ from $\mathbf{u}^\perp$ arises as a hyperplane $W_\mathbf{x}$ of $U_\mathbf{x}$ in $\mathbf{x}^\perp$. Therefore the intersection $W_\mathbf{x} \cap V_\mathbf{x}$ contains a point $p_\mathbf{x}$. If in $\mathbf{x}^\perp$ the line $u_\mathbf{x} v_\mathbf{x}$ does not contain $p_\mathbf{x}$, we can find a hyperplane $H_\mathbf{x} \supset u_\mathbf{x} v_\mathbf{x}$ that does not contain $p_\mathbf{x}$, and $(p_\mathbf{x}, H_\mathbf{x})$ is a vertex of $\mathbf{x}^\perp$ which is adjacent to both $\mathbf{u}$ and $\mathbf{v}$. But inside $\mathbf{u}^\perp$ this vertex also corresponds to some point-hyperplane pair, whose point is contained in $Y_\mathbf{u}$ and whose hyperplane contains $y_\mathbf{u} = x_\mathbf{u}$, whence this vertex is also adjacent to $\mathbf{y}$, and we are done.

So assume we have $p_\mathbf{x} \in u_\mathbf{x} v_\mathbf{x}$ in $\mathbf{x}^\perp$. Then choose any hyperplane $H_\mathbf{x}$ that contains $u_\mathbf{x}$ but not $p_\mathbf{x}$. Then the vertex $\mathbf{t} := (p_\mathbf{x}, H_\mathbf{x})$ is adjacent to $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{y}$, but not $\mathbf{v}$. Inside $\mathbf{t}^\perp$ we have hyperplanes $X_\mathbf{t}$ and $Y_\mathbf{t}$ coming from $\mathbf{x}$ and $\mathbf{y}$. The intersection $X_\mathbf{t} \cap Y_\mathbf{t}$ corresponds to a subspace $S_\mathbf{x}$ of $H_\mathbf{x}$ (the hyperplane of the vertex $\mathbf{t}$) in $\mathbf{x}^\perp$. The intersection $S_\mathbf{x} \cap V_\mathbf{x}$ in $\mathbf{x}^\perp$ contains some point $q_\mathbf{x}$. If $q_\mathbf{x}$ lies on the line $p_\mathbf{x} v_\mathbf{x}$, then $q_\mathbf{x} = p_\mathbf{x} v_\mathbf{x} \cap H_\mathbf{x} = p_\mathbf{x} u_\mathbf{x} \cap H_\mathbf{x} = u_\mathbf{x}$, and we have $u_\mathbf{x} \in V_\mathbf{x}$. But this contradicts $p_\mathbf{x} \in u_\mathbf{x} v_\mathbf{x}$, as $p_\mathbf{x} \in V_\mathbf{x} \cap U_\mathbf{x}$, $v_\mathbf{x} \notin V_\mathbf{x}$ and $u_\mathbf{x} \in V_\mathbf{x} \backslash U_\mathbf{x}$. Therefore we have $q_\mathbf{x} \notin p_\mathbf{x} v_\mathbf{x}$ and we are in the situation of the preceding paragraph with the vertex $\mathbf{t}$ instead of $\mathbf{u}$. □

Finally, we prove the fact that there exists a well-defined notion of *global points* on Γ, which will then allow us to study a geometry on Γ. Moreover, all statements and results about the local relations $\approx_\mathbf{x}^p$ are also true for the local relations $\approx_\mathbf{x}^h$, and we can define a global relation $\approx^h = \bigcup_{\mathbf{x}\in\Gamma} \approx_\mathbf{x}^h$ with the same nice properties on the local intersections. Of course, also the following result is true for $\approx^h$ as well.

Lemma 1.3.10
$\approx^p$ is an equivalence relation.

Proof. Reflexivity and symmetry follow from reflexivity and symmetry of each $\approx^p_{\mathbf{x}}$. To prove transitivity, assume that $\mathbf{x} \approx^p \mathbf{y}$ and $\mathbf{y} \approx^p \mathbf{z}$. Then there exist vertices $\mathbf{u}$, $\mathbf{v}$ with $\mathbf{x} \approx^p_{\mathbf{u}} \mathbf{y}$ and $\mathbf{y} \approx^p_{\mathbf{v}} \mathbf{z}$. By Lemma 1.3.5, there also exists a vertex $\mathbf{a} \in \{\mathbf{x}, \mathbf{z}\}^\perp$. We will prove that $\mathbf{x} \approx^p_{\mathbf{a}} \mathbf{z}$. Two applications of Lemma 1.3.4 (on the chains $\mathbf{a} \perp \mathbf{x} \perp \mathbf{u} \perp \mathbf{y}$ and $\mathbf{a} \perp \mathbf{z} \perp \mathbf{v} \perp \mathbf{y}$) yield vertices $\mathbf{b} \in \{\mathbf{a}, \mathbf{x}, \mathbf{y}\}^\perp$ and $\mathbf{c} \in \{\mathbf{a}, \mathbf{z}, \mathbf{y}\}^\perp$. Lemma 1.3.9 implies $\mathbf{x} \approx^p_{\mathbf{b}} \mathbf{y}$ and $\mathbf{y} \approx^p_{\mathbf{c}} \mathbf{z}$. Set $\mathbf{b} = (b_{\mathbf{a}}, B_{\mathbf{a}})$, $\mathbf{c} = (c_{\mathbf{a}}, C_{\mathbf{a}})$, $\mathbf{x} = (x_{\mathbf{a}}, X_{\mathbf{a}})$, and $\mathbf{z} = (z_{\mathbf{a}}, Z_{\mathbf{a}})$ in $\mathbf{a}^\perp$. Notice that $z_{\mathbf{a}} \in C_{\mathbf{a}}$. We can additionally assume that $x_{\mathbf{a}} \in C_{\mathbf{a}}$ and $c_{\mathbf{a}} \notin b_{\mathbf{a}} x_{\mathbf{a}}$. (Indeed, set $\mathbf{a} = (a_{\mathbf{c}}, A_{\mathbf{c}})$, $\mathbf{y} = (y_{\mathbf{c}}, Y_{\mathbf{c}})$, $\mathbf{z} = (z_{\mathbf{c}}, Z_{\mathbf{c}})$ in $\mathbf{c}^\perp$. The intersection $A_{\mathbf{c}} \cap Y_{\mathbf{c}}$ contains a line $l_{\mathbf{c}}$. Moreover, $y_{\mathbf{c}} = z_{\mathbf{c}}$, as $\mathbf{y} \approx^p_{\mathbf{c}} \mathbf{z}$. Locally in $\mathbf{a}^\perp$ the line $l_{\mathbf{c}}$ arises as a line $l_{\mathbf{a}} \subset C_{\mathbf{a}}$. Fix a hyperplane $H_{\mathbf{a}}$ that contains $\langle c_{\mathbf{a}}, x_{\mathbf{a}}, z_{\mathbf{a}} \rangle$ and fix a point $p_{\mathbf{a}}$ on $l_{\mathbf{a}}$ off $\langle c_{\mathbf{a}}, x_{\mathbf{a}}, z_{\mathbf{a}} \rangle$ and $\langle b_{\mathbf{a}}, x_{\mathbf{a}} \rangle$; such a choice is always possible as $x_{\mathbf{a}} \notin l_{\mathbf{a}}$ and $c_{\mathbf{a}} \notin C_{\mathbf{a}}$ and $l_{\mathbf{a}}$ contains at least three points. This gives a new vertex $\mathbf{c}' = (p_{\mathbf{a}}, H_{\mathbf{a}})$ that is adjacent to $\mathbf{a}$, $\mathbf{c}$, and $\mathbf{y}$. Local analysis of $\mathbf{c}^\perp$ shows that we can find a vertex $\mathbf{z}'$ in $\approx^p_{\mathbf{c}}$ relation to $\mathbf{z}$ that is adjacent to $\mathbf{c}'$ and $\mathbf{a}$.) But now, we can find a vertex $\mathbf{d} = (x_{\mathbf{a}}, D_{\mathbf{a}})$ in $\mathbf{a}^\perp$ that is adjacent to $\mathbf{b} = (b_{\mathbf{a}}, B_{\mathbf{a}})$ and $\mathbf{c} = (c_{\mathbf{a}}, C_{\mathbf{a}})$ (notice that by the above we can assume $c_{\mathbf{a}} \notin b_{\mathbf{a}} x_{\mathbf{a}}$, whence $x_{\mathbf{a}} \notin b_{\mathbf{a}} c_{\mathbf{a}}$). We have $\mathbf{d} \approx^p_{\mathbf{b}} \mathbf{x}$, $\mathbf{d} \approx^p_{\mathbf{a}} \mathbf{x}$, and $\mathbf{d} \approx^p_{\mathbf{b}} \mathbf{y}$. By Lemma 1.3.9, this implies $\mathbf{d} \approx^p_{\mathbf{c}} \mathbf{y}$. Transitivity of $\approx^p_{\mathbf{c}}$ implies $\mathbf{d} \approx^p_{\mathbf{c}} \mathbf{z}$ and, again Lemma 1.3.9, yields $\mathbf{d} \approx^p_{\mathbf{a}} \mathbf{z}$. Finally, transitivity of $\approx^p_{\mathbf{a}}$ gives $\mathbf{x} \approx^p_{\mathbf{a}} \mathbf{z}$, yielding $\mathbf{x} \approx^p \mathbf{z}$, and $\approx^p$ is transitive. □

Definition 1.3.11 A **global point** of Γ is defined as an equivalence class of $\approx^p$. Dually, define a **global hyperplane** as an equivalence class of $\approx^h$.

We already have a local notion of incidence as defined before Definition 1.2.6. Similarly a global point p and a global hyperplane H are **incident** if and only if $p \cap H = \emptyset$. It is easily seen that $p \cap H = \emptyset$ if and only if $p_{\mathbf{x}} \cap H_{\mathbf{x}} = \emptyset$ for all $\mathbf{x}$ for which $p_{\mathbf{x}}$ and $H_{\mathbf{x}}$ exist. One implication is obvious. To prove the other, suppose the exists a vertex $\mathbf{y} \in p \cap H$. Then, any vertex $\mathbf{x}$ for which $p_{\mathbf{x}}$ and $H_{\mathbf{x}}$ exist is at distance at most two to $\mathbf{y}$, by Lemma 1.3.5, and there exists a vertex $\mathbf{z}$ adjacent to both $\mathbf{y}$ and $\mathbf{x}$. The local elements $p_{\mathbf{z}}$ and $H_{\mathbf{z}}$ exist, as $\mathbf{y}$ is a representative of both. But then $p_{\mathbf{x}} \cap z^\perp \neq \emptyset$ as well as $H_{\mathbf{x}} \cap z^\perp \neq \emptyset$, and the question of incidence of $p_{\mathbf{x}}$ and $H_{\mathbf{x}}$ can be decided inside $\mathbf{x}^\perp \cap \mathbf{z}^\perp$. Therefore $p_{\mathbf{x}} \cap H_{\mathbf{x}} \neq \emptyset$.

Definition 1.3.12 Let p and q be distinct global points and let $\mathbf{x}$ be a vertex such that $p_{\mathbf{x}}$ and $q_{\mathbf{x}}$ exist. Then the **global line** of Γ spanned by p and q is the set of those global points a such that $a_{\mathbf{x}}$ exists and is contained in the local line $p_{\mathbf{x}} q_{\mathbf{x}}$. Define a **global hyperline** in analogy to global lines. Moreover, let $\mathbb{P}_\Gamma = (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma, \subset)$ be the point-line geometry consisting of the point set $\mathcal{P}_\Gamma$ of global points of Γ and the line set $\mathcal{L}_\Gamma$ of global lines of Γ.

Lemma 1.3.13
The notion of a global line is well defined.

Proof. Let l be the global line spanned by the global points p and q. Let $\mathbf{x}$ and $\mathbf{y}$ be distinct vertices such that $p_{\mathbf{x}}$, $q_{\mathbf{x}}$, $p_{\mathbf{y}}$, and $q_{\mathbf{y}}$ exist. Choose vertices $\mathbf{a} \in p_{\mathbf{x}}$, $\mathbf{b} \in q_{\mathbf{x}}$, $\mathbf{c} \in q_{\mathbf{y}}$, and $\mathbf{d} \in p_{\mathbf{y}}$. If $\mathbf{x} \perp \mathbf{y}$, then $p_{\mathbf{x}} \cap p_{\mathbf{y}} \neq \emptyset$ and $q_{\mathbf{x}} \cap q_{\mathbf{y}} \neq \emptyset$, and the claim follows from Lemma 1.2.17.

Notice that by Lemma 1.2.13 we can assume that $\mathbf{c}$ and $\mathbf{d}$ are adjacent. There exists a vertex $\mathbf{z}_1$ adjacent to both $\mathbf{x}$ and $\mathbf{c}$ by Lemma 1.3.5. By Lemma 1.3.4 (applied to the path $\mathbf{a}$, $\mathbf{x}$, $\mathbf{z}_1$, $\mathbf{c}$) we can find a vertex $\mathbf{z}_2$ adjacent to $\mathbf{a}$, $\mathbf{x}$, and $\mathbf{c}$ (indeed, inside $\mathbf{z}_1^{\perp}$ the point $c_{\mathbf{z}_1}$ of $\mathbf{c}$ has to lie in the hyperplane $X_{\mathbf{z}_1}$ of $\mathbf{x}$, and we can apply that lemma). Local analysis of $\mathbf{c}$ yields a vertex $\mathbf{z}_3$ that is adjacent to $\mathbf{z}_2$, $\mathbf{c}$, and $\mathbf{d}$. The induced subgraph $\{\mathbf{c},\mathbf{d}\}^{\perp}$ of Γ is isomorphic to $\mathbf{H}_{n-1}(\mathbb{F})$, which is connected by Proposition 1.1.3. Therefore, we can find a path from $\mathbf{y}$ to $\mathbf{z}_3$ inside $\{\mathbf{c},\mathbf{d}\}^{\perp}$. Altogether we have found a path of vertices from $\mathbf{x}$ to $\mathbf{y}$ such that the local points $p_{\mathbf{w}}$ and $q_{\mathbf{w}}$ exist for every vertex $\mathbf{w}$ contained in that path. The lemma follows from the above inspection of the case $\mathbf{x} \perp \mathbf{y}$. □

Proposition 1.3.14
$\mathbb{P}_\Gamma$ is a linear space with thick lines.

Proof. This is an immediate consequence of Lemma 1.3.13. □

As customary in linear spaces, for distinct global points p and q we shall denote by pq the unique global line on p and q.

Proposition 1.3.15
$\mathbb{P}_\Gamma$ is a projective space.

Proof. In view of Proposition 1.3.14 we only have to verify Pasch's Axiom. Let a, b, c, d be four global points such that ab intersects cd in the global point e. Then $ab = ae$ and $cd = ce$. By Lemma 1.3.5 and Lemma 1.2.13, there are vertices $\mathbf{a}$ in a and $\mathbf{e}$ in e such that $\mathbf{a} \perp \mathbf{e}$. Choose a vertex $\mathbf{c}$ in c. Now, by Lemma 1.3.5, there is a vertex $\mathbf{y}$ adjacent to $\mathbf{e}$ and $\mathbf{c}$. After suitable replacements of $\mathbf{e}$ in e and $\mathbf{c}$ in c, we can assume that inside $\mathbf{y}^{\perp}$ we have $\mathbf{c} = (c_{\mathbf{y}}, C_{\mathbf{y}})$ and $\mathbf{e} = (e_{\mathbf{y}}, E_{\mathbf{y}})$ with $C_{\mathbf{y}} \cap E_{\mathbf{y}} \cap c_{\mathbf{y}}e_{\mathbf{y}} = \emptyset$. Lemma 1.3.4 implies the existence of $\mathbf{x} \in \{\mathbf{a},\mathbf{c},\mathbf{e}\}^{\perp}$. The global lines ae and ce meet $\mathbf{x}^{\perp}$ in interior lines. In particular, by Pasch's Axiom applied to the interior projective space of $\mathbf{x}^{\perp}$, there is an interior point $w_{\mathbf{x}}$ on both the interior lines $(ac)_{\mathbf{x}}$ and $(bd)_{\mathbf{x}}$ of $\mathbf{x}^{\perp}$. Consequently, the global lines ac and bd meet in a global point, whence Pasch's Axiom holds. □

Notation 1.3.16 Denote by $\langle \mathbf{x}^{\perp} \rangle$ the set of global points intersecting $\mathbf{x}^{\perp}$. Notice that this set is a subspace of $\mathbb{P}_\Gamma$.

Lemma 1.3.17
Let $\mathbf{x}, \mathbf{y} \in \Gamma$ *with* $\mathbf{x} \approx^h \mathbf{y}$*. Then* $\langle \mathbf{x}^\perp \rangle = \langle \mathbf{y}^\perp \rangle$*.*

Proof. By symmetry of $\approx^h$ it is enough to show $\langle \mathbf{x}^\perp \rangle \subseteq \langle \mathbf{y}^\perp \rangle$. To this end, let $p \in \langle \mathbf{x}^\perp \rangle$. Then there exists a vertex $\mathbf{p} \in p$ with $\mathbf{p} \perp \mathbf{x}$. By Lemma 1.3.5, we can find a vertex $\mathbf{z}$ with $\mathbf{x} \perp \mathbf{z} \perp \mathbf{y}$. If $\mathbf{x} = (x_\mathbf{z}, X_\mathbf{z})$, $\mathbf{y} = (y_\mathbf{z}, Y_\mathbf{z})$ inside $\mathbf{z}^\perp$, we have $X_\mathbf{z} = Y_\mathbf{z}$, as $\mathbf{x} \approx^h \mathbf{y}$. Applying Lemma 1.3.4, we obtain a vertex $\mathbf{a} \in \{\mathbf{p}, \mathbf{x}, \mathbf{y}\}^\perp$. Writing $\mathbf{p} = (p_\mathbf{a}, H_\mathbf{a})$ in $\mathbf{a}^\perp$, we see $p_\mathbf{a} \in X_\mathbf{a}$, whence $p_\mathbf{a} \in Y_\mathbf{a}$ by $\mathbf{x} \approx^h_\mathbf{a} \mathbf{y}$. But now we can find a vertex $\mathbf{p}_1 = (p_\mathbf{a}, H^1_\mathbf{a})$ with $y_\mathbf{a} \in H^1_\mathbf{a}$ and consequently $p \in \langle \mathbf{y}^\perp \rangle$. □

Lemma 1.3.18
$\langle \mathbf{x}^\perp \rangle$ *does not contain the global point that contains* $\mathbf{x}$*.*

Proof. Otherwise $\mathbf{x}^\perp$ contains a vertex $\mathbf{y}$ that belongs to the same global point. But then there exists a third vertex $\mathbf{z}$ adjacent to both $\mathbf{x}$ and $\mathbf{y}$, so $\mathbf{x}$ and $\mathbf{y}$ are two adjacent vertices belonging to the same interior point in $\mathbf{z}^\perp$, a contradiction to the local structure of $\mathbf{z}^\perp$. □

Lemma 1.3.19
Let $\mathbf{x}$ *be a vertex of* Γ*. Then* $\langle \mathbf{x}^\perp \rangle$ *is a hyperplane of* $\mathbb{P}_\Gamma$*. Conversely, any hyperplane* Π *of* $\mathbb{P}_\Gamma$ *is of this form. Moreover, if the global point* y *is not contained in* Π*, then there is a vertex* $\mathbf{y} \in y$ *with* $\langle \mathbf{y}^\perp \rangle = \Pi$*.*

Proof. Suppose l is a global line of Γ. We have to show that it intersects $\langle \mathbf{x}^\perp \rangle$. Let $a \neq b$ be two global points on l and choose vertices $\mathbf{a} \in a$, $\mathbf{b} \in b$. By Lemma 1.2.13 we may assume $\mathbf{a} \perp \mathbf{b}$. By Lemma 1.3.5, there exists a vertex $\mathbf{y}$ with $\mathbf{b} \perp \mathbf{y} \perp \mathbf{x}$. Changing $\mathbf{b}$ inside $b \cap \mathbf{a}^\perp \cap \mathbf{y}^\perp$ and $\mathbf{x}$ inside $\mathbf{y}^\perp$ while leaving $\langle \mathbf{x}^\perp \rangle$ invariant, we can assume $B_\mathbf{y} \cap X_\mathbf{y} \cap b_\mathbf{y} x_\mathbf{y} = \emptyset$ (for $\mathbf{b} = (b_\mathbf{y}, B_\mathbf{y})$, $\mathbf{x} = (x_\mathbf{y}, X_\mathbf{y})$, inside $\mathbf{y}^\perp$); notice that, by Lemma 1.3.17, changing $\mathbf{x}$ as indicated basically means changing the point $x_\mathbf{y}$. Consequently, there exists a vertex $\mathbf{c} \in \{\mathbf{a}, \mathbf{b}, \mathbf{x}\}^\perp$, by Lemma 1.3.4. Now local analysis of $\mathbf{c}^\perp$ shows that l has to intersect $\langle \mathbf{x}^\perp \rangle$. Lemma 1.3.18 shows that $\langle \mathbf{x}^\perp \rangle$ is not the whole space, and $\langle \mathbf{x}^\perp \rangle$ is a hyperplane.

Conversely, let y be a global point off the hyperplane Π. Any global line containing y intersects Π in a point x, say. Choose vertices in x and in y. By Lemma 1.3.5 there exists a third vertex adjacent to those two. Then, by Lemma 1.2.13, there exist adjacent vertices $\mathbf{x} \in x$ and $\mathbf{y} \in y$. The hyperplane $\langle \mathbf{x}^\perp \rangle$ intersects Π in a hyperplane of Π, since the global point x containing $\mathbf{x}$ is not contained in $\langle \mathbf{x}^\perp \rangle$ by Lemma 1.3.18. In $\langle \mathbf{x}^\perp \rangle \cap \Pi$ we find a clique of vertices $\mathbf{x}_i$, $i \in I$ for some index set I, such that the global points x_i of Γ containing them span $\langle \mathbf{x}^\perp \rangle \cap \Pi$ as a projective space; by Lemma 1.2.13. Then x together with the x_i spans Π. Moreover, y and x_i, $i \in I$, span $\langle \mathbf{x}^\perp \rangle$, since the x_i span $\langle \mathbf{x}^\perp \rangle \cap \Pi$ and y is a point of $\langle \mathbf{x}^\perp \rangle \setminus \Pi$. But again by Lemma 1.2.13, in $\mathbf{x}^\perp$ we can find $\mathbf{y} \in y$ such that $\mathbf{y}$ and $\mathbf{x}_i$, $i \in I$, form a clique. But then the hyperplane $\langle \mathbf{y}^\perp \rangle$ contains the points x, x_i, $i \in I$, hence $\langle \mathbf{y}^\perp \rangle = \Pi$, and the lemma is proved. □

Proposition 1.3.20
Let $n \geq 3$, let $\mathbb{F}$ be a division ring, let Γ be a connected, locally $\mathbf{H}_n(\mathbb{F})$ graph, and let $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma, \subset)$ be the projective space consisting of the global points and global lines of Γ. Then the point-hyperplane graph of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma, \subset)$ is isomorphic to Γ.

Proof. Let Δ be the point-hyperplane graph of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma, \subset)$. Consider the map $\Gamma \to \Delta : \mathbf{x} \mapsto (x, \langle \mathbf{x}^\perp \rangle)$ where x is the global point of Γ containing $\mathbf{x}$. We want to show that this is an isomorphism of graphs. Surjectivity follows from Lemma 1.3.19, since any point x of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma, \subset)$ is a global point of Γ and any hyperplane not containing it is of the form $\langle \mathbf{x}^\perp \rangle$ for a vertex $\mathbf{x} \in x$. Injectivity is obtained as follows. Suppose the global point x contains two vertices $\mathbf{x}_1$, $\mathbf{x}_2$ with $\langle \mathbf{x}_1^\perp \rangle = \langle \mathbf{x}_2^\perp \rangle$. By Lemma 1.3.5 there exists a vertex $\mathbf{y}$ adjacent to both $\mathbf{x}_1$ and $\mathbf{x}_2$. Since $\langle \mathbf{x}_1^\perp \rangle = \langle \mathbf{x}_2^\perp \rangle$, both vertices describe the same hyperplane in $\mathbf{y}^\perp$. But they also describe the same point and hence have to be equal, a contradiction. Finally, if $\mathbf{x} \perp \mathbf{y}$, then obviously $x \in \langle \mathbf{y}^\perp \rangle$ and $y \in \langle \mathbf{x}^\perp \rangle$, if x and y are the global points of Γ containing $\mathbf{x}$ and $\mathbf{y}$, respectively. □

This immediately implies the following result.

Theorem 1.3.21 (joint with Cohen, Cuypers)
Let $n \geq 3$, let $\mathbb{F}$ be a division ring, and let Γ be a connected, locally $\mathbf{H}_n(\mathbb{F})$ graph. Then Γ is isomorphic to $\mathbf{H}_{n+1}(\mathbb{F})$. □

The theorem does not hold for $n = 2$ as there is a connected graph that is locally $\mathbf{H}_2(2)$ but not isomorphic to $\mathbf{H}_3(2)$, see the next section.

1.4 Small dimensions

Any connected, locally $\mathbf{H}_0$ graph is isomorphic to a clique of size two. Furthermore, it is easily seen that any connected, locally $\mathbf{H}_1$ graph admits an infinite universal cover, so we obtain infinitely many connected graphs that are locally $\mathbf{H}_1$ but not isomorphic to $\mathbf{H}_2$. The case $n = 2$ proves to be a bit more complicated. We can only provide an example of a connected, locally $\mathbf{H}_2(2)$ graph that is not isomorphic to $\mathbf{H}_3(2)$. The proof of its existence is based on a computation with the computer algebra package GAP [Sch95]. We are indebted to Sergey Shpectorov for pointing out to us the technique used in the proof of the following theorem.

Theorem 1.4.1 (joint with Cohen, Cuypers)
There exists a connected graph on $128 \cdot 120$ vertices that is locally $\mathbf{H}_2(2)$.

Proof. We determine the stabilizers of a vertex, an edge, and a 3-clique of the graph $\mathbf{H}_3(2)$ inside the canonical group $(P)SL_4(2)$ and let GAP determine the order of the universal completion of the amalgam of these groups and their intersections. This universal completion is the group G with a presentation by the generators w,

u, b, a and the relations

$$\begin{aligned}
&w^2 = u^2 = b^2 = a^2 = 1,\\
&(wu)^3 = (ab)^3 = 1,\\
&(bw)^3 = (bu)^4 = 1,\\
&(wub)^7 = (wa)^2 = (ua)^2 = 1.
\end{aligned}$$

The stabilizers of a vertex, an edge, and a 3-clique of $\mathbf{H}_3(2)$, respectively, are of the form

$$\begin{aligned}
\langle w, u, b\rangle &\cong SL_3(2),\\
\langle w, u, a\rangle &\cong SL_2(2) \times C_2,\\
\langle a, b\rangle &\cong Sym_3,
\end{aligned}$$

with the intersections

$$\begin{aligned}
\langle w, u, b\rangle \cap \langle w, u, a\rangle = \langle w, u\rangle &\cong SL_2(2),\\
\langle w, u, a\rangle \cap \langle a, b\rangle = \langle a\rangle &\cong C_2,\\
\langle a, b\rangle \cap \langle w, u, b\rangle = \langle b\rangle &\cong C_2.
\end{aligned}$$

By GAP the order of G is $128 \cdot |SL_4(2)|$, and there exists a normal subgroup $N \cong 2^{1+6}$ of G. Hence $\mathbf{H}_3(2)$ admits a 128-fold universal cover Γ with the same local structure. □

This proposition shows that the bound on n in Theorem 1.3.21 is sharp. Besides the above universal cover of the canonical graph $\mathbf{H}_3(2)$ nothing is known about locally $\mathbf{H}_2(\mathbb{F})$ graphs. The methods that we have presented for $n \geq 3$ do not apply in this case.

1.5 Infinite dimensions

The only place where we ever need finite dimensions in our studies of the graphs $\mathbf{H}_n(\mathbb{F})$, of graphs that are isomorphic to $\mathbf{H}_n(\mathbb{F})$, and graphs that are locally $\mathbf{H}_n(\mathbb{F})$ is when discussing the duality automorphism $(p, H) \mapsto (H, p)$ of $\mathbf{H}_n(\mathbb{F})$. But we do not need this automorphism at all in our proof of Theorem 1.3.21. Actually it is responsible for a couple of problems we encounter during the course of the proof and have to avoid.

Let n be an *infinite* cardinal number and let $\mathbb{F}$ be a division ring. In analogy to Definition 1.1.1, let $\mathbf{H}_n(\mathbb{F})$ be the graph on the non-incident point-hyperplane pairs of $\mathbb{P}_n(\mathbb{F})$, the projective space of a vector space of dimension n over $\mathbb{F}$, with the mutual containment of the point of one pair in the hyperplane of another pair as incidence. Then there does not exist a duality automorphism on $\mathbf{H}_n(\mathbb{F})$, and Sections 1.2 and 1.3 imply the following.

Proposition 1.5.1
Let n be an infinite cardinal number, let $\mathbb{F}$ be a division ring, and let Γ be isomorphic to $\mathbf{H}_n(\mathbb{F})$. Then there exists an isomorphism between the interior projective space on Γ and the exterior projective space on $\mathbf{H}_n(\mathbb{F})$. □

Corollary 1.5.2
Let n be an infinite cardinal number, let $\mathbb{F}$ be a division ring, and let Γ be isomorphic to $\mathbf{H}_n(\mathbb{F})$. The interior projective space on Γ is isomorphic to $\mathbb{P}_n(\mathbb{F})$. □

Corollary 1.5.3
Let n be an infinite cardinal number, let $\mathbb{F}$ be a division ring, and let Γ be isomorphic to $\mathbf{H}_n(\mathbb{F})$. The automorphism group of Γ is isomorphic to $P\Gamma L_n(\mathbb{F})$, the full automorphism group of $\mathbb{P}_n(\mathbb{F})$. □

Theorem 1.5.4 (joint with Cohen, Cuypers)
Let n be an infinite cardinal number, let $\mathbb{F}$ be a division ring, and let Γ be a connected, locally $\mathbf{H}_n(\mathbb{F})$ graph. Then Γ is isomorphic to $\mathbf{H}_n(\mathbb{F})$. □

Now let n be an arbitrary cardinal number. If $\mathbb{F}$ contains more than two elements, there is a one-to-one correspondence between the set of point-hyperplane pairs of a projective space $\mathbb{P}_n(\mathbb{F})$ and the collection of subgroups of axial collineations of $P\Gamma L_{n+1}(\mathbb{F})$ with fixed center and fixed axis. If the axis contains the center, this group is isomorphic to the additive group of the underlying division ring and it is called a **transvection group** or **root group**. Otherwise, this group is isomorphic to the multiplicative group of the underlying division ring and is called a **reflection torus** (even if the group happens to be non-Abelian). The nontrivial elements of a reflection torus are called **reflections**. Of course, even for the field of two elements there is a one-to-one correspondence between the incident point-hyperplane pairs of $\mathbb{P}_n(\mathbb{F})$ and the root subgroups of $P\Gamma L_{n+1}(\mathbb{F})$. However, there are no reflections corresponding to the non-incident point-hyperplane pairs.

Proposition 1.5.5
Let $\mathbb{F} \neq \mathbb{F}_2$. The graph $\mathbf{H}_n(\mathbb{F})$ is isomorphic to the graph on the collection of reflection tori of the group $P\Gamma L_{n+1}(\mathbb{F})$ where two reflection tori are adjacent if and only if they are distinct and commute.

Proof. The required isomorphism is determined by the map sending a reflection torus to the point-hyperplane pair consisting of its center and its axis. The fact that two vertices of $\mathbf{H}_n(\mathbb{F})$ are adjacent if and only if the corresponding reflection tori commute is established as follows. Any axial collineation of $\mathbb{P}_n(\mathbb{F})$ with center p and axis H gets induced by a linear map $V \to V : x \mapsto x + \phi(x)c$ with $\phi \in V^*$, $\phi(c) \neq -1$ and $\langle c \rangle = p$ and $\ker \phi = H$ where V is a left vector space of dimension $n+1$ over $\mathbb{F}$. The axial collineation coming from the map $x \mapsto x + \phi(x)c$ is a non-identity reflection if and only if $\phi(c) \neq 0$. The elements of distinct reflection tori commute if and only if the center of one reflection is contained in the axis of the other and vice versa. For, let $V \to V : x \mapsto x + \phi(x)c$ and $V \to V : x \mapsto x + \psi(x)d$ be two linear maps of V that induce reflections of $\mathbb{P}_n(F)$ with $\phi(d) = 0 = \psi(c)$. Then we have the following chain of equalities: $(\psi\circ\phi)(x) = x+\phi(x)c+\psi(x+\phi(x)c)d = x+\phi(x)c+\psi(x)d+\psi(\phi(x)c)d = x+\phi(x)c+\psi(x)d = x+\phi(x)c+\psi(x)d+\phi(\psi(x)d)c = x+\psi(x)d+\phi(x+\psi(x)d)c = (\phi\circ\psi)(x)$. Conversely, $\psi\circ\phi = \phi\circ\psi$ implies $\psi(\phi(x)c)d = \phi(\psi(x)d)c$ for all $x \in V$, whence $\phi(d) = 0 = \psi(c)$, finishing the proof. □

In view of Proposition 1.5.5, our results are in accordance with the results of Arjeh Cohen, Hans Cuypers, and Hans Sterk in [CCS99] on groups generated by reflection tori. They prove that, with the exception of some counterexamples over small fields, the only irreducible subgroups of $GL_{n+1}(\mathbb{F})$ generated by reflection tori are the groups $R(V, W^*)$ with $V \cong \mathbb{F}^{n+1}$, $W^* \subseteq V^*$ and $\mathrm{Ann}(W^*) = 0$ generated by the reflections with $c \in V$ and $\phi \in W^*$. If $W^* = V^*$, the reflections generate the full finitary general linear group $FGL_{n+1}(\mathbb{F})$, i.e., the subgroup of $GL_{n+1}(\mathbb{F})$ consisting of all elements $g \in GL_{n+1}(\mathbb{F})$ with $[V, g] = \{g(v) - v \mid v \in V\}$ finite dimensional. The graph on the point-hyperplane pairs (p, H) with $p = \langle v\rangle$ for $v \in V$ and $H = \ker\phi$ for $\phi \in W^*$ is a subgraph of $\mathbf{H}_n(\mathbb{F})$. We do recover the whole graph (i.e., $W^* = V^*$), since we locally always have the whole graph.

1.6 Group-theoretic consequences

In this section we study group-theoretic consequences of our local recognition Theorem 1.3.21 of the point-hyperplane graphs $\mathbf{H}_n(\mathbb{F})$. We start with a consequence of the simple connectedness of those graphs and then give theorems whose proofs really need the full local recognition result.

Proposition 1.6.1
Let $n \geq 3$, and let $\mathbb{F}$ be a division ring. Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be mutually adjacent vertices of $\mathbf{H}_{n+1}(\mathbb{F})$, and consider the natural action of $G \cong PSL_{n+2}(\mathbb{F})$ on $\mathbf{H}_{n+1}(\mathbb{F})$. Define A as the stabilizer of $\mathbf{x}$ in G, B as the stabilizer of $\{\mathbf{x}, \mathbf{y}\}$ in G, and C as the stabilizer of $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ in G. Define $\mathcal{A}$ to be the amalgam of the groups A, B, and C together with their intersections as subgroups of G. Then G is the universal completion of $\mathcal{A}$.

Proof. The proposition follows from the flag-transitive action of G on $\mathbf{H}_{n+1}(\mathbb{F})$ together with Lemma 1.3.6 and Tits' Lemma B.2.5. □

Theorem 1.6.2 (joint with Cohen, Cuypers)
Let $n \geq 3$, and let $\mathbb{F}$ be a division ring. Let U_i, $1 \leq i \leq 3$, be distinct subgroups of some group U such that

- *$U = \langle U_1, U_2, U_3\rangle$;*
- *$U_i \cap U_j \not\subseteq U_k \cap U_l$ if $\{i, j\} \not\supseteq \{k, l\}$;*

- *the residue* $(U_3/U_1 \cap U_3, U_3/U_2 \cap U_3)$ *is isomorphic to the coset geometry* $(Sym_3/\langle(23)\rangle, Sym_3/\langle(12)\rangle)$*;*

- *the coset geometry* $(U/U_1, U/U_2)$ *is a graph (i.e.,* $[U_2 : U_1 \cap U_2] = 2$*);*

- *the residue* $(U_1/U_1 \cap U_2, U_1/U_1 \cap U_3)$ *is a graph that is isomorphic to* $\mathbf{H}_n(\mathbb{F})$*; and*

- *the action of* U_1 *on* $(U_1/U_1 \cap U_2, U_1/U_1 \cap U_3)$ *preserves the interior projective space (i.e.,* U_1 *modulo the kernel of its action on the residue can be embedded in* $P\Gamma L_{n+1}(\mathbb{F})$*).*

Denote by $\mathcal{U}$ *the amalgam of the groups* U_i*,* $1 \leq i \leq 3$*, and their intersections in* U*. Then the group* U *is the universal completion of* $\mathcal{U}$*. If, moreover,* N *is the kernel of the action of* U *on* $(U/U_1, U/U_2, U/U_3)$*, then the quotient amalgam* $\mathcal{U}/N := \bigcup_{1\leq i\leq 3} U_i/(N \cap U_i)$ *can be embedded in* $P\Gamma L_{n+2}(\mathbb{F})$ *and* $\langle \mathcal{U}/N\rangle \subset P\Gamma L_{n+2}(\mathbb{F})$ *is isomorphic to* U/N*.*

Proof. The group U acts flag-transitively on the pregeometry $(U/U_1, U/U_2, U/U_3)$. Indeed, U acts transitively on the cosets of U_3 and the residue of U_3 is isomorphic to a triangle on which U_3 acts as the group Sym_3, and, thus, we have proved flag-transitivity for flags that include a coset of U_3. If gU_1, hU_2 is a flag, then there exists an element $x \in gU_1 \cap hU_2$, whence $1 \in x^{-1}gU_1 \cap x^{-1}hU_2$, and $x^{-1}gU_1 = U_1$ and $x^{-1}hU_2 = U_2$, proving flag-transitivity. Moreover, $x^{-1}gU_1 = U_1$, $x^{-1}hU_2 = U_2$, U_3 is a flag. Therefore, the flag gU_1, hU_3 is contained in the chamber gU_1, hU_2, xU_3, which implies that $(U/U_1, U/U_2, U/U_3)$ is a geometry. The graph $(U/U_1, U/U_2)$ is locally the graph $(U_1/U_1 \cap U_2, U_1/U_1 \cap U_3)$—for U_3 describes the triangles in the graph $(U/U_1, U/U_2)$—which is isomorphic to $\mathbf{H}_n(\mathbb{F})$. As $U = \langle U_1, U_2, U_3\rangle$, the geometry $(U/U_1, U/U_2, U/U_3)$ is connected. By connectedness of the residue of U_3 this implies that the graph $(U/U_1, U/U_2)$ is connected, whence it is isomorphic to $\mathbf{H}_{n+1}(\mathbb{F})$ by Theorem 1.3.21. But now, as $\mathbf{H}_{n+1}(\mathbb{F})$ is simply connected, U is the universal completion of the amalgam $\mathcal{U}$ by Tits' lemma B.2.5. Corollaries 1.2.10 and 1.2.11 state that the full automorphism group of $\mathbf{H}_n(\mathbb{F})$ is of the form $P\Gamma L_{n+2}(\mathbb{F}).2$ or $P\Gamma L_{n+2}(\mathbb{F})$, the maximal part of which preserving the interior projective space is $P\Gamma L_{n+2}(\mathbb{F})$. The group U/N acts faithfully on $(U/U_1, U/U_2, U/U_3)$ and preserves the interior projective space on $(U/U_1, U/U_2)$, since $U_1/N \cap U_1$ does so on the residue of U_1. Therefore U/N can be embedded in $P\Gamma L_{n+2}(\mathbb{F})$. The last statement amounts to whether the quotient amalgam $\mathcal{U}/N$ spans U/N. But it does, since the coset geometry $(U/U_1, U/U_2, U/U_3)$ is connected, and N simply is the kernel of action of U on that coset geometry, which means the geometry $(U/U_1, U/U_2, U/U_3)$ is isomorphic to the geometry $(\frac{U}{N}/\frac{U_1}{N\cap U_1}, \frac{U}{N}/\frac{U_2}{N\cap U_2}, \frac{U}{N}/\frac{U_3}{N\cap U_3})$. Therefore the latter geometry is connected as well and $\mathcal{U}/N$ indeed spans U/N. □

We refer the reader to [CK79] for a list of subgroups of $\Gamma L(V)$, V a finite vector space, that act 2-transitively on the set of points of the corresponding finite projective space $\mathbb{P}(V)$ (cf. Theorem I) and for lists of subgroups of $\Gamma L(V)$ that act transitively on the set of non-incident point-hyperplane pairs (called *antiflags*; cf. Theorem II and Theorem III) of the finite projective space $\mathbb{P}(V)$. Notice that the groups arising in our Theorem 1.6.2 act 2-transitively on the set of points of the projective space. For, they act transitively on the set of ordered triangles of $\mathbf{H}_{n+1}(\mathbb{F})$, whence transitively on the set of (ordered) edges. But for every pair of points p, q of a projective space there exist hyperplanes H and I with $p \notin H \ni q$ and $q \notin I \ni p$, so transitivity on edges implies transitivity on points, and—in case of a finite projective space—we can apply Theorem I of [CK79]. We conclude that the finite groups with a faithful action arising from our theorem have to contain the group $PSL_{n+2}(q)$, as $n \geq 3$.

In view of Proposition 1.5.5 we can use Theorem 1.3.21 and Theorem 1.6.2 in order to locally recognize a group G that contains two commuting involutions which locally describe reflections and whose centralizers are central extensions of $PGL_{n+1}(\mathbb{F})$.

Theorem 1.6.3 (joint with Cohen, Cuypers)
Let $n \geq 3$, and let $\mathbb{F}$ be a field of characteristic distinct from 2. Let G be a group with distinct involutions x, y and subgroups $X \cong Y$ such that

- $C_G(x) = X \times K$ *with* $K \cong GL_{n+1}(\mathbb{F})$;
- $C_G(y) = Y \times J$ *with* $J \cong GL_{n+1}(\mathbb{F})$;
- x *is an involutive reflection of* J;
- y *is an involutive reflection of* K; *and*
- *there exists an involution in* $J \cap K$ *that is a reflection of both* J *and* K.

If $G = \langle J, K\rangle$, then $PSL_{n+2}(\mathbb{F}) \leq G/Z(G) \leq PGL_{n+2}(\mathbb{F})$.

Proof. Choose an involution $z \in J \cap K$ that is an involutive reflection in the groups J and K. Note that z commutes with x and y. The elements y and z are conjugate in K by an involution, whence they are conjugate in G. Similarly, x and z are conjugate in J by an involution. Therefore the conjugation action of the group G induces an action as the group Sym_3 on the set $\{x, y, z\}$ and as the group Sym_2 on the set $\{x, y\}$. Consider the graph Γ on all conjugates of x in G. A pair a, b of vertices of Γ is adjacent if there exists a vertex c and an element $g \in G$ such that $(gxg^{-1}, gyg^{-1}, gzg^{-1}) = (a, b, c)$. Since G induces the action of Sym_3 on $\{x, y, z\}$, this definition of adjacency is completely symmetric, and we have defined an undirected graph. The elements x, y, z form a 3-clique of Γ. Define U_1 as the stabilizer in G of the vertex x, define U_2 as the stabilizer in G of the edge $\{x, y\}$, and define U_3 as the stabilizer in G of the triangle $\{x, y, z\}$. The stabilizer of $\{x, y\}$ permutes x and y and therefore interchanges $C_G(x) \geq K$ and $C_G(y) \geq J$. Hence the stabilizer of x together with the stabilizer of $\{x, y\}$ generates G, as $G = \langle J, K\rangle \leq \langle U_1, U_2\rangle$. Consequently, the graph Γ is connected.

Also, Γ is locally $\mathbf{H}_n(\mathbb{F})$ by construction. To prove this, it is enough to show that any triangle in Γ is a conjugate of (x, y, z). Let (a, b, c) be a triangle, which means there exist vertices d, e, f of Γ such that (a, b, d), (a, c, e), and (b, c, f) are conjugates of (x, y, z) in G. Let $g \in G$ with $(gxg^{-1}, gyg^{-1}, gzg^{-1}) = (a, b, d)$. Notice that $b, d \in gKg^{-1}$ are commuting involutive reflections of gKg^{-1}. The triangles (a, b, d) and (a, c, e) are conjugate in $C_G(a) = gXg^{-1} \times gKg^{-1}$. Choose $h \in C_G(a)$ such that $(hah^{-1}, hbh^{-1}, hdh^{-1}) = (a, c, e)$. Then $h = h_X h_K$ with $h_X \in gXg^{-1}$, $h_K \in gKg^{-1}$. The element h_X centralizes b and d, since $b, d \in gKg^{-1}$. Therefore $c = hbh^{-1} = h_K b h_K^{-1} \in gKg^{-1}$ is an involutive reflection of gKg^{-1}. The elements x and y commute and so do b and c because the triangle (b, c, f) is conjugate to the triangle (x, y, z). Hence (a, b, d) and (a, b, c) are conjugate in gKg^{-1}. Therefore (a, b, c) and (x, y, z) are conjugate in G.

By Theorem 1.6.2 the group G is the universal completion of the amalgam on the groups U_i, $1 \leq i \leq 3$, and their intersections. Moreover, there exists a kernel N of the action of G on Γ such that G/N can be embedded in $P\Gamma L_{n+2}(\mathbb{F})$. Let $g \in N$. Then g acts trivially on Γ, in particular it centralizes x and y, so we have $g \in X \times K$ and $g \in Y \times J$. Let $g_X \in X$ and $g_K \in K$ be such that $g = g_X g_K$. The element g_X commutes with K, and therefore also centralizes all neighbors of x. Consequently, also $g_K = g_X^{-1} g$ centralizes all neighbors of x, and hence lies in the center of K. We have proved that g commutes with K. Similarly, g commutes with J. This implies that g commutes with $G = \langle J, K \rangle$, and, thus, $g \in Z(G)$. Certainly, $Z(G)$ acts trivially on Γ, whence $N = Z(G)$. □

Corollary 1.6.4
Let $n \geq 3$, and let $\mathbb{F}$ be a field of characteristic distinct from 2. Let G be a group with distinct involutions x, y, an element g, and a subgroup X such that

- $gxg^{-1} = y$ *and* $gyg^{-1} = x$;
- $C_G(x) = X \times K$ *with* $K \cong GL_{n+1}(\mathbb{F})$;
- y *is an involutive reflection of* K; *and*
- g *centralizes an involutive reflection of* K.

If $G = \langle K, g \rangle$, *then* $PSL_{n+2}(\mathbb{F}) \leq G/Z(G) \leq PGL_{n+2}(\mathbb{F})$. □

1.7 Non-singular points in polarities

So far we have exclusively studied graphs related to the graph $\mathbf{H}_n(\mathbb{F})$. Suppose now that the division ring $\mathbb{F}$ admits an involutory anti-automorphism so that $\mathbb{P}_n(\mathbb{F})$ admits a nondegenerate polarity π. Then we can study the subgraph of $\mathbf{H}_n(\mathbb{F})$ on the fixed elements under the automorphism induces by this polarity. This subgraph consists of precisely those non-incident point-hyperplane pairs (p, H) of $\mathbb{P}_n(\mathbb{F})$ with $p^\sigma = H$, in other words, the vertices (p, p^σ) for all non-singular points p. Since this subgraph is empty if π is a symplectic polarity, we can restrict ourselves to orthogonal and unitary polarities. There exists the following theorem by Hans Cuypers.

Theorem 1.7.1 (Cuypers [Cuyb], Theorem 1.1)
Let Γ be a connected graph on non-singular points of a nondegenerate polarity π of some Desarguesian projective space with $p \perp q$ if and only if $p \in q^\pi$, satisfying the following conditions for each of its vertices p:

(i) *the tangent space on $p^\perp$ is connected;*

(ii) *the polar space at infinity of the tangent space on $p^\perp$ has rank at least three;*

(iii) *tangent lines can be recovered inside $p^\perp$;*

(iv) *in the Veldkamp embedding of $p^\perp$ (with respect to the map π), any codimension two subspace meets $p^\perp$.*

Then Γ is locally recognizable.

Condition (i) of the theorem sometimes forces us not to consider the whole subgraph of $\mathbf{H}_n(\mathbb{F})$ of fixed elements under a polarity, but graphs on smaller sets of vertices. For example, if $\mathbb{F}$ is a finite field of odd order and π is an orthogonal polarity, then there exist non-singular points of $+$ type (all vectors of the point have a square value under the quadratic form) and of $-$ type (all non-zero vectors of the point have a non-square value under the quadratic form). If one considers the graph on all non-singular points, then the perp of a point will not admit a connected tangent space. However, if one restricts oneself to the graph on the non-singular points of $+$ type, respectively $-$ type, then the tangent space on the perp of a point will be connected, cf. [Cuyb] and [CP92].

In the following we will restrict ourselves to finite fields. Let q be a prime power. Denote by $\mathcal{NU}_n(q^2)$ the graph on the non-singular one-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_{q^2}$ endowed with a nondegenerate unitary polarity in which two vertices are adjacent if and only if they are perpendicular. Now let V be an n-dimensional vector space of $\mathbb{F}_q$ for q odd. If n is even, then there exist two non-isometric orthogonal forms on V. For each isometry type of quadratic forms, the tangent lines induce a tangent space on the non-singular points of V, which consists of two isomorphic connected components. Denote the graph on the points of one of those connected components with being perpendicular as adjacency by $\mathcal{NO}_n^\pm(q)$, where $+$ indicates that the quadratic form gives rise to a polar space of rank $\frac{n}{2}$, whereas $-$ stands for the quadratic form yielding a polar space of rank $\frac{n-2}{2}$. If n is odd, then there exists only one isometry class of orthogonal forms on V, but the tangent lines induce a tangent space on the non-singular points of V, which consists of two non-isomorphic connected components. Denote by $\mathcal{N}^+\mathcal{O}_n(q)$ the graph on the projective points of $+$ type (i.e., the points whose perp is a $+$ type orthogonal space), and denote by $\mathcal{N}^-\mathcal{O}_n(q)$ the graph on the projective points of $-$ type (i.e., the points whose perp is a $-$ type orthogonal space), with being perpendicular as adjacency. Notice that $\mathcal{NU}_n(q^2)$ is locally $\mathcal{NU}_{n-1}(q^2)$, whereas $\mathcal{N}^{\pm}\mathcal{O}_n(q)$ is locally $\mathcal{NO}^{\pm}_{n-1}(q)$ and $\mathcal{NO}^{\pm}_n(q)$ is locally $\mathcal{N}^{\mp}\mathcal{O}_{n-1}(q)$, see [Cuyb].

Corollary 1.7.2 (Cuypers [Cuyb], Corollary 1.2)

(i) *The graphs $\mathcal{NU}_n(q^2)$ are locally recognizable for $n \geq 7$.*

(ii) *The graphs $\mathcal{NO}^{\pm}_n(q)$ and $\mathcal{N}^{\pm}\mathcal{O}_n(q)$ with q odd are locally recognizable for $n \geq 8$.*

The above theorem and its corollary heavily depend on the article [CP92] by Hans Cuypers and Antonio Pasini. Combining both [CP92] and [Cuyb], it seems unlikely that one can forfeit Axioms (i) and (iii) of Theorem 1.7.1 without a new approach to the problem. This leaves Axioms (ii) and (iv). Of the graphs in the corollary, the only graph satisfying Axioms (i), (ii), and (iii), but not (iv) is the graph $\mathcal{N}^+\mathcal{O}_7(q)$ for odd q. Since there is a connected graph that is locally $\mathcal{NO}^+_6(3)$ but not isomorphic to $\mathcal{N}^+\mathcal{O}_7(3)$ (but a threefold cover of $\mathcal{N}^+\mathcal{O}_7(3)$ instead; cf. [BCN89]), there is only hope of improving Corollary 1.7.2 for $q \geq 5$. Application of Tits' lemma B.2.5, shows that Cuypers paved the way to results on universal completions of certain amalgams (in the flavor of the corollaries of Theorem 1.7.5) and local recognition results for certain groups (using the techniques developed in Section 1.6).

We will prove the simple connectedness of a connected graph Γ that is locally $\mathcal{NO}^+_6(q)$ for odd $q \geq 5$, which by the existence of the abovementioned three-fold cover of $\mathcal{N}^+\mathcal{O}_7(3)$ is the best we can hope to achieve. We will not prove a local recognition theorem for connected, locally $\mathcal{NO}^+_6(q)$ graphs, although we conjecture it is possible in 1.7.6. Notice that by [CP92], we can reconstruct the polar space $O^+_6(q)$ together with its natural embedding into a six-dimensional vector space over $\mathbb{F}_q$. If, in the sequel, we talk about polar spaces and singular elements on the perp $p^\perp$ of a vertex p of Γ, then we mean precisely this reconstructed polar space.

Lemma 1.7.3
Let $q \geq 5$ be odd, and let Γ be a connected, locally $\mathcal{NO}^+_6(q)$ graph. To any chain $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ of vertices in Γ there exists a vertex $\mathbf{a} \in \{\mathbf{w},\mathbf{y}\}^\perp$ with $\{\mathbf{a},\mathbf{w},\mathbf{y},\mathbf{z}\}^\perp \neq \emptyset$. Moreover, if $\mathbf{a} \neq \mathbf{x}$, then we can assume that, considered inside $\mathbf{y}^\perp$, the two-dimensional subspace spanned by $\mathbf{a}$ and $\mathbf{x}$ is nondegenerate.

Proof. Consider the perp $\mathbf{x}^\perp$. The vertex $\mathbf{y}$ corresponds to a non-singular projective point of the polar space on $\mathbf{x}^\perp$ and, thus, the perp $\mathbf{y}^\perp$ is a nondegenerate subspace of that polar space. Hence the totally singular subspaces of the five-dimensional space $Y := \mathbf{x}^\perp \cap \mathbf{y}^\perp$ have algebraic dimension at most two. Therefore, for $W := \mathbf{x}^\perp \cap \mathbf{w}^\perp$, the four-dimensional space $W \cap Y$ has rank at least two and we find non-singular projective points of $+$ type in $W \cap Y$. Moreover, through each such projective point $\mathbf{p}$ of $+$ type we can find three elliptic lines that span $W \cap Y$ (as a vector space).

Notice that $\mathbf{p}$ is adjacent to $\mathbf{w}$, $\mathbf{x}$, and $\mathbf{y}$. Interchanging the roles of $\mathbf{p}$ and $\mathbf{x}$ and thus considering $\mathbf{p}^\perp$, we also find three such elliptic lines through $\mathbf{x}$.

Now translate the setting from $\mathbf{x}^\perp$ to $\mathbf{y}^\perp$; especially the intersection $W \cap Y$ translates. We have the non-singular projective points $\mathbf{x}$ and $\mathbf{z}$ and their respective perps X and Z, which themselves are nondegenerate five-dimensional orthogonal spaces. They intersect in a four-dimensional space. Another four-dimensional subspace of X is the space $W \cap Y$. The intersection $A := W \cap X \cap Z$ has dimension at least three. If the rank of A is two or three, then we find a non-singular projective point of $+$ type in A and the lemma holds for the choice $\mathbf{a} := \mathbf{x}$.

We can assume that A has rank one. (Indeed, zero is impossible, because A has dimension three inside the nondegenerate five-dimensional space Z.) Therefore, A has a two-dimensional radical l. Varying the projective point $\mathbf{x}$ inside a three-dimensional space implies a variation of its perp X on a three-dimensional space (meaning that the intersection of all variations of X is a three-dimensional space). By the above choice of the elliptic lines on $\mathbf{x}$, there is at least one, k say, whose perp does not contain A. Consequently, when $\mathbf{x}$ moves on the line k, the perp X moves away from that intersection A.

It is not clear whether these variations of X contain the radical l of A or not. However, if they do, then the varied spaces $W \cap X \cap Z$ cannot also have rank one, as otherwise l has to be the radical of all of them and inside the nondegenerate five-dimensional space Z we find a four-dimensional space that is perpendicular to a two-dimensional space, a contradiction. Thus, if l lies in all variations of X, we have found an intersection $W \cap X \cap Z$ of rank two and we are done by the above.

Suppose now that l does not lie in all variations of $W \cap X \cap Z$. Then on the elliptic line k there are $\frac{q+1}{2} \geq 3$ non-isotropic projective points of $+$ type. So, we have three different choices $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ for $\mathbf{x}$ on k and obtain three different spaces $W \cap X_i \cap Z$, $1 \leq i \leq 3$. If one of these has rank at least two, we are done, so assume all three have rank one. Denote the three two-dimensional radicals by l_1, l_2, l_3. If all those three radicals lie in a three-dimensional space, then this three-dimensional space is totally singular, a contradiction to the fact that Z does not contain three-dimensional totally singular subspaces. But if they do not lie in a three-dimensional space, then they span a four-dimensional space, i.e., $X \cap Z$. The intersection of the three spaces $W \cap X_i \cap Z$, $1 \leq i \leq 3$, is two-dimensional and contains a non-singular projective point p. Indeed, if not, then any of $W \cap X_i \cap Z$ is totally singular, a contradiction. But now this projective point p is in the perp of all three radicals l_1, l_2, l_3 and lies in their span, also a contradiction. Choose a vertex $\mathbf{x}_i$, $1 \leq i \leq 3$, with the property that $W \cap X_i \cap Z$ has rank two and denote it by $\mathbf{a}$.

This vertex $\mathbf{a}$ satisfies the first statement of the lemma. If $\mathbf{a} \neq \mathbf{x}$, then $\mathbf{a}$ and $\mathbf{x}$ span the elliptic line k inside $\mathbf{y}^\perp$, and the lemma is proved. □

Lemma 1.7.4
Let $q \geq 5$ be odd, let Γ be a connected, locally $\mathcal{NO}_6^+(q)$ graph, and let $\mathbf{x}$, $\mathbf{y}$ be two vertices of Γ at mutual distance two. Moreover, suppose that there exists a

common neighbor $\mathbf{b}$ *of* $\mathbf{x}$, $\mathbf{y}$ *such that the projective line inside* $\mathbf{b}^\perp$ *spanned by* $\mathbf{x}$, $\mathbf{y}$ *is nondegenerate. Then the induced subgraph* $\{\mathbf{x},\mathbf{y}\}^\perp$ *of* Γ *is connected.*

Proof. It is enough to prove that every other vertex $\mathbf{a}$ adjacent to both $\mathbf{x}$ and $\mathbf{y}$ can be connected to $\mathbf{b}$ via a path in $\{\mathbf{x},\mathbf{y}\}^\perp$. Consider $\mathbf{b}^\perp$. The vertex $\mathbf{x}$ corresponds to a non-singular projective point, likewise $\mathbf{y}$. Their respective perps X and Y are nondegenerate as is their intersection $X \cap Y$ of dimension four. Indeed, $X \cap Y$ is the perp of the nondegenerate two-dimensional space spanned by $\mathbf{x}$ and $\mathbf{y}$.

The intersection $\mathbf{x}^\perp \cap \mathbf{a}^\perp$ corresponds to another four-dimensional subspace $A \cap X$ of X. But since $X \cap Y$ is nondegenerate, the intersection $A \cap X \cap Y$ cannot have rank less than two. But then $A \cap X \cap Y$ contains a non-singular projective point of $+$ type, and we have found a common neighbor of $\mathbf{x}$, $\mathbf{y}$, $\mathbf{a}$, $\mathbf{b}$. □

Theorem 1.7.5
Let $q \geq 5$ *be odd, and let* Γ *be a connected, locally* $\mathcal{NO}_6^+(q)$ *graph. Then* Γ*, viewed as a two-dimensional simplicial complex whose two-simplices are its triangles, is simply connected and its diameter is two.*

Proof. This follows from a combination of Lemma 1.7.3 and Lemma 1.7.4. □

When proving Theorem 1.7.1, Hans Cuypers also proves a result on the simple connectedness of the local graphs, using Axiom (iv) instead of combinatorial arguments. We believe that it is possible to adjust the remainder of Cuypers' proof as well in order to obtain a local recognition result.

Conjecture 1.7.6
Let $q \geq 5$ *be odd, and let* Γ *be a connected, locally* $\mathcal{NO}_6^+(q)$ *graph. Then* Γ *is isomorphic to* $\mathcal{N}^+\mathcal{O}_7(q)$.

Theorem 1.7.5 implies a number of corollaries.

Corollary 1.7.7 (of Theorem 1.7.5)
Let $q \geq 5$ *be odd. Consider the natural action of* $PSO_7(q)$ *on* $\mathcal{N}^+\mathcal{O}_7(q)$*. Let* $\mathcal{A}$ *be the amalgam of the (setwise) stabilizers in* $PSO_7(q)$ *of a vertex, an edge, and a triangle of* $\mathcal{N}^+\mathcal{O}_7(q)$ *and all their intersections. Then* $PSO_7(q)$ *is the universal completion of* $\mathcal{A}$.

Proof. This follows from Theorem 1.7.5 and Tits' lemma B.2.5. □

Corollary 1.7.8 (of Theorem 1.7.5)
Let $q \geq 5$ *be odd. Suppose* G *is a group that contains an amalgam* $\mathcal{B}$ *of three groups* B_1, B_2, B_3 *and all their intersections such that*

- $G = \langle \mathcal{B} \rangle$;
- $B_i \cap B_j \not\subseteq B_k \cap B_l$ *if* $\{i,j\} \not\supseteq \{k,l\}$;

- *the residue* $(B_3/B_1 \cap B_3, B_3/B_2 \cap B_3)$ *is isomorphic to the coset geometry* $(Sym_3/\langle(23)\rangle, Sym_3/\langle(12)\rangle)$;
- *the coset geometry* $(G/B_1, G/B_2)$ *is a graph (i.e.,* $[B_2 : B_1 \cap B_2] = 2$*); and*
- $(B_1/B_1 \cap B_2, B_1/B_1 \cap B_3)$ *is a graph (i.e.,* $[B_1 \cap B_3 : B_1 \cap B_2 \cap B_3] = 2$*) that is isomorphic to* $\mathcal{NO}_6^+(q)$.

Then G *is the universal completion of* $\mathcal{B}$.

Proof. The proof is an analog of the proof of Theorem 1.6.2. □

If Conjecture 1.7.6 were true, then—in the flavor of Theorem 1.6.2—one could extend Corollary 1.7.8 by the statement that the amalgam $\mathcal{B}$ and the group G (modulo a kernel, if necessary) are embeddable in $P\Gamma O_7(q)$ and $\langle\mathcal{B}\rangle$ inside $P\Gamma O_7(q)$ equals G. Of course, there exist such corollaries of Theorem 1.7.1 and Corollary 1.7.2. Also notice the versions of Theorem 1.6.3 that arise in the present context.

In [Cuy92] one can see how to reconstruct the polar space from the graphs $\mathcal{NO}_6^-(q)$ and $\mathcal{NU}_5(q^2)$. With this information we obtain the following theorems.

Theorem 1.7.9
Let $q \geq 5$ *be odd, and let* Γ *be a connected, locally* $\mathcal{NO}_6^-(q)$ *graph. Then* Γ*, viewed as a two-dimensional simplicial complex whose two-simplices are its triangles, is simply connected and its diameter is two.*

Proof. [Cuy92] allows us to reconstruct the polar spaces from the local graphs. Lemmas 1.7.3 and 1.7.4 then prove the theorem. □

Theorem 1.7.10
Let $q \geq 3$*, and let* Γ *be a connected, locally* $\mathcal{NU}_5(q^2)$ *graph. Then* Γ*, viewed as a two-dimensional simplicial complex whose two-simplices are its triangles, is simply connected and its diameter is two.*

Proof. Again by [Cuy92] we can reconstruct the polar spaces from the local graphs. Slight variations of Lemmas 1.7.3 and 1.7.4 prove the theorem. □

The graph $\mathcal{NU}_6(2^2)$ admits a two-fold and a four-fold cover that are both locally $\mathcal{NU}_5(2^2)$ (cf. [BCN89]), whence the bound on q in the preceding theorem is sharp.

Chapter 2

Line-Hyperline Pairs

The contents of this chapter are very similar to those of the preceding chapter. Instead of studying graphs on non-incident point-hyperplane pairs, this chapter is devoted to the investigation of graphs on non-intersecting line-hyperline pairs in which two line-hyperline pairs are adjacent if and only if the line of one pair is contained in the hyperline of the other and vice versa. Naturally, the results are very similar to the ones of the preceding chapter. Nevertheless, we include complete proofs of the results in high dimensions. Theorems for lower dimensions are simply stated with some hints how to overcome the difficulties that are encountered.

As is the case for non-incident point-hyperplane pairs, there is a group-theoretic interpretation of the graphs we are studying. For, there is a one-to-one correspondence of the non-intersecting line-hyperline pairs of the projective space $\mathbb{P}_n(\mathbb{F})$ and the fundamental SL_2's of the group $PSL_{n+1}(\mathbb{F})$ for sufficiently large n, cf. Section 4.1. The adjacency relation of the line-hyperline graph coincides with the commutation relation on the fundamental SL_2's.

In this context it is not surprising that one can deduce a nice geometric characterization of the hyperbolic root group geometry of $PSL_{n+1}(\mathbb{F})$, i.e., the geometry on the root subgroups of $PSL_{n+1}(\mathbb{F})$ as points and the fundamental SL_2's of $PSL_{n+1}(\mathbb{F})$ as lines, from the local recognition theorems of this chapter. For those characterizations the reader is referred to Chapter 4.

2.1 Line-hyperline graphs of projective spaces

Definition 2.1.1 Let $n \in \mathbb{N}$ and let $\mathbb{F}$ be a division ring. Consider the projective space $\mathbb{P}_n(\mathbb{F})$ of (projective) dimension n over $\mathbb{F}$. The **line-hyperline graph** $\mathbf{L}(\mathbb{P}_n(\mathbb{F})) = \mathbf{L}_n(\mathbb{F})$ of $\mathbb{P}_n(\mathbb{F})$ is the graph whose vertices are the non-intersecting line-hyperline pairs of $\mathbb{P}_n(\mathbb{F})$ and in which one vertex (a, A) is adjacent to another vertex (b, B) (in symbols, $(a, A) \perp (b, B)$) if and only if $a \subseteq B$ and $b \subseteq A$.

For a vertex $\mathbf{x} \in \mathbf{L}_n(\mathbb{F})$, we write $\mathbf{x}^\perp$ to denote the set of all vertices of $\mathbf{L}_n(\mathbb{F})$ at distance one from $\mathbf{x}$. Moreover, for a set X of vertices, define the **perp of** X as

$X^{\perp}:= \bigcap_{\mathbf{x}\in X} \mathbf{x}^{\perp}$, with the understanding that $\emptyset^{\perp} = \mathbf{L}_n(\mathbb{F})$, and the **double perp of** X as $X^{\perp\perp}:= (X^{\perp})^{\perp}$. We also write $\mathbf{L}_n$ instead of $\mathbf{L}_n(\mathbb{F})$ if $\mathbb{F}$ is obvious or not relevant.

The projective space $\mathbb{P}_n(\mathbb{F})$ induces a Grassmann space of lines on $\mathbf{L}_n(\mathbb{F})$, i.e., the shadow space on the rank one spaces (i.e., lines) of the projective geometry of rank n over $\mathbb{F}$. The points of this Grassmann space are of the form $v_l = \{(a, A) \in \mathbf{L}_n(\mathbb{F}) \mid a = l\}$ for a line l. A typical line of the Grassmann space is of the form $v_{p,\pi} = \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid p \in l \in \pi\}$ for an incident point-plane pair (p, π). Dually, the point-line geometry on points of the form $v_L = \{(a, A) \in \mathbf{L}_n(\mathbb{F}) \mid A = L\}$ for a hyperline L and lines of the form $v_{K,H} = \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid K \subseteq L \subseteq H\}$ for an incident pair (K, H) of an $(n-3)$-space K and a hyperplane H is also a Grassmann space. The geometrical objects v_l and v_L defined above are called **exterior lines** and **exterior hyperlines**, respectively. If $\mathbb{F} \cong \mathbb{F}^{\mathrm{opp}}$, then there exists a duality δ mapping $\mathbb{P}_n(\mathbb{F})$ to $\mathbb{P}_n(\mathbb{F})^{\mathrm{dual}}$ which induces a graph automorphism $(l, L) \mapsto (\delta(L), \delta(l))$ on $\mathbf{L}_n(\mathbb{F})$ mapping one Grassmann space onto the other. Note that even if $\mathbb{P}_n(\mathbb{F})$ and $\mathbb{P}_n(\mathbb{F})^{\mathrm{dual}}$ are non-isomorphic, still $\mathbf{L}(\mathbb{P}_n(\mathbb{F})) \cong \mathbf{L}(\mathbb{P}_n(\mathbb{F})^{\mathrm{dual}})$ by the map $(l, L) \mapsto (L, l)$.

One can define another point-line geometry on $\mathbf{L}_n(\mathbb{F})$. The lines of that geometry are the exterior lines, and the points are the full line pencils of exterior lines, i.e., a point is of the form $v_p = \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid p \in l\}$ for a point p of $\mathbb{P}_n(\mathbb{F})$. Incidence is symmetrized containment. Note that this point-line geometry is isomorphic to $\mathbb{P}_n(\mathbb{F})$. The points v_p are called **exterior points**, the resulting point-line geometry the **exterior projective space**. Dually, define **exterior hyperplanes** and the resulting **dual exterior projective space**.

One goal of this chapter is to obtain a characterization of a graph $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ as being connected and locally $\mathbf{L}_{n-2}(\mathbb{F})$. In this light, the following two propositions are important.

Proposition 2.1.2
Let $n \geq 3$. The graph $\mathbf{L}_n(\mathbb{F})$ is locally $\mathbf{L}_{n-2}(\mathbb{F})$.

Proof. Let $\mathbf{x} = (x, X)$ be a vertex of $\mathbf{L}_n(\mathbb{F})$. Then $X \cong \mathbb{P}_{n-2}(\mathbb{F})$. After we identify X with $\mathbb{P}_{n-2}(\mathbb{F})$ by means of this isomorphism, we establish an isomorphism $\mathbf{x}^{\perp} \cong \mathbf{L}_{n-2}(\mathbb{F})$. Indeed, for any vertex $\mathbf{y} = (y, Y)$ adjacent to $\mathbf{x}$, we have $x \subseteq Y$, $y \subseteq X \setminus (X \cap Y)$, and $\dim(X \cap Y) = n - 4$, so $(y, X \cap Y)$ belongs to $\mathbf{L}(X) \cong \mathbf{L}_{n-2}(\mathbb{F})$. Conversely, for any vertex of $\mathbf{L}(X)$, i.e., for any non-intersecting pair (z, Z) consisting of a line z and an $(n-4)$-space Z of $\mathbb{P}_n(\mathbb{F})$ with $z \subseteq X$, $Z \subseteq X$, the pair $(z, \langle Z, x\rangle)$ is a vertex of $\mathbf{x}^{\perp}$. (Indeed, $z \cap \langle Z, x\rangle = \emptyset$, since $x \cap X = \emptyset$.) Clearly, the maps $(y, Y) \mapsto (y, X \cap Y)$ and $(z, Z) \mapsto (z, \langle Z, x\rangle)$ are each other's inverses. Moreover, these maps preserve adjacency, whence the claim. $\square$

Proposition 2.1.3
$\mathbf{L}_1$ consists of precisely one point; $\mathbf{L}_2$ is the disjoint union of singletons; $\mathbf{L}_3$ is the disjoint union of cliques of size two; the graphs $\mathbf{L}_4$, $\mathbf{L}_5$, and $\mathbf{L}_6$ are connected; the diameter of $\mathbf{L}_n$, $n \geq 7$, equals two.

Proof. The first four statements are obvious. Let (x, X), (y, Y) be two non-adjacent vertices of $\mathbf{L}_n$, $n \geq 7$. The intersection $X \cap Y$ has dimension at least three. Since $x \cap X = \emptyset$ and $y \cap Y = \emptyset$, the intersection $\langle x, y\rangle \cap X \cap Y$ has at most (projective) dimension one, hence we can find a line $z \subseteq (X \cap Y) \setminus \langle x, y\rangle$. Moreover the dimension of $\langle x, y\rangle$ is at most three, and there is a hyperline $Z \supseteq \langle x, y\rangle$ with $z \cap Z = \emptyset$. □

Later, when dealing with graphs isomorphic to $\mathbf{L}_n$ or locally $\mathbf{L}_n$, many constructions will depend on investigation of double perps of two vertices. Therefore it is important to observe the following.

Lemma 2.1.4
Let $n \geq 4$. Let $\mathbf{x} = (x, X)$, $\mathbf{y} = (y, Y)$ be two vertices of $\mathbf{L}_n$ with $\{\mathbf{x}, \mathbf{y}\}^\perp \neq \emptyset$. Then the double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ equals the set of vertices $\mathbf{z} = (z, Z)$ of $\mathbf{L}_n$ with $z \subseteq \langle x, y\rangle$ and $Z \supseteq X \cap Y$.

Proof. The vertices of $\{\mathbf{x}, \mathbf{y}\}^\perp$ are precisely the non-intersecting line-hyperline pairs (a, A) with $a \subseteq X \cap Y$ and $A \supseteq \langle x, y\rangle$. Let $\left\{(a_i, A_i) \in \{\mathbf{x}, \mathbf{y}\}^\perp \mid i \in I\right\}$ be the set of all these vertices, indexed by some set I. Now $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp} = (\{\mathbf{x}, \mathbf{y}\}^\perp)^\perp$ consists of precisely those vertices (z, Z) with $z \subseteq \bigcap_{i \in I} A_i$ and $Z \supseteq \langle (a_i)_{i \in I}\rangle$. But since $\{\mathbf{x}, \mathbf{y}\}^\perp \neq \emptyset$, we have $\bigcap_{i \in I} A_i = \langle x, y\rangle$ and $\langle (a_i)_{i \in I}\rangle = X \cap Y$. □

The following two lemmas will be used in proofs and constructions of later sections.

Lemma 2.1.5
Let $n \geq 5$, and let $\mathbf{x} = (x, X)$ and $\mathbf{y} = (y, Y)$ be adjacent vertices in $\mathbf{L}_n$. If $\mathbf{x}$ is adjacent to a vertex (a, A_1) and $\mathbf{y}$ is adjacent to a vertex (a, A_2), then there exists a vertex (a, A_3) adjacent to both $\mathbf{x}$ and $\mathbf{y}$.

Proof. We have $a \subseteq X \cap Y$. Since $\mathbf{x}$ and $\mathbf{y}$ are adjacent, $x \subseteq Y$ and $y \subseteq X$ do not intersect and $\langle x, y\rangle$ is a 3-space. Moreover, $\langle x, y\rangle \cap a = \emptyset$. For, $a \cap x = \emptyset = a \cap y$, because $x \cap X = \emptyset = y \cap Y$ and $a \subseteq X \cap Y$. Therefore any intersection $\langle x, y\rangle \cap a$ is off x and y, but $\langle x, a\rangle \subseteq Y$ and $y \cap Y = \emptyset$ imply $\langle x, y\rangle \cap a = \emptyset$. Hence a choice of a hyperline A_3 that contains $\langle x, y\rangle$ and does not intersect a is possible. We have found a vertex (a, A_3) adjacent to both $\mathbf{x}$ and $\mathbf{y}$. □

Lemma 2.1.6
Let $n \geq 4$. Let (l, L) be a vertex of $\mathbf{L}_n$, and let p be a point on l. Furthermore, let (x, X) be a vertex of $\mathbf{L}_n$ adjacent to (l, L). Then there exists a vertex (m, M) adjacent to (x, X) such that l and m are distinct and intersect in p.

Proof. X is a hyperline of $\mathbb{P}_n$ containing l. Hence also $p \in X$. Since $n \geq 4$, the hyperline X contains a plane and we can find a line $m \subseteq X$ distinct from l that contains p. There certainly exists a hyperline M of $\mathbb{P}_n$ that contains x and does not intersect m, whence the claim is proved. $\square$

The rest of this section is dedicated to the development of means to recover the natural projective spaces on a graph Γ isomorphic to $\mathbf{L}_n$ without making use of a particular isomorphism and coordinates. To this end, we have to understand how two vertices $\mathbf{x} = (x, X)$ and $\mathbf{y} = (y, Y)$ of $\mathbf{L}_n$ can lie relative to each other and need to describe these *relative positions* by information contained in the graph Γ rather than by using coordinates. First of all, let us introduce precise terminology.

Definition 2.1.7 Two vertices $\mathbf{x} = (x, X)$ and $\mathbf{y} = (y, Y)$ of $\mathbf{L}_n$ are **in relative position** (i, j), if

$$i = \dim \langle x, y \rangle \text{ and } j = \operatorname{codim}(\mathrm{X} \cap \mathrm{Y}),$$

where dim denotes the projective dimension and codim the projective codimension. Let $\mathbf{x}$, $\mathbf{y}$ be distinct vertices of $\mathbf{L}_n$ with $\{\mathbf{x}, \mathbf{y}\}^\perp \neq \emptyset$. The double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is called n**th minimal** if there exist vertices $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{a}_i \neq \mathbf{b}_i$, $1 \leq i \leq n$, with $\{\mathbf{a}_i, \mathbf{b}_i\}^\perp \neq \emptyset$ for all i and $\{\mathbf{a}_1, \mathbf{b}_1\}^{\perp\perp} \subsetneq \cdots \subsetneq \{\mathbf{a}_n, \mathbf{b}_n\}^{\perp\perp} = \{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ and there does not exist a longer chain of strict inclusions.

It is quite clear that vertices $\mathbf{x}$ and $\mathbf{y}$ of $\mathbf{L}_n$ can only be in relative positions $(1, 1)$, $(1, 2)$, $(2, 1)$, $(2, 2)$, $(1, 3)$, $(3, 1)$, $(2, 3)$, $(3, 2)$, or $(3, 3)$. The following three lemmas will provide us with information on how to distinguish, up to duality, all cases from each other without having to use coordinates. However, it will not be possible to distinguish the case (i, j) from the case (j, i), which was to be expected from the existence of the isomorphism

$$\mathbf{L}(\mathbb{P}_n(\mathbb{F})) \cong \mathbf{L}(\mathbb{P}_n(\mathbb{F})^{\mathrm{dual}}) : (l, L) \mapsto (L, l).$$

Lemma 2.1.8
Let $n \geq 4$, and let $\mathbf{x}$, $\mathbf{y}$ be vertices of $\mathbf{L}_n$. Then the following hold true.

(i) *$\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1, 1)$ if and only if they are equal.*

(ii) *$\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1, 2)$ or $(2, 1)$ if and only if they are distinct, the perp $\{\mathbf{x}, \mathbf{y}\}^\perp$ is non-empty, and the double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is first minimal.*

(iii) *$\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1, 3)$, $(3, 1)$, $(2, 2)$, $(2, 3)$, $(3, 2)$, or $(3, 3)$ if and only if they are distinct and either the perp $\{\mathbf{x}, \mathbf{y}\}^\perp$ is empty or the double perp $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is not first minimal.*

Proof. The first statement is obvious. Let the relative position of $\mathbf{x}$ and $\mathbf{y}$ be $(1, 2)$ or $(2, 1)$. Then $\{\mathbf{x}, \mathbf{y}\}^\perp \neq \emptyset$. Indeed, suppose $\mathbf{x} = (x, X)$ and $\mathbf{y} = (y, Y)$ are in relative position $(1, 2)$. We have $x = y$, since x and y span a line. The intersection $X \cap Y$ contains an $(n-3)$-space (a space of projective codimension 2), which is at least a line since $n \geq 4$. So there exists a common neighbor of $\mathbf{x}$ and $\mathbf{y}$. The relative position $(2,1)$ is handled in the same way. Now let $\mathbf{a}$, $\mathbf{b}$ be distinct vertices contained in $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$. By Lemma 2.1.4, $\mathbf{a}$ and $\mathbf{b}$ are in relative position $(1,2)$ or $(2,1)$ and, thus, $\{\mathbf{a},\mathbf{b}\}^{\perp} \neq \emptyset$. Again by Lemma 2.1.4, the double perps $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ and $\{\mathbf{a},\mathbf{b}\}^{\perp\perp}$ coincide. If $\mathbf{x}$ and $\mathbf{y}$ are in any other relative position and $\{\mathbf{x},\mathbf{y}\}^{\perp}$ is empty, then we are done. So let us assume $\{\mathbf{x},\mathbf{y}\}^{\perp} \neq \emptyset$. Then the double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is given by Lemma 2.1.4 and it follows immediately that it contains vertices $\mathbf{a}$ and $\mathbf{b}$ in relative position $(1,2)$ or $(2,1)$. But, again by Lemma 2.1.4, this gives rise to a strictly smaller double perp. Hence $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is not minimal. Since we have listed all possible relative positions two vertices can be in, Statements (ii) and (iii) follow. □

Lemma 2.1.9
Let $n \geq 5$, and let $\mathbf{x}$ and $\mathbf{y}$ be vertices of $\mathbf{L}_n$ in relative position $(1,3)$ or $(3,1)$. Then $\{\mathbf{x},\mathbf{y}\}^{\perp} \neq \emptyset$.

Proof. We will prove the claim for the relative position $(1,3)$, and the proof for $(3,1)$ follows by duality. Let $\mathbf{x} = (x,X)$ and $\mathbf{y} = (y,Y)$. We have $x = y$, since x and y span a line. The intersection $X \cap Y$ contains an $(n-4)$-space (a space of projective codimension 3), which is at least a line since $n \geq 5$. So there exists a common neighbor of $\mathbf{x}$ and $\mathbf{y}$. □

Lemma 2.1.10
Let $n \geq 5$, and let $\mathbf{x}$ and $\mathbf{y}$ be vertices of $\mathbf{L}_n$. The property '$\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1,3)$ or $(3,1)$' is characterized by

- *the perp $\{\mathbf{x},\mathbf{y}\}^{\perp}$ is non-empty,*
- *the double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is second minimal, and*
- *there do not exist vertices $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ with $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{c} \neq \mathbf{d}$ such that $\{\mathbf{a},\mathbf{b}\}^{\perp\perp} \cap \{\mathbf{c},\mathbf{d}\}^{\perp\perp} = \emptyset$.*

Proof. Let $\mathbf{x}$ and $\mathbf{y}$ be in relative position $(3,1)$. By Lemma 2.1.9, the perp $\{\mathbf{x},\mathbf{y}\}^{\perp}$ is non-empty. The double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is described by Lemma 2.1.4. From that description it is obvious that $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is second minimal. Now let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ be vertices as stated in the present lemma. By Lemma 2.1.4, the vertices $\mathbf{a}$ and $\mathbf{b}$, respectively $\mathbf{c}$ and $\mathbf{d}$, can only be in relative positions $(2,1)$ or $(3,1)$. But then Lemma 2.1.8 and Lemma 2.1.9 show that both $\{\mathbf{a},\mathbf{b}\}^{\perp} \neq \emptyset$ and $\{\mathbf{c},\mathbf{d}\}^{\perp} \neq \emptyset$. There is a common vertex in $\{\mathbf{a},\mathbf{b}\}^{\perp\perp}$ and $\{\mathbf{c},\mathbf{d}\}^{\perp\perp}$ if one pair is in relative position $(3,1)$. So suppose both are in relative position $(2,1)$. Let $\mathbf{a} = (a,A)$, $\mathbf{b} = (b,B)$, $\mathbf{c} = (c,C)$, $\mathbf{d} = (d,D)$. We have $A = B = C = D$, since $\mathbf{x}$ and $\mathbf{y}$ are at relative position $(3,1)$, cf. Lemma 2.1.4. Moreover, both a, b and c, d span a plane inside a

3-space, by Lemma 2.1.4. But these two planes have to have a line in common, and we have found a common vertex of $\{\mathbf{a},\mathbf{b}\}^{\perp\perp}$ and $\{\mathbf{c},\mathbf{d}\}^{\perp\perp}$. By duality, two vertices in relative position $(1,3)$ have the same properties. Conversely, let $\mathbf{x}$ and $\mathbf{y}$ be in any relative position. Suppose $\{\mathbf{x},\mathbf{y}\}^{\perp} \neq \emptyset$. Then another application of Lemma 2.1.4 shows, that $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ only can be second minimal if $\mathbf{x}$ and $\mathbf{y}$ are in relative position $(1,3)$, $(3,1)$, or $(2,2)$. But if they are in relative position $(2,2)$, then we can find vertices $\mathbf{a}=(a,A)$, $\mathbf{b}=(b,B)$ in relative position $(1,2)$ and $\mathbf{c}=(c,C)$, $\mathbf{d}=(d,D)$ in relative position $(1,2)$ contained in $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ and such that $\{\mathbf{a},\mathbf{b}\}^{\perp\perp}\cap\{\mathbf{c},\mathbf{d}\}^{\perp\perp}=\emptyset$. (Note that $\{\mathbf{a},\mathbf{b}\}^{\perp}\neq\emptyset$ and $\{\mathbf{c},\mathbf{d}\}^{\perp}\neq\emptyset$ by Lemma 2.1.8.) Indeed, we have $a=b$ and $c=d$. But since we can choose both $a=b$ and $c=d$ freely in a plane, they only have to intersect in a point, and we have $\{\mathbf{a},\mathbf{b}\}^{\perp\perp}\cap\{\mathbf{c},\mathbf{d}\}^{\perp\perp}=\emptyset$. □

The preceding lemmas essentially provide us with means to distinguish the different cases of relative position, which will allow for a coordinate-free definition of lines. On the other hand the following two lemmas will help to recover the full line pencils, i.e., points.

Lemma 2.1.11
Let $n\geq 5$. Let k, l, and m be distinct exterior lines of $\mathbf{L}_n(\mathbb{F})$. They intersect in a common exterior point (i.e., they are contained in a line pencil), if there exist vertices $\mathbf{a}\in k$, $\mathbf{b}\in l$, $\mathbf{c}\in m$ that are pairwise in relative position $(2,1)$ such that $\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$ contains vertices $\mathbf{x}$, $\mathbf{y}$ in relative position $(3,1)$ with $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}=\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$.

Proof. Suppose $\mathbf{a}=(k,K)$, $\mathbf{b}=(l,L)$, $\mathbf{c}=(m,M)$ with $K=L=M$. The lines k, l, m mutually intersect each other, since (k,K), (l,L), and (m,M) are in mutual position $(2,1)$. But, by Lemma 2.1.4, the lines k, l, and m together span a projective 3-space, because $\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$ contains vertices $\mathbf{x}$, $\mathbf{y}$ in relative position $(3,1)$ with $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}=\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$. The claim now follows because three mutually intersecting lines that span a 3-space necessarily intersect in one point. □

The purpose of the following lemma is to ensure, for every point p of the projective space $\mathbb{P}_n$, the existence of vertices and exterior lines as in the hypothesis of Lemma 2.1.11. Notice that, for any such point p of $\mathbb{P}_n$, one can find two vertices $\mathbf{a}$, $\mathbf{b}$ of $\mathbf{L}_n$ as in Lemma 2.1.12, such that the corresponding lines intersect in that given point.

Lemma 2.1.12
Let $n\geq 5$, and let $\mathbf{a}$ and $\mathbf{b}$ be vertices in relative position $(2,1)$. Then there exists a third vertex $\mathbf{c}$ in relative position $(2,1)$ to both $\mathbf{a}$ and $\mathbf{b}$ such that $\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$ contains vertices $\mathbf{x}$, $\mathbf{y}$ in relative position $(3,1)$ with $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}=\{\mathbf{a},\mathbf{b},\mathbf{c}\}^{\perp\perp}$.

Proof. Suppose $\mathbf{a}=(a,A)$, $\mathbf{b}=(b,B)$ with $A=B$. The lines a and b intersect in a point, p say. Let q be a point outside the plane $\langle a,b\rangle$ such that the line pq does not intersect the hyperline A. The vertex $\mathbf{c}=(pq,A)$ has the required properties. □

2.2 The interior projective space

This section will concentrate on graphs Γ that are isomorphic to $\mathbf{L}_n(\mathbb{F})$. One can define a projective space on Γ by fixing an isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ that induces the exterior projective space of $\mathbf{L}_n(\mathbb{F})$ on Γ. However, when studying graphs that are locally $\mathbf{L}_n(\mathbb{F})$ one might not always want to have to choose an isomorphism, whence it is useful to recover this projective spaces on $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ from the graph Γ without making use of the coordinization of $\mathbf{L}_n(\mathbb{F})$. Up to duality this is what will be achieved in this section. The construction of the projective spaces on Γ heavily relies on the concept of relative position in $\mathbf{L}_n(\mathbb{F})$ and its characterizations in terms of double perps from the preceding section. Abusing notation, we will sometimes speak of relative positions on Γ, but *only if we have fixed a particular isomorphism* $\Gamma \cong \mathbf{L}_n(\mathbb{F})$.

Definition 2.2.1 Let $n \geq 5$. Define a reflexive relation $\approx$ on the vertex set of a graph $\Gamma \cong \mathbf{L}_n$. For distinct vertices $\mathbf{x}$, $\mathbf{y}$ with $\{\mathbf{x},\mathbf{y}\}^\perp \neq \emptyset$ we have $\mathbf{x} \approx \mathbf{y}$ if

- the double perp $\{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ is first or second minimal, and
- there do not exist vertices $\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d} \in \{\mathbf{x},\mathbf{y}\}^{\perp\perp}$ with $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{c} \neq \mathbf{d}$ such that $\{\mathbf{a},\mathbf{b}\}^{\perp\perp} \cap \{\mathbf{c},\mathbf{d}\}^{\perp\perp} = \emptyset$.

Moreover, if $n \geq 7$, then for a vertex $\mathbf{x}$ of Γ, write $\approx_\mathbf{x}$ for the relation $\approx$ defined on $\mathbf{x}^\perp \cong \mathbf{L}_{n-2}$.

For a fixed isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ that coordinatizes Γ, Lemma 2.1.8(i), (ii) and Lemma 2.1.10 imply that the relation $\approx$ on Γ coincides with the relation 'being equal or in relative positions $(1,2)$, $(1,3)$, $(2,1)$, or $(3,1)$'.

Lemma 2.2.2
Let $n \geq 7$. Let $\mathbf{x}$ be a vertex of $\Gamma \cong \mathbf{L}_n(\mathbb{F})$. Then $\approx_\mathbf{x}$ is the restriction of $\approx$ to $\mathbf{x}^\perp$.

Proof. Fix an isomorphism $\phi : \Gamma \cong \mathbf{L}_n(\mathbb{F})$ and let $(x,X) \in \mathbf{L}_n(\mathbb{F})$ be the image of $\mathbf{x} \in \Gamma$. Moreover, let $\mathbf{a},\mathbf{b} \in \mathbf{x}^\perp$ with $\mathbf{a} \approx_\mathbf{x} \mathbf{b}$, and denote their images under ϕ by (a,A), respectively (b,B). In $\mathbf{L}_n(\mathbb{F})$ we have $a,b \subseteq X$ and $x \subseteq A,B$. The statements $a \cap X = b \cap X$ and $a = b$ (respectively $A \cap X = B \cap X$ and $A = B$), are equivalent (the former obviously and the latter by $A = \langle A \cap X, x\rangle$ and $B = \langle B \cap X, x\rangle$). But this implies the equivalence of $\mathbf{a} \approx_\mathbf{x} \mathbf{b}$ and $\mathbf{a} \approx \mathbf{b}$ for neighbors $\mathbf{a}$, $\mathbf{b}$ of $\mathbf{x}$. □

Lemma 2.2.3
Let $n \geq 5$. On the vertex set of $\Gamma \cong \mathbf{L}_n(\mathbb{F})$, there are unique equivalence relations $\approx^l$ and $\approx^h$ such that $\approx = \approx^l \cup \approx^h$ and $\approx^l \cap \approx^h$ is the identity relation. Moreover, for a fixed isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$, we either have

- *$\approx^l$ is the relation 'equal, in relative position $(1,2)$, or in relative position $(1,3)$' and $\approx^h$ is the relation 'equal, in relative position $(2,1)$, or in relative position $(3,1)$', or*

- *$\approx^l$ is the relation 'equal, in relative position $(2,1)$, or in relative position $(3,1)$' and $\approx^h$ is the relation 'equal, in relative position $(1,2)$, or in relative position $(1,3)$'.*

In other words, for a fixed isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ and up to interchange of $\approx^l$ and $\approx^h$, we can assume that $\approx^l$ corresponds to the relation 'equal, in relative position $(1,2)$, or in relative position $(1,3)$'.

Proof. As we have noticed after Definition 2.2.1, two vertices $\mathbf{x}$, $\mathbf{y}$ of Γ are in relation $\approx$ if and only if their images (x,X) and (y,Y) under an isomorphism to $\mathbf{L}_n(\mathbb{F})$ are equal or in relative positions $(1,2)$, $(1,3)$, $(2,1)$, or $(3,1)$. Let us consider equivalence relations that are subrelations of $\approx$. Obviously, the identity relation is an equivalence relation. Moreover, the relation 'equal, in relative position $(1,2)$, or in relative position $(1,3)$' and the relation 'equal, in relative position $(2,1)$, or in relative position $(3,1)$' are equivalence relations. Now let us assume we have vertices $\mathbf{x} = (x,X)$, $\mathbf{y} = (y,Y)$, $\mathbf{z} = (z,Z)$ of $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ such that $\mathbf{x}$, $\mathbf{y}$ are in relative position $(1,\cdot)$ and $\mathbf{x}$, $\mathbf{z}$ are in relative position $(\cdot,1)$. Then $y \neq z$ and $Y \neq Z$ and $\mathbf{y}$, $\mathbf{z}$ cannot be in relative position $(1,\cdot)$ or $(\cdot,1)$. Consequently, if we want to find two sub-equivalence relations $\approx^l$ and $\approx^h$ of $\approx$ whose union equals $\approx$, then either of $\approx^l$ and $\approx^h$ has to be a subrelation of the relation 'equal, in relative position $(1,2)$, or in relative position $(1,3)$' or of the relation 'equal, in relative position $(2,1)$, or in relative position $(3,1)$'. The lemma follows. $\square$

Convention 2.2.4 From now on, we will always assume that, as soon as we fix an isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$, there is the correspondence of $\approx^l$ to the relation 'equal, in relative position $(1,2)$, or in relative position $(1,3)$'.

Definition 2.2.5 Let $n \geq 5$, and let $\mathbf{x}$ be a vertex of $\Gamma \cong \mathbf{L}_n$. With $\approx^l$ and $\approx^h$ on Γ as in Lemma 2.2.3, we shall write $[\mathbf{x}]^l$ to denote the equivalence class of $\approx^l$ containing $\mathbf{x}$ and similarly $[\mathbf{x}]^h$ to denote the equivalence class of $\approx^h$ containing $\mathbf{x}$. We refer to $[\mathbf{x}]^l$ as the **interior line** on $\mathbf{x}$ and to $[\mathbf{x}]^h$ as the **interior hyperline** on $\mathbf{x}$ of Γ.

Lemma 2.2.6
Let $n \geq 5$. For a fixed isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$, an interior line of Γ is the image of an exterior line by means of this isomorphism, and vice versa. In particular, there is a one-to-one correspondence between interior lines of Γ and exterior lines of $\mathbf{L}_n(\mathbb{F})$. A similar statement is true for interior hyperlines of Γ.

Proof. This is direct from the above. $\square$

Now that we know how to construct lines on a graph $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ without making use of coordinates, it is time to focus our attention on recovering points. In view of the definition of exterior points on $\mathbf{L}_n(\mathbb{F})$ it might be useful to study pencils of interior lines. However, before we can do this, there are some definitions to be made.

Definition 2.2.7 Let $n \geq 5$ and choose $\approx^l$ and $\approx^h$ as in Lemma 2.2.3. For distinct vertices $\mathbf{x}$, $\mathbf{y}$ of Γ with $\mathbf{x} \approx^h \mathbf{y}$, denote by $\mathbf{x} \approx_1^h \mathbf{y}$ that $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is minimal and by $\mathbf{x} \approx_2^h \mathbf{y}$ that $\{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$ is second minimal.

For a fixed isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ the vertices $\mathbf{x}$, $\mathbf{y}$ satisfy $\mathbf{x} \approx_1^h \mathbf{y}$ if and only if they are in relative position $(2,1)$ and $\mathbf{x} \approx_2^h \mathbf{y}$ if and only if they are in relative position $(3,1)$, cf. Lemma 2.1.8 and Lemma 2.2.3.

Definition 2.2.8 Let $n \geq 5$. A set S of mutually intersecting interior lines of $\Gamma \cong \mathbf{L}_n$ is called **full** if

(i) $|S| \geq 2$,

(ii) for any two distinct interior lines $k, l \in S$ there exist vertices $\mathbf{a} \in k$, $\mathbf{b} \in l$ with $\mathbf{a} \approx_1^h \mathbf{b}$,

(iii) for any two vertices $\mathbf{a}$, $\mathbf{b}$ with $[\mathbf{a}]^l$, $[\mathbf{b}]^l \in S$ and $\mathbf{a} \approx_1^h \mathbf{b}$, there exists a third vertex $\mathbf{c}$ satisfying $\mathbf{a} \approx_1^h \mathbf{c}$ and $\mathbf{b} \approx_1^h \mathbf{c}$ such that $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}^{\perp\perp}$ contains vertices $\mathbf{x}$, $\mathbf{y}$ with $\mathbf{x} \approx_2^h \mathbf{y}$ and $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}^{\perp\perp} = \{\mathbf{x}, \mathbf{y}\}^{\perp\perp}$, and

(iv) any interior line $[\mathbf{c}]^l$ containing a vertex $\mathbf{c}$ as in (iii) is also contained in S.

A full set of interior lines essentially is all we need to define points.

Lemma 2.2.9
Let $n \geq 5$. For a fixed isomorphism $\Gamma \cong \mathbf{L}_n$, a full set of interior lines of Γ is the image of a full line pencil of exterior lines of $\mathbf{L}_n$ under this isomorphism and vice versa.

Proof. Let $\phi : \Gamma \cong \mathbf{L}_n$ be the fixed isomorphism. By Lemma 2.1.11, the image $\phi(S)$ of a full set of interior lines in Γ is contained in a pencil of exterior lines of $\mathbf{L}_n$, through the point p, say. Let l be an exterior line of $\mathbf{L}_n$ incident with p. The full set S contains at least two distinct lines a and b; notice that $\phi(a)$, $\phi(b)$ are incident with p. If $\phi(a)$, $\phi(b)$, and l span a 3-space, then $\phi^{-1}(l)$ is contained in the full set by definition. So suppose l lies in the plane $\langle \phi(a), \phi(b) \rangle$. Then the full set has to contain a third line c such that $\phi(a)$, $\phi(b)$, $\phi(c)$ span a 3-space, by Lemma 2.1.12 and the definition of a full set. But then also l, $\phi(b)$ and $\phi(c)$ span a 3-space, and $\phi^{-1}(l)$ is contained in the full set. $\square$

Definition 2.2.10 Let $n \geq 5$. Let S be a full set of interior lines of $\Gamma \cong \mathbf{L}_n$. The **interior point** $p(S)$ of Γ is the union $\bigcup_{l \in S} l$ over all interior lines in the full set S. The geometry of interior points and interior lines with symmetrized containment as incidence is called the **interior projective space on** Γ. Dually, define **interior hyperplanes** and the **dual interior projective space**.

The isomorphism—for a fixed isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$—between the interior projective space on Γ and the exterior projective space on $\mathbf{L}_n(\mathbb{F})$ (and their duals), by Lemmas 2.2.6 and 2.2.9, on one hand, and the isomorphism between the exterior projective space on $\mathbf{L}_n(\mathbb{F})$ and $\mathbb{P}_n(\mathbb{F})$ (and their duals) on the other hand, imply the following theorem and corollaries:

Proposition 2.2.11
Let $n \geq 5$. Up to interchanging $\approx^l$ and $\approx^h$ every isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ induces an isomorphism between the interior projective space on Γ and the exterior projective space on $\mathbf{L}_n(\mathbb{F})$. The same statement holds true for their duals. $\square$

Corollary 2.2.12
Let $n \geq 5$, and let Γ be isomorphic to $\mathbf{L}_n(\mathbb{F})$. Then the interior projective space on Γ is isomorphic to $\mathbb{P}_n(\mathbb{F})$ or $\mathbb{P}_n(\mathbb{F})^{\mathrm{dual}}$. $\square$

Corollary 2.2.13
Let $n \geq 5$, and let Γ be isomorphic to $\mathbf{L}_n(\mathbb{F})$. If $\mathbb{F}$ admits an anti-automorphism, then the automorphism group of Γ is of the form $P\Gamma L_{n+1}(\mathbb{F}).2$. $\square$

Corollary 2.2.14
Let $n \geq 5$, and let Γ be isomorphic to $\mathbf{L}_n(\mathbb{F})$. If $\mathbb{F}$ does not admit an anti-automorphism, the automorphism group of Γ is isomorphic to $P\Gamma L_{n+1}(\mathbb{F})$. $\square$

Having achieved the goal of this section, let us state some facts that will be applied in proofs of later sections. We will start with a straightforward lemma: For a vertex $\mathbf{x} \in \Gamma \cong \mathbf{L}_n(\mathbb{F})$, the induced subgraph $\mathbf{x}^\perp$ of Γ gives rise to a subspace of the interior projective space on Γ:

Lemma 2.2.15
Let $n \geq 7$, and let Γ be isomorphic to $\mathbf{L}_n(\mathbb{F})$. Let $\mathbf{x}$ be a vertex of Γ. If l is an interior line of Γ with $l \cap \mathbf{x}^\perp \neq \emptyset$, then $l \cap \mathbf{x}^\perp$ is an interior line or an interior hyperline of $\mathbf{x}^\perp$. Conversely, if m is an interior line of $\mathbf{x}^\perp$, then there exists an interior line or hyperline m' of Γ with $m' \cap \mathbf{x}^\perp = m$.

Proof. Fix an isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$, and identify Γ with $\mathbf{L}_n(\mathbb{F})$ by means of this isomorphism. If l is an interior line of Γ such that $\mathbf{a} = (a, A) \in l \cap \mathbf{x}^\perp$, then either the $(\approx^l)_\mathbf{x}$ class of $\mathbf{a}$ or the $(\approx^h)_\mathbf{x}$ class of $\mathbf{a}$ coincides with $l \cap \mathbf{x}^\perp$, by Lemma 2.2.2 and Lemma 2.2.3. Likewise, an interior line of $\mathbf{x}^\perp$ is an $(\approx_\mathbf{x})^l$ equivalence class of $\mathbf{x}^\perp$ and hence, again by Lemma 2.2.2 and Lemma 2.2.3, is the intersection with $\mathbf{x}^\perp$ of a single $\approx^l$ or $\approx^h$ equivalence class of Γ. $\square$

In view of the lemma, for any vertex $\mathbf{x} \in \Gamma \cong \mathbf{L}_n(\mathbb{F})$, $n \geq 7$, one can choose the equivalence relation $\approx^l_\mathbf{x}$ on $\mathbf{x}^\perp \cong \mathbf{L}_{n-2}(\mathbb{F})$ in such a way that $(\approx_\mathbf{x})^l = (\approx^l)_\mathbf{x}$. In that case, there is no harm in writing $\approx^l_\mathbf{x}$ to denote this relation. In particular, one can identify interior lines of $\mathbf{x}^\perp$ with interior lines of Γ. The same holds true for interior hyperlines, interior points, and interior hyperplanes.

Lemma 2.2.16
Let $n \geq 5$, and let Γ be isomorphic to $\mathbf{L}_n(\mathbb{F})$. Any two interior points of Γ intersect.

Proof. Fix an isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$. The images of interior points p, q of Γ are of the form $v_p = \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid p \in l\}$ and $v_q = \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid q \in l\}$. The statement is obviously true for $p = q$, so assume $p \neq q$. Then any vertex (pq, L) with $L \cap pq = \emptyset$ lies in the intersection. □

The following two facts give insight in the behavior of certain subspaces of the interior projective space of $\Gamma \cong \mathbf{L}_n$.

Lemma 2.2.17
Let $n \geq 7$, let Γ be isomorphic to $\mathbf{L}_n(\mathbb{F})$, and let $\mathbf{x}$ be a vertex of Γ. Then the interior projective space on $\mathbf{x}^\perp$ is a hyperline of the interior projective space on Γ.

Proof. By Proposition 2.2.11 and a choice of an isomorphism $\phi : \Gamma \cong \mathbf{L}_n$, the image under ϕ of any interior point of Γ is of the form $v_p = \{(l, L) \in \mathbf{L}_n \mid p \in l\}$, and any interior line has an image of the form $v_l = \{(a, A) \in \mathbf{L}_n \mid a = l\}$. Suppose $\phi(\mathbf{x}) = (x, X)$. For $l \subseteq X$ we can identify $v_l = \left\{(a, A) \in \phi(\mathbf{x})^\perp \mid a = l\right\}$ with $v_l = \{(a, A) \in \mathbf{L}_n \mid a = l\}$. If for $p \in X$ we also identify $v_p = \left\{(l, L) \in \phi(\mathbf{x})^\perp \mid p \in l\right\}$ with $v_p = \{(l, L) \in \mathbf{L}_n \mid p \in l\}$, then we have essentially identified $\mathbf{x}^\perp$ with X and the claim follows from the fact that X is a hyperline of $\mathbb{P}_n$. □

Lemma 2.2.18
Let $n \geq 5$, let Γ be isomorphic to $\mathbf{L}_n(\mathbb{F})$, and let U be a subspace of the interior projective space on Γ of odd projective dimension m. Then there exists a clique of $\frac{m+1}{2}$ vertices in Γ such that the interior lines containing these vertices span U.

Proof. A projective basis B of U consists of $m + 1$ interior points, say $B = \{x_0, \ldots, x_m\}$. By Corollary 2.2.12, the interior projective space on Γ is isomorphic to $\mathbb{P}_n(\mathbb{F})$. Now, we can identify the interior points of Γ with points of $\mathbb{P}_n(\mathbb{F})$—and Γ with $\mathbf{L}_n(\mathbb{F})$—and the $\frac{m+1}{2}$ pairs $\left(x_i x_{i+\frac{m+1}{2}}, \left\langle B \backslash \left\{x_i, x_{i+\frac{m+1}{2}}\right\}\right\rangle\right)$, $0 \leq i \leq \frac{m-1}{2}$, are vertices of Γ as needed. □

2.3 Geometries on interior root points

Besides the exterior projective space, the dual exterior projective space, and the Grassmann spaces on lines and hyperlines, the graph $\mathbf{L}_n(\mathbb{F})$ admits two further natural geometries induced by coordinization, one isomorphic to the root group geometry of $PSL_{n+1}(\mathbb{F})$ and the other isomorphic to the hyperbolic root group geometry of the same group. For fixed coordinates, the set $v_{p,H} = \{(l, L) \in \mathbf{L}_n(\mathbb{F}) \mid p \in l, L \subseteq H\}$, for a point p and a hyperplane $H \ni p$ of $\mathbb{P}_n(\mathbb{F})$, is called an **exterior root point** of $\mathbf{L}_n(\mathbb{F})$. Likewise, an **exterior root line** is defined as the union $v_{l,H} = \bigcup_{p \in l} v_{p,H}$, for a line l and a hyperplane $H \supseteq l$, or as the union $v_{p,L} = \bigcup_{H \supseteq L} v_{p,H}$, for a hyperline L and a point $p \in L$. The geometry of the exterior root points and the exterior root lines of $\mathbf{L}_n(\mathbb{F})$, with symmetrized inclusion as incidence, is called the **exterior root group geometry** on $\mathbf{L}_n(\mathbb{F})$ and is isomorphic to the root group geometry of $PSL_{n+1}(\mathbb{F})$. Notice that the exterior root group geometry does not change when we consider the graph $\mathbf{L}_n(\mathbb{F})$ as the graph $\mathbf{L}_n(\mathbb{F}^{\mathrm{opp}})$, instead. Similarly, consider the geometry on the exterior root points of $\mathbf{L}_n(\mathbb{F})$ as points and the vertices of $\mathbf{L}_n(\mathbb{F})$ as lines with symmetrized inclusion as incidence. That geometry is isomorphic to the hyperbolic root group geometry of $PSL_{n+1}(\mathbb{F})$ and is called the **exterior hyperbolic root group geometry** on $\mathbf{L}_n(\mathbb{F})$. This geometry, too, is independent of whether we consider the graph as $\mathbf{L}_n(\mathbb{F})$ or $\mathbf{L}_n(\mathbb{F}^{\mathrm{opp}})$.

In this section, we will give constructions of the exterior (hyperbolic) root group geometries on $\mathbf{L}_n(\mathbb{F})$ in terms of exterior points, exterior lines, exterior hyperplanes, and exterior hyperlines. Then Lemmas 2.2.6 and 2.2.9 will give us means to describe these geometries coordinate-freely on any graph $\Gamma \cong \mathbf{L}_n(\mathbb{F})$.

Lemma 2.3.1
Let $n \geq 5$, and let $v_{p,H}$ and $v_{q,I}$ be distinct exterior root points of $\mathbf{L}_n(\mathbb{F})$. Then we have $|v_{p,H} \cap v_{q,I}| \leq 1$.

Proof. Suppose $|v_{p,H} \cap v_{q,I}| \neq \emptyset$, i.e., in $\mathbb{P}_n(\mathbb{F})$ there exists a line l and a hyperline L with $l \supseteq \langle p, q\rangle$, $L \subseteq H \cap I$ and $l \cap L = \emptyset$. Assume there exists another line-hyperline pair (m, M) satisfying these conditions. If it is distinct from (l, L), then $l \neq m$ or $L \neq M$. Up to duality, we may assume $L \neq M$. Then immediately $H = I$, whence $p \neq q$. But then $l = m = pq$ is contained in $H = I$ (recall that $v_{p,H}$ and $v_{q,I}$ being root points means $p \in H$ and $q \in I$), which have hyperplanes L and M. Hence $l \cap L \neq \emptyset$, a contradiction. □

Lemma 2.3.2
Let $n \geq 5$, and let $v_{p,H}$ and $v_{q,I}$ be distinct exterior root points of $\mathbf{L}_n(\mathbb{F})$. The points p and q, respectively the hyperplanes H and I are distinct and the line pq does not intersect the hyperline $H \cap I$ if and only if $|v_{p,H} \cap v_{q,I}| = 1$.

Proof. By the preceding lemma, we always have $|v_{p,H} \cap v_{q,I}| \leq 1$. If $p \neq q$, $H \neq I$ and $pq \cap H \cap I = \emptyset$, then $(pq, H \cap I)$ is a vertex of $\mathbf{L}_n(\mathbb{F})$ contained in $v_{p,H} \cap v_{q,I}$. Conversely, suppose there exists such a vertex. In the proof of the preceding lemma, we have seen that this implies $p \neq q$ and $H \neq I$. But then the only candidate for being contained in $v_{p,H} \cap v_{q,I}$ is $(pq, H \cap I)$, whence $pq \cap H \cap I = \emptyset$. □

For the next lemma, we would like to point out that an exterior hyperplane of $\mathbf{L}_n(\mathbb{F})$ is *not* a hyperplane of the exterior projective space on $\mathbf{L}_n(\mathbb{F})$. However, there is an obvious one-to-one correspondence between exterior hyperplanes and hyperplanes of the exterior projective space, by the map

$$v_H = \{(l,L) \in \mathbf{L}_n(\mathbb{F}) \mid L \subseteq H\} \mapsto \bigcup_{p \in H} v_p = \bigcup_{p \in H} \{(l,L) \in \mathbf{L}_n(\mathbb{F}) \mid p \in l\}\,.$$

Therefore there is no harm done if we speak of **incidence** between exterior points and exterior hyperplanes and rather mean incidence between exterior points and the images of exterior hyperplanes under this map.

Lemma 2.3.3
Let $n \geq 5$. An exterior point v_p and an exterior hyperplane v_H of $\mathbf{L}_n(\mathbb{F})$ are non-incident if and only if any exterior line v_l incident with v_p contains a vertex contained in an exterior hyperline v_L incident with v_H and vice versa.

Proof. Suppose v_p and v_H are non-incident and let v_l be an exterior line incident with v_p. The set v_l consists of all vertices of $\mathbf{L}_n(\mathbb{F})$ having l as the first coordinate. The second coordinate ranges over all hyperlines L that do not intersect l. By the isomorphism between the exterior projective space on $\mathbf{L}_n(\mathbb{F})$ and $\mathbb{P}_n(\mathbb{F})$ that maps v_p onto p, v_l onto l, v_H onto H, also p is incident with l and non-incident with H. Hence l intersects H in a single point. But then there exists a hyperline M not intersecting l that is contained in H. The vertex (l, M) is contained in the exterior hyperline v_M, which is incident with v_H. Similarly, any exterior hyperline incident with v_H contains a vertex contained in an exterior line incident with v_p. Conversely, suppose v_p and v_H are incident. Choose an exterior line v_l through v_p such that l is contained in H. Now, a hyperline that does not intersect l cannot be contained in H. □

In view of the preceding lemma, in a graph $\Gamma \cong \mathbf{L}_n(\mathbb{F})$, an interior point p and an interior hyperplane H are called **non-incident** if and only if any interior line l incident with p contains a vertex of Γ contained in an interior hyperline L incident with H and vice versa. Conversely, an interior point and an interior hyperplane are **incident** if they are not non-incident.

Definition 2.3.4 Let $n \geq 5$, and let $\Gamma \cong \mathbf{L}_n(\mathbb{F})$. An **interior root point** of Γ is the intersection of an interior point with an incident interior hyperplane. Notice that this definition is independent of the choice of the relations $\approx^l$ and $\approx^h$ in Lemma 2.2.3. Recall that an exterior root line is defined as the union $v_{l,H} = \bigcup_{p \in l} v_{p,H}$, for a line l and a hyperplane $H \supseteq l$, or as the union $v_{p,L} = \bigcup_{H \supseteq L} v_{p,H}$, for a hyperline L and a point $p \in L$. To show that a line l is contained in a hyperplane H, one just has to find distinct points on l contained in H. By Lemma 2.3.3, we know when an exterior point is incident with an exterior hyperplane, and we can define an exterior root line from the knowledge of exterior points, line, hyperlines, hyperplanes, and incidence. Consequently, an **interior root line** is of the form

$$\bigcup_{\text{interior point } p \in l} p \cap H$$

for an interior line l contained in the interior hyperplane H, or

$$\bigcup_{\text{interior hyperplane } H \supseteq L} p \cap H$$

for an interior hyperline L containing the interior point p.

We are now ready to state the main result of this section.

Proposition 2.3.5
Let $n \geq 5$, and let Γ be isomorphic to $\mathbf{L}_n(\mathbb{F})$. The following hold true.

(i) *The geometry of exterior root points and exterior root lines on $\mathbf{L}_n(\mathbb{F})$ with symmetrized containment as incidence is isomorphic to the root group geometry of $PSL_{n+1}(\mathbb{F})$.*

(ii) *The geometry of exterior root points and vertices of $\mathbf{L}_n(\mathbb{F})$ with symmetrized containment as incidence is isomorphic to the hyperbolic root group geometry of $PSL_{n+1}(\mathbb{F})$.*

(iii) *The geometry of interior root points and interior root lines on Γ with symmetrized containment as incidence is isomorphic to the root group geometry of $PSL_{n+1}(\mathbb{F})$.*

(iv) *The geometry of interior root points and vertices of Γ with symmetrized containment as incidence is isomorphic to the hyperbolic root group geometry of $PSL_{n+1}(\mathbb{F})$.*

More precisely, there is a one-to-one correspondence between the vertices of Γ and the lines of the hyperbolic root group geometry of $PSL_{n+1}(\mathbb{F})$. The interior root points of Γ correspond one-to-one to the full line pencils of the (hyperbolic) root group geometry.

Proof. By Lemmas 2.2.6 and 2.2.9 plus the Definition 2.3.4 of interior root points and interior root lines, Statement (i) is equivalent to Statement (iii) and Statement (ii) is equivalent to Statement (iv). However, the first two statements are satisfied by definition. The additional claim about the vertices of Γ follows from the isomorphism $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ and the fact that hyperbolic lines of the hyperbolic root group geometry of $PSL_{n+1}(\mathbb{F})$ correspond to non-intersecting line-hyperline pairs of the projective space $\mathbb{P}_n(\mathbb{F})$. The claim about the line pencils follows from the fact that interior points correspond to full line pencils of interior lines (cf. Lemma 2.2.9), similarly interior hyperplanes correspond to full pencils of interior hyperlines. Intersecting an interior point with an incident interior hyperplane, we obtain a full line pencil of the hyperbolic root group geometry. □

The geometries on the interior objects in Proposition 2.3.5 are called the **interior (hyperbolic) root group geometry on** Γ, respectively. We conclude this section with a lemma that is needed later.

Lemma 2.3.6
Let $n \geq 5$, and let $v_{p,H}$ be an exterior root point of $\mathbf{L}_n(\mathbb{F})$. Any vertex adjacent to a vertex of $v_{p,H}$ is adjacent to another vertex of $v_{p,H}$.

Proof. Let $\mathbf{x}$ be a vertex contained in $v_{p,H}$, and let $\mathbf{z}$ be adjacent to $\mathbf{x}$. The vertex $\mathbf{z}$ corresponds to a non-intersecting line-hyperline pair (z, Z), likewise $\mathbf{x}$ corresponds to (x, X). Adjacency implies $z \subseteq X$ and $x \subseteq Z$. Thus we have $p \in Z$ and $z \subseteq H$. Now we easily find a pair $(y, Y) \neq (x, X)$ consisting of a line $y \subseteq Z$ containing p and a non-intersecting hyperline $Y \subseteq H$ containing z, and obtain a vertex $\mathbf{y} = (y, Y)$ as claimed. □

2.4 Locally line-hyperline graphs

We can now shift our attention to graphs that are locally $\mathbf{L}_n(\mathbb{F})$. It is enough to investigate connected graphs with that local property. Throughout the whole section let $n \geq 11$, let $\mathbb{F}$ be a division ring, and let Γ be a connected, locally $\mathbf{L}_n(\mathbb{F})$ graph. Since we are considering graphs Γ that are locally $\mathbf{L}_n(\mathbb{F})$, the notions of interior points and lines are defined on perps $\mathbf{x}^\perp$ of vertices $\mathbf{x}$ in Γ rather than on the whole graph Γ. To emphasize this, any interior object will be indexed by the vertex in whose perp it is defined and called **local**, e.g., an interior point p of $\mathbf{x}^\perp$ will be denoted by $p_\mathbf{x}$ and called a **local point of $\mathbf{x}^\perp$**; likewise, define the **local** relations $\approx^l_\mathbf{x}$ and $\approx^h_\mathbf{x}$.

Lemma 2.4.1
Let $\mathbf{x}$ and $\mathbf{y}$ be any two adjacent vertices of Γ. Then there is a choice of local equivalence relations $\approx^l_\mathbf{x}$ and $\approx^l_\mathbf{y}$ such that the intersections of $\approx^l_\mathbf{x}$ and $\approx^l_\mathbf{y}$ to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ coincide.

Proof. This follows immediately from application of Lemma 2.2.15 (and the discussion after that lemma) first to $\mathbf{x}^\perp \cong \mathbf{L}_n(\mathbb{F})$ and $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cong \mathbf{L}_{n-2}(\mathbb{F})$ and then to $\mathbf{y}^\perp \cong \mathbf{L}_n(\mathbb{F})$ and $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cong \mathbf{L}_{n-2}(\mathbb{F})$. □

Having Lemma 2.2.17 in mind, the preceding lemma allows us to view $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ as a subspace of the interior projective spaces on both $\mathbf{x}^\perp$ and $\mathbf{y}^\perp$. We will immediately apply this knowledge in the proof of the following lemma.

Lemma 2.4.2
Let $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ be a path of vertices in Γ. Then $\mathbf{w}^\perp \cap \mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp \neq \emptyset$. In particular, the diameter of Γ is two and Γ, viewed as a two-dimensional simplicial complex whose two-simplices are its triangles, is simply connected.

Proof. Choose local equivalence relations $\approx^l_\mathbf{w}$, $\approx^l_\mathbf{x}$, $\approx^l_\mathbf{y}$, and $\approx^l_\mathbf{z}$ such that $\approx^l_\mathbf{w}$ and $\approx^l_\mathbf{x}$ coincide on $\mathbf{w}^\perp \cap \mathbf{x}^\perp$, that $\approx^l_\mathbf{x}$ and $\approx^l_\mathbf{y}$ coincide on $\mathbf{x}^\perp \cap \mathbf{y}^\perp$, and that $\approx^l_\mathbf{y}$ and $\approx^l_\mathbf{z}$ coincide on $\mathbf{y}^\perp \cap \mathbf{z}^\perp$. This is possible by Lemma 2.4.1. Application of Lemma 2.2.17 to the interior projective space of $\mathbf{y}^\perp \cong \mathbf{L}_n(\mathbb{F})$ shows that the interior projective spaces of $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ and of $\mathbf{y}^\perp \cap \mathbf{z}^\perp$ correspond to hyperlines of $\mathbf{y}^\perp \cong \mathbf{L}_n(\mathbb{F})$. We have to investigate $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$. Suppose we have $\mathbf{x} = (x_\mathbf{y}, X_\mathbf{y})$ and $\mathbf{z} = (z_\mathbf{y}, Z_\mathbf{y})$ inside $\mathbf{y}^\perp$. Then the graph $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$ (considered inside $\mathbf{y}^\perp$) consists of the non-intersecting line-hyperline pairs whose lines are contained in $X_\mathbf{y} \cap Z_\mathbf{y}$ and whose hyperlines contain the space $\langle x_\mathbf{y}, z_\mathbf{y}\rangle$. Since $n \geq 11$, the space $X_\mathbf{y} \cap Z_\mathbf{y}$ has (projective) dimension at least 7. The intersection of $\langle x_\mathbf{y}, z_\mathbf{y}\rangle$ with $X_\mathbf{y} \cap Z_\mathbf{y}$ is at most a line, as $x_\mathbf{y} \cap X_\mathbf{y} = \emptyset = z_\mathbf{y} \cap Z_\mathbf{y}$. Suppose this intersection actually is a line and denote that line by $l_\mathbf{y}$. By the choice of local equivalence relations we can consider this configuration also in $\mathbf{x}^\perp$. There the vertices $\mathbf{w}$ and $\mathbf{y}$ correspond to line-hyperline pairs $(w_\mathbf{x}, W_\mathbf{x})$ and $(y_\mathbf{x}, Y_\mathbf{x})$, respectively. The space $X_\mathbf{y} \cap Z_\mathbf{y}$ of $\mathbf{y}^\perp$ corresponds to a space $U_\mathbf{x}$ in $\mathbf{x}^\perp$ (of the same dimension as $X_\mathbf{y} \cap Z_\mathbf{y}$), which lies inside $Y_\mathbf{x}$. Of the subspace $\langle x_\mathbf{y}, z_\mathbf{y}\rangle$ of $\mathbf{y}^\perp$, in $\mathbf{x}^\perp$ we can only see the intersection with $X_\mathbf{y}$, which is a line $l_\mathbf{x}$ induced by the line $l_\mathbf{y}$. The intersection of $U_\mathbf{x}$ and $W_\mathbf{x}$ is a space of dimension at least 5. The space $\langle l_\mathbf{x}, y_\mathbf{x}, w_\mathbf{x}\rangle$ has at most dimension 5, and the intersection $U_\mathbf{x} \cap W_\mathbf{x} \cap \langle l_\mathbf{x}, y_\mathbf{x}, w_\mathbf{x}\rangle$ has at most dimension 3, because $w_\mathbf{x} \cap W_\mathbf{x} = \emptyset$. Hence we can find a line $a_\mathbf{x} \subseteq U_\mathbf{x} \cap W_\mathbf{x} \setminus \langle l_\mathbf{x}, y_\mathbf{x}, w_\mathbf{x}\rangle$ and a non-intersecting hyperline $A_\mathbf{x}$ containing $\langle l_\mathbf{x}, y_\mathbf{x}, w_\mathbf{x}\rangle$. The pair $(a_\mathbf{x}, A_\mathbf{x})$ corresponds to a vertex of $\mathbf{x}^\perp$ adjacent to $\mathbf{w}$ (as $w_\mathbf{x} \subseteq A_\mathbf{x}$ and $a_\mathbf{x} \subseteq W_\mathbf{x}$) as well as adjacent to $\mathbf{y}$ (as $y_\mathbf{x} \subseteq A_\mathbf{x}$ and $a_\mathbf{x} \subseteq U_\mathbf{x} \subseteq Y_\mathbf{y}$) and to $\mathbf{z}$ (as translated to $\mathbf{y}^\perp$ the line $a_\mathbf{y}$ lies in $X_\mathbf{y} \cap Z_\mathbf{y}$ and the hyperline $A_\mathbf{y}$ contains $l_\mathbf{y}$ and $x_\mathbf{y}$, whence also $z_\mathbf{y}$). Our assumptions made during the proof resemble the most difficult case; all other cases run along the same lines.

The remaining statements of the lemma are obvious consequences. □

The next lemma is a generalization of Lemma 2.4.1 to a choice of local equivalence relations for all vertices of Γ.

Lemma 2.4.3
There is a choice of local equivalence relations $\approx^l_\mathbf{x}$ for $\mathbf{x}$ running over the vertices of Γ such that, for any two adjacent vertices $\mathbf{x}$ and $\mathbf{y}$, the restrictions of $\approx^l_\mathbf{x}$ and $\approx^l_\mathbf{y}$ to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ coincide.

Once a choice of local equivalence relations $\approx^l_\mathbf{x}$ is given, the same statement holds for the dual choice of local equivalence relations $\approx^h_\mathbf{x}$. In particular, everything that will be shown for the relations $\approx^l_\mathbf{x}$ immediately also holds for the relation $\approx^h_\mathbf{x}$.

Proof. Suppose that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ is a triangle. In view of Lemma 2.4.1, we may assume that $\approx^l_\mathbf{x}$ and $\approx^l_\mathbf{y}$ have the same restriction to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ and that $\approx^l_\mathbf{x}$ and $\approx^l_\mathbf{z}$ have the same restriction to $\mathbf{x}^\perp \cap \mathbf{z}^\perp$. Let $l_\mathbf{x}$ be an interior line of $\mathbf{x}^\perp$ such that $l_\mathbf{x} \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp \neq \emptyset$. By analysis of $\mathbf{x}^\perp$, we can find two vertices, say $\mathbf{u}$ and $\mathbf{v}$, in $l_\mathbf{x} \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$. Now the above choices of local equivalence relations imply that $\mathbf{u}$ and $\mathbf{v}$ belong to $\approx^l_\mathbf{y} \cap \approx^l_\mathbf{z}$. This forces that $\approx^l_\mathbf{y}$ and $\approx^l_\mathbf{z}$ have the same restriction to $\mathbf{y}^\perp \cap \mathbf{z}^\perp$ by Lemma 2.2.2 and Lemma 2.2.3. Since Γ is simply connected (cf. Lemma 2.4.2), the lemma follows immediately from the triangle analysis. □

Notation 2.4.4 Fix a choice of $\approx^l_{\mathbf{x}}$ as in Lemma 2.4.3, and set $\approx^l = \bigcup_{\mathbf{x}\in\Gamma} \approx^l_{\mathbf{x}}$.

Lemma 2.4.5
Let $\mathbf{x}$ be a vertex of Γ. Then the restriction of $\approx^l$ to $\mathbf{x}^\perp$ equals $\approx^l_{\mathbf{x}}$. In particular, suppose that $\mathbf{x}$ and $\mathbf{y}$ are vertices of Γ such that $\mathbf{x} \approx^l_{\mathbf{u}} \mathbf{y}$ for some vertex $\mathbf{u}$ in $\{\mathbf{x},\mathbf{y}\}^\perp$. Then $\mathbf{x} \approx^l_{\mathbf{v}} \mathbf{y}$ for every vertex $\mathbf{v}$ in $\{\mathbf{x},\mathbf{y}\}^\perp$.

Proof. Obviously, $\approx^l_{|\mathbf{x}^\perp\times\mathbf{x}^\perp} \supseteq \approx^l_{\mathbf{x}}$. Assume there exist $\mathbf{y},\mathbf{z} \in \mathbf{x}^\perp$ with $\mathbf{y} \approx^l \mathbf{z}$ but $\mathbf{y} \not\approx^l_{\mathbf{x}} \mathbf{z}$. Since $\approx^l = \bigcup_{\mathbf{x}\in\Gamma} \approx^l_{\mathbf{x}}$, there exists a vertex $\mathbf{a}$ adjacent to $\mathbf{y}$ and $\mathbf{z}$ with $\mathbf{y} \approx^l_{\mathbf{a}} \mathbf{z}$. By Lemma 2.4.2, there exists a vertex $\mathbf{b}$ adjacent to $\mathbf{a} \perp \mathbf{y} \perp \mathbf{x} \perp \mathbf{z}$. But both $\mathbf{y} \approx^l_{\mathbf{b}} \mathbf{z}$ and $\mathbf{y} \not\approx^l_{\mathbf{b}} \mathbf{z}$ yield a contradiction to Lemma 2.4.3. □

Lemma 2.4.6
$\approx^l$ is an equivalence relation.

Proof. Reflexivity and symmetry follow from reflexivity and symmetry of each $\approx^l_{\mathbf{x}}$. To prove transitivity, suppose that $\mathbf{x} \approx^l_{\mathbf{u}} \mathbf{y}$ and $\mathbf{y} \approx^l_{\mathbf{v}} \mathbf{z}$. By Lemma 2.4.2 there is a vertex $\mathbf{w}$ in $\{\mathbf{x},\mathbf{u},\mathbf{y},\mathbf{v}\}^\perp$. By the same lemma, there is a vertex $\mathbf{a}$ in $\{\mathbf{x},\mathbf{w},\mathbf{v},\mathbf{z}\}^\perp$. A third application of the lemma yields a vertex $\mathbf{b}$ in $\{\mathbf{a},\mathbf{w},\mathbf{y},\mathbf{v}\}^\perp$. By analysis of $\mathbf{w}^\perp$, we find a vertex $\mathbf{c}$ in $\{\mathbf{a},\mathbf{b},\mathbf{w}\}^\perp$ in $\approx^l_{\mathbf{w}}$ relation with both $\mathbf{x}$ and $\mathbf{y}$ by Lemma 2.1.5. This gives $\mathbf{x} \approx^l_{\mathbf{a}} \mathbf{c}$ by Lemma 2.4.5. Similarly, using an analysis of $\mathbf{v}^\perp$, we find a final vertex $\mathbf{d} \in \{\mathbf{a},\mathbf{b},\mathbf{v}\}^\perp$ in $\approx^l_{\mathbf{v}}$ relation with both $\mathbf{y}$ and $\mathbf{z}$. This gives $\mathbf{d} \approx^l_{\mathbf{a}} \mathbf{z}$, again by Lemma 2.4.5. In $\mathbf{b}^\perp$, we have $\mathbf{c} \approx^l_{\mathbf{b}} \mathbf{y}$ as well as $\mathbf{d} \approx^l_{\mathbf{b}} \mathbf{y}$ (Lemma 2.4.5), whence by transitivity of $\approx^l_{\mathbf{b}}$, also $\mathbf{c} \approx^l_{\mathbf{b}} \mathbf{d}$. This establishes $\mathbf{c} \approx^l_{\mathbf{a}} \mathbf{d}$. We have found a chain $\mathbf{x},\mathbf{c},\mathbf{d},\mathbf{z}$ of $\approx^l_{\mathbf{a}}$ related vertices in $\mathbf{a}^\perp$. By transitivity of $\approx^l_{\mathbf{a}}$, this gives $\mathbf{x} \approx^l_{\mathbf{a}} \mathbf{z}$, whence $\mathbf{x} \approx^l \mathbf{z}$, proving transitivity of $\approx^l$. □

Definition 2.4.7 A **global line** of Γ is an equivalence class of $\approx^l = \bigcup_{\mathbf{x}\in\Gamma} \approx^l_{\mathbf{x}}$. Dually, define a **global hyperline** as an equivalence class of $\approx^h = \bigcup_{\mathbf{x}\in\Gamma} \approx^h_{\mathbf{x}}$. The set of global lines of Γ is denoted by $\mathcal{L}_\Gamma$. Notice that, by Lemma 2.4.5, for a global line l and a vertex $\mathbf{x}$ we either have $l \cap \mathbf{x}^\perp = \emptyset$ or $l \cap \mathbf{x}^\perp = l_{\mathbf{x}}$, a local line of $\mathbf{x}^\perp$.

The next task is to define global points. This will prove to be a bit more complicated than the definition of global lines. Let us first study how 'intersecting' global lines behave.

Lemma 2.4.8
Let l and m be global lines of Γ and let $\mathbf{x}$ be a vertex of Γ with $l\cap\mathbf{x}^\perp \neq \emptyset \neq m\cap\mathbf{x}^\perp$ and such that the local lines $l_{\mathbf{x}}$ and $m_{\mathbf{x}}$ intersect in an interior point of $\mathbf{x}^\perp$. Then, for any vertex $\mathbf{y}$ of Γ with $l\cap\mathbf{y}^\perp \neq \emptyset \neq m\cap\mathbf{y}^\perp$, the local lines $l_{\mathbf{y}}$ and $m_{\mathbf{y}}$ intersect in an interior point of $\mathbf{y}^\perp$.

Proof. First we will show that it is enough to prove the claim for adjacent $\mathbf{x}$ and $\mathbf{y}$. Indeed, let $\mathbf{x}$ and $\mathbf{y}$ be distinct vertices of Γ such that there exist vertices $\mathbf{l}_1 \perp \mathbf{x}$,

$\mathbf{l}_2 \perp \mathbf{y}$ contained in the global line l and vertices $\mathbf{m}_1 \perp \mathbf{x}$, $\mathbf{m}_2 \perp \mathbf{y}$ contained in the global line m. By Lemma 2.4.2, we can find a vertex $\mathbf{a}$ adjacent to both $\mathbf{m}_1$ and $\mathbf{l}_2$. By Lemma 2.4.2 again, there is a vertex $\mathbf{b}$ adjacent to $\mathbf{x}$, $\mathbf{m}_1$, $\mathbf{a}$, and $\mathbf{l}_2$. Finally, again by Lemma 2.4.2 there is a vertex $\mathbf{c}$ adjacent to $\mathbf{b}$, $\mathbf{l}_2$, $\mathbf{y}$, and $\mathbf{m}_2$. Hence now suppose $\mathbf{x} \perp \mathbf{y}$. Since none of $l \cap \mathbf{x}^\perp$, $m \cap \mathbf{x}^\perp$, $l \cap \mathbf{y}^\perp$, and $m \cap \mathbf{y}^\perp$ is empty, neither are $l \cap \mathbf{x}^\perp \cap \mathbf{y}^\perp$ and $m \cap \mathbf{x}^\perp \cap \mathbf{y}^\perp$, which are equal to $l_\mathbf{x} \cap l_\mathbf{y}$, respectively $m_\mathbf{x} \cap m_\mathbf{y}$, by Lemma 2.4.5 and Lemma 2.4.3. But now the local lines $l_\mathbf{x}$ and $m_\mathbf{x}$ and the local lines $l_\mathbf{y}$ and $m_\mathbf{y}$ intersect if and only if the restrictions of l and m to $\mathbf{x}^\perp \cap \mathbf{y}^\perp$ intersect, which we can determine in $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cong \mathbf{L}_{n-2}(\mathbb{F})$ by Lemma 2.1.11 and Lemma 2.1.12. □

Let $\mathbf{a}$ and $\mathbf{b}$ be distinct vertices of Γ. Suppose we have vertices $\mathbf{l}$ and $\mathbf{m}$ adjacent to $\mathbf{a}$ and vertices $\mathbf{n}$ and $\mathbf{o}$ adjacent to $\mathbf{b}$. By Lemma 2.4.2, there exists a vertex $\mathbf{c}$ adjacent to $\mathbf{m}$ and $\mathbf{n}$. Two further applications of Lemma 2.4.2 give rise to a vertex $\mathbf{d}$ adjacent to $\mathbf{a}$, $\mathbf{m}$, $\mathbf{c}$, and $\mathbf{n}$ and to a vertex $\mathbf{e}$ adjacent to $\mathbf{m}$, $\mathbf{d}$, $\mathbf{n}$, and $\mathbf{b}$. Suppose $\mathbf{l}$ and $\mathbf{m}$ are contained in two intersecting local lines $l_\mathbf{a}$ and $m_\mathbf{a}$, likewise $\mathbf{n}$ and $\mathbf{o}$ in two intersecting local lines $n_\mathbf{b}$ and $o_\mathbf{b}$. Suppose the local lines in $\mathbf{d}^\perp$ containing $\mathbf{m}$ and $\mathbf{n}$ intersect in a point. Then we can see whether the local point $l_\mathbf{a} \wedge m_\mathbf{a}$ (the intersection of $l_\mathbf{a}$ and $m_\mathbf{a}$) and the local point $m_\mathbf{d} \wedge n_\mathbf{d}$ both actually lie in $\mathbf{a}^\perp \cap \mathbf{d}^\perp$ and are equal or not. By the above lemma, this then automatically holds for the respective lines in $\mathbf{d}^\perp$ and $\mathbf{e}^\perp$. Now we can see whether the points $n_\mathbf{b} \wedge o_\mathbf{b}$ and $m_\mathbf{e} \wedge n_\mathbf{e}$ both lie in $\mathbf{b}^\perp \cap \mathbf{e}^\perp$ and describe the same local point. Two local points $l_\mathbf{a} \wedge m_\mathbf{a}$ and $n_\mathbf{b} \wedge o_\mathbf{b}$ are **kin**, if they are equal or one can find a chain of vertices as described above where the respective local points mutually are equal.

Lemma 2.4.9
The notion of kinship is well defined, i.e., it is independent of the chosen path in Γ.

Proof. It is obvious that for mutually adjacent vertices $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ kinship is independent of the path chosen in the triangle $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$.

Let $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be a path of vertices such that there exists a local point $p_\mathbf{w}$ that induces a local point $p_\mathbf{x}$ which in turn induces a local point $p_\mathbf{y}$ that induces a local point $p_\mathbf{z}$. Using the notation of Lemma 2.4.2, we have a subspace $\langle l_\mathbf{x}, w_\mathbf{x}, y_\mathbf{x} \rangle$ of $\mathbf{x}^\perp$. If that space $\langle l_\mathbf{x}, w_\mathbf{x}, y_\mathbf{x} \rangle$ does not contain the point $p_\mathbf{x}$, by Lemma 2.4.2 we can find a vertex $\mathbf{a} \in \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}^\perp$ whose line $a_\mathbf{x}$ in $\mathbf{x}^\perp$ contains the point $p_\mathbf{x}$. Locally in $\mathbf{a}^\perp$ we find a common neighbor $\mathbf{v}$ of $\mathbf{w}$ and $\mathbf{z}$. It is easily seen that the kinship of the two interior points $p_\mathbf{w}$ and $p_\mathbf{z}$ is independent of our choice of the path $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ or the path $\mathbf{w}$, $\mathbf{v}$, $\mathbf{z}$. For, it is possible to find a triangulization of the circuit $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{v}$, $\mathbf{w}$ inside $\mathbf{a}^\perp \cong \mathbf{L}_n(\mathbb{F})$. So assume that $p_\mathbf{x} \in \langle l_\mathbf{x}, w_\mathbf{x}, y_\mathbf{x} \rangle$. Then we find a vertex $\mathbf{a} \in \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}^\perp$ that locally admits the point $p_\mathbf{a}$ that is kin to $p_\mathbf{w}$ and $p_\mathbf{z}$. Since kinship is independent of paths in triangles, the kinship of $p_\mathbf{w}$ and $p_\mathbf{z}$ is independent of the chosen path $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ or $\mathbf{w}$, $\mathbf{a}$, $\mathbf{z}$.

Now let the local points $p_\mathbf{x}$ and $p_\mathbf{y}$ be kin. As yet we have proved that we can assume that there is a path $\mathbf{x}$, $\mathbf{z}_1$, $\mathbf{y}$ establishing the kinship of $p_\mathbf{x}$ and $p_\mathbf{y}$. Take any other path from $\mathbf{x}$ to $\mathbf{y}$. By our findings above we can assume this path is of length two, $\mathbf{x}$, $\mathbf{z}_2$, $\mathbf{y}$ say. But now we can study the graph $\{\mathbf{x}, \mathbf{z}_1, \mathbf{y}, \mathbf{z}_2\}^\perp$ and we find, as above, that the kinship of $p_\mathbf{x}$ and $p_\mathbf{y}$ is independent of the choice of $\mathbf{x}$, $\mathbf{z}_1$, $\mathbf{y}$ or $\mathbf{x}$, $\mathbf{z}_2$, $\mathbf{y}$. □

Definition 2.4.10 Let $\mathbf{x}$ be a vertex of Γ and let $p_\mathbf{x}$ be a local point of $\mathbf{x}^\perp$. The set

$$\bigcup_{l \in \mathcal{L}_\Gamma,\ l \cap p_\mathbf{x} \neq \emptyset} l$$

is called an **almost local point** of $\mathbf{x}^\perp$ and is denoted by $p^\mathbf{x}$. Note that a local point is the union of some local lines. An almost local point is nothing more than the union of the corresponding global lines. Two almost local points are **kin** if the corresponding local points are kin. Define a relation $\approx^p$ on the set of almost local points of Γ by $p^\mathbf{x} \approx^p q^\mathbf{y}$ if and only if they are kin.

Lemma 2.4.11
The relation $\approx^p$ is an equivalence relation on the almost local points of Γ.

Proof. This is obvious. □

Definition 2.4.12 The equivalence classes of the relation $\approx^p$ are called **global points**. The set of global points of Γ is denoted by $\mathcal{P}_\Gamma$. Dually, define **global hyperplanes**. Moreover, let $\mathbb{P}_\Gamma = (\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ be the point-line geometry consisting of the point set $\mathcal{P}_\Gamma$ of global points of Γ and the line set $\mathcal{L}_\Gamma$ of global lines of Γ with symmetrized containment as incidence.

Like global lines, also global points have nice local behavior:

Lemma 2.4.13
Let $\mathbf{x}$ and $\mathbf{y}$ be distinct vertices of Γ, and let p be a global point. Then $p \cap \mathbf{x}^\perp$ is either empty or a local point of $\mathbf{x}^\perp$.

Proof. Assume $\mathbf{m} \in p \cap \mathbf{x}^\perp$. Then there is a vertex $\mathbf{z}$ such that $\mathbf{m} \in q^\mathbf{z}$ for an almost local point $q^\mathbf{z}$ of $\mathbf{z}^\perp$ with $[q^\mathbf{z}]_{\approx^p} = p$. Hence there exist vertices $\mathbf{l}$ and $\mathbf{m}_1$ adjacent to $\mathbf{z}$ belonging to different global lines l and m contained in the almost local point $q^\mathbf{z}$. By Lemma 2.4.2 there exists a vertex $\mathbf{a}$ adjacent to $\mathbf{l}$ and $\mathbf{m}$ and, again by Lemma 2.4.2, there exists a vertex $\mathbf{b}$ adjacent to $\mathbf{l}$, $\mathbf{a}$, $\mathbf{m}$, and $\mathbf{x}$. Lemma 2.4.8 implies that the local lines $l_\mathbf{b}$ and $m_\mathbf{b}$ intersect in an interior point, and local analysis of $\mathbf{b}^\perp$ together with Lemma 2.1.6 implies the existence of a vertex $\mathbf{l}_1 \in l$ adjacent to $\mathbf{x}$. □

Proposition 2.4.14
$\mathbb{P}_\Gamma$ is a projective space.

Proof. First we have to show that $\mathbb{P}_\Gamma$ is a linear space. This means that any two global points are incident with a global line and that any two global lines intersecting in two distinct global points are equal. Let p and q be distinct global points, and let $\mathbf{p} \in p$ and $\mathbf{q} \in q$. By Lemma 2.4.2, there exists a vertex $\mathbf{a}$ adjacent to both $\mathbf{p}$ and $\mathbf{q}$, and Lemma 2.4.13 shows that $p \cap \mathbf{a}^\perp$ and $q \cap \mathbf{a}^\perp$ are local points of $\mathbf{a}^\perp$. By Lemma 2.2.16 there exists a vertex $\mathbf{l} \in p \cap q \cap \mathbf{a}^\perp$. The global line l containing $\mathbf{l}$ is incident with both p and q. Now suppose we have two global lines l and m intersecting in distinct global points p and q. Arguments similar to the arguments before yield a vertex $\mathbf{x}$ with local points $p_\mathbf{x}$, $q_\mathbf{x}$ and local lines $l_\mathbf{x}$, $m_\mathbf{x}$. The local lines have to be distinct, since the global lines l and m are distinct and contain both local points, a contradiction to the fact that the interior projective space on $\mathbf{x}^\perp$ is a linear space.

The validity of Pasch's axiom remains to be shown. To this end, let l and m be two intersecting global lines. There exist vertices $\mathbf{l} \in l$ and $\mathbf{m} \in m$ and, by Lemma 2.4.2, a vertex $\mathbf{x}$ adjacent to both. Now, since Pasch's axiom holds in the interior projective space on $\mathbf{x}^\perp$ it also holds for all configurations involving the lines l and m. Since the lines l and m have been chosen arbitrarily, Pasch's axiom holds in the geometry $\mathbb{P}_\Gamma$. □

Notation 2.4.15 For a vertex $\mathbf{x}$ of Γ, denote by $\langle \mathbf{x}^\perp \rangle$ the set of global points that intersect $\mathbf{x}^\perp$.

Lemma 2.4.16
$\langle \mathbf{x}^\perp \rangle$ does not intersect the global line containing $\mathbf{x}$.

Proof. Otherwise there exist a global point p with $\mathbf{x} \in p$ and a vertex $\mathbf{y} \in p$ with $\mathbf{y} \perp \mathbf{x}$. But then there exists a third vertex $\mathbf{z}$ adjacent to both $\mathbf{x}$ and $\mathbf{y}$, and local analysis of $\mathbf{z}^\perp$ yields a point $p_\mathbf{z}$ that lies on either of the local lines described by two adjacent vertices, a contradiction. □

Lemma 2.4.17
Let $\mathbf{x}$ be a vertex of Γ. Then $\langle \mathbf{x}^\perp \rangle$ is a hyperline of $\mathbb{P}_\Gamma$. Conversely, any hyperline Λ of $\mathbb{P}_\Gamma$ is of this form. Moreover, if the global line y does not intersect Λ, then there is a vertex $\mathbf{y} \in y$ with $\langle \mathbf{y}^\perp \rangle = \Lambda$.

Proof. Let l be a global line of Γ incident with a global point p contained in $\langle \mathbf{x}^\perp \rangle$. The intersection of that global point with $\mathbf{x}^\perp$ is a local point of $\mathbf{x}^\perp$ since it cannot be empty, by Lemma 2.4.13. For any other global point q incident with l, the line l is the unique global line connecting both points (by Proposition 2.4.14). The intersection $q \cap \mathbf{x}^\perp$ is also a local point of $\mathbf{x}^\perp$, distinct from $p \cap \mathbf{x}^\perp$. Hence there exists a local line of $\mathbf{x}^\perp$ connecting both. But this local line has to be equal to $l \cap \mathbf{x}^\perp$, thus all global points incident with l are contained in $\langle \mathbf{x}^\perp \rangle$, and $\langle \mathbf{x}^\perp \rangle$ is a subspace of $\mathbb{P}_\Gamma$. Now let U be a three-dimensional space. There exist two non-intersecting global lines l, m that span U. Take vertices $\mathbf{l} \in l$, $\mathbf{m} \in m$. By Lemma 2.4.2, there exists a vertex $\mathbf{a}$ adjacent to both $\mathbf{l}$ and $\mathbf{m}$. By Lemma 2.2.18 applied to $\mathbf{a}^\perp$, we can assume that $\mathbf{l}$ and $\mathbf{m}$ are adjacent. Again by Lemma 2.4.2 there exists a vertex $\mathbf{b}$ adjacent to $\mathbf{x}$ and $\mathbf{l}$. A third application of Lemma 2.4.2 yields the existence of a vertex $\mathbf{c}$ adjacent to $\mathbf{x}$, $\mathbf{b}$, $\mathbf{l}$, $\mathbf{m}$. In $\mathbf{c}^\perp$ the local lines containing $\mathbf{l}$ and $\mathbf{m}$ span a 3-space U'. For any global point $p \in U$ we have $p \cap \mathbf{c}^\perp \in U'$. By Lemma 2.2.17, $\mathbf{x}^\perp \cap \mathbf{c}^\perp$ is a hyperline of $\mathbf{c}^\perp \cong \mathbf{L}_n(\mathbb{F})$, which has to intersect U' in at least a line. But this proves that any three-dimensional subspace of $\mathbb{P}_\Gamma$ intersects $\langle \mathbf{x}^\perp \rangle$ in at least a line, whence $\langle \mathbf{x}^\perp \rangle$ is a hyperline of $\mathbb{P}_\Gamma$ (it cannot be a hyperplane by Lemma 2.4.16).

Conversely, let y be a global line that does not intersect the hyperline Λ. Any three-dimensional space containing y intersects Λ in a global line, x, say. Choose a vertex in both lines. By Lemma 2.4.2 there exists a third vertex adjacent to both. Then by Lemma 2.2.18, there exist adjacent vertices $\mathbf{x} \in x$ and $\mathbf{y}_1 \in y$. The hyperline $\langle \mathbf{x}^\perp \rangle$ intersects Λ in a hyperline of Λ, since the global line x containing $\mathbf{x}$ does not intersect $\langle \mathbf{x}^\perp \rangle$ by Lemma 2.4.16. Let V be a subspace of $\langle \mathbf{x}^\perp \rangle \cap \Lambda$ of maximal odd dimension. We find a clique of vertices $\mathbf{x}_i$, $i \in I$ (for some index set I), such that the global points x_i of Γ containing them span V as a projective space, by Lemma 2.2.18. Then x together with the x_i spans Λ or a hyperplane of Λ, of odd dimension. Moreover, y and x_i, $i \in I$, span $\langle \mathbf{x}^\perp \rangle$ or a hyperplane W, of odd dimension, since the x_i span V and y is a line of $\langle \mathbf{x}^\perp \rangle \backslash V$. But again by Lemma 2.2.18, in $\langle \mathbf{x}^\perp \rangle$ (respectively W) we can find $\mathbf{y} \in y$ such that $\mathbf{y}$ and $\mathbf{x}_i$, $i \in I$, form a clique. But then the hyperline $\langle \mathbf{y}^\perp \rangle$ contains the lines x, x_i, $i \in I$, whence $\langle \mathbf{y}^\perp \rangle \cap \Lambda$ equals Λ or a hyperplane of Λ. In the former case the lemma is proved. In the latter the claim follows immediately by suitable variation of the subspace V. □

Proposition 2.4.18
Let $n \geq 11$, let $\mathbb{F}$ be a division ring, let Γ be a connected, locally $\mathbf{L}_n(\mathbb{F})$ graph, and let $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma, \supset)$ be the projective space consisting of the global points and global lines of Γ. Then the line-hyperline graph of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma, \supset)$ is isomorphic to Γ.

Proof. Let Δ be the line-hyperline graph of $\mathbb{P}_\Gamma$. Consider the map $\Gamma \to \Delta : \mathbf{x} \mapsto (x, \langle \mathbf{x}^\perp \rangle)$ where x is the global line of Γ containing $\mathbf{x}$. We want to show that this is an isomorphism of graphs. Surjectivity follows from Lemma 2.4.17, since any line x of $\mathbb{P}_\Gamma$ is a global line of Γ and any hyperline not intersecting it is of the form $\langle \mathbf{x}^\perp \rangle$ for a vertex $\mathbf{x} \in x$. Injectivity is obtained as follows. Suppose the global line x contains two vertices $\mathbf{x}_1$, $\mathbf{x}_2$ with $\langle \mathbf{x}_1^\perp \rangle = \langle \mathbf{x}_2^\perp \rangle$. By Lemma 2.4.2 there exists a vertex $\mathbf{y}$ adjacent to both $\mathbf{x}_1$ and $\mathbf{x}_2$. Since $\langle \mathbf{x}_1^\perp \rangle = \langle \mathbf{x}_2^\perp \rangle$, both vertices describe the same hyperline in $\mathbf{y}^\perp$. But they also describe the same line and hence have to be equal, a contradiction. Finally, if $\mathbf{x} \perp \mathbf{y}$, then obviously $x \in \langle \mathbf{y}^\perp \rangle$ and $y \in \langle \mathbf{x}^\perp \rangle$, if x and y are the global lines of Γ containing $\mathbf{x}$ and $\mathbf{y}$, respectively. □

Theorem 2.4.19
Let $n \geq 11$, let $\mathbb{F}$ be a division ring, and let Γ be a connected, locally $\mathbf{L}_n(\mathbb{F})$ graph. Then Γ is isomorphic to $\mathbf{L}_{n+2}(\mathbb{F})$.

Proof. The theorem follows immediately from Proposition 2.4.14, Lemma 2.4.17, and Proposition 2.4.18. □

The graph on the fundamental SL_2's of the group $E_6(\mathbb{F})$ provides an example of a graph that is connected and locally $\mathbf{L}_5(\mathbb{F})$ but not isomorphic to $\mathbf{L}_7(\mathbb{F})$.

2.5 More results without proofs

We didn't achieve the best result possible in the preceding section. A bit more work, similar to the extra work we had in the cases $n = 3, 4$ for the point-hyperplane graphs in Chapter 1, yields the following result.

Theorem 2.5.1
Let $n \geq 7$, let $\mathbb{F}$ be a division ring, and let Γ be a connected, locally $\mathbf{L}_n(\mathbb{F})$ graph. Then Γ is isomorphic to $\mathbf{L}_{n+2}(\mathbb{F})$.

Sketch of proof. First prove the line-hyperline equivalents of Lemmas 1.3.4, 1.3.5, and 1.3.6, which imply simple connectedness of Γ. The proof of the existence of global lines is then exactly the same as the proof of the existence of global points in Section 1.3. However, in the case of line-hyperline graphs we cannot easily construct global points. Instead we define a global plane as we defined a global line in Definition 1.3.12. Then one proves that the notion of a global plane is well defined. In the notation of Lemma 1.3.13 we have the problem that we cannot assume $\mathbf{c}$ and $\mathbf{d}$ to be connected. Nevertheless one can find the vertex $\mathbf{z}_1$ adjacent to both $\mathbf{x}$ and $\mathbf{c}$, and there is a $\mathbf{z}_2$ adjacent to $\mathbf{a}$, $\mathbf{x}$, and $\mathbf{c}$. Then, as in Lemma 1.3.4 and after replacement of $\mathbf{a}$ inside its local line of $\mathbf{z}_2^\perp$ if necessary, one can find a path of vertices inside $\mathbf{a}^\perp \cap \mathbf{c}^\perp$ from $\mathbf{z}_2$ to a vertex $\mathbf{v}$ that is also adjacent to $\mathbf{y}$, proving that global planes are well defined. Similarly, one defines global 3-spaces and proves that they are well defined. Now we can define global points as pencils of global lines. Two global lines intersect if and only if they span a global plane. Three global lines intersect in a single point if they mutually intersect and together span a global 3-space. It remains to prove, essentially by proving the validity of Pasch's Axiom, that the geometry on global points and global lines is a projective space and that the graph we are considering is the line-hyperline graph of that geometry. ■

We also have a result for infinite dimensions. Notice that, although we used finite dimensions in the proof of Lemma 2.4.17, we can modify that lemma to a statement about infinite dimensional vector spaces. Then we can just plug in that modified lemma into the proof of Proposition 2.4.18 without making use of finite dimensions. Altogether we obtain the following.

Theorem 2.5.2
Let n be a infinite cardinal number, let $\mathbb{F}$ be a division ring, and let Γ be a connected, locally $\mathbf{L}_n(\mathbb{F})$ graph. Then Γ is isomorphic to $\mathbf{L}_n(\mathbb{F})$. □

Moreover, there exist group-theoretic consequences of Theorem 2.5.1 similar to the consequences of Theorem 1.3.21, as given in Section 1.6. For example, the following is true.

Theorem 2.5.3
Let $n \geq 8$ and let $\mathbb{F}$ be a field of characteristic distinct from 2. Let G be a group with subgroups A and B isomorphic to $SL_2(\mathbb{F})$, and denote the central involution of A by x and the central involution of B by y. Furthermore, assume the following holds:

- *$C_G(x) = A' \times K$ with $K \cong GL_n(\mathbb{F})$, $A \leq A'$;*
- *$C_G(y) = B' \times J$ with $J \cong GL_n(\mathbb{F})$, $B \leq B'$;*
- *A is a fundamental SL_2 of J;*
- *B is a fundamental SL_2 of K; and*
- *there exists an involution in $J \cap K$ that is the central involution of a fundamental SL_2 of both J and K.*

If $G = \langle J, K \rangle$, then $PSL_{n+2}(\mathbb{F}) \leq G/Z(G) \leq PGL_{n+2}(\mathbb{F})$. ■

Let us now generalize our findings for point-hyperplane graphs and line-hyperline graphs to arbitrary decompositions of vector spaces. To this end let $\mathbb{F}$ be a division ring and let $n \geq 0$. For any $k \leq n$, the graph $\mathbf{SC}_{n,k}$ consists of the pairs of a dimension k space and a non-intersecting codimension k space of $\mathbb{P}_n(\mathbb{F})$ where a vertex (a, A) is adjacent to another vertex (b, B) (in symbols $(a, A) \perp (b, B)$) if and only if $a \subset B$ and $b \subset A$. (The **S** stands for *space*, the **C** for *complement*.) Notice that $\mathbf{SC}_{n,0} = \mathbf{H}_n$ (cf. Chapter 1) and $\mathbf{SC}_{n,1} = \mathbf{L}_n$. Using the methods of Section 1.3 and Section 2.4 one can prove the following analogs of Theorem 1.3.21 and Theorem 2.5.1.

Theorem 2.5.4
Let $k \geq 0$, let $n \geq 4(k+1) - 1$, let $\mathbb{F}$ be a division ring, and let Γ be a connected, locally $\mathbf{SC}_{n,k}(\mathbb{F})$ graph. Then Γ is isomorphic to $\mathbf{SC}_{n+k+1,k}(\mathbb{F})$. ■

This result is related to a result of Hall's in [Hal87] on Kneser graphs. Kneser graphs are in some sense thin versions of our graphs $\mathbf{SC}_{n,k}$. Indeed, consider the $(n+1)$-simplex which from a geometric point of view is a thin projective space. In this setting a dimension k space coincides with a subset of the $(n+1)$-simplex of size $k+1$, whereas a codimension k space coincides with a subset of size $n-k$. Notice that by a choice of a dimension k space the non-intersecting codimension k space is uniquely determined. Therefore the set of vertices can be described as the set of dimension k spaces. Now we can define adjacency of two vertices as being disjoint, and we obtain the Kneser graphs. Denote the Kneser graph on the $(k+1)$-subsets of a set of cardinality $n+1$ by $\mathbf{SC}_{n,k}(1)$. Then Jon Hall's Theorem 1 of [Hal87] states that a connected, locally $\mathbf{SC}_{n,k}(1)$ graph is isomorphic to $\mathbf{SC}_{n+k+1,k}(1)$ provided n

be sufficiently large. For example, a connected, locally $\mathbf{SC}_{n,1}(1)$ graph is isomorphic to $\mathbf{SC}_{n+2,1}(1)$, if $n \geq 6$. Hall's bound in the case of $k = 1$ is sharp. Indeed, the Kneser graph $\mathbf{SC}_{5,1}(1)$ is isomorphic to the collinearity graph of the generalized quadrangle on 15 points and 15 lines. By a result of Buekenhout and Hubaut (Theorem 2 of [BH77]), there are the following three isomorphism types of graphs that are locally $\mathbf{SC}_{5,1}(1)$. First, let $\mathbb{A}_5(2)$ be the five-dimensional affine space over the field of two elements and consider $\mathbb{P}_4(2)$ as its projective space at infinity. Let Q be the nondegenerate orthogonal quadric of rank 2 in $\mathbb{P}_4(2)$ (this quadric is isomorphic to the generalized quadrangle on 15 points and 15 lines). Define a graph Γ whose vertices are the points of the affine space $\mathbb{A}_5(2)$ in which distinct vertices p, q are adjacent if and only if the line pq intersects the projective space at infinity in a point of Q. To construct the two other examples let $\mathbb{P}_5(2)$ be the five-dimensional projective space over the field of two elements, and let Q be a nondegenerate orthogonal quadric of $\mathbb{P}_5(2)$. Define a graph Γ whose vertices are the non-singular points of $\mathbb{P}_5(2)$ in which distinct vertices p, q are adjacent if and only if the line pq is a tangent line to the quadric Q. The two possible isometry types of Q give rise to two isomorphism types for Γ. The graph Γ is isomorphic to the Kneser graph $\mathbf{SC}_{7,1}(1)$ in case Q is of $+$ type.

For k greater than one Jon Hall has the following result. If $k \geq 2$, $n \geq 3(k+1)$, and Γ is a connected, locally $\mathbf{SC}_{n,k}(1)$ graph, then Γ is isomorphic to $\mathbf{SC}_{n+k+1,k}(1)$, by Theorem 2 of [Hal87]. There is some difference between Hall's bound for the thin case and our bound for the thick case, as given in Theorem 2.5.4, and it might still be possible to improve the bound given in our theorem.

Let us finish this chapter with a group-theoretic consequence of Jon Hall's work on locally Kneser graphs. Actually, this implication has been a motivation for Hall to pursue the local recognition of Kneser graphs. Notice the similarities to the 'thick' case stated in Theorem 2.5.3.

Theorem 2.5.5 (e.g., Gorenstein et al. [GLS94], Theorem 27.1)
Let $m \geq 7$, and let G be a group with distinct involutions x, y such that

- $C_G(x) = \langle x \rangle \times K$ *with* $K \cong Sym_m$;
- $C_G(y) = \langle y \rangle \times J$ *with* $J \cong Sym_m$;
- x *is a transposition in* J;
- y *is a transposition in* K; *and*
- *there exists an involution in* $J \cap K$ *that is a transposition in both* J *and* K.

If $G = \langle J, K \rangle$, *then* $G \cong Sym_{m+2}$. □

Chapter 3

Curtis-Phan-Tits Theorems

This chapter starts with a discussion of the famous Curtis-Tits theorem and Phan's theorems. Both are very important recognition tools in the classification of finite simple groups. Recent developments have shown that there is a uniform geometric approach to both theorems by looking at certain sub-chamber systems of the opposites chamber system of spherical twin buildings as defined by Tits in [Tit90]; see also [Müh] or Definition 3.1.3. The Curtis-Tits theorem is a direct consequence of Mühlherr's result [Müh] on the 2-simple connectedness of this opposites chamber system after invoking Tits' lemma B.2.5. However, Mühlherr's approach in [Müh] fails to work in some small cases, which are covered by the Curtis-Tits theorem. Section 3.1 offers a discussion of the different versions of the Curtis-Tits theorem, including Mühlherr's. In [BS] Curt Bennett and Sergey Shpectorov, on the other hand, obtained a new proof of Phan's theorem from [Pha77a], see also Theorem 3.2.1. They use the geometry on the nondegenerate subspaces with respect to a nondegenerate unitary form on a vector space. The 2-simple connectedness of this geometry yields the fact that Phan's amalgam has the group $SU_{n+1}(q^2)$ as its universal completion. Bennett and Shpectorov also study related amalgams and obtain uniqueness of the amalgam. The chamber system of this geometry arises as a sub-chamber system of the above-mentioned opposites chamber system in a natural way as explained in Section 3.3.

The main purpose of this chapter is to present a detailed proof of a Phan-type theorem. That theorem is joint work with Corneliu Hoffman and Sergey Shpectorov and originates from [GHS]. This illustrates that the methods described in Section 3.3 give rise to several theorems, some of which have already been proved by Phan in [Pha77a], [Pha77b]; others are new. The geometry related to this Phan-type theorem can be described as the set of all subspaces of some vector space V over $\mathbb{F}_{q^2}$ that are totally isotropic with respect to some nondegenerate symplectic form and nondegenerate with respect to some related unitary form. Connectivity properties will imply that the group $Sp_{2n}(q)$ is the universal completion of some amalgam of groups isomorphic to $SU_2(q^2)$ that pairwise generate groups isomorphic to either

$SU_3(q^2)$, $Sp_4(q)$, or the direct product of two copies of $SU_2(q^2)$, see Theorem 3.8.4. The pursuit of this theorem was done at the request of Richard Lyons and Ronald Solomon, who were interested in a theorem in this flavor on the symplectic groups for the revision of the classification of finite simple groups. Refer to Section 3.8 for a precise statement of the group-theoretic results.

3.1 Curtis-Tits by Mühlherr

In Section C.3 we have given Steinberg's theorem on defining relations for the universal Chevalley group constructed from an indecomposable root system Σ of rank at least two and a field $\mathbb{F}$. The Curtis-Tits theorem states that several of the Steinberg relations are redundant. We present the Curtis-Tits theorem in several guises. Let us start with the version of Curtis [Cur65], which illustrates which Steinberg relations are superfluous.

Curtis-Tits Theorem 3.1.1 (Curtis [Cur65], Corollary 1.8)
Let Σ be an indecomposable root system of rank at least two, and let Π be a fundamental system of Σ. Furthermore, let $\mathbb{F}$ be an arbitrary field. Define G to be the abstract group with generators $\{x_r(t) \mid r \in \Sigma, t \in \mathbb{F}\}$ and defining relations

$$x_r(t)x_r(u) = x_r(t+u), r \in \Sigma, t, u \in \mathbb{F}, \tag{3.1.1}$$

and for independent roots r, s,

$$[x_r(t), x_s(u)] = \prod x_{hr+ks}(C_{hkrs}t^h u^k), \tag{3.1.2}$$

with $h, k > 0$, $hr + ks \in \Sigma$ (if there are no such numbers, then $[x_r(t), x_s(u)] = 1$), and structure constants $C_{hkrs} \in \{\pm 1, \pm 2, \pm 3\}$.

Let $A = \bigcup A_{ij}$, where A_{ij} is the set of all roots which are linear combinations of the fundamental roots $r_i, r_j \in \Pi$. Let G^* be the abstract group with generators $\{x_r(t) \mid r \in \Sigma, t \in \mathbb{F}\}$ and defining relations (3.1.1), for $r \in A$, and (3.1.2) for independent roots r, s belonging to some A_{ij}. Then the natural epimorphism $G^* \to G$ is an isomorphism.

By Theorem C.3.2 and the above theorem, the group G^* is isomorphic to the universal Chevalley group constructed from Σ and $\mathbb{F}$, if $\mathbb{F}$ is an algebraic extension of a finite field. Otherwise one can obtain the universal Chevalley group by adding the set (iii) of Steinberg relations from Section C.3 to the above theorem. Rephrasing Curtis' version of the Curtis-Tits theorem in amalgam language yields the following:

Curtis-Tits Theorem 3.1.2 (Gorenstein et al. [GLS98], Theorem 2.9.3)
Let Σ be an indecomposable root system of rank at least two with a fundamental system Π, and let $\mathbb{F}$ be a field. Let G be the universal Chevalley group constructed from Σ and $\mathbb{F}$. For each $r \in \Sigma$ denote by x_r the root subgroups $\{x_r(t) \mid t \in \mathbb{F}\}$, and for each $J \subset \Pi$ let G_J be the subgroup of G generated by all root subgroups x_r,

$\pm r \in J$. Let D be the set of all subsets of Π with at most two elements. Then G is the universal completion of the amalgam $\bigcup_{J \in D} G_J$.

The set (iii) of the Steinberg relations from Section C.3 only involves x_r and x_{-r} for each root r and, thus, the elements lie in the fundamental SL_2's generated by x_r and x_{-r}. Tits' version of the Curtis-Tits theorem as can be found in [Tit62], Theorem 2.12, is very similar to Theorem 3.1.2.

Bernhard Mühlherr's approach to that problem of defining amalgams is a completely different one based on chamber systems. For the definition of and results related to twin buildings refer to Section A.3.

Definition 3.1.3 Let B be a spherical Tits building, and let $\mathcal{B} = (B_+, B_-, d^*)$ be the twin building as given in Proposition A.3.2. Let $\mathcal{C}(B_+) = (\mathcal{C}_+, (\sim_i)_{i \in I})$ and $\mathcal{C}(B_-) = (\mathcal{C}_-, (\sim_i)_{i \in I})$ be the respective chamber systems associated to B_+ and B_-. Then the chamber system $Opp(\mathcal{B}) = (\mathcal{C}, (\sim_i)_{i \in I})$ consists of the set of chambers $\mathcal{C} = \{(c_+, c_-) \in \mathcal{C}_+ \times \mathcal{C}_- \mid c_+ \text{ opp } c_-\}$ where, for $i \in I$ and $c = (c_+, c_-), d = (d_+, d_-) \in \mathcal{C}$, the chamber c is defined to be i-adjacent to the chamber d if $c_+ \sim_i d_+$ and $c_- \sim_i d_-$. The chamber system $Opp(\mathcal{B})$ is called the **opposites chamber system** of the twin building $\mathcal{B}$.

Mühlherr has proved the following result.

Theorem 3.1.4 (Mühlherr [Müh], Main Theorem)
Let $\mathcal{B} = (B_+, B_-, \delta^*)$ be a thick spherical twin building of rank at least two such that B_+ (whence also B_-) does not contain a rank 2 residue which is isomorphic to the building associated to $B_2(2)$. Then $Opp(\mathcal{B})$ is 2-simply connected.

Actually, Mühlherr does not require $\mathcal{B}$ to be spherical. Instead, he demands that neither B_+ nor B_- have tree residues (i.e., a residue isomorphic to a generalized ∞-gon) or residues isomorphic to buildings of type $B_2(2)$, $G_2(2)$, $G_2(3)$, or ${}^2F_4(2)$, and concludes 2-simple connectedness of $Opp(\mathcal{B})$. In case $\mathcal{B}$ is non-spherical, this implies a result on defining amalgams of certain Kac-Moody groups. In case of a spherical $\mathcal{B}$, however, we obtain the Curtis-Tits theorem:

Curtis-Tits Theorem 3.1.5 (Mühlherr [Müh], Application 1)
Let $\mathcal{B}$ be a thick spherical twin building of rank at least two such that B_+ (whence also B_-) does not contain a rank 2 residue which is isomorphic to the building associated to $B_2(2)$. Let G be a group of automorphisms of $\mathcal{B}$ that acts transitively on the set of pairs of opposite chambers, and let (c_+, c_-) be a pair of opposite chambers of $\mathcal{B}$. Moreover, let D denote the set of all subsets of the type set I that have cardinality at most two, and for each $J \in D$ let G_J denote the subgroup of G which stabilizes the J-cell $(c_+, c_-)J$ of (c_+, c_-). Then the group G is the universal completion of the amalgam $\bigcup_{J \in D} G_J$.

Proof. The claim follows from the theorem and Tits' lemma B.2.5. □

The exceptions in Theorem 3.1.4 arise from the proof rather than being actual exceptions. For, in a spherical twin building that admits a residue of type $B_2(2)$, the set of chambers opposite to a fixed chamber need not be connected and Mühlherr's methods of proof fail to work. The same holds for residues of type $B_2(2)$, $G_2(2)$, $G_2(3)$, ${}^2F_4(2)$ in the non-spherical case. See Section 3.2 of the PhD thesis of Rieuwert Blok [Blo99] for a discussion of connectedness of chamber systems far from a residue. Especially notice Theorem 3.12, which can also be found as Proposition 7 in Peter Abramenko's lecture notes [Abr96]. Furthermore, in [AV99] Peter Abramenko and Hendrik Van Maldeghem treat the problem of non-connectedness of the set of chambers opposite to a chamber of a Moufang polygon and obtain exactly the list $B_2(2)$, $G_2(2)$, $G_2(3)$, ${}^2F_4(2)$ as above.

3.2 Phan by Bennett and Shpectorov

Phan's theorems have the same flavor as the Curits-Tits theorem. The main objective is to find nice defining amalgams for certain groups of Lie type. In addition, Phan's theorems state the uniqueness of those amalgams. He studies groups G that admit what is now called a **Phan system of type Δ over $\mathbb{F}_{q^2}$**, i.e., Δ is one of the diagrams A_n, D_n, E_6, E_7, E_8, and for each vertex i of Δ there is a pair of subgroups L_i, H_i of G such that the following conditions hold.

(i) G is generated by the subgroups L_i,

(ii) $L_i \cong SU_2(q^2)$, $H_i \leq L_i$, $|H_i| = q + 1$,

(iii) $[L_i, L_j] = 1$ whenever (i, j) is not an edge of Δ,

(iv) $\langle L_i, L_j \rangle \cong SU_3(q^2)$, $\langle L_i, H_j \rangle \cong GU_2(q^2) \cong \langle H_i, L_j \rangle$ whenever (i, j) is an edge of Δ,

(v) $\langle H_i, H_j \rangle = H_i \times H_j$ whenever $i \neq j$.

Then Phan's first theorem as given in [Pha77a] reads as follows:

Theorem 3.2.1 (Phan [Pha77a], Theorem 2.3)
Let $n \geq 2$, and let G be a group that admits a Phan system of type A_n over $\mathbb{F}_{q^2}$. If $q \geq 5$, then G is a homomorphic image of $SU_{n+1}(q^2)$ (the universal Chevalley group of type ${}^2A_n(q^2)$).

Like the Curtis-Tits theorem, Phan's theorem has been used as a recognition tool in the classification of finite simple groups. Theorem 3.2.1 has been investigated from a geometric point of view by Kaustuv Mukul Das and Michael Aschbacher, and later by Curt Bennett and Sergey Shpectorov. Let us review here the work of Bennett and Shpectorov, see [BS]. They study an incidence system $\mathcal{N} = \mathcal{N}(n, q^2)$ defined as follows.

Definition 3.2.2 Let $n \geq 2$, and let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_{q^2}$ endowed with a nondegenerate unitary form. Then the pregeometry $\mathcal{N}$ is the set of all nondegenerate subspaces of V with respect to the unitary form with symmetrized containment as incidence.

It is easily seen that $\mathcal{N}$ actually is a geometry. Moreover, Bennett and Shpectorov obtain the following.

Theorem 3.2.3 (Bennett, Shpectorov)
Let $n \geq 2$. The geometry $\mathcal{N}$ is connected unless $(n,q) = (2,2)$. If $n \geq 3$, the geometry $\mathcal{N}$ is simply connected unless $(n,q) = (3,2)$ or $(3,3)$.

Corollary 3.2.4 (Bennett, Shpectorov)
For $n \geq 3$, the geometry $\mathcal{N}$ is 3-simply connected. For $n \geq 2$, the geometry $\mathcal{N}$ is 2-simply connected if $q \geq 4$.

With the 2-simple connectedness of $\mathcal{N}$ in case of $q \geq 4$ and yet another application of Tits' lemma B.2.5, Bennett and Shpectorov prove that the amalgam of subgroups of $SU_{n+1}(q^2)$ as given in Phan's theorem has $SU_{n+1}(q^2)$ as its universal completion. It remains to prove the uniqueness of this amalgam, which can also be found in [BS], and Phan's theorem follows. There even is a slight improvement of the bound on q. In case $q = 2, 3$, Bennett and Shpectorov obtain a similar result under the additional assumption that three copies of $SU_2(q^2)$ amalgamate nicely. Phan's other theorems read as follows. It is conjectured in [GHS] that they can also be proved, case by case, in Bennett-Shpectorov style.

Theorem 3.2.5 (Phan [Pha77b], Theorem 1.9)
Let $n \geq 4$, let $q \geq 5$ be odd, and let G be a group that admits a Phan system of type D_n over $\mathbb{F}_{q^2}$. If n is even then G is a homomorphic image of $Spin^+_{2n}(q)$ (the universal Chevalley group of type $D_n(q)$), and if n is odd then G is a homomorphic image of $Spin^-_{2n}(q)$ (the universal Chevalley group of type ${}^2D_n(q^2)$).

Theorem 3.2.6 (Phan [Pha77b], Theorem 2.6)
Let $q \geq 5$ be odd, and let G be a group that admits a Phan system of type E_6 over $\mathbb{F}_{q^2}$. Then G is a homomorphic image of the universal Chevalley group of type ${}^2E_6(q^2)$.

Theorem 3.2.7 (Phan [Pha77b], Theorem 2.7)
Let $q \geq 5$ be odd, and let G be a group that admits a Phan system of type E_7 over $\mathbb{F}_{q^2}$. Then G is a homomorphic image of the universal Chevalley group of type $E_7(q)$.

Theorem 3.2.8 (Phan [Pha77b], Theorem 2.8)
Let $q \geq 5$ be odd, and let G be a group that admits a Phan system of type E_8 over $\mathbb{F}_{q^2}$. Then G is a homomorphic image of the universal Chevalley group of type $E_8(q)$.

The phenomenon that in Theorems 3.2.1 and 3.2.6 we obtain twisted groups instead of the non-twisted universal Chevalley groups belonging to the diagram can be explained by the existence of non-trivial diagram automorphisms. To be more precise, let Δ be a spherical Coxeter diagram, let $W(\Delta)$ be the corresponding Weyl group, and let w_0 be the longest word in $W(\Delta)$. Then, if G is a group that admits a Phan system of type Δ over $\mathbb{F}_{q^2}$, the group is a homomorphic image of the universal group of type Δ over $\mathbb{F}_q$ if conjugation by w_0 acts as the identity on Δ, and a homomorphic image of the universal group of (twisted) type ${}^2\Delta$ over $\mathbb{F}_{q^2}$ if conjugation by w_0 acts as an (involutive) non-identity automorphism on Δ. This observation is underscored by Phan's Theorem 3.2.5. See also the next section.

3.3 The opposites chamber system approach

Let $\mathbb{F}$ be a field, let $\mathbb{K}$ be a quadratic extension of $\mathbb{F}$. Furthermore, let B be a spherical Tits building over $\mathbb{K}$, let $\mathcal{B} = (B_+, B_-)$ be the corresponding twin building, and let σ either be the identity or be an involutive bijection $\mathcal{B} \to \mathcal{B}$ such that

- $\sigma(B_\epsilon) = B_{-\epsilon}$,
- $\delta_\epsilon(c, d) = \delta_{-\epsilon}(c^\sigma, d^\sigma)$,
- $\delta^*(x, y) = \delta^*(x^\sigma, y^\sigma)$,

for $\epsilon = \pm$ and $c, d \in B_\epsilon$, $x \in B_\epsilon$, $y \in B_{-\epsilon}$. The map σ is called a **flip**. Define $\mathcal{C}(B,\sigma) = \left\{(c_+, c_-) \in Opp(B) \mid \{c_+, c_-\} = \left\{c_+^\sigma, c_-^\sigma\right\}\right\}$. We also write $\mathcal{C}_\sigma$ instead of $\mathcal{C}(B,\sigma)$, if B is obvious.

Notice that $\mathcal{C}(B, \mathrm{id}) = Opp(\mathcal{B})$, the opposites chamber system of Definition 3.1.3, which has been used by Mühlherr to re-prove the Curtis-Tits theorem. For B the projective geometry $\mathbb{P}_n(q^2)$ and σ the composition of the contragredient automorphism (the diagram automorphism) and the involutive field automorphism, the chamber system $\mathcal{C}(B,\sigma)$ equals the one Bennett and Shpectorov have studied to obtain a new proof of Phan's theorem, cf. Section 3.2. Similarly, it is expected that it is possible to re-prove Phan's theorems of [Pha77b] by looking at buildings of type D_n and E_n and suitable maps σ, as conjectured in [GHS]. In case of Phan's theorem for E_6 over $\mathbb{F}_{q^2}$, for example, the flip would be the composition of a correlation of the building geometry of type E_6 over $\mathbb{F}_{q^2}$ that interchanges points and symplecta (and, thus, acts as the non-trivial involutive diagram automorphism on the diagram E_6) and the involutive field automorphism.

We want to point out that strictly speaking our examples are maps on spherical buildings and not maps on spherical twin buildings. If we want to view a flip σ of the twin building $\mathcal{B}$ as a map on the building B, then from $\delta_\epsilon(c, d) = \delta_{-\epsilon}(c^\sigma, d^\sigma)$ from the definition of a flip and $\delta_+ = \delta$, $\delta_- = w_0\delta w_0$ from Proposition A.3.2, we obtain the property $\delta(c, d) = w_0\delta(c^\sigma, d^\sigma)w_0$. Conversely, every involutive correlation σ of B with $\delta(c, d) = w_0\delta(c^\sigma, d^\sigma)w_0$ for every opposite pair c, d of chambers of B can be considered as a flip on the twin building $\mathcal{B}$. In particular, the action of σ on the diagram D of the building B equals the conjugation action of w_0 on D.

From the fact that σ is involutive (in the sense that $\sigma = \mathrm{id}$ or $\sigma^2 = \mathrm{id}$) it follows that $\delta(c^{\sigma^{-1}}, d) = \delta(c^\sigma, d) = w_0\delta(c, d^\sigma)w_0$, whence from c opp d^σ we find c^σ opp d. By Proposition A.3.4, we obtain the following.

Proposition 3.3.1
Let $\mathcal{B} = ((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta^*)$ be the twin building obtained from the spherical building $B = (\mathcal{C}, \delta)$ as in Proposition A.3.2, and let w_0 be the longest word in the Weyl group of B. A non-identity bijection $\mathcal{B} \to \mathcal{B}$ is a flip if and only if $\sigma(\mathcal{C}_+) = \mathcal{C}_-$ and $\delta(c^\sigma, d) = w_0\delta(c, d^\sigma)w_0$ for all $c, d \in \mathcal{C}$.

Proof. The only condition that remains to be checked is that under the assumption $\delta(c^\sigma, d) = w_0\delta(c, d^\sigma)w_0$ we have $\delta^*(c, d) = \delta^*(c^\sigma, d^\sigma)$. But by Proposition A.3.2, this is equivalent to $w_0\delta(c, d) = \delta(c^\sigma, d^\sigma)w_0$ which in turn is equivalent to $\delta(c, d) = w_0\delta(c^\sigma, d^\sigma)w_0$. □

In all the examples mentioned above, the chamber system $\mathcal{C}(B, \sigma)$ actually gives rise to a geometry. It is currently unknown whether it is true that any non-empty chamber system $\mathcal{C}(B, \sigma)$ is geometrizable, though we conjecture that to be the case. If $\mathcal{C}(B, \sigma)$ happens to be geometrizable, denote the resulting geometry by $\mathcal{G}(B, \sigma)$, or $\mathcal{G}_\sigma$, if B is obvious. Another interesting problem would be to find all possible chamber systems $\mathcal{C}(B, \sigma)$. Also, while there is a uniform idea which chamber systems and geometries one has to study in order to obtain and state Curtis-Phan-Tits theorems, there is still need for a uniform proof.

3.4 Flips and forms

In the remainder of this chapter, which is taken from the preprint [GHS] by Corneliu Hoffman, Sergey Shpectorov, and the author, we present a Phan-type theorem by studying a subgeometry of the symplectic polar geometry over a finite field of square order. We only investigate connectivity properties and do not discuss uniqueness of the amalgam.

Let V be a $2n$-dimensional nondegenerate symplectic space over $\mathbb{F}_{q^2}$ and let $(\cdot, \cdot)$ be the corresponding alternating bilinear form. Let the bar denote the involutive automorphism of $\mathbb{F}_{q^2}$. In this section we study semilinear transformations σ of V satisfying

(T1) $(\lambda v)^\sigma = \bar{\lambda} v^\sigma$;

(T2) $(u^\sigma, v^\sigma) = \overline{(u, v)}$; and

(T3) $\sigma^2 = -\mathrm{id}$.

Notice that σ induces a flip on the symplectic geometry. We will refer to σ as a **symplectic flip**. An example of a symplectic flip can be constructed as follows. Choose a basis $\mathcal{B} = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ in V such that, for $1 \le i, j \le n$, we have that $(e_i, e_j) = (f_i, f_j) = 0$ and $(e_i, f_j) = \delta_{ij}$. This corresponds to the Gram matrix

$$A = \begin{pmatrix} 0 & \mathrm{id}_{n\times n} \\ -\mathrm{id}_{n\times n} & 0 \end{pmatrix}.$$

Here $\mathrm{id}_{n\times n}$ is the identity matrix of size $n\times n$, whereas 0 stands for the all-zero matrix of the same size. A basis like $\mathcal{B}$ is called a **hyperbolic** basis. Let ϕ be the linear transformation of V whose matrix (acting from the left on column vectors) with respect to the basis $\mathcal{B}$ coincides with A and let ψ be the semilinear transformation of V that applies the bar automorphism to the $\mathcal{B}$-coordinates of every vector. If $\sigma_0 = \phi \circ \psi$, then for a vector

$$u = \sum_{i=1}^{n} x_i e_i + \sum_{i=1}^{n} y_i f_i$$

we compute that

$$u^{\sigma_0} = -\sum_{i=1}^{n} \bar{y}_i e_i + \sum_{i=1}^{n} \bar{x}_i f_i.$$

One easily verifies that (T1) and (T3) are satisfied for σ_0. To check (T2), consider

$$v = \sum_{i=1}^{n} x'_i e_i + \sum_{i=1}^{n} y'_i f_i.$$

Then

$$(u^{\sigma_0}, v^{\sigma_0}) = \sum_{i=1}^{n} (-\bar{y}_i)\bar{x}'_i - \bar{x}_i(-\bar{y}'_i) = \overline{(u,v)},$$

yielding (T2). Thus, σ_0 is a symplectic flip. Notice that $\sigma = \sigma_0$ can be characterized as the unique semilinear transformation such that (T1) holds and

$$e_i^{\sigma} = f_i, \quad f_i^{\sigma} = -e_i, \quad \text{for } 1 \le i \le n.$$

Whenever these latter conditions are satisfied for a symplectic flip σ and a hyperbolic basis $\mathcal{B} = \{e_1, \ldots, f_n\}$, we will say that $\mathcal{B}$ is a **canonical** basis for σ. Let $G \cong Sp_{2n}(q^2)$ be the group of all linear transformations of V preserving the form $(\cdot,\cdot)$. One of the principal results of this section is the following.

Proposition 3.4.1
Every symplectic flip admits a canonical basis.

In other words, every symplectic flip σ is conjugate to σ_0 by an element of G. We start by discussing the general properties of symplectic flips. Let σ be a symplectic flip. Define

$$((x, y)) = (x, y^{\sigma}).$$

Lemma 3.4.2
The form $((\cdot,\cdot))$ *is a nondegenerate Hermitian form satisfying* $((u^\sigma, v^\sigma)) = \overline{((u,v))}$ *for* $u, v \in V$.

Proof. Clearly, $((\cdot,\cdot))$ is a sesquilinear form. Also, $((v,u)) = (v, u^\sigma) = -(u^\sigma, v) = -\overline{(u^{\sigma^2}, v^\sigma)} = -\overline{(-u, v^\sigma)} = \overline{(u, v^\sigma)} = \overline{((u,v))}$. Thus, $((\cdot,\cdot))$ is Hermitian. If u is in the radical of $((\cdot,\cdot))$ then for any $v \in V$, $0 = ((u, v^{\sigma^3})) = (u, v^{\sigma^4}) = (u,v)$. Therefore, $u = 0$, as $(\cdot,\cdot)$ is nondegenerate. Finally, $((u^\sigma, v^\sigma)) = (u^\sigma, -v) = (v, u^\sigma) = ((v,u)) = \overline{((u,v))}$. $\square$

In what follows we will work with both $(\cdot,\cdot)$ and $((\cdot,\cdot))$. This calls for two different perpendicularity symbols. We will use $\perp$ for the form $(\cdot,\cdot)$, while $\perp\!\!\!\perp$ will be used for $((\cdot,\cdot))$.

Proof of Proposition 3.4.1. Let σ be a symplectic flip. Pick a vector $u \in V$ such that $((u,u)) = 1$. Such a vector exists since $((\cdot,\cdot))$ is nondegenerate by Lemma 3.4.2. Set $e_n = u$ and $f_n = u^\sigma$. Since $(\cdot,\cdot)$ is an alternating form we have $(e_n, e_n) = (f_n, f_n) = 0$. Furthermore, $(e_n, f_n) = ((e_n, f_n^{\sigma^{-1}})) = ((e_n, e_n)) = 1$. In particular, the subspace $U = \langle e_n, f_n \rangle$ is nondegenerate with respect to $(\cdot,\cdot)$. Consider now $V' = U^\perp$. Notice that U is invariant under σ. Together with (T2), this implies that V' is also invariant under σ. It is easy to see that the restriction of σ to V' is a symplectic flip of V'. By induction, there exists a hyperbolic basis $e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1}$ in V', such that $e_i^\sigma = f_i$ for $1 \le i \le n-1$. (Since $\sigma^2 = -\mathrm{id}$, this automatically implies $f_i^\sigma = -e_i$.) Clearly, $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is a canonical basis for σ. $\square$

Next, we discuss the behavior of σ, $(\cdot,\cdot)$, and $((\cdot,\cdot))$ with respect to subspaces of V.

Lemma 3.4.3
For a subspace U *of* V, *we have* $U^{\perp\!\!\!\perp} = (U^\sigma)^\perp = (U^\perp)^\sigma$. *Similarly,* $U^\perp = (U^\sigma)^{\perp\!\!\!\perp} = (U^{\perp\!\!\!\perp})^\sigma$.

Proof. The first equality in the first claim immediately follows from the definition of $((\cdot,\cdot))$. If $u \in (U^\perp)^\sigma$ (say, $u = (u')^\sigma$ for $u' \in U^\perp$) and $v \in U$ then $((u,v)) = ((u')^\sigma, v^\sigma) = \overline{(u', v)} = 0$. The second claim follows by an application of σ to the equalities in the first claim. $\square$

Lemma 3.4.4
The form $(\cdot,\cdot)$ *has the same rank on* U *and on* U^σ; *likewise, it has the same rank on* $U^\perp$ *and on* $U^{\perp\!\!\!\perp} = (U^\perp)^\sigma$. *The same statements hold for* $((\cdot,\cdot))$.

Proof. The first claim follows from (T2) for $(\cdot,\cdot)$, and from Lemma 3.4.2 for $((\cdot,\cdot))$. The second claim follows from the first one and Lemma 3.4.3. $\square$

If U is σ-invariant then we can say more. It follows from Lemma 3.4.3 that $U^\perp = U^{\perp\!\!\!\perp}$. In other words, for a σ-invariant subspace U, the orthogonal complement (and hence also the radical) of U is the same with respect to $(\cdot,\cdot)$ and $((\cdot,\cdot))$. It also follows from Lemma 3.4.3 that both the orthogonal complement and the radical of U are σ-invariant.

It was noticed in the proof of Proposition 3.4.1 that the properties (T1)–(T3) are inherited by the restrictions of σ to all σ-invariant subspaces $U \subset V$. If U is nondegenerate—it does not matter with respect to which form—then the restriction of σ to U is a symplectic flip of U. We should now discuss what happens when U has a nontrivial radical. First of all, by the above comment, the radical of U is σ-invariant.

Lemma 3.4.5
If U is σ-invariant then the radical of U has a σ-invariant complement in U.

Proof. The proof is analogous to that of Proposition 3.4.1. If U is totally singular then there is nothing to prove. Otherwise, choose $u \in U$ such that $((u,u)) = 1$. Then $W = \langle u, u^\sigma \rangle$ is a σ-invariant nondegenerate subspace. Hence $U = (U \cap W^\perp) \oplus W$ and the radical of U coincides with the radical of $U_0 = U \cap W^\perp$. Clearly, U_0 is σ-invariant, and so induction applies. $\square$

Notice that the σ-invariant complement in the above lemma is automatically nondegenerate. Next, let us study the "eigenspaces" of σ on V. For $\lambda \in \mathbb{F}_{q^2}$, define $V_\lambda = \{u \in V \mid u^\sigma = \lambda u\}$. Note that V_λ is not a true eigenspace, because σ is not linear.

Lemma 3.4.6
The following hold.

(i) *For $0 \neq \mu \in \mathbb{F}_{q^2}$, we have $\mu V_\lambda = V_{\lambda'}$, where $\lambda' = \frac{\bar\mu}{\mu}\lambda$; in particular, V_λ is an $\mathbb{F}_q$-subspace of V.*

(ii) *$V_\lambda \neq 0$ if and only if $\lambda\bar\lambda = -1$; furthermore, if $V_\lambda \neq 0$ then V_λ contains a basis of V.*

Proof. Suppose $u \in V_\lambda$. Then $(\mu u)^\sigma = \bar\mu u^\sigma = \bar\mu\lambda u = \frac{\bar\mu}{\mu}\lambda(\mu u)$. This proves (i). Also, $-u = u^{\sigma^2} = \bar\lambda\lambda u$. Thus, if $u \neq 0$ then $\lambda\bar\lambda = -1$. This proves the 'only if' part of (ii). To prove the 'if' part, choose a canonical basis $\{e_1, \ldots, f_n\}$ for σ. Fix a $\lambda \in \mathbb{F}_{q^2}$ such that $\lambda\bar\lambda = -1$, which is possible by surjectivity of the norm function. Define $u_i = e_i - \bar\lambda f_i$ and $v_i = \bar\lambda e_i + f_i$ for $1 \leq i \leq n$. A simple check shows that u_i and v_i are in V_λ. This shows that $V_\lambda \neq 0$. Furthermore, u_i and v_i are not proportional unless $\bar\lambda = \lambda$, that is, $\lambda \in \mathbb{F}_q$. Thus, if $\lambda \notin \mathbb{F}_q$ then $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ is a basis of V. If $\lambda \in \mathbb{F}_q$ then consider $\lambda' = \frac{\bar\mu}{\mu}\lambda$, where μ is chosen so that $\frac{\bar\mu}{\mu} \notin \mathbb{F}_q$. By (i), $V_{\lambda'} = \mu V_\lambda$. Also, since $\lambda' \notin \mathbb{F}_q$, we have that $V_{\lambda'}$ contains a basis of V, and hence so does V_λ. $\square$

Consider an $\mathbb{F}_q$-linear map $\phi : v \mapsto v - \bar\lambda v^\sigma$, where $\lambda \in \mathbb{F}_{q^2}$ and $\lambda\bar\lambda = -1$. It can be checked that ϕ maps V onto V_λ, and its kernel is $V_{\bar\lambda}$. The above vectors u_i and v_i are obtained by applying ϕ to the vectors in the canonical basis $\{e_1, \ldots, f_n\}$.

Notation 3.4.7 Fix a $\lambda \in \mathbb{F}_{q^2}$ such that $\lambda\bar{\lambda} = -1$. Also, fix a $\mu \in \mathbb{F}_{q^2}$ with $\bar{\mu} = -\mu$.

Lemma 3.4.8
The restriction of $\mu\lambda(\cdot,\cdot)$ to V_λ is a nondegenerate alternating $\mathbb{F}_q$-bilinear form.

Proof. Clearly, the form $\mu\lambda(\cdot,\cdot)$ is $\mathbb{F}_q$-bilinear and alternating. Since V_λ contains a basis of V by Lemma 3.4.6 (ii), the form is nondegenerate. It remains to see that it takes values in $\mathbb{F}_q$. However, if $u, v \in V_\lambda$, then $\overline{\mu\lambda(u,v)} = \bar{\mu}\bar{\lambda}(u^\sigma, v^\sigma) = -\mu\bar{\lambda}\lambda^2(u,v) = \mu\lambda(u,v)$. □

Observe that conjugation by σ is an automorphism of G. Let G_σ be the centralizer of σ in G. The above setup gives us a means to identify G_σ. Let $H \cong Sp_{2n}(q)$ be the group of all linear transformations of V_λ preserving the (restriction of the) form $\mu\lambda(\cdot,\cdot)$. Since V_λ contains a basis of V, we can use $\mathbb{F}_{q^2}$-linearity to extend the action of the elements of H to the entire V. This allows us to identify H with a subgroup of G. Clearly, since $h \in H$ preserves $\mu\lambda(\cdot,\cdot)$, it must also preserve $(\cdot,\cdot)$.

Proposition 3.4.9
$G_\sigma = H$.

Proof. Choose a basis $\{w_1, \dots, w_{2n}\}$ in V_λ. Then this set is also a basis of V. Let $h \in H$. If $u = \sum_{i=1}^{2n} x_i w_i \in V$ then $u^{\sigma h} = (\sum_{i=1}^{2n} \bar{x}_i \lambda w_i)^h = \lambda \sum_{i=1}^{2n} \bar{x}_i w_i^h$. On the other hand, $u^{h\sigma} = (\sum_{i=1}^{2n} x_i w_i^h)^\sigma = \sum_{i=1}^{2n} \bar{x}_i \lambda w_i^h$. Therefore, $H \leq G_\sigma$. Now take $g \in G_\sigma$. If $u \in V_\lambda$ then $(u^g)^\sigma = (u^\sigma)^g = (\lambda u)^g = \lambda u^g$. This proves that g leaves V_λ invariant. It remains to see that g preserves $\mu\lambda(\cdot,\cdot)$. However, this is clear, because g is $\mathbb{F}_{q^2}$-linear and it preserves $(\cdot,\cdot)$. □

3.5 The flipflop geometry $\mathcal{G}$

We will be using the notation from the previous section. In particular, V is a nondegenerate symplectic $\mathbb{F}_{q^2}$-space of dimension $2n$ with a form $(\cdot,\cdot)$, σ is a symplectic flip and $((\cdot,\cdot))$ is the corresponding Hermitian form. Also, $G \cong Sp_{2n}(q^2)$ is the group of linear transformations preserving $(\cdot,\cdot)$ and $G_\sigma = C_G(\sigma)$. Throughout this section, we assume $n \geq 2$. Let B be the building geometry associated with G. Its elements are all the $(\cdot,\cdot)$-totally singular subspaces of V. Two elements U and U' of B are **opposite** whenever $V = U' \oplus U^\perp$, i.e., U, U' have the same dimension and $U' \cap U^\perp = 0$. Two chambers (maximal flags) F and F' are opposite whenever for each subspace $U \in F$ there is a $U' \in F'$ such that U and U' are opposite. Using this observation, it can be shown that the opposites chamber system $Opp(\mathcal{B})$ related to B is geometrizable and the elements of the corresponding geometry $\mathcal{G}(B, \mathrm{id})$ are all pairs (U, U') that are opposite totally singular subspaces of V. Turning to $\mathcal{C}(B,\sigma)$, let F be a maximal flag of B such that F and F^σ are opposite. Then, for every $U \in F$, the space U^σ must be the element of F^σ that is opposite U. Indeed, this follows from the fact that opposite elements have the same dimension. Thus,

$(F, F^\sigma) \in \mathcal{C}(B, \sigma)$ if and only if U^σ is opposite U for every element $U \in F$ (that is, $(U, U^\sigma) \in \mathcal{G}(B, \mathrm{id})$).

Our first goal is to show that $\mathcal{C}_\sigma$ is geometrizable, that is, its chambers arise as maximal flags of a suitable geometry. The natural candidate for this geometry is the following subset of $\mathcal{G}_{\mathrm{id}}$:

$$\{(U, U') \in \mathcal{G}_{\mathrm{id}} \mid U' = U^\sigma\}.$$

(For convenience, we will refer to this set as $\mathcal{G}_\sigma$, anticipating that this is a geometry.) It suffices to show that $\mathcal{G}_\sigma$ is a full rank (that is, rank n) subgeometry of $\mathcal{G}_{\mathrm{id}}$. In order to avoid cumbersome notation, let us project every pair $(U, U') \in \mathcal{G}_\sigma$ to its first coordinate U. Since $U' = U^\sigma$, this establishes a bijection (in fact, an isomorphism of pregeometries) between $\mathcal{G}_\sigma$ and the following subset of B:

$$\mathcal{G} = \{U \in B \mid U^\sigma \text{ is opposite } U\}.$$

The definition of $\mathcal{G}$ can be nicely restated in terms of the forms $(\cdot, \cdot)$ and $((\cdot, \cdot))$.

Proposition 3.5.1
The elements of $\mathcal{G}$ are precisely the subspaces $U \subset V$ which are totally isotropic with respect to $(\cdot, \cdot)$ and nondegenerate with respect to $((\cdot, \cdot))$.

Proof. By Lemma 3.4.3, $U^{\perp\!\!\perp} = (U^\sigma)^\perp$. Hence U and U^σ are opposite if and only if $U \cap U^{\perp\!\!\perp} = 0$. □

We use $\{1, \ldots, n\}$ as the type set of B. In particular, the type function assigns to an element of $\mathcal{G}$ its linear (rather than its projective) dimension. We will use the customary geometric terminology. In particular, **points**, **lines**, and **planes** are elements of type 1, 2, and 3, respectively. We stress again that we will mostly work with $\mathcal{G}$, using the fact that $\mathcal{G}$ and $\mathcal{G}_\sigma$ are isomorphic. We also notice that the isomorphism between $\mathcal{G}$ and $\mathcal{G}_\sigma$ commutes with the action of $H = G_\sigma$.

Proposition 3.5.2
The pregeometry $\mathcal{G}$ is a geometry. Moreover, H acts flag-transitively on $\mathcal{G}$.

Proof. Let $V_1 \leq V_2 \leq \cdots \leq V_k$ be a maximal flag. Let $\mathcal{B} = \{e_1, \ldots, e_t\}$ be an orthonormal basis of V_k with respect to $((\cdot, \cdot))$. (This exists since V_k is nondegenerate with respect to $((\cdot, \cdot))$.) Then $\mathcal{B} \cup \mathcal{B}^\sigma$ forms a canonical basis of $V_k \oplus V_k^\sigma$. If V_k is not a maximal totally isotropic subspace of V with respect to $(\cdot, \cdot)$, there exists a nontrivial $u \in (V_k \oplus V_k^\sigma)^\perp = (V_k \oplus V_k^\sigma)^{\perp\!\!\perp}$ such that $((u, u)) = 1$. Then $\langle V_k, u\rangle$ is totally isotropic for $(\cdot, \cdot)$ and nondegenerate with respect to $((\cdot, \cdot))$, contradicting maximality of the flag. Hence we can assume V_k is a maximal totally isotropic subspace with respect to $(\cdot, \cdot)$. Induction shows that V_{i-1} is a codimension one subspace in V_i for $2 \leq i \leq k$, proving that the maximal flag is a chamber.

Let $V_1 \leq V_2 \leq \cdots \leq V_n$ and $V_1' \leq V_2' \leq \cdots \leq V_n'$ be two chambers. Choose bases $\mathcal{B} = \{e_1, \ldots, e_n\}$, $\mathcal{B}' = \{e_1', \ldots, e_n'\}$ for V_n, respectively V_n' such that they are orthonormal with respect to $((\cdot,\cdot))$ and $V_i = \langle e_1,\dots,e_i\rangle$, $V_i' = \langle e_1',\dots,e_i'\rangle$. Define $g \in G$ such that $e_i^g = e_i'$ and $(e_i^\sigma)^g = (e_i')^\sigma$. Such a g obviously exists, since $G \cong Sp_{2n}(q^2)$ acts flag-transitively on the symplectic polar geometry $(V,(\cdot,\cdot))$. It is also clear that g maps one chamber onto the other. Moreover notice that $\sigma\circ g = g\circ\sigma$ on the basis $\mathcal{B}\cup\mathcal{B}^\sigma$. Therefore $g \in G_\sigma$. □

The following lemma will prove to be very useful throughout this whole chapter.

Lemma 3.5.3
Let p be a point of $\mathcal{G}$, and let $\Pi \supset p$ be a three-dimensional subspace of V of rank at least two with respect to $((\cdot,\cdot))$ such that p is in the radical of Π with respect to $(\cdot,\cdot)$. Then any two-dimensional subspace of Π that does not contain p is incident with at least q^2-q-1 (respectively, q^2-2q-1) points of $\mathcal{G}$ collinear to p if its rank is one (respectively, two) with respect to $((\cdot,\cdot))$.

Proof. Since p is in the radical of Π with respect to $(\cdot,\cdot)$, all lines passing through p will be totally isotropic with respect to $(\cdot,\cdot)$ so we only need to consider $((\cdot,\cdot))$. Notice that if l is a two-dimensional subspace of V that is not totally isotropic with repect to $((\cdot,\cdot))$ then l contains at least q^2-q points of $\mathcal{G}$. (If the rank of l is one then its radical is the only nontrivial isotropic subspace of l and if the rank of l is two then l contains $q+1$ distinct nontrivial isotropic subspaces.) Consider $l_1 = p^{\perp\!\!\perp}\cap\Pi$. Then by the above, there are at least q^2-q lines of $\mathcal{G}$ through p that intersect l_1 in a point of $\mathcal{G}$. If l is any other not totally isotropic (with respect to $((\cdot,\cdot))$) two-dimensional subspace of Π that does not contain p, at most 1, respectively $q+1$ of these q^2-q lines will intersect l in isotropic subspaces. □

Actually, we also showed the following:

Lemma 3.5.4
Let p be a point of $\mathcal{G}$, and let $\Pi \supset p$ be a three-dimensional subspace of V of rank at least two with respect to $((\cdot,\cdot))$. Then any two-dimensional subspace of Π that does not contain p is incident with at least q^2-q-1 (respectively, q^2-2q-1) points of $\mathcal{G}$ that generate a $((\cdot,\cdot))$-nondegenerate two space with p if its $((\cdot,\cdot))$-rank is one (respectively, two). □

We need to prove that the geometry $\mathcal{G}$ is connected. This is equivalent to proving the connectivity of the point shadow space of $\mathcal{G}$ which in turn is equivalent with the connectivity of the collinearity graph of $\mathcal{G}$.

Lemma 3.5.5
Suppose $n \geq 3$. Then, if $(n,q) \neq (3,2)$, the collinearity graph of the geometry $\mathcal{G}$ has diameter two. If $(n,q) = (3,2)$, then the collinearity graph of $\mathcal{G}$ has diameter three. In particular, $\mathcal{G}$ is connected in all cases.

Proof. If $(n,q) = (3,2)$ then the claim can be checked computationally (say, in GAP [Sch95]). So suppose $(n,q) \neq (3,2)$. Let p_1, p_2 be two points in the geometry.

Consider $W_i := p_i^{\perp} \cap p_i^{\perp\!\!\perp}$, $i \in \{1,2\}$. Then $\dim W_i = 2n-2$ so $\dim W_1 \cap W_2 \geq 2n-4$. If $2n-4 > n-1$ then the space $W_1 \cap W_2$ cannot be totally isotropic for $((\cdot,\cdot))$ (it lies inside the $(2n-1)$-dimensional nondegenerate space $p_1^{\perp\!\!\perp}$). Therefore if $n > 3$ we can find a point q of the geometry lying in $W_1 \cap W_2$. In this case q connects p_1 and p_2. If $n = 3$, the space $U = p_1^{\perp} \cap p_2^{\perp} \cap p_2^{\perp\!\!\perp}$ is at least three-dimensional inside the four-dimensional space $p_2^{\perp} \cap p_2^{\perp\!\!\perp}$, which is nondegenerate with respect to both forms. Actually, U has rank at least two with respect to $((\cdot,\cdot))$, because if it had a two-dimensional radical, this radical would be a maximal totally isotropic subspace of $p_2^{\perp} \cap p_2^{\perp\!\!\perp}$ and had to be equal to its own polar in $p_2^{\perp} \cap p_2^{\perp\!\!\perp}$ with respect to $((\cdot,\cdot))$. Hence we can find a $((\cdot,\cdot))$-nondegenerate two-dimensional subspace l of U, all points of which actually are collinear to p_2. Applying Lemma 3.5.3 to the plane $\langle p_1, l\rangle$, we find a common neighbor of p_1 and p_2. □

Lemma 3.5.6
If $n = 2$ and $q \neq 2$, then $\mathcal{G}$ is connected.

Proof. Fix a point p of $\mathcal{G}$. Then p is collinear to $(q^2-q)(q^2-q-1)$ points of $\mathcal{G}$ (there are q^2-q lines through p, each of which contains q^2-q-1 points of $\mathcal{G}$ besides p). Now let us estimate the number of points at distance two to p. Each point q at distance one to p is incident with q^2-q-1 lines that do not contain p. Each of these lines contains q^2-q-1 points other than q. Moreover, if r is a point at distance two from p, then there are at most q^2 common neighbors of p and r (indeed, $\langle p, r\rangle^{\perp}$ is a two-dimensional space which is not totally isotropic with respect to $((\cdot,\cdot))$, whence it contains either q^2 or q^2-q points of $\mathcal{G}$). Hence there are at least $\frac{(q^2-q)(q^2-q-1)^3}{q^2}$ points at distance two from p. On the other hand, $\mathcal{G}$ contains $\frac{q^8-1}{q^2-1} - (q^2+1)(q^3+1)$ points (the number of points of the projective space minus the number of points of the unitary generalized quadrangle).

By Proposition 3.5.2 and Proposition 3.4.9, the group $G_\sigma \cong Sp_4(q)$ acts flag-transitively on $\mathcal{G}$. In particular, it permutes the connected components of $\mathcal{G}$. More precisely, the number of connected components is equal to the index of the stabilizer in G_σ of one component. By [Coo78], Table 5.2.A, the index of a maximal subgroup of $Sp_4(q)$ is at least 27, if $q > 2$. Hence, to show connectivity, it is enough to prove that $1 + (q^2-q)(q^2-q-1) + \frac{(q^2-q)(q^2-q-1)^3}{q^2}$ is greater than $\frac{1}{27}\left(\frac{q^8-1}{q^2-1} - (q^2+1)(q^3+1)\right)$, which is true for all $q \geq 3$. □

We summarize Lemmas 3.5.5 and 3.5.6 as the following result.

Proposition 3.5.7
Let $n \geq 2$. Then $\mathcal{G}$ is connected, provided that $(n,q) \neq (2,2)$. □

Combined with the results of [BS], also Theorem 3.2.3 in the present thesis, this yields residual connectedness:

Corollary 3.5.8
If $q \neq 2$, then $\mathcal{G}$ is residually connected. □

Finally, let us discuss the diagram of the geometry $\mathcal{G}_\sigma$. Notice that it is a linear (string) diagram. Furthermore, it follows from Proposition 3.5.1 that the residue of an element of maximal type $n-1$ is the geometry of all nondegenerate subspaces of a nondegenerate n-dimensional unitary space. The residue of a point is a geometry similar to $\mathcal{G}$ but with rank $n-1$. This leads to the diagram

$$\underset{q^2-q}{\circ}\overset{U}{\rule{2em}{0.4pt}}\underset{q^2-q}{\circ}\overset{U}{\rule{2em}{0.4pt}}\underset{q^2-q}{\circ}\ \cdots\ \underset{q^2-q}{\circ}\overset{U}{\rule{2em}{0.4pt}}\underset{q^2-q}{\circ}\overset{S}{\rule{2em}{0.4pt}}\underset{q^2-q}{\circ}.$$

The exact meaning of the edges $\circ\overset{U}{\rule{2em}{0.4pt}}\circ$ and $\circ\overset{S}{\rule{2em}{0.4pt}}\circ$ is as follows. The first one represents the geometry of all one and two-dimensional nondegenerate subspaces of a three-dimensional unitary space. It appears in [BS]; see also Section 3.2. The second edge represents our flipflop geometry in the case of rank two. We note that both geometries are disconnected for $q = 2$ and connected for $q \geq 3$. See [BS] (or again Theorem 3.2.3) for $\circ\overset{U}{\rule{2em}{0.4pt}}\circ$ and Lemma 3.5.6 for $\circ\overset{S}{\rule{2em}{0.4pt}}\circ$.

3.6 Simple connectedness. Part I

In this and the next section we will prove that, apart from a few exceptional cases, the geometry $\mathcal{G}$ is simply connected. Here we collect some general statements and then complete the case $n \geq 4$. The next section handles the case $n = 3$, which is somewhat more complicated. In order to prove simple connectedness, we pick the base element b of all cycles to be a point of $\mathcal{G}$.

Lemma 3.6.1
Unless $(n, q) = (3, 2)$, every cycle based at the point b is homotopically equivalent to a cycle passing only through points and lines.

Proof. We will induct on the number of elements of the path that are not points or lines. If this number is zero there is nothing to prove. Take an arbitrary cycle $\gamma := x_0 x_1 \ldots x_{k-1} x_k$ with $x_0 = b = x_k$. Let x_i be the first element that is not a point or a line. Clearly $i \notin \{0, k\}$. There are two cases to consider:

If the type of x_{i+1} is bigger than the type of x_i then x_{i-1} and x_{i+1} are incident and γ is homotopically equivalent to the cycle $b x_1 \ldots x_{i-1} x_{i+1} \ldots b$. Suppose the type of x_{i+1} is smaller than the type of x_i. Let y be an element of type n which is incident to x_i, then y is incident to both x_{i-1} and x_{i+1} (the type of x_{i-1} is clearly smaller than the type of x_i). Therefore γ is homotopically equivalent to the path $b x_1 \ldots x_{i-1} y x_{i+1} \ldots b$. Now pick two points w, z such that w is incident to x_{i-1} and z is incident to x_{i+1}; in case x_{i-1} or x_{i+1} is a point, choose w, respectively z to be x_{i-1}, respectively x_{i+1}. Using Lemma 3.5.5 and Lemma 3.5.6 we can connect w and z with a path $w w_1 \ldots w_t z$ of only points and lines incident to y. Then γ

is homotopically equivalent to $bx_1 \dots x_{i-1} w_1 \dots w_t z x_{i+2} \dots b$ which contains fewer elements that are not points and lines. □

We can therefore restrict our attention to the point-line incidence graph of $\mathcal{G}$ and, thus, to the collinearity graph of $\mathcal{G}$.

The first step is the analysis of triangles (i.e., 3-cycles in the collinearity graph). We will call (p, q, r) a **good triangle** if the points p, q, and r are noncollinear but pairwise collinear in $\mathcal{G}$ and incident to a common plane of the geometry. All triangles that are not good (and are not a line) are called **bad**. A good triangle is homotopically trivial, since it is in the residue of the plane in which it is contained. We are to prove that all bad triangles are homotopically trivial, i.e., they can be decomposed into good triangles or are contained in elements of $\mathcal{G}$ of higher rank.

Lemma 3.6.2
Let (p, q, r) be a bad triangle. Then the plane $\langle p, q, r\rangle$ contains a one-dimensional radical with respect to $((\cdot, \cdot))$.

Proof. It is clear that the plane $\Pi = \langle p, q, r\rangle$ is totally isotropic with respect to $(\cdot, \cdot)$. Since p, q, r is a bad triangle, Π is degenerate with respect to $((\cdot, \cdot))$. Also, the rank of Π with respect to $((\cdot, \cdot))$ is at least two (it contains the nondegenerate projective line $\langle p, q\rangle$), so the radical is obviously one-dimensional. □

Lemma 3.6.3
Let (p, q, r) be a bad triangle, and let x be the radical of the plane $\langle p, q, r\rangle$. If $x^\sigma = x$, then the triangle can be decomposed into triangles in which two of the vertices are perpendicular with respect to $((\cdot, \cdot))$.

Proof. If two of p, q, r are already perpendicular with respect to $((\cdot, \cdot))$, then there is nothing to show. So assume this is not the case. Consider the unique projective point r_1 of the line $\langle p, q\rangle$ with $r \perp\!\!\!\perp r_1$. It is sufficient to prove that r_1 is a point of $\mathcal{G}$, because then (p, r_1, r) and (q, r_1, r) are triangles as required. Suppose r_1 is not a point of $\mathcal{G}$. Then $\langle r, r_1\rangle = r_1^{\perp\!\!\!\perp} \cap \langle p, q, r\rangle$ and so it contains x. Therefore $\langle r, r_1\rangle$ is a totally isotropic space with respect to $((\cdot, \cdot))$ that contains r, contradicting the fact that r is a point of $\mathcal{G}$. □

Lemma 3.6.4
Let (p, q, r) be a bad triangle with $p \perp\!\!\!\perp q$ and let x be the radical of the plane $\langle p, q, r\rangle$. If $x^\sigma = x$, then we can find a canonical basis $e_1, \dots, e_n, f_1, \dots, f_n$ of V for σ such that (p, q, r) equals $(\langle e_1\rangle, \langle e_2\rangle, \langle ye_1 + ze_2 + (ce_3 + f_3)\rangle)$ with $c\bar{c} = -1$ and $yz \neq 0$ and $y\bar{y} + z\bar{z} \neq 0$.

Proof. Choose a canonical basis $e_1, \dots, e_n, f_1, \dots, f_n$ of V such that $p = \langle e_1\rangle$, $q = \langle e_2\rangle$. Then $x \in U := \langle e_1, e_2\rangle^\perp \cap \langle e_1, e_2\rangle^{\perp\!\!\!\perp} = \langle e_1, e_2, f_1, f_2\rangle^{\perp\!\!\!\perp}$, which is a nondegenerate space with respect to both forms. Pick a, a' such that $e_3 + af_3, e_3 +$

$a'f_3, \ldots, e_n + af_n, e_n + a'f_n$ are isotropic with respect to $((\cdot,\cdot))$, σ-invariant and form a basis of U. (This is equivalent to a choice $a \neq a'$ with $a\bar{a} = -1 = a'\bar{a}'$.) The radical x cannot be orthogonal to all of these vectors, so there exists one vector u in this basis such that $((u,x)) \neq 0$. The space $\langle u, x\rangle$ is nondegenerate and σ-invariant so it will contain a vector e such that $((e,e)) = 1$ and therefore $\langle u, x\rangle = \langle e, e^\sigma\rangle$. Choosing a new canonical basis of U that starts with e we can assume that the bad triangle is contained in the space $\langle e_1, e_2, e_3, f_3\rangle$ and $x = \langle ce_3 + f_3\rangle$. $\square$

For the rest of this section assume $n \geq 4$.

Lemma 3.6.5
Let (p, q, r) be a bad triangle. Then the triangle is homotopically trivial.

Proof. Let x be the radical of the plane $\langle p, q, r\rangle$ with respect to $((\cdot,\cdot))$, which exists by Lemma 3.6.2. Suppose $x = x^\sigma$. By Lemma 3.6.3 and Lemma 3.6.4 we can assume that our triangle has the form $p = \langle e_1\rangle$, $q = \langle e_2\rangle$, $r = \langle xe_1 + ye_2 + (ce_3 + f_3)\rangle$ where $c\bar{c} = -1$ and $x\bar{x} + y\bar{y} \neq 0$. Now one can add the point $\langle e_4\rangle$ and form a tetrahedron in which all triangles but the initial one are good. Now, if $x \neq x^\sigma$, then consider the line $l = pq$ of $\mathcal{G}$. Let $V' = l^{\perp\!\!\perp} \cap (l^\sigma)^{\perp\!\!\perp}$. Clearly, V' is a nondegenerate (with respect to both forms), σ-invariant space of dimension $2n-4$. Moreover, $x \in V'$. Nondegeneracy of V' and $x \neq x^\sigma$ imply the existence of a vector $v \in V'$ with $(x,v) = 0$ and $((x,v)) = 1$. Hence $\langle x, v\rangle$ is a line of $\mathcal{G}$, and $\langle p, q, x, v\rangle$ is totally isotropic with respect to $(\cdot,\cdot)$ and nondegenerate with respect to $((\cdot,\cdot))$, whence it is an element of $\mathcal{G}$ that contains the triangle (p, q, r), finishing the proof. $\square$

The next task is to prove that all quadrangles are homotopically trivial. We denote a quadrangle (a, b, c, d) by its vertices where consecutive points lie on a line of $\mathcal{G}$.

Lemma 3.6.6
If U is a σ-invariant, nondegenerate subspace of V of dimension $2k \geq 4$ and p is a point of $\mathcal{G}$, then p is collinear with a point of U or we have $2k = 4$ and $q = 2$.

Proof. Consider the decomposition $V = U \oplus U^{\perp\!\!\perp}$. Let $p_1 \in U$ be the projection of p onto U (with respect to this decomposition). If we find a point q of $\mathcal{G}$ in $p_1^\perp \cap p_1^{\perp\!\!\perp} \cap U$, then we are done. Indeed, $q \perp p_1$, $q \perp\!\!\perp p_1$ implies $q \perp p$, $q \perp\!\!\perp p$ by our choice of the projection. In particular this holds, if $k > 2$; then $2k - 2 > k$ and $p_1^\perp \cap p_1^{\perp\!\!\perp} \cap U$ cannot be totally isotropic. (Notice, that we are also done, if p_1 itself is non-singular with respect to $((\cdot,\cdot))$.) Thus, consider the case $k = 2$. The space $U \cap p_1^\perp$ is three-dimensional and has rank at least two with respect to $((\cdot,\cdot))$. Choose a projective line l of $((\cdot,\cdot))$-rank two in $U \cap p_1^\perp$. Notice that $p \perp l$, whence by Lemma 3.5.3, the projective line l contains $q^2 - 2q - 1$ points of $\mathcal{G}$ collinear to p, giving at least one, if $q > 2$. $\square$

A pair p, q of points of $\mathcal{G}$ will a be called **solid** if the space $p^\perp \cap p^{\perp\!\!\perp} \cap q^\perp \cap q^{\perp\!\!\perp}$ is nondegenerate with respect to both forms; notice that nondegeneracy of one form implies nondegeneracy of the other.

Lemma 3.6.7
Let a, b be two distinct points of $\mathcal{G}$ with $b \notin \langle a, a^\sigma \rangle$. The pair a, b is solid if and only if the projection of b onto $\langle a, a^\sigma \rangle^\perp$ (via the decomposition $V = \langle a, a^\sigma \rangle \oplus \langle a, a^\sigma \rangle^\perp$) is non-singular.

Proof. Let $b' = pr_{\langle a,a^\sigma\rangle^\perp}(b)$ be the projection of b onto $\langle a, a^\sigma \rangle^\perp$. Notice that $b' \neq 0$. We have $\langle a, a^\sigma, b \rangle = \langle a, a^\sigma, b' \rangle$ which is of rank three with respect to $((\cdot,\cdot))$ if and only if b' is non-singular with respect to $((\cdot,\cdot))$. But if the rank of this space is three, then the rank of $\langle a, a^\sigma, b, b^\sigma \rangle$ has to be four, since its radical with respect to $((\cdot,\cdot))$ equals the radical with respect to $(\cdot,\cdot)$ and it contains a subspace of rank three with respect to $((\cdot,\cdot))$. (Notice that an alternating form always has even rank.) This settles the 'if'-part of the lemma. Now, suppose b' is singular with respect to $((\cdot,\cdot))$. Then $\langle a, a^\sigma, b, b^\sigma \rangle = \langle a, a^\sigma, b', (b')^\sigma \rangle$ and b' is obviously contained in the radical of the latter space. □

Lemma 3.6.8
If $n \geq 5$ or $n = 4$ and $q \neq 2$, any quadrangle (p, q, r, s) with a solid pair p, r is null homotopic.

Proof. Assume p, r is a solid pair and let $U = p^\perp \cap p^{\perp\!\!\perp} \cap r^\perp \cap r^{\perp\!\!\perp}$. U is a σ-invariant, nondegenerate $(2n-4)$-space and all points of $\mathcal{G}$ in U are collinear to both p and r. By Lemma 3.6.6, q and s are collinear to points in U unless $n = 4$ and $q = 2$. Also, because of Lemma 3.5.5 and Lemma 3.5.6, the intersection of U with the geometry $\mathcal{G}$ is connected unless $n = 4$, $q = 2$. This finishes the proof. □

Lemma 3.6.9
If $n \geq 5$ or $n = 4$ and $q \neq 2, 3$, then any quadrangle is homotopically trivial.

To prove this lemma we will need some results from linear algebra:

Lemma 3.6.10
Let $n \geq 2$, $q \geq 3$, and let W be an $\mathbb{F}_{q^2}$-vector space of dimension n. Suppose f_1 and f_2 are two nontrivial Hermitian forms on W. Then there exists a vector of W which is non-singular with respect to both f_1 and f_2.

Proof. First suppose that f is a Hermitian form on W and l is a two-dimensional subspace in W that is not totally singular with respect to f. Then, if l is nondegenerate with respect to f, out of the total number of q^2+1 one-dimensional subspaces of l exactly $q+1$ are singular. Similarly, if f has rank one on l, then l contains exactly one singular one-dimensional subspace. Now, any f_1-singular one-dimensional subspace of W is contained in a two space l which is not totally isotropic with respect to f_1, since f_1 is nontrivial. If l is not totally isotropic with respect to f_2, then it contains at least $q^2 + 1 - q - 1 - q - 1 \geq 2$ one-dimensional subspaces that are non-singular with respect to both f_1 and f_2. On the other hand, if any such l is totally isotropic with respect to f_2, then any one-dimensional subspace that is singular with respect to f_1, is also singular with respect to f_2. But since f_2 is nontrivial on W, there exists a vector that is non-singular with respect to f_2, and hence with respect to f_1, too. $\square$

Lemma 3.6.11
Let $n \geq 3$, $q \geq 3$, and let W be an $\mathbb{F}_{q^2}$-vector space of dimension n. Suppose f_1, f_2, f_3 are three nontrivial Hermitian forms on W, and, furthermore, assume that f_1 is nondegenerate. Then there exists a vector of W which is non-singular with respect to all three forms.

Proof. As f_1 is nondegenerate and $n \geq 3$, any one-dimensional subspace that is singular with respect to f_1 is contained in a two-dimensional subspace l of f_1-rank one. Notice that l contains exactly q^2 one-dimensional subspaces that are non-singular with respect to f_1. If l is totally isotropic with respect to neither f_2 nor f_3, then there are at least $q^2 - q - 1 - q - 1 \geq 1$ one-dimensional subspaces that are non-singular with respect to all three forms. Therefore, suppose that any such subspace l is totally singular with respect to f_2 or f_3. However, this means that the set of f_1-singular one-dimensional subspaces is contained in the union of singular one-dimensional subspaces with respect to f_2, respectively f_3. But by Lemma 3.6.10, there is a vector $w \in W$ that is non-singular with respect to both f_2 and f_3. Consequently, w is also non-singular with respect to f_1. $\square$

Lemma 3.6.12
Let $n \geq 3$, $q \geq 4$, and let W be an $\mathbb{F}_{q^2}$-vector space of dimension n. Suppose f_1, f_2, f_3, f_4 are four nontrivial Hermitian forms on W, and, furthermore, assume that f_1 is nondegenerate. Then there exists a vector of W which is non-singular with respect to all four forms.

Proof. The proof of this lemma is similar to the proofs of the preceding two lemmas. The bound on q arises from the condition $q^2 - 3(q+1) \geq 1$. $\square$

Proof of Lemma 3.6.9. Let (a, b, c, d) be a quadrangle. If it contains a solid pair a, c or b, d, then we are done by Lemma 3.6.8. It is enough to show that any other quadrangle can be decomposed into triangles and quadrangles of that former case. We can assume a and c to be contained in $\langle b, b^\sigma \rangle^\perp$ by decomposing the quadrangle (a, b, c, d) into two quadrangles (a, b, c, b') and (a, d, c, b') where b' is a point of $\mathcal{G}$ in $\langle a, c, a^\sigma, c^\sigma \rangle^\perp$, a space which is not totally isotropic with respect to $((\cdot, \cdot))$. If $n \geq 5$ consider the space $U := \langle a, b, a^\sigma, b^\sigma \rangle^\perp$ of dimension $2n - 4$, which is nondegenerate (with respect to both forms). We want to find a point x of $\mathcal{G}$ in U that forms a solid pair with both c and d. Besides $((\cdot, \cdot))$ consider two more forms $f_2 : U \times U \to \mathbb{F}_{q^2} : f_2(u, v) = ((u', v'))$ and $f_3 : U \times U \to \mathbb{F}_{q^2} : f_3(u, v) = ((u'', v''))$ where u', v' are the projections onto $U \cap \langle c, c^\sigma \rangle^\perp$ and u'', v'' are the projections onto $U \cap \langle d, d^\sigma \rangle^\perp$, via the decomposition as given in Lemma 3.6.7. The forms f_2

and f_3 are nontrivial, as $U \cap \langle c, c^\sigma \rangle^\perp$ and $U \cap \langle d, d^\sigma \rangle^\perp$ both contain points of $\mathcal{G}$. For, by our assumption $a, c \in \langle b, b^\sigma \rangle^\perp$, and we can restrict our considerations to the subspace $\langle b, b^\sigma \rangle^\perp$ of $\mathcal{G}$. The dimension of $\langle a, c, a^\sigma, c^\sigma \rangle^\perp \cap \langle b, b^\sigma \rangle^\perp$ is at least four, so $\langle a, c, a^\sigma, c^\sigma \rangle^\perp \cap \langle b, b^\sigma \rangle^\perp$ cannot be totally isotropic with respect to $((\cdot,\cdot))$. Therefore we have found a point of $\mathcal{G}$ in $\langle a, c, a^\sigma, c^\sigma \rangle^\perp \cap \langle b, b^\sigma \rangle^\perp$, so f_2 is nontrivial. Similarly, f_3 is nontrivial. By Lemma 3.6.11, with $f_1 = ((\cdot,\cdot))_{|U \times U}$, there exists a point x of U such that its projections onto both $\langle c, c^\sigma \rangle^\perp$ and $\langle d, d^\sigma \rangle^\perp$ are non-singular. Hence, by Lemma 3.6.7, the point x forms a solid pair with both c and d, as we wanted. Now, let $W := \langle c, d, c^\sigma, d^\sigma \rangle^\perp$, which is also of dimension $2n-4$ and nondegenerate. By Lemma 3.6.6, W contains a point y of $\mathcal{G}$ collinear to x. We have accomplished the following: the quadrangle (a, b, c, d) has been decomposed into the triangles (a, b, x), (c, d, y) and the quadrangles (c, b, x, y), (a, d, y, x), both of which contain a solid pair c, x, respectively d, x.

In case $n = 4$ let $U := \langle a, a^\sigma \rangle^\perp$. Besides $((\cdot,\cdot))$ consider three more forms $f_2 : U \times U \to \mathbb{F}_{q^2} : f_2(u,v) = ((u', v'))$, $f_3 : U \times U \to \mathbb{F}_{q^2} : f_3(u,v) = ((u'', v''))$, and $f_4 : U \times U \to \mathbb{F}_{q^2} : f_4(u,v) = ((u''', v'''))$ where u', v' are the projections onto $U \cap \langle b, b^\sigma \rangle^\perp$, u'', v'' are the projections onto $U \cap \langle c, c^\sigma \rangle^\perp$, and u''', v''' are the projections onto $U \cap \langle d, d^\sigma \rangle^\perp$, via the decomposition as given in Lemma 3.6.7. The forms f_2, f_3, and f_4 are easily seen to be nontrivial. The remainder of the proof is as in the preceding paragraph with the only difference that we invoke Lemma 3.6.12 instead of Lemma 3.6.11 to find a suitable point x that is collinear with a and forms solid pairs with b, c, and d. Applications of Lemma 3.6.6 yield points y_1 collinear to x, b, c and y_2 collinear to x, c, d. We have decomposed the quadrangle (a, b, c, d) into the triangles (b, c, y_1) and (c, d, y_2) and the quadrangles (a, b, y_1, x), (a, d, y_2, x), and (c, y_1, x, y_2) with solid pairs. □

Finally, the decomposition of pentagons is now easy:

Lemma 3.6.13
If $n \geq 5$ or $n = 4$ and $q \neq 2, 3$, then any pentagon is homotopically trivial.

Proof. Let (a, b, c, d, e) be a pentagon. Consider $U := \langle a, b, a^\sigma, b^\sigma \rangle^\perp$ of dimension $2n-4$, which is nondegenerate (with respect to both forms). By Lemma 3.6.6, the point d is collinear to a point f of $\mathcal{G}$ inside U, decomposing the pentagon into triangles and quadrangles. □

We can summarize the results of this section as follows.

Proposition 3.6.14
If $n \geq 4$, then the geometry $\mathcal{G}$ is simply connected, unless $(n, q) \in \{(4, 2), (4, 3)\}$. □

It is unknown to us whether the cases $(n, q) \in \{(4, 2), (4, 3)\}$ are true exceptions.

3.7 Simple connectedness. Part II

In this section we assume $n = 3$. We will prove that the geometry $\mathcal{G}$ is simply connected for $q \geq 8$.

Lemma 3.7.1
Let (p, q, r) be the bad triangle $(\langle e_1 \rangle, \langle e_2 \rangle, \langle ye_1 + ze_2 + (ce_3 + f_3)\rangle)$ with $c\bar{c} = -1$ and $yz \neq 0$ and $y\bar{y} + z\bar{z} \neq 0$. Furthermore, assume that $y\bar{y} \neq 1$, $z\bar{z} \neq 1$, $y\bar{y} + z\bar{z} \neq 1$, $y\bar{y} + z\bar{z} \neq 2$, $(y\bar{y} - 1)(y\bar{y} + z\bar{z} - 1) \neq 1$, $(z\bar{z} - 1)(y\bar{y} + z\bar{z} - 1) \neq 1$. Then (p, q, r) can be decomposed into good triangles.

Proof. Consider the plane $\langle f_1, f_2, f_3 \rangle$ and fix the points $a = \langle f_3 \rangle$, $b = \langle -yf_3 + cf_1 \rangle$, $c = \langle -zf_3 + cf_2 \rangle$. These are uniquely determined by the conditions that $a \perp \langle p, q \rangle$, $b \perp \langle q, r \rangle$ and $c \perp \langle p, r \rangle$. Notice that all of a, b, c are points of $\mathcal{G}$ if and only if $y\bar{y} \neq 1$ and $z\bar{z} \neq 1$ which is satisfied by assumption. The projective lines ap, aq, bq, and cp are lines of $\mathcal{G}$ because the two points on them are perpendicular with respect to $((\cdot, \cdot))$. Also ab and ac are in fact the projective lines $\langle f_1, f_3 \rangle$, respectively $\langle f_2, f_3 \rangle$, so they are lines of $\mathcal{G}$. Next we have to investigate the conditions under which the projective lines bc, br, and cr are lines in $\mathcal{G}$. We need to see that $((\cdot, \cdot))$ is nondegenerate on each of these two-dimensional spaces, so we will investigate the Gram matrices and find their determinants. In the case of bc we get

$$\det \begin{pmatrix} y\bar{y} - 1 & z\bar{z} \\ \bar{y}z & z\bar{z} - 1 \end{pmatrix} = -y\bar{y} - z\bar{z} + 1.$$

The space br yields

$$\det \begin{pmatrix} y\bar{y} - 1 & -y \\ -\bar{y} & y\bar{y} + z\bar{z} \end{pmatrix} = (y\bar{y} - 1)(y\bar{y} + z\bar{z} - 1) - 1.$$

In the case of cr we get

$$\det \begin{pmatrix} z\bar{z} - 1 & -z \\ -\bar{z} & y\bar{y} + z\bar{z} \end{pmatrix} = (z\bar{z} - 1)(y\bar{y} + z\bar{z} - 1) - 1.$$

Now we compute conditions such that (a, b, c), (a, b, q), (a, c, p), (a, p, q), (b, c, r), (b, q, r), and (c, p, r) are good triangles. Notice that the triangles (a, b, c), (a, b, q), (a, p, q), and (a, c, p) are automatically good. Moreover, the case of (b, q, r) gives

$$\det \begin{pmatrix} y\bar{y} - 1 & 0 & -y \\ 0 & 1 & \bar{z} \\ -\bar{y} & z & y\bar{y} + z\bar{z} \end{pmatrix} = y\bar{y}(y\bar{y} - 1).$$

In the case of (b, c, r) we get

$$\det \begin{pmatrix} y\bar{y} - 1 & y\bar{z} & -y \\ \bar{y}z & z\bar{z} - 1 & -z \\ -\bar{y} & -\bar{z} & y\bar{y} + z\bar{z} \end{pmatrix} = (y\bar{y} + z\bar{z})(2 - y\bar{y} - z\bar{z}).$$

Finally, for (c, p, r) we have

$$\det \begin{pmatrix} z\bar{z}-1 & 0 & -z \\ 0 & 1 & \bar{y} \\ -\bar{z} & y & y\bar{y}+z\bar{z} \end{pmatrix} = z\bar{z}(z\bar{z}-1).$$

We have obtained the conditions listed in the hypothesis of the lemma. □

Lemma 3.7.2
Let $q = p^e$, and let $c, d \in \mathbb{F}_{q^2}$ such that $c\bar{c} = -1$, $d \neq 0$. Then the system of equations $y\bar{y} + z\bar{z} = 1$ and $\bar{y} - \bar{z}c = d$ has exactly q solutions.

Proof. The pair (y, z) is a solution of the first equation if and only if the matrix $A_{y,z} := \begin{pmatrix} y & -\bar{z} \\ z & \bar{y} \end{pmatrix}$ has determinant one, thus the solutions of the first equation are parametrized by the elements of the group $SU_2(q^2)$. Observe that

$$(c, 1)A_{y,z} = (yc + z, \bar{y} - \bar{z}c) = \left(c\overline{(\bar{y} - \bar{z}c)}, \bar{y} - \bar{z}c\right).$$

Therefore two pairs (y, z), (y', z') are solutions for the system of equations if and only if the matrix $A_{y,z}A^{-1}_{y',z'}$ stabilizes the vector $(c, 1)$, which is of norm 0 with respect to the unitary form. The stabilizer such a vector $(c, 1)$ is the p-Sylow subgroup of the unitary group. (For, a matrix that stabilizes $(c, 1)$ has 1 as an eigenvalue. As its determinant is 1 as well, the other eigenvalue also has to be 1. But any such matrix has either order 1 or order p, as can be seen from its Jordan normal form.) So, if the above system has a solution, then it has exactly q solutions, for a fixed d. Since the order of $SU_2(q^2)$ is $q(q^2 - 1)$, the above system has q solutions for each $d \neq 0$. (Indeed, there are $q^2 - 1$ possible d's.) □

Lemma 3.7.3
Let (p, q, r) be a bad triangle, and let x be the radical of the plane $\langle p, q, r\rangle$ with respect to $((\cdot, \cdot))$. Then $x^\sigma = x$.

Proof. Suppose $x^\sigma \neq x$. Then the $(\cdot, \cdot)$-totally isotropic planes $\langle p, q, r\rangle$ and $\langle p^\sigma, q^\sigma, r^\sigma\rangle$ do not intersect. Indeed, if they did, then the the radical of $\langle p, q, r\rangle$ were contained in the intersection. Hence, by symmetry, $\langle p, q, r\rangle \cap \langle p^\sigma, q^\sigma, r^\sigma\rangle$ had to contain the two space $\langle x, x^\sigma\rangle$, which on one hand were contained in the radical of $\langle p, q, r\rangle$ and on the other hand is totally isotropic with respect to $((\cdot, \cdot))$, contradicting the fact that the rank with respect to $((\cdot, \cdot))$ of $\langle p, q, r\rangle$ equals two. Consequently, $V = \langle p, q, r, p^\sigma, q^\sigma, r^\sigma\rangle$, which has a radical with respect to $(\cdot, \cdot)$ containing x, contradicting nondegeneracy of $(\cdot, \cdot)$. □

Lemma 3.7.4
Let $q \geq 8$, and let (p, q, r) be a bad triangle. Then the triangle can be decomposed into good triangles.

Proof. Let x be the radical of the plane $\langle p, q, r\rangle$. By the preceding lemma we have $x = x^\sigma$. Now, by Lemmas 3.6.3 and 3.6.4, we can assume $(p, q, r) = (\langle e_1\rangle, \langle e_2\rangle, \langle ye_1 + ze_2 + (ce_3 + f_3)\rangle)$ satisfying $c\bar{c} = -1$ and $yz \neq 0$ and $y\bar{y} + z\bar{z} \neq 0$. It is enough to show that this triangle is conjugate to a triangle satisfying the hypothesis of Lemma 3.7.1. Let $g \in G_\sigma$ fixing e_1, e_2, f_1, f_2 pointwise. Then Lemma 3.7.2 shows that, for any nontrivial $d \in \mathbb{F}_{q^2}$, the element g can be chosen such that $(ce_3 + f_3)^g = d(c\frac{\bar{d}}{d}e_3 + f_3)$, and we have conjugated (p, q, r) to $(p^g, q^g, r^g) = \left(\langle e_1\rangle, \langle e_2\rangle, \left\langle \frac{y}{d}e_1 + \frac{z}{d}e_2 + (c\frac{\bar{d}}{d}e_3 + f_3)\right\rangle\right)$. It remains to be seen that we can pick d such that $y' = \frac{y}{d}$, $z' = \frac{z}{d}$ satisfy the conditions of 3.7.1. Then, by that lemma, we can decompose (p^g, q^g, r^g) (and hence its conjugate (p, q, r)) into good triangles. Notice that $yz \neq 0$ if and only if $\frac{y}{d}\frac{z}{d} \neq 0$, and $c\bar{c} = -1$ if and only if $c\frac{\bar{d}}{d}\overline{\left(c\frac{\bar{d}}{d}\right)} = -1$. The same holds for the condition $y\bar{y} + z\bar{z} \neq 0$.

If there are five different values of $d\bar{d}$ in $\mathbb{F}_q$, then we are able to modify $y\bar{y}$ and $z\bar{z}$ (to $\frac{y\bar{y}}{d\bar{d}}$ respectively $\frac{z\bar{z}}{d\bar{d}}$) such that the conditions $y\bar{y} \neq 1$, $z\bar{z} \neq 1$, $y\bar{y} + z\bar{z} \neq 1$, $y\bar{y} + z\bar{z} \neq 2$ are satisfied for the modified parameters. Furthermore, if there are four more values of $d\bar{d}$, we can additionally modify $y\bar{y}$ and $z\bar{z}$ for $(y\bar{y}-1)(y\bar{y}+z\bar{z}-1) \neq 1$, $(z\bar{z}-1)(y\bar{y}+z\bar{z}-1) \neq 1$ to hold. This is the case for $q \geq 11$, which leaves $q \in \{8, 9\}$. A straightforward check by hand or in GAP [Sch95] will show that any pair $y\bar{y}$, $z\bar{z}$ can be scaled by $d\bar{d}$ to satisfy all conditions. □

Now we will shift our attention to quadrangles. By the preceding results, it is enough to decompose quadrangles into triangles, regardless whether they are good or bad.

Lemma 3.7.5
Let $q \geq 5$. Then any quadrangle such that no three points lie on a common line, can be decomposed into quadrangles that do not lie in a totally isotropic subspace of V with respect to $(\cdot,\cdot)$ and, furthermore, contain two opposite vertices that span a nondegenerate two space with respect to $((\cdot,\cdot))$.

Proof. Let (a, b, c, d) be a quadrangle such that $\langle a, b, c, d\rangle$ is totally isotropic with respect to $(\cdot,\cdot)$. This implies $(a, c) = 0$, whence $c \neq a^\sigma$, because $(a, a^\sigma) = ((a, a)) \neq 0$. Choose a non-singular vector $v \in a^{\perp\!\!\perp} \cap b^\perp \cap d^\perp$ that does not lie in $a^\perp \cup c^\perp$. The vector v exists because, firstly, $a^{\perp\!\!\perp} \cap b^\perp \cap d^\perp$ is not totally isotropic with respect to $((\cdot,\cdot))$ (since it is a three space contained in the nondegenerate five space $a^{\perp\!\!\perp}$) and, secondly, because $a^{\perp\!\!\perp} \cap b^\perp \cap d^\perp \not\subset a^\perp$ and $a^{\perp\!\!\perp} \cap b^\perp \cap d^\perp \not\subset c^\perp$. (First recall from Lemma 3.4.3 that $(a^\sigma)^\perp = a^{\perp\!\!\perp}$. Then, indeed, $(a^\sigma)^\perp \cap b^\perp \cap d^\perp \subset c^\perp$ implies $\langle a^\sigma, b, d\rangle = ((a^\sigma)^\perp \cap b^\perp \cap d^\perp)^\perp \supset (c^\perp)^\perp = c$, and c can be written as a linear combination of a^σ, b, and d. By $(a, c) = 0$ and $(a, a^\sigma) = ((a, a)) \neq 0$, the point c has to lie on the projective line bd, making it a line of Γ. However this cannot be the case by hypothesis. The same arguments work for a instead of c.) Now the projective line $l = \langle a, v\rangle$ has rank two with respect to $((\cdot,\cdot))$ and it contains neither b nor d. Using Lemma 3.5.4, l contains $q^2 - 2q - 1$ points of $\mathcal{G}$ that are collinear with b, respectively d, and at least $q^2 - 2q - 1$ points of $\mathcal{G}$ that generate a nondegenerate two space with c. Since $q \geq 5$ and since l contains $q^2 - q$ points of $\mathcal{G}$, the space l has to contain a point p of $\mathcal{G}$ that generates a nondegenerate two space with c and that is collinear to both b and d. Moreover, $(p, a) \neq 0 \neq (p, c)$ and $((p, a)) = 0$, so we are done. □

Lemma 3.7.6
Let $q \geq 7$. Then any quadrangle can be decomposed into triangles.

Proof. Denote the quadrangle by (a, b, c, d), as in the proof of the preceding lemma. By that lemma, we can assume that $(a, c) \neq 0$ and that $\langle a, c\rangle$ is nondegenerate with respect to $((\cdot, \cdot))$. Set $W := a^\perp \cap c^\perp$ and $U_1 := W \cap b^\perp$ and $U_2 := W \cap d^\perp$.

If $l = U_1 \cap U_2$ is of rank two with respect to $((\cdot, \cdot))$, then we can apply Lemma 3.5.3 to the planes $\langle a, l\rangle$, $\langle b, l\rangle$, $\langle c, l\rangle$, and $\langle d, l\rangle$ to obtain $q^2 - 5q - 4$ points of $\mathcal{G}$ on l collinear to all of a, b, c, d. Notice that this is a positive number for $q \geq 7$.

Suppose now that $l = U_1 \cap U_2$ is of rank one. Then the plane $\langle b, l\rangle$ has rank at least one. However, it cannot have rank one, since it lies inside the $((\cdot, \cdot))$-nondegenerate four-dimensional space $a^\perp \cap b^\perp = (a^\sigma)^{\perp\perp} \cap (b^\sigma)^{\perp\perp}$. Indeed, a two-dimensional radical would be maximal totally isotropic inside $a^\perp \cap b^\perp$ and could not have a polar of dimension three. Similar arguments hold for the points a, c, d instead of b. Applying Lemma 3.5.3 as in the above paragraph gives a point of $\mathcal{G}$ collinear to all of a, b, c, d.

Suppose now l is totally isotropic with respect to $((\cdot, \cdot))$. Then l has to contain the radicals r_1 and r_2 (with respect to $((\cdot, \cdot))$) of the planes U_1 and U_2. These radicals cannot coincide as otherwise we would obtain a radical for the $((\cdot, \cdot))$-nondegenerate space $a^\perp \cap c^\perp$. Notice that $r_2^\perp \cap U_1 = br_2$. Choose a line of $\mathcal{G}$ through b inside U_1. (This exists since the rank with respect to $((\cdot, \cdot))$ of U_1 is two.) This line contains a point p collinear to both a and c, by Lemma 3.5.3. Now $p^\perp \cap W$ intersects U_2 in a line that does not contain r_2. Hence its rank with respect to $((\cdot, \cdot))$ is two. The arguments given in the second paragraph of this proof settle the claim. □

As in the $n \geq 4$ case, pentagons are easy to handle.

Lemma 3.7.7
Let $q \geq 5$. Then any pentagon is null homotopic.

Proof. Let (a, b, c, d, e) be a pentagon. Consider the space $U := \langle a, b, d\rangle^\perp$ of dimension three. Its rank with respect to $((\cdot, \cdot))$ has to be at least two, as the rank of $\langle a, b\rangle$ is two. Choosing a $((\cdot, \cdot))$-nondegenerate projective line l in U and applying Lemma 3.5.3 in turn on the planes $\langle a, l\rangle$, $\langle b, l\rangle$, $\langle d, l\rangle$, we will find $q^2 - 2q - 1 - q - 1 - q - 1 = q^2 - 4q - 3 > 0$ points on l collinear to all of a, b, d, decomposing the pentagon. □

We summarize the results of this section as follows.

Proposition 3.7.8
If $n = 3$ and $q \geq 8$, then $\mathcal{G}$ is simply connected. □

It is easy to see that $\mathcal{G}$ is *not* simply connected if $(n, q) = (3, 2)$. We do not know whether this is the case for $7 \geq q \geq 3$. Altogether, we have proved the following:

Theorem 3.7.9 (joint with Hoffman, Shpectorov)
Let $n \geq 2$. The following hold.

(i) *$\mathcal{G}_\sigma$ is a rank n geometry admitting a flag-transitive group of automorphisms $G_\sigma \cong Sp_{2n}(q)$.*

(ii) *$\mathcal{G}_\sigma$ is connected unless $(n, q) = (2, 2)$; it is residually connected if $q > 2$.*

(iii) *$\mathcal{G}_\sigma$ is simply connected if $n \geq 5$, or $n = 4$ and $q \geq 4$, or $n = 3$ and $q \geq 8$.*

Proof. Part (i) follows from Propositions 3.4.9 and 3.5.2. Part (ii) follows from Proposition 3.5.7 and Corollary 3.5.8. Finally, part (iii) is proved in Propositions 3.6.14 and 3.7.8. □

Corollary 3.7.10
Let $n \geq 3$. Fix a maximal flag F of $\mathcal{G}_\sigma$, and let $\mathcal{A}$ be the amalgam on the stabilizers in G_σ of all non-empty subflags of F. If (n, q) is distinct from $(3, 2)$, $(3, 3)$, $(3, 4)$, $(3, 5)$, $(3, 7)$, $(4, 2)$, and $(4, 3)$, then G_σ is the universal completion of $\mathcal{A}$.

Proof. The claim follows from the theorem and Tits' lemma B.2.5. □

3.8 Consequences of simple connectedness

Theorem 3.7.9 has some group-theoretic implications along the lines of Phan's theorems.

Theorem 3.8.1 (joint with Hoffman, Shpectorov)
Let $n \geq 3$. Let F be a maximal flag of $\mathcal{G}_\sigma$. For $2 \leq s \leq n-1$, let $\mathcal{A}_{(s)}$ be the amalgam of all rank s parabolics, i.e., the stabilizers in G_σ of all subflags of F of corank s. Then the following hold.

(i) *If $q \geq 8$ and $n \geq 3$, then G_σ is the universal completion of $\mathcal{A}_{(2)}$.*

(ii) *If $7 \geq q \geq 4$ and $n \geq 4$, then G_σ is the universal completion of $\mathcal{A}_{(3)}$.*

(iii) *If $q = 2, 3$ and $n \geq 5$, then G_σ is the universal completion of $\mathcal{A}_{(4)}$.*

Recall our notation from Definition B.2.4. By G_i we denote the stabilizer in G_σ of the element of F of type i, and G_J, J a subset of the type set I, stands for $\bigcap_{i \in J} G_i$. This includes $G_\emptyset = G_\sigma$.

Proof. Let $s \geq 2$ if $q \geq 8$, $s \geq 3$ if $7 \geq q \geq 4$, and $s \geq 4$ if $q = 2, 3$. Suppose that $n \geq s+1$. We will proceed by induction and show that the universal completion of

$\mathcal{A}_{(s)}$ coincides with the universal completion of $\mathcal{A}_{(s+1)}$. Denote by $H_{(s)}$ the universal completion of $\mathcal{A}_{(s)}$. Let $J \subset I$ and $|I \setminus J| = s+1$. Let $F_J \subset F$ be of type J, so that G_J is the stabilizer of F_J in H. Observe that the residue of $\mathcal{G}$ with respect to F_J (denoted by $\mathcal{G}_J$) is connected. Indeed, if $q > 2$ then $\mathcal{G}$ is residually connected by Corollary 3.5.8. In particular, $\mathcal{G}_J$ is connected. If $q = 2$ then either the diagram of $\mathcal{G}_J$ is disconnected, or the diagram is connected. In the first case, $\mathcal{G}_J$ is connected. In the second case, $\mathcal{G}_J$ is either our flipflop geometry of rank $s+1$, or the geometry as in [BS]. The connectedness follows from Proposition 3.5.7 and [BS] (see Theorem 3.2.3). Observe also that $\mathcal{G}_J$ is simply connected. Indeed, either the diagram of $\mathcal{G}_J$ is disconnected, or it is connected. In the first case, the simple connectedness follows from Lemma A.7.7. The connectivity assumption in that lemma holds because one of $\mathcal{G}_1$ and $\mathcal{G}_2$ has sufficient rank (rank at least two, if $q \geq 3$, and rank at least three, if $q = 2$) to be connected. If the diagram of $\mathcal{G}_J$ is connected then $\mathcal{G}_J$ is simply connected by Theorem 3.7.9 (iii) or [BS] (see Theorem 3.2.3), depending on its diagram.

The universal completion $H_{(s+1)}$ of $\mathcal{A}_{(s+1)}$ is also a completion of $\mathcal{A}_{(s)}$. Indeed, if $n = s+1$, then $H_{(n)} = H = G_\sigma$, which certainly is a completion of $\mathcal{A}_{(n-1)}$. In case $n > s+1$, the amalgam $\mathcal{A}_{(s+1)}$ is the union of all G_J with J of corank $s+1$ and we have a map $\pi : \mathcal{A}_{(s+1)} \to H_{(s+1)}$ such that $\pi_{|G_J} : G_J \to H_{(s+1)}$ is a homomorphism. Consequently, also $\pi_{|G_J \cap G_{J'}} : G_J \cap G_{J'} \to H_{(s+1)}$ is a homomorphism. It remains to show that the set of all images $\pi(G_J \cap G_{J'})$ with $|I \setminus (J \cup J')| = s$ actually generate $H_{(s+1)}$. But since $\mathcal{G}_J$ is connected, the group $\pi(G_J) \leq H_{(s+1)}$ is generated by all those images for a fixed J (because the $G_J \cap G_{J'}$ are maximal parabolics in G_J). Thus, $H_{(s+1)}$ is a completion of $\mathcal{A}_{(s)}$, as it is generated by the $\pi(G_J)$. Therefore there is a canonical homomorphism ϕ from $H_{(s)}$ onto $H_{(s+1)}$ whose restriction to $\mathcal{A}_{(s)}$ is the identity. Let ψ be the inverse of the restriction of ϕ to $\mathcal{A}_{(s)}$. Let $J \subset I$ be such that $|I \setminus J| = s+1$ and let $\hat{G}_J$ be defined as $\langle \psi(G_J \cap \mathcal{A}_{(s)}) \rangle$. By simple connectedness of $\mathcal{G}_J$ and Tits' lemma B.2.5, ϕ induces an isomorphism of $\hat{G}_J$ onto G_J. Therefore, ψ extends to an isomorphism of $\mathcal{A}_{(s+1)} \subset H_{(s+1)}$ onto

$$\hat{\mathcal{A}}_{s+1} = \bigcup_{J \subset I, |I \setminus J| = s+1} \hat{G}_J \qquad \subset H_{(s)}.$$

Hence the universal completion of $\mathcal{A}_{(s)}$ coincides with the universal completion of $\mathcal{A}_{(s+1)}$. The fact $H_{(n)} = G_\sigma$ finishes the proof. □

Corollary 3.8.2
Let $n \geq 3$. The geometry $\mathcal{G}_\sigma$ is 4-simply connected. It is 2-simply connected if $q \geq 8$ and 3-simply connected if $q \geq 4$. □

Notation 3.8.3 The maximal parabolics G_i with respect to F are semisimple subgroups of $G_\sigma \cong Sp_{2n}(q)$ of the form $GU_i(q^2) \times Sp_{2n-2i}(q)$, $i = 1, \ldots, n$. Each G_i stabilizes a $2i$-dimensional nondegenerate subspace U_i of the natural symplectic module U of G_σ. It induces $GU_i(q^2)$ on U_i and $Sp_{2n-2i}(q)$ on $U_i^\perp$. The intersection of all G_i

(also known as the Borel subgroup arising from the action of G_σ on $\mathcal{G}_\sigma$) is a maximal torus T of G_σ of order $(q+1)^n$. Let G_i^0 be the subgroup $SU_i(q^2) \times Sp_{2n-2i}(q)$ of G_i. For an arbitrary parabolic $G_J = \bigcap_{i\in J} G_i$ define $G_J^0 = \bigcap_{i\in J} G_i^0$. Here J is a subset of the type set $I = \{1,\dots,n\}$ of $\mathcal{G}_\sigma$. It can be shown that $G_J = G_J^0 T$.

In case of a minimal parabolic $G_{I\setminus\{i\}}$, we have that $L_i := G^0_{I\setminus\{i\}} \cong SL_2(q)$. In fact, if $1 \le i \le n-1$ then L_i arises as $SU_2(q^2) \cong SL_2(q)$, while L_n arises as $Sp_2(q) \cong SL_2(q)$. Notice that $T_i = L_i \cap T$ is a torus in L_i of size $q+1$. Notice also that the subgroups T_i generate T. If $q \ne 2$ then $\langle L_i, L_j\rangle = G^0_{I\setminus\{i,j\}}$. In particular, the subgroups L_i have the following properties:

(i) $L_i \cong SU_2(q^2)$, if $i = 1,\dots,n-1$;

(ii) $L_n \cong Sp_2(q)$;

(iii) $\langle L_i, L_j\rangle \cong \begin{cases} L_i \times L_j, & \text{if } |i-j| > 1; \\ SU_3(q^2), & \text{if } |i-j| = 1 \text{ and } \{i,j\} \ne \{n-1,n\}; \\ Sp_4(q), & \text{if } \{i,j\} = \{n-1,n\}. \end{cases}$

These properties are similar to Phan's original description of his Phan systems as given in 3.2 with the difference that Phan only considers simply laced diagrams.

Define $\mathcal{A}^0_{(s)}$ to be the amalgam formed by the subgroups G_J^0 for all parabolics G_J of rank s. The following Theorem 3.8.4 is a "stripped-of-T" (Phan-type) version of Theorem 3.8.1. Let $\{e_1,\dots,f_n\}$ be a canonical basis for σ. For the purposes of proving that theorem, we will assume that the flag F consists of the subspaces $\langle e_1\rangle$, $\langle e_1, e_2\rangle$, …, $\langle e_1,\dots,e_n\rangle$. With respect to this basis, T consists of all diagonal matrices $diag(a_1,\dots,a_n,a_1^{-1},\dots,a_n^{-1})$, where each a_i is of order dividing $q+1$. Furthermore, T_i, $1 \le i < n$, consists of matrices from T, for which $a_i = a_{i+1}^{-1} = a_{n+i}^{-1} = a_{n+i+1}$, with all other a_j equal to one. If $i = n$ then $a_n = a_{2n}^{-1}$ and $a_j = 1$ for all other j. Manifestly, T is the direct product of all T_i's.

Theorem 3.8.4 (joint with Hoffman, Shpectorov)
Retain the notation of 3.8.3. Then the following hold.

(i) *If $q \ge 8$ and $n \ge 3$, then G_σ is the universal completion of $\mathcal{A}^0_{(2)}$.*

(ii) *If $7 \ge q \ge 4$ and $n \ge 4$, then G_σ is the universal completion of $\mathcal{A}^0_{(3)}$.*

(iii) *If $q = 2,3$ and $n \ge 5$, then G_σ is the universal completion of $\mathcal{A}^0_{(4)}$.*

Proof. Let $s = 2$ if $q \ge 8$, $s = 3$ if $7 \ge q \ge 4$, and $s = 4$ if $q = 2,3$, and suppose that $n \ge s+1$. Let $\hat{H}$ be the universal completion of the amalgam $\mathcal{A}^0_{(s)}$. Let ϕ be the canonical homomorphism of $\hat{H}$ onto H, that exists due to the fact that H is a completion of $\mathcal{A}^0_{(s)}$. Denote by $\hat{\mathcal{A}}^0_{(s)}$ the copy of $\mathcal{A}^0_{(s)}$ in $\hat{H}$, so that ϕ induces an isomorphism of $\hat{\mathcal{A}}^0_{(s)}$ onto $\mathcal{A}^0_{(s)}$. As in the proof of Theorem 3.8.1, let $\psi : \mathcal{A}^0_{(s)} \to \hat{\mathcal{A}}^0_{(s)}$ be the inverse of $\phi_{|\hat{\mathcal{A}}^0_{(s)}}$. Additionally, define $\hat{T}_i = \psi(T_i)$

and $\hat{T} = \left\langle \hat{T}_1, \ldots, \hat{T}_n \right\rangle$. Observe that $T_i, T_j \le G^0_{I\setminus\{i,j\}} = \langle L_i, L_j\rangle \subset \mathcal{A}^0_{(s)}$. Since ψ restricted to the latter group is an isomorphism to $\psi(G^0_{I\setminus\{i,j\}})$, the groups $\hat{T}_i$ and $\hat{T}_j$ commute elementwise. Because T is the direct product of T_i's, the map ϕ establishes an isomorphism between $\hat{T}$ and T. Let J be a subset of I with $|I\setminus J| = s$. Observe that $G_J = G^0_J T$. Accordingly, we would like to define $\hat{G}_J$ as $\hat{G}^0_J \hat{T}$, where $\hat{G}^0_J = \psi(G^0_J)$. For this definition to make sense, we need to show that $\hat{T}$ normalizes $\hat{G}^0_J$. Assume first that $q > 2$. Since G^0_i is normal in G_i and since $T \le G_i$, we have that T normalizes all G_i and therefore T normalizes every $L_i = \cap_{j\in I\setminus\{i\}} G^0_J$. Observe that $T_j \le L_j$ and $L_i, L_j \le G^0_{I\setminus\{i,j\}} = \langle L_i, L_j\rangle$. Since ψ is an isomorphism from $\mathcal{A}^0_{(s)}$ to $\hat{\mathcal{A}}^0_{(s)}$, the group $\hat{T}_j$ normalizes $\hat{L}_i$ for all i and j. It is clear that G^0_J is generated by L_i, $i \in I \setminus J$. The same must be true for $\hat{G}^0_J$ and $\hat{L}_i$'s. Therefore every $\hat{T}_j$ will normalize every $\hat{G}^0_J$ which means that also $\hat{T}$ normalizes $\hat{G}^0_J$. If $q = 2$ the same result can be achieved by using $G^0_{I\setminus\{i,j\}}$'s in place of L_i's; recall that in this case we assume $s = 4$. Since $\hat{T}$ normalizes G^0_J and since $\hat{T} \cap \hat{G}^0_J = \left\langle \hat{T}_j \mid j \in I \setminus J \right\rangle$ is isomorphic (via ϕ) to $T \cap G^0_J$, the map ϕ establishes an isomorphism between $\hat{G}_J$ and G_J, and, thus, ϕ extends to an isomorphism

$$\hat{\mathcal{A}}_{(s)} = \bigcup_{J\subset I, |I\setminus J| = s} \hat{G}_J \qquad \longrightarrow \qquad \mathcal{A}_{(s)} \quad .$$

Therefore, the universal completions of $\mathcal{A}_{(s)}$ and $\mathcal{A}^0_{(s)}$ are isomorphic, and the claim follows from Theorem 3.8.1. □

Notation 3.8.5 Note that G_i, $i \ne n$, is not a maximal semisimple subgroup of G_σ. Namely, G_i is contained in the full stabilizer H_i of the decomposition $U = U_i \oplus U_i^\perp$. The subgroup H_i is isomorphic to $Sp_{2i}(q) \times Sp_{2n-2i}(q)$. It is a maximal parabolic with respect to the action of G_σ on the rank $n-1$ pregeometry Δ of all proper nondegenerate subspaces of U.

Proposition 3.8.6
Δ is a connected geometry, and the natural action of $G_\sigma \cong Sp_{2n}(q)$ on it is flag-transitive.

Proof. Let $U_1 \le \cdots \le U_t$ be a maximal flag. If the dimension of U_t is not $2n-2$, then the dimension of $U_t^\perp$ is at least four and we can find a proper nondegenerate two-dimensional subspace U of $U_t^\perp$. But now $U_t \oplus U$ is still a proper nondegenerate subspace of V and $U_1 \le \cdots \le U_t \le U_t \oplus U$ is a flag of Δ, a contradiction. Hence U_t has dimension $2n-2$. Similarly one can show that U_{i-1} has codimension 2 in U_i for $2 \le i \le n-1$. Therefore, Δ is a geometry. Given any maximal flag $U_1 \le \cdots \le U_{n-1}$, we can choose a hyperbolic basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V such that $U_i = \langle e_1, \ldots, e_i, f_1, \ldots, f_i\rangle$, $1 \le i \le n-1$. Flag-transitivity of the group

$Sp_{2n}(q)$ now follows from transitivity of $Sp_{2n}(q)$ on the set of hyperbolic bases of V.

It remains to show connectedness of Δ. Let U and U' be two nondegenerate two-dimensional subspaces of V. If U and U' are orthogonal then $\langle U, U' \rangle$ is nondegenerate and so U and U' are adjacent in the collinearity graph of Δ. If U and U' meet in a one-dimensional space then $\langle U, U' \rangle$ is of dimension three and rank two. Therefore it is contained in a nondegenerate 4 space. Thus again U and U' are adjacent. Finally if U and U' are disjoint and not perpendicular, we can find vectors $u \in U$ and $u' \in U'$ such that $\langle u, u' \rangle$ is nondegenerate. Clearly the latter subspace is adjacent to both U and U' so they are at distance two. We have shown that the collinearity graph of Δ has diameter two. In particular, it is connected. □

Note that $\{U_i \mid 1 \leq i \leq n-1\}$ is a maximal flag of Δ, and H_i's are the corresponding maximal parabolics.

Corollary 3.8.7
Δ is residually connected. □

The following results will be derived from Theorem 3.8.1 and the results from [BS].

Proposition 3.8.8
Let $n \geq 4$. Then Δ is simply connected provided that $(n, q) \notin \{(4,2), (4,3)\}$.

Proof. Suppose that $n \geq 4$ and $n \geq 5$ if $q = 2$ or 3. Let $\mathcal{B} = \bigcup_{1 \leq i \leq n-1} H_i$. According to Tits' lemma B.2.5, the conclusion of the theorem is equivalent to $U(\mathcal{B}) \cong G_\sigma$. Let $\mathcal{A} = \bigcup_{1 \leq i \leq n} G_i$ be, as before, the amalgam of maximal parabolics related to the action of $H = G_\sigma$ on the flipflop geometry $\mathcal{G}$. Let $\mathcal{A}' = \bigcup_{1 \leq i \leq n-1} G_i$. Then $\mathcal{A}'$ is contained in $\mathcal{B}$, since $G_i \leq H_i$ for $1 \leq i \leq n-1$. The claim of the theorem will follow from Theorem 3.7.9 (iii) and Tits' lemma B.2.5, once we show that $U(\mathcal{B}) \cong U(\mathcal{A}')$ and $U(\mathcal{A}') \cong U(\mathcal{A})$. We will start with the second isomorphism. Let $\hat{H} = U(\mathcal{A}')$. Let also ψ be the canonical embedding of $\mathcal{A}'$ into $\hat{H}$ and define $\hat{G}_i = \psi(G_i)$, $1 \leq i \leq n-1$, and $\hat{\mathcal{A}}' = \psi(\mathcal{A}')$. Notice that $G_n \cap \mathcal{A}'$ is the amalgam of maximal parabolics in G_n acting on the residue $\mathcal{G}_{\{n\}}$ of $\mathcal{G}$. By [BS] (see Theorem 3.2.3), $\mathcal{G}_{\{n\}}$ is simply connected. Therefore, $\psi(G_n \cap \mathcal{A}')$ generates in $\hat{H}$ a subgroup $\hat{G}_n$ isomorphic to G_n. Clearly, $\hat{\mathcal{A}}' \cup \hat{G}_n$ is isomorphic to $\hat{\mathcal{A}}$ and hence $U(\mathcal{A}') \cong U(\mathcal{A})$. Turning to the isomorphism $U(\mathcal{B}) \cong U(\mathcal{A}')$, we let $\hat{H} = U(\mathcal{B})$ and let ψ to be the embedding of $\mathcal{B}$ into $\hat{H}$. We claim that $\psi(\mathcal{A}')$ generates $\hat{H}$. Indeed, since Δ is residually connected (cf. the preceding corollary), any two $\psi(H_i)$ generate $\hat{H}$. Take $i = n-1$ or $n-2$. Then $H_i = L \times R$, where $L \cong Sp_{2i}(,q)$ and $R \cong Sp_{2n-2i}(q)$. Observe that $R \leq G_J$ for $1 \leq j \leq i$ and that $\bigcup_{1 \leq j \leq i} (L \cap G_J)$ is the amalgam of maximal parabolics for L acting on its corresponding flipflop geometry (of rank i). Since that geometry is connected, $\psi(H_i) \leq \langle \psi(\mathcal{A}') \rangle$. Thus, $\psi(\mathcal{A}')$ indeed generates $\hat{H}$. Consequently, $\hat{H}$ must be a quotient of $U(\mathcal{A}') \cong U(\mathcal{A}) \cong H$. Since also, H is isomorphic to a quotient of $\hat{H}$, we finally obtain $U(\mathcal{B}) \cong H \cong U(\mathcal{A}')$. □

Corollary 3.8.9
Δ is 2-simply connected for $q \geq 4$, $n \geq 3$ and 3-simply connected if q equals 2 or 3 and $n \geq 4$. □

Theorem 3.8.10 (joint with Hoffman, Shpectorov)
Let $n \geq 4$, and use the notation in 3.8.5. If $q \geq 4$ then the amalgam of any three subgroups H_i has G_σ as its universal completion. If $n \geq 5$ and q equals 2 or 3, then the same holds for the amalgam of any four subgroups H_i.

Proof. Let $s = 2$ if $q \geq 4$ and $s = 3$ if $q = 2$ or 3. Let $\mathcal{B}_{(s)}$ be the subamalgam of $\mathcal{B}$ (see the proof of Proposition 3.8.8) consisting of all rank s parabolics. As in the proof of Theorem 3.8.1, we can show that $U(\mathcal{B}_{(s)}) \cong H = G_\sigma$. (Like before, this also implies 2-simple connectedness, respectively 3-simple connectedness of Δ, as claimed after Proposition 3.8.8 in the introduction.) Since the union of any three (four, if $q = 2$ or 3) H_i contains $\mathcal{B}_{(s)}$ and since $H_i \cap \mathcal{B}_{(s)}$ generates H_i for all i, we are done. □

Notice that if $n \geq 5$ and $q = 2$ or 3 then G_σ can still be recovered from some triples of subgroups H_i. Namely, among others, every amalgam $H_1 \cup H_i \cup H_{n-1}$, $1 < i < n-1$, has G_σ as its universal completion. Indeed, let $H_J = \bigcap_{i \in J} H_i$. By Theorem 3.8.10, the amalgam of rank three parabolics (i.e., the amalgam of all subgroups H_J with $|I \setminus J| = 3$) has G_σ as its universal completion. The only rank 3 parabolic that cannot be found inside the amalgam $H_1 \cup H_i \cup H_{n-1}$ is $H_{I \setminus \{1,i,n-1\}}$. Since $n \geq 5$, $i \neq 2$ or $i \neq n-2$. In the first case $H_{I \setminus \{1,i,n-1\}}$ is isomorphic to $H_{I \setminus \{1\}} \times H_{I \setminus \{i,n-1\}}$. In the second case it is isomorphic to $H_{I \setminus \{1,i\}} \times H_{I \setminus \{n-1\}}$. Let us assume we are in the first case. By connectivity (see Proposition 3.8.6), the rank two parabolic $H_{I \setminus \{i,n-1\}}$ is generated by the two minimal parabolics $H_{I \setminus \{i\}}$ and $H_{I \setminus \{n-1\}}$. It remains to notice that both $H_{I \setminus \{1\}}$ and $H_{I \setminus \{i\}}$ are contained in H_{n-1}, while both $H_{I \setminus \{1\}}$ and $H_{I \setminus \{n-1\}}$ are contained in H_i. So $H_{I \setminus \{1,i,n-1\}}$ does not contain any new relations.

Chapter 4

Hyperbolic Root Geometries

In this chapter we describe an approach to the area of long root group geometries of Chevalley groups by use of fundamental SL_2's. The usual geometries on long root subgroups that have been studied consist of the long root subgroups as points and the spans of two strongly commuting root subgroups as lines. Lacking strongly commuting pairs in the symplectic and unitary groups, in these groups one usually takes the spans of two polar long root subgroups as lines. The latter geometries are precisely the symplectic and unitary polar spaces. The long root group geometries have been studied by a number of people for quite some time. Some references are [Coo76], [Coo79], [Coh82], [Coo83], [Coh83], [CC83], [CC89], [Shu89], [KS]. A good survey is Cohen's Chapter 12 of [Bue95].

Our approach is slightly different. The lines we are using are the fundamental SL_2's of the Chevalley groups. This idea is not new, as Jon Hall and Hans Cuypers already studied those geometries in [Hal88], [Cuy94], and [Cuya] for the symplectic and unitary groups. However, they based their investigations on whether or not the geometries under consideration contain certain planar geometrical configurations. We instead will exploit the commutation relation of the long root groups and the fundamental SL_2's. To this end we introduce a concept of geometries, the *perp spaces*, consisting of a partial linear space together with a relation $\perp$ subject to certain conditions. This relation $\perp$ will serve as the commutation relation. See Section 4.2 for more details. Our characterizations then will be based on the shape of the centralizer of the fundamental SL_2's. The author learned recently that this approach has actually been suggested by Bill Kantor to Arjeh Cohen in the 1980's.

In case of the symplectic and unitary groups we do not attempt to improve the results of Hall and Cuypers. They are optimal already with beautiful proofs. We rather try to look at their geometries in another way. Contrary to the work of Hall and Cuypers our approach via centralizers of fundamental SL_2's of unitary or symplectic groups has another class of examples to take care of. Indeed, in a Chevalley group of type F_4 the centralizer of a fundamental SL_2 is a group of type C_3, cf. Proposition C.5.1, and so a group of type F_4 gives rise to a geometry on long root subgroups whose fundamental SL_2's admit centralizers that give rise to geometries on long root subgroups of a group of type C_3. Besides groups of type C_n we focus on groups of type A_n. Here, fundamental SL_2's correspond to non-intersecting line-hyperline pairs of the corresponding projective spaces, and we can use the results of Chapter 2 to our benefit.

4.1 Geometries on long root subgroups

Linear groups

A typical root subgroup of the group $PSL_{n+1}(\mathbb{F})$ can be described as the following one-parameter subgroup. Consider the projective space $\mathbb{P}_n(\mathbb{F})$ and choose an incident point-hyperplane pair (p, H). The set of all axial collineations of $\mathbb{P}_n(\mathbb{F})$ with center p and axis H form a group $T_{p,H}$ that is isomorphic to $(\mathbb{F}, +)$, a root subgroup. Therefore, we can parametrize the set of all root subgroups of $PSL_{n+1}(\mathbb{F})$ by the incident point-hyperplane pairs. Notice that two root subgroups commute if and only if the center of one is contained in the axis of the other and vice versa.

Two root subgroups $T_{p,H}$ and $T_{q,I}$ form a strongly commuting, polar, special, and hyperbolic pair if

- $p = q$ or $H = I$,
- $p \in I$, $q \in H$ and $p \neq q$, $H \neq I$,
- $p \in I$, $q \notin H$ or $p \notin I$, $q \in H$, or
- $p \notin I$, $q \notin H$, respectively.

A **line** (arising from strongly commuting pairs) (p, L) or, dually, (l, H) is a set of root subgroups that all have the same point p of $\mathbb{P}_n(\mathbb{F})$ with hyperplanes running through all hyperplanes containing some fixed hyperline $L \ni p$ or, dually, a set of root subgroups that all have the same hyperplane H of $\mathbb{P}_n(\mathbb{F})$ with points running through all points on some fixed line $l \subset H$. A **hyperbolic line** (arising from hyperbolic pairs) (l, L) is a set of root subgroups whose points are contained in the projective line l of $\mathbb{P}_n(\mathbb{F})$ and whose hyperplanes contain the hyperline L with $l \cap L = \emptyset$. Note that the notion of a hyperbolic line already exists in this context, cf. Section C.4. The only difference between the two notions is that a hyperbolic line as defined here is the set of all root subgroups contained in a hyperbolic line as defined in Section C.4. This boils down to the same ambiguity in synthetic geometry whether a line is something abstract or just the set of all points incident with it.

Generally, also for other Chevalley groups, let us call the geometry on long root groups and lines the **long root group geometry** and the geometry on long root groups and hyperbolic lines the **hyperbolic long root group geometry**.

There is a one-to-one correspondence between the hyperbolic lines of $PSL_{n+1}(\mathbb{F})$ on one hand and the vertices of the graph $\mathbf{L}_n(\mathbb{F})$ of Chapter 2 on the other hand.

Therefore, the claim of Proposition C.5.1 on groups of type A_n follows immediately from Proposition 2.1.2. Let us investigate the adjacency relation of $\mathbf{L}_n(\mathbb{F})$ in our new terms:

Proposition 4.1.1
Let $\mathbb{F}$ be a field, and let $n \geq 3$. Suppose that l, m are distinct fundamental SL_2's of the group $PSL_{n+1}(\mathbb{F})$, and let (x, X), (y, Y) be the corresponding non-intersecting line-hyperline pairs of $\mathbb{P}_n(\mathbb{F})$. Then $[l, m] = 1$ if and only if $x \subseteq Y$ and $y \subseteq X$.

Proof. The hyperbolic lines l and m can be considered as fundamental SL_2's. Those groups commute if and only if any root subgroup $a \leq l$ (with center $p \in x$ and axis $H \supseteq X$) commutes with every root subgroup $b \leq m$ (center $q \in y$, axis $I \supseteq Y$). The root subgroups a and b commute if and only if $p \in I$ and $q \in H$. Variation of $a \leq l$ and $b \leq m$ yields $p \in Y$ for all $p \in x$ and $q \in X$ for all $q \in y$. Consequently, $x \subseteq Y$ and $y \subseteq X$ is indeed equivalent to $[l, m] = 1$. □

Symplectic and unitary groups

In case of the groups $PSp_{2n}(q)$ and $PSU_n(q^2)$ acting on nondegenerate polar spaces of rank at least two, the long root subgroups correspond to the so-called *isotropic transvection subgroups*, see, e.g., Example 1.4 of Chapter 2 of [Tim01]. Therefore there is a one-to-one correspondence between the long root subgroups on one hand and the singular points of the corresponding polar geometry on the other hand. Note that the relations *strongly commuting* and *special* are trival. So, in Timmesfeld's terminology this set of long root subgroups is a set of abstract transvection groups, see Section C.4, and nontrivial lines do not exist. However, the hyperbolic lines of the long root geometry correspond to hyperbolic lines of the polar space and the *polar* relation gives rise to the lines of the polar space. Studying geometries on the long root subgroups of these groups therefore is the same as studying symplectic and unitary polar spaces. The polar of a hyperbolic line gives rise to a polar space of rank one smaller, which proves the corresponding claim of Proposition C.5.1; see also Lemma 4.4.5.

Orthogonal groups

The root subgroups of the orthogonal groups correspond to *Siegel transvections* and can be described as follows (see, e.g., Example 1.5 in Chapter 2 of [Tim01]). Let (V, q) be a nondegenerate orthogonal space of Witt index at least three. The long root subgroups of $PSO(V)$ correspond precisely to the singular lines of the polar space. Indeed, the root subgroups are the groups

$$\chi(l) = \left\{\tau : \mathbb{P}(V) \to \mathbb{P}(V) \in PSO(V) \mid [\tau, V] \subset l, \left[\tau, l^{\perp}\right] = 0\right\}$$

for totally singular projective lines l of V. Here $[\tau, v]$ stands for $\tau(v) - v$; also, $\chi(l)$ acts trivially on l. Two root subgroups commute if and only if the corresponding singular projective lines l, m satisfy $l \subset m^\perp$ or $l \cap m \neq \emptyset$. They form a strongly commuting pair when the respective lines intersect and span a singular plane. This defines the lines of the long root group geometry as the planar line pencils of singular lines of the polar space, and we obtain a so-called polar Grassmann space, i.e., the line shadow space of a polar geometry. Two root subgroups form a hyperbolic pair if the corresponding singular lines span (as a subspace of the polar space) a grid of the polar space. The hyperbolic line spanned by a hyperbolic pair of long root subgroups consists of the class of singular lines of that grid, which contains the two spanning singular lines.

The statement of Proposition C.5.1 can be obtained as follows. Let l, m be two totally singular lines that span a grid. Notice that the two classes of singular lines in a grid commute with each other, since any two lines from different classes intersect. So, if we fix one class of singular lines of the grid, the other class of that grid corresponds to the A_1 in statements (ii) and (iv) of the proposition. On the other hand, the hyperbolic lines in the space $l^\perp \cap m^\perp$ correspond to the B_{n-2} or D_{n-2} in that proposition.

Exceptional groups

To describe long root group geometries of type F_4, E_6, E_7, and E_8, simply take the corresponding building geometry. There is a unique node in the extended (Dynkin) diagram which is adjacent to the node corresponding to the root of highest weight. The long root group geometry then is isomorphic to the shadow space on that particular node of the diagram of the building. Actually, the long root group geometry of any other type except the linear groups can be obtained in that way as well.

Consider the exceptional group $E_6(\mathbb{F})$. Let Γ be the graph on the fundamental SL_2's (hyperbolic lines) of $E_6(\mathbb{F})$ with commuting being adjacency. Then Γ is locally $\mathbf{L}_5(\mathbb{F})$, by Proposition C.5.1. This is precisely the example mentioned after Theorem 2.4.19.

4.2 Perp spaces

Definition 4.2.1 A **perp space** is a partial linear space $(\mathcal{P}, \mathcal{L})$ endowed with a symmetric relation $\perp \subseteq \mathcal{P} \times \mathcal{P}$ such that for every point x, whenever $p \neq q$ are points on a line l, the fact $x \perp p$ and $x \perp q$ implies $x \perp y$ for all $y \in l$.

Notice that we neither demand nor forbid reflexivity of $\perp$. There are examples of perp spaces with $x \perp x$ for some points x, but not for all, see, e.g., Example 4.2.3 (ii). As usual, denote by $x^\perp$ the set of all points $y \in \mathcal{P}$ with $x \perp y$. For a set $X \subseteq \mathcal{P}$ of points, we have $X^\perp = \bigcap_{x \in X} x^\perp$ with the understanding that $\emptyset^\perp = \mathcal{P}$. If $X, Y \subseteq \mathcal{P}$ are sets of points, then $X \perp Y$ means $x \perp y$ for all $x \in X$, $y \in Y$. This allows the definitions of the **point perp graph** $(\mathcal{P}, \perp)$ and the **line perp graph** $(\mathcal{L}, \perp)$ (if there are points or lines x with $x \perp x$, then the edge $\{x\}$ will be disregarded). Especially the concept of the line perp graph will prove to be very useful, as it is the main tool for the characterizations of the hyperbolic root group geometries given in Theorem 4.3.6 and its corollaries.

Lemma 4.2.2
Let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space.

(i) *For any set $X \subseteq \mathcal{P}$, the set $X^\perp$ is a subspace of $(\mathcal{P}, \mathcal{L})$.*

(ii) *Let k, l, m be lines. If $k \cap l \neq \emptyset \neq k \cap m$ and $k \cap l \neq k \cap m$, then $k^\perp \supseteq \{l, m\}^\perp$.*

Proof.

(i) Let $l \in \mathcal{L}$ contain points $p \neq q$. If x is a point, then $x \perp p$ and $x \perp q$ implies $x \perp l$, whence $x^\perp$ is a subspace of $(\mathcal{P}, \mathcal{L})$. If X is a set of points, then $X^\perp = \bigcap_{x \in X} x^\perp$ is an intersection of subspaces of $(\mathcal{P}, \mathcal{L})$, whence $X^\perp$ is a subspace itself.

(ii) Set $p := k \cap l$ and $q := k \cap m$. Then, for any point x satisfying $x \perp l$ and $x \perp m$, we have $x \perp p$ and $x \perp q$, whence $x \perp k$. $\square$

The preceding proposition tells us that one could also define a perp space as follows. Let $(\mathcal{P}, \mathcal{L})$ be a partial linear space and let $(S_p)_{p \in \mathcal{P}}$ be a collection of subspaces of $(\mathcal{P}, \mathcal{L})$ such that $p \in S_q$ if and only if $q \in S_p$. The relation $\perp$ is obtained by setting $p \perp q$ if $p \in S_q$. A list of examples of perp spaces $(\mathcal{P}, \mathcal{L}, \perp)$ follows. This list is highly non-exhaustive, and its purpose is to indicate that the concept of perp spaces is fairly general.

Examples 4.2.3 (i) Let $(\mathcal{P}, \mathcal{L})$ be a Γ-space. Set $p \perp q$ if p and q are collinear.

(ii) Let $(\mathcal{P}, \mathcal{L})$ be a Δ-space. Set $p \perp q$ if p and q are not collinear.

(iii) Let $(\mathcal{P}, \mathcal{L})$ be a projective space admitting a polarity π. Set $p \perp q$ if $p \in \pi(q)$.

(iv) Let $(\mathcal{P}, \mathcal{L})$ be the (hyperbolic) long root group geometry of a Chevalley group. Set $p \perp q$ if p, q form a strongly commuting or a polar pair, i.e., set $p \perp q$ if they commute.

(v) Let $(\mathcal{P}, \mathcal{L})$ be the (hyperbolic) long root group geometry of a Chevalley group. Set $p \perp q$ if p, q form a strongly commuting pair.

Inspired by Proposition C.5.2, we make the following definitions.

Definition 4.2.4 Let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space such that for any line $l \in \mathcal{L}$ each point of $l^\perp$ is contained in a line of $l^\perp$. A pair (p, q) of distinct points is called

- **strongly commuting** if and only if $p \perp q$ and there does not exist a line on p in relation $\perp$ to q;
- **polar** if and only if $p \perp q$ and there exists a line on p in relation $\perp$ to q;

- **special** if and only if $p \not\perp q$ and there does not exist a line containing p and q;
- **hyperbolic** if and only if $p \not\perp q$ and there exists a line containing p and q.

The condition that $l^\perp$ be a rank two geometry is needed to ensure symmetry of the notions of strongly commuting and polar in $(\mathcal{P}, \mathcal{L})$:

Proposition 4.2.5
Let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space such that, for any line $l \in \mathcal{L}$, the subspace $l^\perp$ is a geometry of rank two. Then (p, q) is a strongly commuting, polar, special, or hyperbolic pair if and only if (q, p) is a strongly commuting, polar, special, respectively hyperbolic pair.

Proof. Obviously, the relations 'special' and 'hyperbolic' are symmetric. Let (p, q) be a polar pair. Then there exists a line $l \ni p$ with $l \perp q$. But by assumption $l^\perp$ is a rank two geometry, whence there exists a line $m \in l^\perp$ on q. Clearly, $m \perp p$ and (q, p) are a polar pair. But now the relation 'strongly commuting' also has to be symmetric. □

Notice that, even if for any line l the space $l^\perp$ is a rank two geometry, the space $p^\perp$ does not have to be a rank two geometry for any point p. Indeed, suppose there exists a point q in strongly commuting relation to p (most of the hyperbolic root group geometries admit such points). Then there exists no line on q that is in $\perp$ relation to p, and q is isolated in $p^\perp$.

4.3 Hyperbolic lines in type A_n geometries

In this section we will continue to discuss geometries on graphs Γ that are locally $\mathbf{L}_n(\mathbb{F})$. However, now we will study geometries on Γ that coincide with the interior hyperbolic root group geometry on the perps $\mathbf{x}^\perp$ (here the symbol $\perp$ is again used to denote adjacency in a graph) for vertices $\mathbf{x}$ of Γ rather than the interior projective space as in Section 2.4. Throughout the whole section let Γ be a connected, locally $\mathbf{L}_n(\mathbb{F})$ graph for a division ring $\mathbb{F}$ and an $n \geq 5$. Moreover, we will freely use the terminology introduced in Chapter 2, especially Section 2.3.

Definition 4.3.1 Let $\Gamma = (\mathcal{V}, \perp)$ be a connected, locally $\mathbf{L}_n(\mathbb{F})$ graph. Γ is **geometrizable** if there exists a family $\mathcal{S}$ of subsets of $\mathcal{V}$ such that

- for any $S \in \mathcal{S}$ and any vertex $\mathbf{x} \in \mathcal{V}$ the intersection $S \cap \mathbf{x}^\perp$ is either empty or an interior root point of $\mathbf{x}^\perp$, and
- for any interior root point $p_\mathbf{x}$ of $\mathbf{x}^\perp$, $\mathbf{x} \in \mathcal{V}$, there exists a unique set $S \in \mathcal{S}$ that contains $p_\mathbf{x}$.

The point-line geometry $(\mathcal{S}, \mathcal{V})$ with symmetrized containment as incidence is called a **geometrization of** Γ. An element of $\mathcal{S}$ is called a **global root point**.

Notice that the notion of geometrizability is very similar to the 'geometrizability' (the existence of global points and global lines) in Section 2.4. But, in fact, geometrizability in this section is possibly weaker. Indeed, if a graph admits global points and global hyperplanes in the sense of Section 2.4 (such that the local restrictions are interior points and interior hyperplanes, respectively), then by intersecting the vertex sets of an incident global point-hyperplane pair, one obtains global root points in the sense of Definition 4.3.1. The converse need not be true. The graph on the hyperbolic lines of the exceptional group $E_6(\mathbb{F})$ with commuting as adjacency may be a candidate for a graph that is geometrizable in the sense of Definition 4.3.1, but not in the sense of Section 2.4. Notice that in order to construct global points in the sense of Section 2.4 an argument using the simple connectnedness of the universal cover of that graph will not suffice, because triangle analysis as in Lemma 2.4.3 fails.

Lemma 4.3.2
Let Γ be geometrizable and let $\mathbf{x}$ and $\mathbf{y}$ be two vertices of Γ. If $\mathbf{p}$, $\mathbf{q}$ are two vertices adjacent to both $\mathbf{x}$ and $\mathbf{y}$ that belong to a common interior root point of $\mathbf{x}^\perp$, then they also belong to a common interior root point of $\mathbf{y}^\perp$.

Proof. Let $(\mathcal{S}, \mathcal{V})$ be a geometry on Γ. Then there is an $S \in \mathcal{S}$ containing $\mathbf{p}$ and $\mathbf{q}$. But since $\mathbf{p}, \mathbf{q} \in S \cap \mathbf{y}^\perp$, they also belong to an interior root point of $\mathbf{y}^\perp$. □

Lemma 4.3.3
Let $n \geq 5$ and let Γ be a connected, locally $\mathbf{L}_n(\mathbb{F})$ graph. Suppose Γ is geometrizable. Then there is at most one geometrization on Γ with the property that any two vertices contained in the same global point are at distance two in Γ.

Proof. Suppose such a geometry on Γ exists. Fix a vertex $\mathbf{x}$ and consider the interior hyperbolic root group geometry on $\mathbf{x}^\perp \cong \mathbf{L}_n(\mathbb{F})$. Let p be an interior root point of $\mathbf{x}^\perp$, let S be the unique set of $\mathcal{S}$ that contains p, and let $\mathbf{a}$ be a vertex of p. Now let $\mathbf{y}$ be an arbitrary vertex of Γ. The proposition is proved, if it can be determined whether $\mathbf{y}$ does or does not belong to the set $S \in \mathcal{S}$ that contains p. We may assume that there exists a vertex $\mathbf{z}$ adjacent to $\mathbf{y}$ and $\mathbf{a}$, since otherwise $\mathbf{y}$ cannot be contained in S by the hypothesis of this lemma. By Proposition 2.1.3 there exists a chain of vertices in $\mathbf{a}^\perp \cong \mathbf{L}_n(\mathbb{F})$ connecting $\mathbf{x}$ and $\mathbf{z}$. Denote the vertex closest to $\mathbf{x}$ by $\mathbf{w}$. By local analysis of $\mathbf{x}^\perp$ using Lemma 2.3.6 we find another vertex $\mathbf{c}$ in $\mathbf{x}^\perp \cap \mathbf{w}^\perp$ that belongs to the interior root point p besides $\mathbf{a}$. By Lemma 4.3.2 the vertices $\mathbf{a}$ and $\mathbf{c}$ are contained in a common interior root point q of $\mathbf{w}^\perp$. Obviously the interior root point q of $\mathbf{w}^\perp$ has also to be contained in S. Using induction on the length of the chain from $\mathbf{x}$ to $\mathbf{z}$, we see that it can be determined whether $\mathbf{y}$ is contained in the set S or not. □

Examples 4.3.4 (i) Let $\Gamma \cong \mathbf{L}_n(\mathbb{F})$ for $n \geq 7$. Then the interior hyperbolic root group geometry on Γ is the unique global geometry satisfying the property of Lemma 4.3.3. Indeed, the diameter of Γ is two by Proposition 2.1.3.

(ii) The hyperbolic root group geometry of $E_6(\mathbb{F})$ fails to induce a geometry as given in Lemma 4.3.3 on the graph; it does induce a geometry but there exist intersecting hyperbolic lines that are not at distance two in the graph. For example, let $\mathbb{F} \neq \mathbb{F}_2$ be a finite field. Then the isomorphism classes of subgroups of $E_6(\mathbb{F})$ spanned by two intersecting hyperbolic lines are given in Table V of [Coo79]. Consider the last line of Table V. The isomorphism class of the subgroup is $SU_3(\mathbb{F})$, which by [McL67] cannot be a subgroup of $SL_6(\mathbb{F})$ generated by root subgroups. But it would have to be, if there existed a hyperbolic line that commutes with the two hyperbolic lines that span the $SU_3(\mathbb{F})$. We are indebted to Bruce Cooperstein for this argument.

Finally, we are ready to state and prove our characterization theorems.

Proposition 4.3.5
Let $n \geq 5$, let $\mathbb{F}$ be a division ring, and let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space. If

(i) *for any line $k \in \mathcal{L}$ the space $k^\perp$ is isomorphic to the hyperbolic root group geometry of $PSL_{n+1}(\mathbb{F})$ with $l \perp m$ if and only if the corresponding fundamental SL_2's of $PSL_{n+1}(\mathbb{F})$ commute for lines l, m inside $k^\perp$, and*

(ii) *any two intersecting lines of $(\mathcal{P}, \mathcal{L})$ are at distance two in $(\mathcal{L}, \perp)$,*

then $(\mathcal{L}, \perp)$ is geometrizable, a geometrization of $(\mathcal{L}, \perp)$ satisfying the conditions of Lemma 4.3.3 exists, and $(\mathcal{P}, \mathcal{L})$ is isomorphic to that geometrization.

Proof. Note that $(\mathcal{L}, \perp)$ is locally $\mathbf{L}_n(\mathbb{F})$. Consider the family of all full line pencils of $(\mathcal{P}, \mathcal{L})$. This family gives rise to a geometry on $(\mathcal{L}, \perp)$ in the sense of Definition 4.3.1. Indeed, any intersection of a full line pencil with $k^\perp$ for an arbitrary line k is either empty or a full line pencil of the subspace $k^\perp$. But by Proposition 2.3.5 a full line pencil of $k^\perp$ corresponds to an interior root point. Conversely, any interior root point of a perp of a line corresponds to a full line pencil of this perp, which is contained in a unique full line pencil of the whole geometry. Hence $(\mathcal{L}, \perp)$ is geometrizable. Moreover, since any two intersecting lines are required to be at distance two in $(\mathcal{L}, \perp)$, the global geometry on $(\mathcal{L}, \perp)$ we just have constructed satisfies the hypothesis of Lemma 4.3.3. The last claim follows from the fact that $(\mathcal{P}, \mathcal{L})$ is isomorphic to the geometry on the full line pencils as points and the line set $\mathcal{L}$. □

Theorem 4.3.6
Let $n \geq 5$, let $\mathbb{F}$ be a division ring, and let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space satisfying Hypothesis (i) *of Proposition 4.3.5. If the graph $(\mathcal{L}, \perp)$ is isomorphic to $\mathbf{L}_{n+2}(\mathbb{F})$, then $(\mathcal{P}, \mathcal{L})$ is isomorphic to the hyperbolic root group geometry of $PSL_{n+3}(\mathbb{F})$.*

Proof. This follows from Example 4.3.4(i), Proposition 2.1.3, and Proposition 4.3.5. □

Corollary 4.3.7
Let $n \geq 7$, let $\mathbb{F}$ be a division ring, and let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space satisfying Hypothesis (i) *of Proposition 4.3.5. If the graph $(\mathcal{L}, \perp)$ is connected, then $(\mathcal{P}, \mathcal{L})$ is isomorphic to the hyperbolic root group geometry of $PSL_{n+3}(\mathbb{F})$.*

Proof. This follows from Theorem 2.5.1 and Theorem 4.3.6. ■

The requirement that $(\mathcal{L}, \perp)$ be connected in Corollary 4.3.7 is not a very restrictive one. Instead, one could require $(\mathcal{P}, \mathcal{L})$ to be connected. Then the union of two copies of the hyperbolic root group geometry where a unique point of one copy is identified with a unique point of the other copy would also be an example. Generally, one can take cocliques (of the same size) C and C' of points in the respective copies and identify the points of C and C' by an arbitrary bijection. Of course, one can also do this for the union of an arbitrary number of copies of hyperbolic root group geometries. However, these are the only examples that would have to be added to the conclusion of Corollary 4.3.7 if one replaces connectedness of $(\mathcal{L}, \perp)$ by connectedness of $(\mathcal{P}, \mathcal{L})$. Dropping connectedness of $(\mathcal{P}, \mathcal{L})$, too, only adds another set of obvious examples.

Corollary 4.3.8
Let n be an infinite cardinal number, let $\mathbb{F}$ be a division ring, let V be a vector space over $\mathbb{F}$ of dimension n, and let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space in which for any line $k \in \mathcal{L}$ the space $k^\perp$ is isomorphic to the hyperbolic root group geometry of $PSL(V)$ with $l \perp m$ if and only if $[l, m] = 1$ for lines l, m inside $k^\perp$. If the graph $(\mathcal{L}, \perp)$ is connected, then $(\mathcal{P}, \mathcal{L})$ is isomorphic to the hyperbolic root group geometry of $PSL(V)$.

Proof. This follows from Theorem 2.5.2 and Theorem 4.3.6. □

The group $PSL(V)$ for an infinite-dimensional vector space V is defined to be the span of all transvections of $\mathbb{P}(V)$, i.e., all axial collineations whose centers are contained in the axes.

4.4 Hyperbolic lines in type C_n geometries

There already exist beautiful characterizations of the geometry on long root subgroups and hyperbolic lines of symplectic groups by Jon Hall and Hans Cuypers. They take advantage of the fact that, as indicated in Section 4.1, there only exist the relations 'polar' and 'hyperbolic' between distinct long root subgroups. Hence, if one knows one relation, one knows the other, and there is no need of using Proposition C.5.2. Let us review the existing results.

Theorem 4.4.1 (Hall [Hal88], Main Theorem)
Let $(\mathcal{P}, \mathcal{L})$ be a finite, connected partial linear space in which each pair of intersecting lines lies in a subspace isomorphic to a dual affine plane. Assume that $(\mathcal{P}, \mathcal{L})$ contains at least two such planes. Then either

(i) *for some prime power q and some integer n at least 3, the space $(\mathcal{P}, \mathcal{L})$ is isomorphic to the partial linear space of hyperbolic lines of a symplectic polar space embedded in the projective space $\mathbb{P}_n(q)$; or*

(ii) *all lines of $\mathcal{L}$ contain exactly three points.*

Since any line is contained in a dual affine plane, we can conclude that each line contains at least three points. The partial linear spaces satisfying (ii) of Theorem 4.4.1 are called **cotriangular spaces**. Theorem 1 of [Hal89] provides a complete classification of the spaces occuring in (ii), which reads as follows.

Theorem 4.4.2 (Hall [Hal89], Theorem 1, Theorem 4)
Let $(\mathcal{P}, \mathcal{L})$ be a connected partial linear space all of whose lines contain exactly three points and in which every pair of intersecting lines lies in a subspace isomorphic to a dual affine plane (of order two). Then $(\mathcal{P}, \mathcal{L})$ is isomorphic to one of the following partial linear spaces:

(i) *the geometry on the non-radical points and the hyperbolic lines of some symplectic space over $\mathbb{F}_2$;*

(ii) *the subgeometry of a space as in (i) on its non-singular points with respect to some quadratic form q such that the symplectic form f is obtained as $f(x, y) = q(x) + q(y) + q(x, y)$; or*

(iii) *the geometry defined as follows. Let Ω be a set of cardinality at least two and let Ω' be a set disjoint from Ω. The points of the geometry are the finite subsets of $\Omega \cup \Omega'$ that intersect Ω in a set of cardinality two. The lines of the geometry are those triples x_1, x_2, x_3 of points with empty symmetric difference, i.e., $\left(\bigcup_{1\leq i\leq 3} x_i\right) \setminus \bigcup_{1\leq i,j\leq 3} (x_i \cap x_j) = \emptyset$.*

If additionally $\{y\} \cup \{p \in \mathcal{P} \mid p \in l, x \in l \in \mathcal{L}\} = \{x\} \cup \{p \in \mathcal{P} \mid p \in l, y \in l \in \mathcal{L}\}$ implies $x = y$ for points x, y, then $(\mathcal{P}, \mathcal{L})$ is isomorphic to a geometry of Case (i) or (ii) with respect to a nondegenerate form or of Case (iii) with $\Omega' = \emptyset$.

Hans Cuypers has proved a version of Theorem 4.4.1 that includes infinite point orders.

Theorem 4.4.3 (Cuypers [Cuy94], Theorem 1.1)
Let $(\mathcal{P}, \mathcal{L})$ be a connected partial linear space in which any pair of intersecting lines is contained in a subspace isomorphic to a dual affine plane. Assume that $(\mathcal{P}, \mathcal{L})$ contains at least two such planes and a line with more than three points. Then $(\mathcal{P}, \mathcal{L})$ is isomorphic to the geometry on the non-radical points and the hyperbolic lines of a symplectic polar space embedded in some projective space of dimension at least 3.

In the remainder of this section, we will indicate how one can prove a similar statement by concentrating on hyperbolic lines and their perps. We should point out, however, that our assumptions on the dimension have to be stronger than in the results of Hall and Cuypers, as the centralizer of a fundamental SL_2 in $F_4(\mathbb{F})$ is isomorphic to $Sp_6(\mathbb{F})$, cf. Proposition C.5.1 or 7.18 of the third chapter of [Tim01]. Our approach has to take care of this example, while the hyperbolic long root group geometry of $F_4(\mathbb{F})$ does not occur as a counterexample to the theorems of Cuypers and Hall. In the course of our proof we invoke the results by Hall and Cuypers and, thus, our result depends on theirs.

The hyperbolic line graph

Definition 4.4.4 Let $n \geq 1$ and let $\mathbb{F}$ be a field. Let $\mathbb{W}_{2n}(\mathbb{F})$ denote the polar space of a nondegenerate symplectic polarity of $\mathbb{P}_{2n-1}(\mathbb{F})$. The **hyperbolic line graph** $\mathbf{S}(\mathbb{W}_{2n}(\mathbb{F})) = \mathbf{S}_{2n}(\mathbb{F})$ is the graph on the hyperbolic lines of $\mathbb{W}_{2n}(\mathbb{F})$ where hyperbolic line l and m are adjacent (in symbols $l \perp m$) if and only if all singular points of l are collinear (in $\mathbb{W}_{2n}(\mathbb{F})$) to all singular points of m.

Equivalently, one can define the graph $\mathbf{S}_{2n}(\mathbb{F})$ as the graph on the fundamental SL_2's of the group $PSp_{2n}(\mathbb{F})$ where two vertices are adjacent if and only if they commute.

Lemma 4.4.5
Let $n \geq 2$. The graph $\mathbf{S}_{2n}(\mathbb{F})$ is locally $\mathbf{S}_{2n-2}(\mathbb{F})$.

Proof. This is immediate from the fact that the set of singular points of $\mathbb{W}_{2n}(\mathbb{F})$ that are collinear to a given hyperbolic line l spans a subspace isomorphic to $\mathbb{W}_{2n-2}(\mathbb{F})$, whose hyperbolic lines are precisely those hyperbolic lines of $\mathbb{W}_{2n}(\mathbb{F})$ that are in relation $\perp$ to the hyperbolic line l. □

Lemma 4.4.6
Let $n \geq 3$, and let l, m be distinct hyperbolic lines of $\mathbb{W}_{2n}(\mathbb{F})$ with $\{l,m\}^\perp \neq \emptyset$. Then any hyperbolic line contained in $\{l,m\}^{\perp\perp}$ is also contained in $\langle l,m\rangle_\mathbb{P}$ and vice versa.

Proof. Let $p \in \langle l,m\rangle_\mathbb{P}$ be a singular point of $\mathbb{W}_{2n}(\mathbb{F})$. Then a vector that spans p can be expressed as a linear combination of vectors spanning singular points on l and m. But then points collinear to these are also collinear to p. Hence a hyperbolic line contained in $\langle l,m\rangle_\mathbb{P}$ is also contained in $\{l,m\}^{\perp\perp}$. Conversely, let q be a singular point not contained in $\langle l,m\rangle_\mathbb{P}$. Note that in a symplectic space any hyperplane is singular, i.e., there exists a singular point having that hyperplane as polar. The space $\langle l,m\rangle_\mathbb{P}$ has at most (projective) dimension three. Since $n \geq 3$, hyperplanes have at least dimension 4. Now consider the hyperplanes Π_i, $i \in I$ for some index set, of $\mathbb{P}_{2n-1}(\mathbb{F})$ containing $\langle l,m\rangle_\mathbb{P}$. Denote the corresponding points by p_i. If all of the p_i were contained in $\langle l,m\rangle_\mathbb{P}$, then $\{l,m\}^\perp = \emptyset$ (for, a hyperbolic line of $\{l,m\}^\perp \cap \langle l,m\rangle_\mathbb{P}$ would have to be contained in the radical of $\langle l,m\rangle_\mathbb{P}$, which does not contain hyperbolic lines), whence there exists a p_i outside $\langle l,m\rangle_\mathbb{P}$. Fix such a

p_i and choose a hyperline $\Lambda_i \subset \Pi_i$ with $\langle l, m\rangle_{\mathbb{P}} \subset \Lambda_i$ and $p_i, q \notin \Lambda_i$. Let Π_j be any other hyperplane of $\mathbb{P}_{2n-1}(\mathbb{F})$ containing Λ_i. Since $p_i \notin \Pi_j$ we have $p_j \notin \Pi_i$ and p_i, p_j are noncollinear. Moreover, at least one of p_i and p_j is not collinear with q (because $q \notin \Pi_i \cap \Pi_j = \Lambda_i$) and we have found a hyperbolic line p_ip_j contained in $\{l, m\}^\perp$ that ensures that no hyperbolic line containing q is contained in $\{l, m\}^{\perp\perp}$. This finishes the proof, because q has been chosen arbitrarily outside $\langle l, m\rangle_{\mathbb{P}}$. □

Notation 4.4.7 Let X be a subspace of $\mathbb{W}_{2n}(\mathbb{F})$. Denote the set of all hyperbolic lines of $\mathbb{W}_{2n}(\mathbb{F})$ contained in X by $\mathbf{S}(X)$.

Lemma 4.4.8
Let $n \geq 3$. Let k, l, m be three hyperbolic lines of $\mathbb{W}_{2n}(\mathbb{F})$ with $\{k, l, m\}^\perp \neq \emptyset$ that intersect in a common point. Then $\mathbf{S}(\langle k, l, m\rangle) = \{k, l, m\}^{\perp\perp}$.

Proof. There exist hyperbolic lines a and b with $\langle a, b\rangle_{\mathbb{P}} = \langle k, l, m\rangle_{\mathbb{P}}$. Then by the preceding lemma we have $\mathbf{S}(\langle a, b\rangle_{\mathbb{P}}) = \{a, b\}^{\perp\perp}$. Finally, $\{a, b\}^{\perp\perp} = \{k, l, m\}^{\perp\perp}$ by $\langle a, b\rangle_{\mathbb{P}} = \langle k, l, m\rangle_{\mathbb{P}}$ and linear algebra. □

Similar to Sections 1.2 and 2.2 we want to reconstruct the underlying polar space from the graph $\mathbf{S}_{2n}(\mathbb{F})$. The main task we have to accomplish is to reconstruct the points. The polar space then is easily obtained from the points and the hyperbolic lines, by Proposition A.6.2.

Lemma 4.4.9
Let $n \geq 3$. Distinct hyperbolic lines l and m of $\mathbb{W}_{2n}(\mathbb{F})$ intersect if and only if the perp $\{l, m\}^\perp$ in $\mathbf{S}_{2n}(\mathbb{F})$ is non-empty and the double perp $\{l, m\}^{\perp\perp}$ in $\mathbf{S}_{2n}(\mathbb{F})$ does not contain adjacent vertices (with respect to $\perp$).

Proof. Let l and m be two intersecting hyperbolic lines. First we will show that $\{l, m\}^\perp \neq \emptyset$. The space $\langle l, m\rangle_{\mathbb{P}}$ has (projective) dimension two. Hence its polar $\langle l, m\rangle_{\mathbb{P}}^\pi$ has dimension two or bigger, since $n \geq 3$. If $n \geq 4$, then $\langle l, m\rangle_{\mathbb{P}}^\pi$ is not totally isotropic, so we find two noncollinear points in $\langle l, m\rangle_{\mathbb{P}}^\pi$, whence we find also a hyperbolic line adjacent to both l and m. Now suppose $n = 3$. If $\langle l, m\rangle_{\mathbb{P}}^\pi$ does not contain a hyperbolic line, then it is totally singular and, because of the dimensions, equal to $\langle l, m\rangle_{\mathbb{P}}$. But $\langle l, m\rangle_{\mathbb{P}}$ is not totally singular, as it contains hyperbolic lines, a contradiction. The space $\langle l, m\rangle_{\mathbb{P}}$ is a projective plane, and the hyperbolic lines contained in which are precisely those of $\{l, m\}^{\perp\perp}$, by Lemma 4.4.6. If this plane contains two adjacent hyperbolic lines $a\mathbb{F} + b\mathbb{F}$ and $c\mathbb{F} + d\mathbb{F}$ then $(a, b) = (a, a\alpha_1 + c\alpha_2 + d\alpha_3) = (a, a)\alpha_1 + (a, c)\alpha_2 + (a, d)\alpha_3 = 0$ (where $(\cdot, \cdot)$ denotes the bilinear form), a contradiction to the fact that $a\mathbb{F} + b\mathbb{F}$ is a hyperbolic line. Conversely, suppose l and m are non-intersecting hyperbolic lines. Then $\langle l, m\rangle_{\mathbb{P}}$ is a projective 3-space and $\langle l, m\rangle_{\mathbb{P}} \cap \mathbb{W}_{2n}(\mathbb{F})$ is a nondegenerate symplectic space (the direct sum of two disjoint hyperbolic lines) or has a projective line as its radical (and hence the space is the direct sum of a hyperbolic line and a non-intersecting singular line). In both cases $\langle l, m\rangle_{\mathbb{P}}$ contains adjacent hyperbolic lines. We may assume $\{l, m\}^{\perp} \neq \emptyset$, and the claim follows from Lemma 4.4.6. $\square$

We now want to recover the points of the polar space as pencils of hyperbolic lines similar to Section 2.2. More precisely, we will copy the method of Lemma 2.1.11: three mutually intersecting hyperbolic lines k, l, m do intersect in one point if there exists a fourth hyperbolic line j that intersects with the first three and spans a projective 3-space with two of them. In terms of double perps this means that k, l and m are intersecting in one point if there exists a hyperbolic line j with $\{k, l\}^{\perp\perp} = \mathbf{S}(\langle k, l\rangle_{\mathbb{P}}) \subsetneq \mathbf{S}(\langle j, k, l\rangle_{\mathbb{P}}) = \{j, k, l\}^{\perp\perp}$. The former equality is due to Lemma 4.4.6, the latter is due to Lemma 4.4.8. The only problem is to ensure that $\{k, l\}^{\perp} \neq \emptyset \neq \{j, k, l\}^{\perp}$. The first inequality has been shown in Lemma 4.4.9, the second will be handled by the following lemma. More precisely, we show that we can choose j in such a way that $\{j, k, l\}^{\perp} \neq \emptyset$ holds.

Lemma 4.4.10
Let $n \geq 3$. For distinct intersecting hyperbolic lines l and m of $\mathbb{W}_{2n}(\mathbb{F})$ there exists a hyperbolic line j that intersects l and m such that $\langle j, l, m\rangle_{\mathbb{P}}$ has projective dimension 3 and $\{j, l, m\}^{\perp}$ in $\mathbf{S}_{2n}(\mathbb{F})$ is non-empty.

Proof. Consider the plane $\langle l, m\rangle_{\mathbb{P}}$. It contains a point x as radical, which lies on neither l nor m. The space l^{π} in $\mathbb{W}_{2n}(\mathbb{F})$ is isomorphic to $\mathbb{W}_{2n-2}(\mathbb{F})$ and contains a point y that is not collinear with x, because $\mathbb{W}_{2n}(\mathbb{F})$ is nondegenerate. Therefore $\langle l, xy\rangle_{\mathbb{P}}$ is a nondegenerate symplectic 3-space, a symplectic generalized quadrangle, and $\{l, xy\}^{\perp} \neq \emptyset$ as $n \geq 3$. Thus we are done, if we can find a point p of $\langle l, xy\rangle_{\mathbb{P}}$ with $\langle l, m, p\rangle_{\mathbb{P}} = \langle l, xy\rangle_{\mathbb{P}}$ that is not collinear with $q := l \cap m$. But this point p exists since $\langle l, m\rangle_{\mathbb{P}} \subset \langle l, xy\rangle_{\mathbb{P}}$ and $\langle l, xy\rangle_{\mathbb{P}}$ is nondegenerate, so we can choose j to be the hyperbolic line pq. $\square$

Definition 4.4.11 Let $n \geq 3$. Following Lemma 4.4.9, distinct vertices l, m of a graph Γ isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$ are said to **intersect** if $\{l, m\}^{\perp} \neq \emptyset$ and the double perp $\{l, m\}^{\perp\perp}$ in Γ does not contain adjacent vertices. In view of the paragraph before Lemma 4.4.10 three mutually intersecting vertices k, l, m of $\Gamma \cong \mathbf{S}_{2n}(\mathbb{F})$ are said to **intersect in one point** if there exists a vertex j of Γ that intersects k, l, and m and that has the property that $\{j, k, l\}^{\perp} \neq \emptyset$ and $\{k, l\}^{\perp\perp} = \mathbf{S}(\langle k, l\rangle_{\mathbb{P}}) \subsetneq \mathbf{S}(\langle j, k, l\rangle_{\mathbb{P}}) = \{j, k, l\}^{\perp\perp}$.

An **interior point** of a graph Γ isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$ is a maximal set of mutually intersecting vertices of Γ any three elements of which intersect in one point. Denote the set of all interior points of Γ by $\mathcal{P}$. Furthermore, an **interior hyperbolic line** of $\Gamma \cong \mathbf{S}_{2n}(\mathbb{F})$ is a vertex of Γ. The set of interior hyperbolic lines of Γ is denoted by $\mathcal{H}$.

By definition we have the following.

Proposition 4.4.12
Let $n \geq 3$, and let Γ be isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$. The geometry $(\mathcal{P}, \mathcal{H}, \supset)$ on the interior points and interior hyperbolic lines of Γ is isomorphic to the geometry on points and hyperbolic lines of the symplectic space $\mathbb{W}_{2n}(\mathbb{F})$. □

Definition 4.4.13 Let $\sim \subset \mathcal{P} \times \mathcal{P}$ be a relation that denotes the fact that two interior points p, q do *not* lie on a common interior hyperbolic line, and denote by $p^{\sim}$ the set of all interior points in $\sim$ relation to p. Then for any two points $p \sim q$, the **interior singular line** pq is defined as the set $(\{p, q\}^{\sim})^{\sim}$. Denote the set of all interior singular lines by $\mathcal{L}$.

Corollary 4.4.14 (of Proposition 4.4.12)
Let $n \geq 3$, and let Γ be isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$. The geometry $(\mathcal{P}, \mathcal{L}, \subset)$ on the interior points and the interior singular lines of Γ is isomorphic to the symplectic polar space $\mathbb{W}_{2n}(\mathbb{F})$.

Proof. This follows from Proposition 4.4.12 and Proposition A.6.2. □

The space $(\mathcal{P}, \mathcal{L}, \subset)$ of Corollary 4.4.14 is the **interior polar space** $(\mathcal{P}, \mathcal{L})$ on $\Gamma \cong \mathbf{S}_{2n}(\mathbb{F})$ and space $(\mathcal{P}, \mathcal{H}, \supset)$ is the **interior hyperbolic space** $(\mathcal{P}, \mathcal{H})$ on $\Gamma \cong \mathbf{S}_{2n}(\mathbb{F})$. We will mainly work with the interior hyperbolic space.

Local recognition

We now understand the graph $\mathbf{S}_{2n}(\mathbb{F})$ well enough to prove a local recognition result. Let $n \geq 6$, let $\mathbb{F}$ be a field, and let Γ be a connected graph that is locally $\mathbf{S}_{2n}(\mathbb{F})$. It will turn out that Γ is isomorphic to $\mathbf{S}_{2n+2}(\mathbb{F})$. To obtain this result we will use methods similar to those in Section 2.4. From the interior hyperbolic spaces on the perps we will construct a global geometry on Γ, which will be shown to be isomorphic to the hyperbolic long root group geometry of a symplectic group (using the characterizations offered in the beginning of this section), whose hyperbolic line graph is isomorphic to Γ.

Lemma 4.4.15
Let $\mathbf{w} \perp \mathbf{x} \perp \mathbf{y} \perp \mathbf{z}$ be a chain of vertices in Γ. Then $\mathbf{w}^{\perp} \cap \mathbf{x}^{\perp} \cap \mathbf{y}^{\perp} \cap \mathbf{z}^{\perp} \neq \emptyset$. In particular, the diameter of Γ is two, and Γ, viewed as a two-dimensional simplicial complex whose two-simplices are its triangles, is simply connected.

Proof. The perp $\mathbf{y}^{\perp}$ is isomorphic to $\mathbf{S}_{2n}(\mathbb{F})$, which can be endowed with the interior polar space isomorphic to $\mathbb{W}_{2n}(\mathbb{F})$ living in a projective space $\mathbb{P}_{2n-1}(\mathbb{F})$. By Lemma 4.4.5, the intersections $\mathbf{x}^{\perp} \cap \mathbf{y}^{\perp}$ and $\mathbf{y}^{\perp} \cap \mathbf{z}^{\perp}$ are isomorphic to $\mathbf{S}_{2n-2}(\mathbb{F})$ and can be endowed with interior polar spaces isomorphic to $\mathbb{W}_{2n-2}(\mathbb{F})$ that are subspaces of the interior polar space on $\mathbf{y}^{\perp}$. These subspaces live in hyperlines of the projective space $\mathbb{P}_{2n-1}(\mathbb{F})$. Therefore the intersection $\mathbf{x}^{\perp} \cap \mathbf{y}^{\perp} \cap \mathbf{z}^{\perp}$ is a subspace of the interior polar space on $\mathbf{y}^{\perp}$ living in a subspace of $\mathbb{P}_{2n-1}(\mathbb{F})$ of projective codimension at most three. The polar space on $\mathbf{x}^{\perp} \cap \mathbf{y}^{\perp} \cap \mathbf{z}^{\perp}$ can also be considered as a subspace of the interior polar space on $\mathbf{x}^\perp$. The intersection $\mathbf{w}^\perp \cap \mathbf{x}^\perp$ admits an interior polar space isomorphic to $\mathbb{W}_{2n-2}(\mathbb{F})$, as above. Now $\mathbf{w}^\perp \cap \mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$ can be consdered as the intersection of the interior polar space on $\mathbf{w}^\perp \cap \mathbf{x}^\perp$ with the polar space on $\mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$. The dimensions of the projective spaces are at least $2n-3$ and $2n-5$, whence the dimension of the intersection is at least $2n-7 \geq n-1$, since $n \geq 6$. But the largest totally isotropic subspace of the interior projective space on $\mathbf{w}^\perp \cap \mathbf{x}^\perp$ has (projective) dimension $n-2$ and we can find a hyperbolic line in $\mathbf{w} \cap \mathbf{x}^\perp \cap \mathbf{y}^\perp \cap \mathbf{z}^\perp$, proving the first claim. The other claims are immediate consequences. □

We do not have to worry about how to define global hyperbolic lines as we did in Section 2.4 for locally line-hyperline graphs. Here, our **global hyperbolic lines** are simply the vertices of the graph Γ. A bit more complicated is the definition of **global points**. However this can be done in exactly the same way as in Section 2.4 by intersections of hyperbolic lines. We omit the details.

Notation 4.4.16 Denote the set of global hyperbolic lines of Γ by $\mathcal{H}_\Gamma$ and the set of global points of Γ by $\mathcal{P}_\Gamma$.

Proposition 4.4.17
$(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ is a connected partial linear space.

Proof. Let p and q be two global points. Fix a vertex in each point, $\mathbf{p}$ and $\mathbf{q}$, say. By Lemma 4.4.15, there exists a vertex $\mathbf{x}$ adjacent with both $\mathbf{p}$ and $\mathbf{q}$. Hence there exist local counterparts $p_\mathbf{x}$ and $q_\mathbf{x}$, where the index $\mathbf{x}$ indicates that we are considering interior points on $\mathbf{x}^\perp$. Connectedness of $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ now follows from connectedness of the interior hyperbolic space on $\mathbf{x}^\perp$. Moreover, two global points p, q cannot intersect in more than one vertex, whence $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ is a partial linear space. For, if two global points would intersect in two vertices $\mathbf{x}$, $\mathbf{y}$, then there exists a vertex $\mathbf{z}$ adjacent to both $\mathbf{x}$, $\mathbf{y}$ by Lemma 4.4.15. But then $p \cap \mathbf{z}^\perp$ and $q \cap \mathbf{z}^\perp$ are two local points that intersect in two vertices, a contradiction. □

Proposition 4.4.18
The space $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ is isomorphic to the geometry of hyperbolic lines of a symplectic polar space $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ embedded in some projective space of dimension at least 3.

Proof. Let $\mathbf{l}$ and $\mathbf{m}$ be two intersecting hyperbolic lines of $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$. By Lemma 4.4.15, there exists a vertex $\mathbf{k}$ of Γ adjacent to both $\mathbf{l}$ and $\mathbf{m}$. Local analysis of $\mathbf{k}^\perp$ (or rather the interior hyperbolic space on it) shows that the intersecting lines $\mathbf{l}$ and $\mathbf{m}$ are contained in a dual affine plane. Certainly, $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ contains two such planes. If $\mathbb{F} \neq \mathbb{F}_2$, the space $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ contains a line with more than three points and the claim follows from Theorem 4.4.3 with Proposition 4.4.17. If $\mathbb{F} = \mathbb{F}_2$, we can invoke Theorem 4.4.2. It remains to show that the geometry $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ does not belong to Cases (ii) or (iii). Case (ii) is easily excluded, as locally all symplectic points occur, not only a subset of the symplectic points. Case (iii) is a bit more difficult. However, by the second statement of Theorem 4.4.2, we obtain $\Omega' = \emptyset$.

Indeed, for any pair x, y of points of $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$, we find hyperbolic lines $\mathbf{l}$ incident with x and $\mathbf{m}$ incident with y. By Lemma 4.4.15 there exists a hyperbolic line $\mathbf{k}$ that is adjacent to both $\mathbf{l}$ and $\mathbf{m}$ in Γ. Therefore, we can consider x, y in some local space isomorphic to $\mathbb{W}_{2n}(\mathbb{F})$. But if $x \neq y$, then we find a point that lies on a common hyperbolic line with x, but not with y. Hence, in $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$, the equality $\{y\} \cup \{p \in \mathcal{P} \mid p \ni \mathbf{l}, x \ni \mathbf{l} \in \mathcal{L}\} = \{x\} \cup \{p \in \mathcal{P} \mid p \ni \mathbf{l}, y \ni \mathbf{l} \in \mathcal{L}\}$ implies $x = y$, and $\Omega' = \emptyset$. It follows from the size of n that $(\mathcal{P}_\Gamma, \mathcal{H}_\Gamma)$ cannot belong to Case (iii) either. The proposition is proved. □

Proposition 4.4.19
The hyperbolic line graph of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ (as defined in Proposition 4.4.18) is isomorphic to Γ.

Proof. By definition the elements of $\mathcal{H}_\Gamma$ are precisely the vertices of Γ. The preceding proposition tells us that the elements of $\mathcal{H}_\Gamma$ are also precisely the hyperbolic lines of the symplectic space $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$, and we have a natural bijection between the hyperbolic lines of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ and the vertices of Γ, which preserves adjacency. □

Proposition 4.4.20
The space $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ is isomorphic to the symplectic polar space $\mathbb{W}_{2n+2}(\mathbb{F})$.

Proof. By Proposition 4.4.19, the hyperbolic line graph of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ is isomorphic to Γ. Since Γ is locally $\mathbf{S}_{2n}(\mathbb{F})$, this means for any hyperbolic line $\mathbf{l}$ that the subspace of $(\mathcal{P}_\Gamma, \mathcal{L}_\Gamma)$ consisting of all points collinear with all points of l is isomorphic to $\mathbb{W}_{2n}(\mathbb{F})$. But the only symplectic polar space with that property is $\mathbb{W}_{2n+2}(\mathbb{F})$. The claim follows. □

We have proved the following.

Theorem 4.4.21
Let $n \geq 6$, let $\mathbb{F}$ be a field, and let Γ be a connected graph that is locally $\mathbf{S}_{2n}(\mathbb{F})$. Then Γ is isomorphic to $\mathbf{S}_{2n+2}(\mathbb{F})$. □

Notice that the dimensions of the vector spaces belonging to Theorem 4.4.21 and Theorem 2.4.19 coincide. Actually, also the following holds, which is an analogue of Theorem 2.5.1.

Theorem 4.4.22
Let $n \geq 4$, let $\mathbb{F}$ be a field, and let Γ be a connected graph that is locally $\mathbf{S}_{2n}(\mathbb{F})$. Then Γ is isomorphic to $\mathbf{S}_{2n+2}(\mathbb{F})$.

Sketch of proof. Global hyperbolic lines are simply the vertices of Γ. As sketched for Theorem 2.5.1 we can define global 'hyperbolic' planes and global 'hyperbolic' 3-spaces. Global points are defined as follows. Two global hyperbolic lines intersect if and only if they span a global hyperbolic plane. Three global hyperbolic lines intersect in a single point if they mutually intersect and together span a global hyperbolic 3-space. From the structure of the global hyperbolic planes it is immediately clear that the geometry on global points and global hyperbolic lines satisfies the hypothesis of Theorem 4.4.1. It remains to prove that the graph we are considering is the hyperbolic line graph of the corresponding symplectic space. ■

The bound on n in Theorem 4.4.22 is the lowest possible, as a fundamental SL_2 of the group $F_4(\mathbb{F})$ centralizes a group isomorphic to $Sp_6(\mathbb{F})$ (see Proposition C.5.1 or 7.18 in Chapter 3 of [Tim01]), thus yielding a counterexample for $n = 3$. Finally, a discussion of geometrizability as in Section 4.3 yields a result on perp spaces.

Theorem 4.4.23
Let $n \geq 4$, let $\mathbb{F}$ be a field, and let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space in which for any line $k \in \mathcal{L}$ the space $k^\perp$ is isomorphic to the hyperbolic long root group geometry of $PSp_{2n}(\mathbb{F})$ with $l \perp m$ if and only if $[l, m] = 1$ for lines l, m inside $k^\perp$. If the graph $(\mathcal{L}, \perp)$ is connected, then $(\mathcal{P}, \mathcal{L}, \perp)$ is isomorphic to the hyperbolic long root group geometry of $PSp_{2n+2}(\mathbb{F})$. ■

Corollary 4.4.24
Let $n \geq 4$, let $(\mathcal{P}, \mathcal{L})$ be a partial linear space, and let $\perp$ denote non-collinearity in that space. Assume there exists a nondegenerate symplectic space of rank n such that for all $k \in \mathcal{L}$ the set $k^\perp$ of all points and lines of $(\mathcal{P}, \mathcal{L})$ not collinear to k is a subspace of $(\mathcal{P}, \mathcal{L})$ that is isomorphic to the hyperbolic geometry of that symplectic polar space. If the graph $(\mathcal{L}, \perp)$ is connected, then the space $(\mathcal{P}, \mathcal{L})$ is isomorphic to the geometry on the points and hyperbolic lines of a nondegenerate symplectic polar space of rank $n + 1$. ■

Finally, we have a group-theoretic consequence of our findings.

Theorem 4.4.25
Let $n \geq 4$, and let $\mathbb{F}$ be a field of characteristic distinct from 2. Let G be a group with subgroups A and B isomorphic to $SL_2(\mathbb{F})$, and denote the central involution of A by x and the central involution of B by y. Furthermore, assume the following holds:

- *$C_G(x) = A \times K$ with $K \cong Sp_{2n}(\mathbb{F})$;*
- *$C_G(y) = B \times J$ with $J \cong Sp_{2n}(\mathbb{F})$;*
- *A is a fundamental SL_2 of J;*
- *B is a fundamental SL_2 of K;*
- *there exists an involution in $J \cap K$ that is the central involution of a fundamental SL_2 of both J and K.*

If $G = \langle J, K \rangle$, then $G/Z(G) \cong PSp_{2n+2}(\mathbb{F})$. ■

Corollary 4.4.26
Let $n \geq 4$, and let $\mathbb{F}$ be a field of characteristic distinct from 2. Let G be a group with a subgroup A isomorphic to $SL_2(\mathbb{F})$, let x be the central involution of A, and let g be an element in G such that

- $gxg^{-1} \neq x$ *and* $g^2xg^{-2} = x$;
- $C_G(x) = A \times K$ *with* $K \cong Sp_{2n}(\mathbb{F})$;
- gAg^{-1} *is a fundamental* SL_2 *of* K; *and*
- g *centralizes an involution of* K *that is the central involution of a fundamental* SL_2 *of both* K *and* gKg^{-1}.

If $G = \langle K, g\rangle$, *then* $G/Z(G) \cong PSp_{2n+2}(\mathbb{F})$. ∎

4.5 Hyperbolic lines in type ^2A_n geometries

It is possible to obtain results on unitary spaces similar to the ones on symplectic spaces in the previous section. Let us just briefly mention our findings. The starting point is again a theorem of Hans Cuypers'. He has characterized the geometry on the hyperbolic lines of a finite unitary polar geometry in the flavor of Theorem 4.4.3:

Theorem 4.5.1 (Cuypers [Cuya], Theorem 1.3)
Let $(\mathcal{P}, \mathcal{L})$ be a non-linear, planar (i.e., every pair of intersecting lines is contained in a unique plane) and connected partial linear space of finite order $q \geq 3$. Suppose the following hold in $(\mathcal{P}, \mathcal{L})$:

(i) *all planes are finite and either linear or isomorphic to a dual affine plane;*

(ii) *in a linear plane no four lines intersect in six points;*

(iii) *if $x^\perp \subseteq y^\perp$, then $x = y$; and*

(iv) *if π is a linear plane and x a point, then $x^\perp \cap \pi \neq \emptyset$.*

Then q is a prime power and $(\mathcal{P}, \mathcal{L})$ is isomorphic to the geometry of hyperbolic lines in a nondegenerate symplectic or unitary polar space over the field $\mathbb{F}_q$, respectively $\mathbb{F}_{q^2}$.

In the preceding theorem the symbol $\perp$ stands for non-collinearity in the partial linear space $(\mathcal{P}, \mathcal{L})$. Therefore, Cuypers' symbol $\perp$ of non-collinearity conincides with our symbol $\perp$ for commuting, as distinct root subgroups of the unitary group either commute or form a hyperbolic pair.

The main difference to the case of hyperbolic lines in symplectic polar spaces is, that one has to be a bit more careful when recovering the points. In a symplectic space all points are singular, whereas in a unitary space we have to develop a method to distinguish the singular and non-singular points. One can handle that problem with the following lemma.

Lemma 4.5.2
Let P be a nondegenerate unitary polar space of rank ≥ 3, and let l and m be two hyperbolic lines of P. If l and m intersect (possibly in a non-singular point), then there is at most one point of $\langle l, m\rangle_{\mathbb{P}}$ that does not lie on a hyperbolic line contained in $\langle l, m\rangle_{\mathbb{P}}$. Moreover, if such a point exists, then it is singular.

Proof. The projective plane spanned by two intersecting hyperbolic lines either contains a unital or a degenerate plane of P. In case of a unital, each projective point of the plane lies on a hyperbolic line contained in the plane. In case of a degenerate plane, any projective point of the projective plane but the radical of the polar plane lies on a hyperbolic line contained in the plane. The lemma is now proved. □

Notice that this lemma precisely resembles Condition (i) of Theorem 4.5.1. The linear spaces mentioned in the theorem come from the unitals, whereas the dual affine planes come from the degenerate planes with a point as radical. By counting the number of projective points that are contained in a hyperbolic line of that plane we can decide whether the plane contains a unital or a single point as a radical. If there is a radical, then all other points of that plane are non-singular. In this way we can identify all non-singular points. The remaining ones are singular.

Then the following holds:

Theorem 4.5.3
Let $n \geq 8$, let $\mathbb{F}$ be a finite field distinct from $\mathbb{F}_2$, let $\mathbb{K}$ be a quadratic extension of $\mathbb{F}$, and let Γ be a connected graph that is locally the graph on the hyperbolic lines of a nondegenerate unitary polar space $U_n(\mathbb{K})$ with the commutation relation as adjacency. Then Γ is isomorphic to the graph on the hyperbolic lines of a nondegenerate unitary polar space $U_{n+2}(\mathbb{K})$ with commuting as adjacency. ■

As in the case of symplectic groups, there exists a counterexample for $n = 6$ coming from a group of type F_4, leaving the case $n = 7$ as an open problem. Indeed, let $\mathbb{K}$ be a quadratic extension of some field $\mathbb{F}$. Then the centralizer of a fundamental SL_2 in $^2E_6(\mathbb{K})$ is isomorphic to $SU_6(\mathbb{K})$, see Proposition C.5.1 or 7.18 in Chapter 3 of [Tim01].

The consequence for perp spaces reads as follows.

Theorem 4.5.4
Let $n \geq 8$, let $\mathbb{F}$ be a finite field distinct from $\mathbb{F}_2$, let $\mathbb{K}$ be a quadratic extension of $\mathbb{F}$, and let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space in which for any line $k \in \mathcal{L}$ the space $k^\perp$ is isomorphic to the hyperbolic long root group geometry of $PSU_n(\mathbb{K})$ with $l \perp m$ if and only if $[l, m] = 1$ for lines l, m inside $k^\perp$. If the graph $(\mathcal{L}, \perp)$ is connected, then the space $(\mathcal{P}, \mathcal{L}, \perp)$ is isomorphic to the hyperbolic long root group geometry of $PSU_{n+2}(\mathbb{K})$. ■

Finally, there is a group-theoretic implication as well.

Theorem 4.5.5
Let $n \geq 8$, let $\mathbb{F}$ be a finite field of characteristic distinct from 2, and let $\mathbb{K}$ be a quadratic extension of $\mathbb{F}$. Let G be a group with subgroups A and B isomorphic to $SL_2(\mathbb{F})$, and denote the central involution of A by x and the central involution of B by y. Furthermore, assume the following holds:

- $C_G(x) = A' \times K$ *with* $K \cong GU_n(\mathbb{K})$, $A \leq A'$;
- $C_G(y) = B' \times J$ *with* $J \cong GU_n(\mathbb{K})$, $B \leq B'$;
- A *is a fundamental* SL_2 *of* J;
- B *is a fundamental* SL_2 *of* K; *and*
- *there exists an involution in* $J \cap K$ *that is the central involution of a fundamental* SL_2 *of both* J *and* K.

If $G = \langle J, K\rangle$, *then* $PSU_{n+2}(\mathbb{K}) \leq G/Z(G) \leq PGU_{n+2}(\mathbb{K})$. ■

Corollary 4.5.6
Let $n \geq 8$, let $\mathbb{F}$ be a finite field of characteristic distinct from 2, and let $\mathbb{K}$ be a quadratic extension of $\mathbb{F}$. Let G be a group with a subgroup A isomorphic to $SL_2(\mathbb{F})$, let x be the central involution of A, and let g be an element in G such that

- $gxg^{-1} \neq x$ *and* $g^2xg^{-2} = x$;
- $C_G(x) = A' \times K$ *with* $K \cong GU_n(\mathbb{K})$, $A \leq A'$;
- gAg^{-1} *is a fundamental* SL_2 *of* K; *and*
- g *centralizes an involution of* K *that is the central involution of a fundamental* SL_2 *of both* K *and* gKg^{-1}.

If $G = \langle K, g\rangle$, *then* $PSU_{n+2}(\mathbb{K}) \leq G/Z(G) \leq PGU_{n+2}(\mathbb{K})$. ■

4.6 Hyperbolic lines in type F_4 geometries

In this section we study hyperbolic long root geometries of type F_4. To be precise, we provide a direct consequence of a theorem by Arjeh Cohen. A **metasymplectic space**, as defined in 10.13 of [Tit74], is the point-shadow space of a building geometry of type F_4 (with the four types **point**, **line**, **plane**, and **symplecton**) such that the point shadows of two distinct symplecta intersect in either the empty set, a point, a line, or a plane. The metasymplectic space is called **thick** if the symplecta are thick (as polar spaces) and every plane is contained in at least three symplecta. Then there exists the following characterization.

Theorem 4.6.1 (Cohen [Coh82], Theorem 2.3)
Let $(\mathcal{P}, \mathcal{L})$ be a connected partial linear space. Then $\mathcal{P}$ and $\mathcal{L}$ can be identified with the point set and line set of a connected metasymplectic space or a polar space if and only if $(\mathcal{P}, \mathcal{L})$ satisfies the following axioms:

(F1) *For each $p \in \mathcal{P}$ and each $l \in \mathcal{L}$ either none, precisely one or all points of l are collinear to p (i.e., $(\mathcal{P}, \mathcal{L})$ is a* **Γ-space***).*

(F2) *For each pair $p, q \in \mathcal{P}$ with p collinear to q, the collinearity graph on the common neighbors of p and q is not a clique.*

(F3) *For each pair $p, q \in \mathcal{P}$ with p not collinear to q, such that there are at least two distinct points collinear to both p and q, the set of all points collinear to both p and q together with the induced lines is a polar space of rank at least two.*

(F4) *There are no minimal 5-circuits, i.e., given points p_1, p_2, p_3, p_4, p_5 with p_i collinear p_{i+1}, indices taken modulo five, there is at least one i for which x_i is collinear with a point on the line through x_{i+2} and x_{i+3}.*

(F5) *If $p, q, r \in \mathcal{P}$ with q collinear to r are such that there are at least two distinct points collinear to both p and q, then there exists a point collinear to both p and r.*

Using this theorem, we can characterize the hyperbolic long root group geometry of Chevalley groups of type F_4 as perp spaces satisfying a number of further axioms. We would like to express our gratitude to Gábor Ivanyos for a fruitful discussion on metasymplectic and parapolar spaces.

Proposition 4.6.2
Let $(\mathcal{P}, \mathcal{L}, \perp)$ be a perp space such that for any hyperbolic line $l \in \mathcal{L}$ the space $l^\perp$ is a rank two geometry and

(i) *for any strongly commuting pair p, q and distinct $x, y \in \{p, q\}^{\perp\perp}$, we have $\{x, y\}^{\perp\perp} = \{p, q\}^{\perp\perp}$;*

(ii) *to any special pair p, q, there exists a unique point x that strongly commutes with both p and q; moreover, $a \perp p, q$ implies $a \perp x$ for any point a;*

(iii) *to any polar pair x, y, there exists a strongly commuting pair a, b such that the pairs xa, xb, ya, yb strongly commute, and vice versa;*

(iv) *if x, y form a polar pair and a, b form a strongly commuting pair such that xa, xb, ya, yb strongly commute, then x, y are strongly commuting to all of $\{a, b\}^{\perp\perp}$;*

(v) *the following configuration does not exist: $x_i \in \mathcal{P}$, $1 \leq i \leq 5$, with x_i, x_{i+1} strongly commuting and x_i, x_{i+2} special, indices taken modulo 5.*

(vi) *the following configuration does not exist: $a, b, p \in \mathcal{P}$ with a, b strongly commuting and p polar to all points of $\{a, b\}^{\perp\perp}$;*

(vii) *the graph $(\mathcal{P}, \perp)$ is connected.*

Then $(\mathcal{P}, \mathcal{L}, \perp)$ *is the hyperbolic root group geometry of a metasymplectic space.*

Proof. Let $\mathcal{X} := \left\{ \{p,q\}^{\perp\perp} \mid p \text{ and } q \text{ are strongly commuting} \right\}$ be the set of *singular lines*. We will prove this proposition by showing that the geometry $(\mathcal{P}, \mathcal{X})$ satisfies the axioms of Theorem 4.6.1.

$(\mathcal{P}, \mathcal{X})$ is connected by Axiom (vii) and Axiom (iii). It satisfies (F1) by the following: Suppose p, q are strongly commuting and assume that $a \notin \{p,q\}^{\perp\perp}$ is strongly commuting with distinct $x, y \in \{p,q\}^{\perp\perp}$. By $a \perp x, y$ and Axiom (i) we have $a \perp \{p,q\}^{\perp\perp}$. Suppose a is polar to $z \in \{p,q\}^{\perp\perp}$. Then z has to be strongly commuting with a by Axiom (iv), a contradiction. Hence if a is strongly commuting with two points of $\{p,q\}^{\perp\perp}$, then it is strongly commuting with all points on that singular line and, thus, $(\mathcal{P}, \mathcal{X})$ is a Γ-space.

The validity of Axiom (F2) immediately follows from Axiom (iii).

For Axiom (F3), let x, y be two points that are not strongly commuting such that the set S of points strongly commuting with both x and y contains at least two points. By Axiom (ii), the points x and y cannot form a special pair. They cannot form a hyperbolic pair either, by the definition of strongly commuting. Hence (x, y) is a polar pair. We have to prove that S is a polar space of rank at least two. By Axiom (iii) there exists a strongly commuting pair a, b in S, whence S contains lines. Now let l be any line of S and p be any point of S off l. The point p cannot be special with any point on l since both x and y are strongly commuting with all points in S, contradicting Axiom (ii). They cannot form a hyperbolic line either, by the definition of strongly commuting. Hence $p \perp l$. But now Axiom (vi) implies the existence of a point on l that strongly commutes with p, proving the validity of Axiom (F3), since $(\mathcal{P}, \mathcal{X})$ is a Γ-space.

To prove Axiom (F4), suppose $x_1, x_2, x_3, x_4, x_5 \in \mathcal{P}$ where x_i, x_{i+1}, indices taken modulo 5, are strongly commuting and form a minimal circuit of length 5. Then x_1 cannot be polar to both x_3 and x_4, since otherwise we find a point on the line x_3x_4 strongly commuting with x_1 by Axiom (vi), contradicting the minimality of the circuit. Hence without loss of generality x_1 and x_4 form a special pair (strongly commuting is impossible since it would contradict the minimality of the circuit and hyperbolic is impossible by the definition of strongly commuting in view of the point x_5) and the point x_5 is the unique point strongly commuting with both x_1 and x_4. Now x_2 cannot be polar to x_4, as otherwise $x_2 \perp x_4$ and $x_2 \perp x_1$ yield $x_2 \perp x_5$, whence x_2, x_5 being strongly commuting or polar, both cases contradicting the minimality of the circuit (strongly commuting immediately, polar by the above). Continuing arguments like the above, we end up with the configuration that x_i, x_{i+1} are strongly commuting and x_i, x_{i+2} are special, indices taken modulo 5, a configuration that does not exist by Axiom (v). Hence a minimal circuit of length 5 does not exist.

Now let us consider Axiom (F5). Obviously, p and q can neither be hyperbolic nor special. If they are strongly commuting, the axiom is obviously satisfied. So suppose p and q are polar. Then any point r collinear to q forms either a strongly commuting, a polar, or a special pair with p. So $(\mathcal{P}, \mathcal{X})$ satisfies (F5).

Theorem 4.6.1 implies that $(\mathcal{P}, \mathcal{X})$ is a connected metasymplectic space or a polar space. However it can not be a polar space. Indeed, let x, y be a hyperbolic pair. Then there does not exist any point strongly commuting with both x and y, by the definition of strongly commuting. Hence there exists a line through x no point on which is strongly commuting with y, violating the Buekenhout-Shult Axiom. It is immediate that $(\mathcal{P}, \mathcal{L}, \perp)$ is the geometry on the points and hyperbolic lines of $(\mathcal{P}, \mathcal{X})$ with $x \perp y$ if and only if x and y commute. The theorem is now proved. $\square$

It remains to show that the hyperbolic long root groups geometries of metasymplectic spaces satisfy the hypothesis of Proposition 4.6.2. Certainly, those geometries are connected. Moreover, the centralizers of hyperbolic lines are listed in 7.18 in Chapter 3 of [Tim01], all of which give rise to rank two geometries on long root subgroups and hyperbolic lines. A typical line of the metasymplectic space on distinct strongly commuting long root subgroups A, B is defined as the group $\langle A, B\rangle$. Axiom (i) states that any two distinct root subgroups contained in a line span the same group and that the double centralizer of $\langle A, B\rangle$ in G (the group belonging to the F_4 building) equals $\langle A, B\rangle$, which is true by [Coo79]. Axiom (ii) follows from Theorem C.4.1; everything that commutes with two root subgroups A, B, also commutes with their span and the center of the latter. One implication of Axiom (iii) means that for a polar pair x, y, the space of all common neighbors of x and y contains a line. This follows from the fact that metasymplectic spaces have polar rank 3, whence the space of all common neighbors of x and y is a generalized quadrangle. The other implication holds, e.g., by Axiom (F2) of Theorem 4.6.1. Axiom (iv) follows from the fact that a metasymplectic space is a Γ-space, which is Axiom (F1) of Theorem 4.6.1. Axiom (v) and Axiom (vi) hold by the proof of Theorem 12.11 of [Tim91]. Axiom (vii) follows from the connectedness of a metasymplectic space and the fact that $\perp$ corresponds to commuting.

It would be interesting to obtain a characterization of hyperbolic long root geometries of type F_4 by means of centralizers of fundamental SL_2's.

Appendix A

Synthetic Geometry

In this appendix we collect the relevant definitions of notions and concepts that are used throughout the main body of this thesis. We refer to the literature for more information and a systematic approach.

A.1 Chamber systems and geometries

Definition A.1.1 A **chamber system over** I is a pair $\mathcal{C} = (C, \{\sim_i \mid i \in I\})$ consisting of a set C, whose members are called **chambers**, and a collection of equivalence relations $\sim_i$ on C indexed by $i \in I$. Two chambers c, d are called i**-adjacent** if $c \sim_i d$. The **rank** of $\mathcal{C}$ is $|I|$. The pair $(C, \sim_I)$ is called the **graph of** $\mathcal{C}$. A path in this graph is called a **gallery**. A path of $(C, \sim_J)$ is called a J**-gallery**. A gallery is called **closed**, if its starting chamber and end chamber are equal. It is called **simple**, if it does not contain repetitions, i.e., no two consecutive chambers are equal.

The chamber system $\mathcal{C}$ is called **connected** if its graph is connected. For $J \subset I$, a connected component of $(C, (\sim_i)_{i \in J})$ is called a J**-cell** of $\mathcal{C}$. For $i \in I$, the $(I \setminus \{i\})$-cells are called i**-panels**. For a chamber $c \in \mathcal{C}$ let cJ denote the J-cell that contains c. The chamber system $\mathcal{C}$ is called **residually connected** if, for every subset J of I and every family of j-panels Z_j, one for each $j \in J$, with the property that any two have a non-empty intersection, it follows that $\bigcap_{j \in J} Z_j$ is an $(I \setminus J)$-cell.

Definition A.1.2 A **pregeometry over** I is a triple $\mathcal{G} = (X, *, typ)$ where X is a set (its elements are called the **elements of** $\mathcal{G}$), $*$ is a symmetric and reflexive relation defined on X which is called the **incidence relation of** $\mathcal{G}$, and typ is a map from X to I (the set I is called the **type set of** $\mathcal{G}$) such that $typ(x) = typ(y)$ and $x * y$ implies $x = y$. The pregeometry $\mathcal{G}$ is called **connected** if the graph $(X, *)$ is connected.

If $A \subseteq X$, then A is of the **type** $typ(A)$, of **rank** $|typ(A)|$, and of **corank** $|I \setminus typ(A)|$. The cardinality $|I|$ of I is called the **rank of** $\mathcal{G}$. A **flag** of $\mathcal{G}$ is a set of mutually incident elements of $\mathcal{G}$. Flags of type I are called **chambers**. If F is a flag of $\mathcal{G}$, then the **residue** of F in $\mathcal{G}$ is the triple $\mathcal{G}_F = (X_F, *_F, typ_F)$ where $X_F = F^* \backslash F$, i.e., the set of elements of X that are incident with but distinct from all elements of F, and $*_F$, typ_F are the restrictions of $*$ and typ to $X_F \times X_F$ and X_F, respectively. The pregeometry $\mathcal{G}$ is called **residually connected** if $(X_F, *_F)$ is a connected graph for each flag F of $\mathcal{G}$ such that $|I \backslash typ(F)| \geq 2$ and non-empty for each flag F such that $|I \backslash typ(F)| = 1$.

A **geometry over** I is a pregeometry $\mathcal{G}$ over I in which every maximal flag is a chamber. A geometry $\mathcal{G}$ is **thick** if every flag of type distinct from I is contained in at least three distinct chambers of $\mathcal{G}$.

Proposition A.1.3 (e.g., Buekenhout, Cohen [BC], Lemma 1.6.4)
A residually connected pregeometry is a geometry.

Definition A.1.4 If $\mathcal{C}$ is a chamber system over I, the **pregeometry of** $\mathcal{C}$, notation $\mathcal{G}(\mathcal{C})$, is the pregeometry over I determined as follows. Its elements of type i are the i-panels. Two panels x, y are incident if and only if $x \cap y \neq \emptyset$ in $\mathcal{C}$, i.e., x and y have a chamber in common.

Definition A.1.5 If $\mathcal{G}$ is a geometry over I, then the **chamber system of** $\mathcal{G}$, notation $\mathcal{C}(\mathcal{G})$, is the chamber system over I determined as follows. Its elements are the chambers of $\mathcal{G}$ where two chambers x, y are i-adjacent if and only if $x = y$ or x and y differ in precisely the element of type i.

Proposition A.1.6 (e.g., Buekenhout, Cohen [BC], Proposition 3.6.5)
Let I be a finite index set.

(i) *If $\mathcal{G}$ is a residually connected geometry over I, then $\mathcal{C}(\mathcal{G})$ is a residually connected chamber system over I.*

(ii) *If $\mathcal{C}$ is a residually connected chamber system over I, then $\mathcal{G}(\mathcal{C})$ is a residually connected geometry over I.*

A.2 Tits buildings

The standard reference for Tits buildings is of course Tits' work [Tit74]. See also [Bue95], [BC], [Ron89], [Bro89].

Definition A.2.1 A chamber system of Coxeter type is called a **Tits building** if every simple closed gallery with minimal type is trivial, i.e., consists of a single chamber. A Tits building is **spherical** if its Coxeter diagram is spherical.

Proposition A.2.2 (e.g., Buekenhout, Cohen [BC], Corollary 13.4.5)
Tits buildings are residually connected.

With view to Propositions A.1.6 and A.2.2, we can consider a spherical Tits building also as a geometry, the so-called **building geometry**.

Definition A.2.3 (alternative definition) Let D be a Coxeter diagram. A **Tits building of type** D is a pair $\mathcal{B} = (C, \delta)$ where C is a set and $\delta : C \times C \to W(D)$ is a **distance function** satisfying the following axioms for $x, y \in \mathcal{C}$, $w = \delta(x, y)$, and the set S of generators of $W(D)$.

(i) $w = 1$ if and only if $x = y$;

(ii) if $z \in C$ is such that $\delta(y, z) = s \in S$, then $\delta(x, z)$ equals w or ws, and if, furthermore, $l(ws) = l(w) + 1$, then $\delta(x, z) = ws$;

(iii) if $s \in S$, there exists a $z \in C$ such that $\delta(y, z) = s$ and $\delta(x, z) = ws$.

In case of a spherical Tits building, for $x, y \in \mathcal{C}$ denote by x opp y the fact that $\delta(x, y) = w_0$, the longest word of the Weyl group $W(D)$.

Theorem A.2.4 (Tits [Tit74], Theorem 4.1.2)
Let B be a thick spherical Tits building of rank at least two with irreducible diagram. Then B is 2-simply connected.

Corollary A.2.5
Let B be a thick spherical Tits building of rank at least three with irreducible diagram. Then B is simply connected. □

A Tits building of rank two, a generalized polygon, cf. Section A.6, is not simply connected.

Corollary A.2.6 (Tits [Tit74], Theorem 13.32)
Let B be a thick spherical Tits building of rank at least two with irreducible diagram, and let G be the corresponding Chevalley group. Let $\mathcal{A}$ be the amalgam of all stabilizers in G of flags of B of corank at most two. Then G is the universal completion of $\mathcal{A}$.

Proof. This follows from Theorem A.2.4 and Tits' lemma B.2.5. □

Notice the similarity of Corollary A.2.6 and the different versions of the Curtis-Tits theorem.

A.3 Twin buildings

Peter Abramenko's lecture notes [Abr96] may serve as an introduction to the subject.

Definition A.3.1 Let D be a Coxeter diagram and let $\mathcal{B}_+ = (C_+, \delta_+)$, $\mathcal{B}_- = (C_-, \delta_-)$ be two Tits buildings of type D. A **codistance** (or a **twinning**) between $\mathcal{B}_+$ and $\mathcal{B}_-$ is a map $\delta^* : (C_+ \times C_-) \cup (C_- \times C_+) \to W(D)$ satisfying the following axioms, where $\epsilon \in \{+,-\}$, $x \in C_\epsilon$, $y \in C_{-\epsilon}$, $w = \delta^*(x,y)$, and S is the set of generators of $W(D)$.

(i) $\delta^*(y,x) = w^{-1}$;

(ii) if $z \in C_{-\epsilon}$ is such that $\delta_{-\epsilon}(y,z) = s \in S$ and $l(ws) = l(w) - 1$, then $\delta^*(x,z) = ws$;

(iii) if $s \in S$, there exists a $z \in C_{-\epsilon}$ such that $\delta_{-\epsilon}(y,z) = s$ and $\delta^*(x,z) = ws$.

The triple $(\mathcal{B}_+, \mathcal{B}_-, \delta^*)$ is a **twin building**. For $x \in \mathcal{C}_\pm$, $y \in \mathcal{C}_\mp$ denote by x opp y the fact that $\delta^*(x,y) = 1$.

Proposition A.3.2 (Tits [Tit92], Proposition 1)
Let $B = (\mathcal{C}, \delta)$ be a spherical Tits building of type D, and let w_0 be the longest word of its Weyl group $W = W(D)$. Let $\mathcal{C}_\epsilon$, with $\epsilon = +$ or $-$, be two copies of $\mathcal{C}$ and let the functions $\delta_\epsilon : \mathcal{C}_\epsilon \times \mathcal{C}_\epsilon \to W$, $\delta^* : (\mathcal{C}_+ \times \mathcal{C}_-) \cup (\mathcal{C}_- \times \mathcal{C}_+) \to W$ be defined by $\delta_+ = \delta$, $\delta_- = w_0 \delta w_0$, while $\delta^* = w_0\delta$ on $\mathcal{C}_+ \times \mathcal{C}_-$ and $\delta^* = \delta w_0$ on $\mathcal{C}_- \times \mathcal{C}_+$. Then $((\mathcal{C}_+, \delta_+), (\mathcal{C}_-, \delta_-), \delta^*)$ is a twinning and all twinnings of (spherical) type D are obtained in that way up to isomorphism.

Proposition A.3.3 (Abramenko, Van Maldeghem [AV00], Corollary 1.5)
Let $\mathcal{B} = (B_+, B_-, \delta^*)$ be a thick twin building without any non-spherical rank 2 residues. Then the distance and codistance functions are completely determined by the opposition relation. Hence a thick 2-spherical twin building (in particular any spherical building) is completely determined by its set of chambers and the opposition relation.

Proposition A.3.4 (Abramenko, Van Maldeghem [AV00], Corollary 5.5)
Let B be a thick spherical building. A permutation σ of the set of chambers of B satisfies the implication c opp $\sigma(d) \Rightarrow \sigma(c)$ opp d, for all chambers c and d of B, if and only if σ extends to an involutive automorphism of B.

A.4 Diagrams

See [Pas94] or [BC]. A lot of intuition can be gained by studying—besides diagram geometry—Lie algebras and Lie groups or Coxeter groups and reflections in Euclidian space.

Definition A.4.1 Let I be a set of types. A **diagram** D over I consists of a map $\mathcal{D}$ defined on $\binom{I}{2} = \{\{i,j\} \subseteq I \mid i \neq j\}$ assigning to every pair $\{i,j\}$ of distinct elements of I some class $\mathcal{D}(i,j) = \mathcal{D}(j,i)$ of rank two geometries over $\{i,j\}$. A diagram D is a **Coxeter diagram** if the map $\mathcal{D}$ takes values only in the class of generalized polygons. A diagram D over I is **reducible** if there exists a partition $I_1 \cup I_2 = I$ such that $\mathcal{D}(i,j)$ is the class of generalized 2-gons for all $i \in I_1$, $j \in I_2$. A geometry $\mathcal{G}$ over I **belongs to the diagram** D over I if for every pair of distinct types $i, j \in I$ and every flag F sucht that the residue $\mathcal{G}_F$ is of type $\{i, j\}$ one has $\mathcal{G}_F \in \mathcal{D}(i, j)$. In this case one also says that $\mathcal{G}$ is **of type** D. A chamber system $\mathcal{C}$ over I **belongs to the diagram** D over I if, for each subset J of I of size two, every J-cell of $\mathcal{C}$ is the chamber system of a residually connected geometry over J belonging to $\mathcal{D}(J)$. A geometry or a chamber system belonging to some diagram D are **of Coxeter type** if D is a Coxeter diagram.

Definition A.4.2 Let D be a Coxeter diagram over I. For distinct $i, j \in I$ denote by $d(i, j)$ the gonality of the class of generalized polygons $\mathcal{D}(i, j)$. The **Coxeter group** or **Weyl group** $W = W(D)$ of D is the group presented as $\langle x_i \mid i \in I, (x_i)^2 = 1, (x_i x_j)^{d(i,j)} = 1 \rangle$ with the understanding that there is no relation between x_i and x_j if $d(i, j) = \infty$. A Coxeter diagram D is **spherical** if the group $W(D)$ is finite.

A.5 Constructions of new geometries

The **direct sum** of two geometries $\mathcal{G}_1$ and $\mathcal{G}_2$ is defined as follows. The type set (respectively, element set) of $\mathcal{G}_1 \oplus \mathcal{G}_2$ is the disjoint union of the type sets (respectively, element sets) of $\mathcal{G}_1$ and $\mathcal{G}_2$. The incidence relation on $\mathcal{G}_1 \oplus \mathcal{G}_2$ is the combination of the incidence relations on $\mathcal{G}_1$ and $\mathcal{G}_2$ and the condition that every element of $\mathcal{G}_1$ is incident with every element of $\mathcal{G}_2$.

The **Veldkamp space** $\mathcal{V}(S)$ of a point-line space S is the space in which a **point** is a geometric hyperplane of S and a **line** is the collection H_1H_2 of all geometric hyperplanes H of S such that $H_1 \cap H_2 = H_1 \cap H = H_2 \cap H$ or $H = H_i$, $i = 1, 2$, where H_1, H_2 are distinct points of $\mathcal{V}(S)$. If S is a space with a fixed injection ϕ from the set of points of S into the set of geometric hyperplanes of S (e.g., the map assigning to each point of a nondegenerate polar space the space perpendicular to p), then the space on those geometric hyperplanes of S which are an image under ϕ and the induced lines of $\mathcal{V}(S)$ is called the **Veldkamp embedding** of S with respect to ϕ.

A.6 Point-line spaces

Good introductions into the area of projective and polar spaces are the books [Cam91] and [Tay92]. The book on generalized polygons is of course [Van98]. Generalized quadrangles, which are both polar spaces and generalized polygons, are treated in [PT84].

Definition A.6.1 A **point-line space** is a geometry of rank two. A **partial linear space** is a point-line space in which two points are incident with at most one common line, the **connecting line** of those two points. Any two points that admit a connecting line are **collinear**. A **linear space** is a partial linear space in which any pair of points admit a connecting line.

A **projective space** is a linear space in which Pasch's axiom holds: Suppose a, b, c, d are distinct points. Then the lines ab and cd intersect if and only if the lines ac and bd intersect.

A **polar space** is a partial linear space in which the Buekenhout-Shult axiom holds: Suppose p is a point and l a non-incident line. Then either one or all points on l are collinear to p.

Let $n \geq 2$. A **generalized n-gon** is a point-line space that does not contain an ordinary k-gon for $k < n$ and that has the property that any two of its elements are contained in an ordinary n-gon. A generalized n-gon is also called a **generalized polygon**.

Proposition A.6.2 (e.g. Buekenhout, Cohen [BC], Corollary 9.5.6)
Let x, y be distinct collinear points of a nondegenerate polar space. Denote collinearity in the polar space by $\perp$. Then $l = (\{x, y\}^\perp)^\perp$ is the unique line on x and y.

A **geometric hyperplane** of a point-line space S is a proper subspace of S that meets every line of S.

A.7 Coverings and simple connectedness

Definition A.7.1 Let $\mathcal{G}$ be a geometry. A **path** of length k in the geometry is a sequence of elements $x_0, \ldots, x_k$ such that x_i and x_{i+1} are incident, $0 \leq i \leq k-1$. We do not allow repetitions; hence $x_i \neq x_{i+1}$. A **cycle** based at an element x is a path in which $x_0 = x_k = x$. Two paths are **homotopically equivalent** if one can be obtained from the other via the following operations (called **elementary homotopies**): inserting or deleting a **return** (i.e., a cycle of length 2) or a **triangle** (i.e., a cycle of length 3). The equivalence classes of cycles based at an element x form a group under the operation induced by concatenation of cycles. This group is called the **fundamental group** of $\mathcal{G}$ and denoted by $\pi_1(\mathcal{G}, x)$. A connected geometry is called **simply connected** if its fundamental group is trivial. A cycle that is homotopically equivalent to the cycle of length 0 is called **null homotopic**, or **homotopically trivial**.

Definition A.7.2 Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are two geometries over the same type set and suppose $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ is a **morphism** of geometries, i.e., ϕ preserves the type and sends incident elements to incident elements. The morphism ϕ is called a **covering** if and only if ϕ is surjective and for every non-empty flag F_1 in $\mathcal{G}_1$ the mapping ϕ induces an isomorphism between the residue of F_1 in $\mathcal{G}_1$ and the residue of $F_2 = \phi(F_1)$ in $\mathcal{G}_2$.

Proposition A.7.3
Let $\mathcal{G}$ be a simply connected geometry. Then every covering of $\mathcal{G}$ is an isomorphism.

Proof. Suppose $\gamma := x_0 x_1 \ldots x_k = x_0$ is a null homotopic k-cycle based at x_0 in $\mathcal{G}$. If $\pi : \mathcal{G}' \to \mathcal{G}$ is a covering of $\mathcal{G}$, then, for any element $a \in \pi^{-1}(x_0)$, there exists a unique k-cycle in $\mathcal{G}'$ based at a that has γ as its image. (This is clear for triangles. On the other hand, any null homotopic cycle can be filled up with triangles.) Now, if $\mathcal{G}$ is simply connected, but admits a non-injective covering $\pi : \mathcal{G}' \to \mathcal{G}$, then there exist distinct elements a and b of $\mathcal{G}'$ with $\pi(a) = \pi(b)$. Choose a path from a to b in $\mathcal{G}'$. Then its image under π is a cycle in $\mathcal{G}$, which has to lift to both a cycle in $\mathcal{G}'$ and the path from a to b in $\mathcal{G}'$ we have chosen before, a contradiction. □

Definition A.7.4 A morphism $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ of geometries is a k**-covering**, if ϕ is surjective and for any flag F_1 of corank at most k of $\mathcal{G}_1$, the induced mapping from the residue of F_1 in $\mathcal{G}_1$ onto the residue of $\phi(F_1)$ in $\mathcal{G}_2$ is an isomorphism. A connected geometry is k**-simply connected** if it admits no proper k-coverings. Note that every k-covering is a $(k-1)$-covering, so $(k-1)$-simple connectedness of a geometry implies k-simple connectedness. Moreover, if the rank of $\mathcal{G}$ is n, then $(n-1)$-simple connectedness of $\mathcal{G}$ coincides with simple connectedness of $\mathcal{G}$.

Let Γ be a connected graph such that every vertex is contained in an edge and any edge is contained in a 3-clique. Then we can consider Γ as a two-dimensional simplicial complex whose two-simplices are its triangles. Consequently we can define a fundamental group as we did for geometries above.

Definition A.7.5 Let Γ be a connected graph such that every vertex is contained in an edge and any edge is contained in a 3-clique, and let $\mathbf{x}$ be a vertex of Γ, **the base vertex**. Construct a graph $\hat{\Gamma}$, the **universal cover of** Γ, as follows. The vertices of $\hat{\Gamma}$ are the paths in Γ starting at $\mathbf{x}$ where two vertices γ_1, γ_2 are equal if and only if they have the same end vertex in Γ and the cycle $\gamma_1\gamma_2^{-1}$ is null homotopic. Two vertices γ_1, γ_2 are adjacent if and only if the end vertices are adjacent in Γ and $\gamma_1^{-1}\gamma_2$ is homotopic to a path of length 1.

Proposition A.7.6
Let Γ be a connected graph such that every vertex is contained in an edge and any edge is contained in a 3-clique. Then the universal completion $\hat{\Gamma}$ is independent of the choice of the base vertex. Moreover, $\hat{\Gamma}$ is simply connected and admits a covering onto Γ. In particular, if Γ is locally homogeneous, then so is $\hat{\Gamma}$.

Proof. The proof is straightforward. □

Lemma A.7.7
Let $\mathcal{G}$ be a geometry. Assume that $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ can be decomposed as the direct sum of geometries $\mathcal{G}_1$ and $\mathcal{G}_2$ with $\mathcal{G}_1$ connected of rank at least two. Then $\mathcal{G}$ is simply connected.

Proof. Certainly, $\mathcal{G}$ is connected. Choose a base point $x \in \mathcal{G}_1$. We first prove that any cycle originating at x is homotopic to a cycle fully contained in $\mathcal{G}_1$. Let $xx_1 \dots x_{n-1}x$ be a cycle. Proceed by induction on the number of elements on the cycle which are not in $\mathcal{G}_1$. Suppose x_s is the first element in the cycle which is not in $\mathcal{G}_1$. Let $y \in \mathcal{G}_1$ such that $y \neq x_{s+1}$ and y is incident with x_{s+1}. Notice that y is incident with x_s. Since the residue of x_s contains $\mathcal{G}_1$, we can connect x_{s-1} with y via a path $x_{s-1}y_1 \dots y_{k-1}y$ fully contained in $\mathcal{G}_1$. Furthermore, this path is homotopic to the path $x_{s-1}x_s y$. Thus, our original path is homotopic to the path $xx_1 \dots x_{s-1}y_1 \dots y_{k-1}yx_{s+1} \dots x_{n-1}x$. This path has fewer elements outside $\mathcal{G}_1$, and our claim is proved. Choosing an element $z \in \mathcal{G}_2$ we see that this z is incident to all elements in $\mathcal{G}_1$, so any cycle in $\mathcal{G}_1$ is null homotopic. □

Appendix B

Flag-transitive Geometries

This appendix provides definitions and basic informations about amalgams of groups and coverings of geometries, both concepts joined by Tits' lemma B.2.5. The definitions as well as a lot of technique can be found in Sasha Ivanov's book [Iva99].

B.1 Coset geometries

Definition B.1.1 Let $\mathcal{G}$ be a geometry. The geometry is called **flag-transitive** if there exists a group G of automorphisms of $\mathcal{G}$ such that, whenever F, F' are flags of $\mathcal{G}$ with $typ(F) = typ(F')$, then there exists a $g \in G$ with $g(F) = F'$.

Proposition B.1.2 (e.g., Ivanov [Iva99], Proposition 1.2.1)
Let $\mathcal{G}$ be a geometry of rank n over the set $I = \{1, 2, \cdots, n\}$ of types, and let G be a flag-transitive group of automorphisms of $\mathcal{G}$. Let $F = x_1, x_2, \ldots, x_n$ be a maximal flag in $\mathcal{G}$, and let G_{x_i} be the stabilizer of x_i in G. Let $\mathcal{G}(G)$ be the incidence system whose elements of type i are the left cosets of G_i in G and in which two elements are incident if and only if the intersection of the corresponding cosets is non-empty. Then $\mathcal{G}(G)$ is a geometry and the map $y \mapsto gG_i$, for $typ(y) = typ(x_i)$ and $g(x_i) = y$, establishes an isomorphism of $\mathcal{G}$ onto $\mathcal{G}(G)$.

The preceding result actually can be found in any serious book treating group-related geometries, e.g., [BC], [Iva99]. The original idea is contained in [Kle72].

Definition B.1.3 The incidence system $\mathcal{G}(G)$ of Proposition B.1.2 is called the **coset geometry** of $\mathcal{G}$ in G. We suppress the flag F.

B.2 Amalgams

Definition B.2.1 An **amalgam** $\mathcal{A}$ of groups is a set with a partial operation of multiplication and a collection of subsets $\{G_i\}_{i \in I}$, for some index set I, such that the following hold:

(i) $\mathcal{A} = \bigcup_{i \in I} G_i$;

(ii) the product ab is defined if and only if $a, b \in G_i$ for some $i \in I$;

(iii) the restriction of the multiplication to each G_i turns G_i into a group; and

(iv) $G_i \cap G_j$ is a subgroup in both G_i and G_j for all $i, j \in I$.

It follows that the groups G_i share the same identity element, which is then the only identity element in $\mathcal{A}$, and that $a^{-1} \in \mathcal{A}$ is well-defined for every $a \in \mathcal{A}$. We will call the groups G_i the **members** of the amalgam $\mathcal{A}$. Notice that our definition is a special case of the general definition of an amalgam as found, say, in [Ser77].

Definition B.2.2 A group G is called a **completion** of an amalgam $\mathcal{A}$ if there exists a map $\pi : \mathcal{A} \to G$ such that

(i) for all $i \in I$ the restriction of π to G_i is a homomorphism of G_i to G; and

(ii) $\pi(\mathcal{A})$ generates G.

Among all completions of $\mathcal{A}$ there is one *largest* which can be defined as the group having the following presentation:

$$U(\mathcal{A}) = \langle t_h \mid h \in \mathcal{A},\ t_x t_y = t_z, \text{ whenever } xy = z \text{ in } \mathcal{A} \rangle.$$

$U(\mathcal{A})$ is called the **universal completion**.

Proposition B.2.3 (e.g., Ivanov [Iva99])
Let $\mathcal{A}$ be an amalgam. Then $U(\mathcal{A})$ is a completion of $\mathcal{A}$. Furthermore, every completion of $\mathcal{A}$ is isomorphic to a quotient of $U(\mathcal{A})$.

Definition B.2.4 Suppose a group $G \leq \mathrm{Aut}\,\mathcal{G}$ acts flag-transitively on a geometry $\mathcal{G}$ of rank n. A **rank k parabolic** is the stabilizer in G of a flag of corank k from $\mathcal{G}$. Parabolics of rank $n-1$ are called **maximal parabolics**. They are exactly the stabilizers in G of single elements of $\mathcal{G}$.

Let F be a maximal flag in $\mathcal{G}$, and let G_x denote the stabilizer in G of $x \in \mathcal{G}$. The amalgam $\mathcal{A} = \mathcal{A}(F) = \bigcup_{x \in F} G_x$ is called the **amalgam of maximal parabolics in** G. For a fixed flag F we also use the notation G_i for the maximal parabolic G_x, where $x \in F$ is of type i. For a subset $J \subset I = \{0, 1, \ldots, n-1\}$, define G_J to be $\bigcap_{j \in J} G_j$, including $G_\emptyset = G$. Similarly to $\mathcal{A}$, we define the amalgam $\mathcal{A}_{(s)}$ as the union of all rank s parabolics. With this notation we have $\mathcal{A} = \mathcal{A}_{(n-1)}$. Moreover, according to our definition, $\mathcal{A}_{(n)} = G$.

Tits' Lemma B.2.5 (Tits [Tit86], Corollaire 1)
Suppose a group G acts flag-transitively on a geometry $\mathcal{G}$, and let $\mathcal{A}$ be the amalgam of parabolics associated with some maximal flag F of $\mathcal{G}$. Then G is the universal completion of the amalgam $\mathcal{A}$ if and only if $\mathcal{G}$ is simply connected.

Appendix C

Groups of Lie Type

This appendix offers some definitions and facts about algebraic groups, groups of Lie type, and Chevalley groups. This appendix is to be seen as a motivation for our geometric investigations and as a stackpile of results we are referring to in the text. The interested reader is refered to [Ste68] or [Car72] about Chevalley groups and groups of Lie type and to [Tim01] about abstract root groups.

C.1 Algebraic groups

Definition C.1.1 The **Zariski topology** on $GL_n(\overline{\mathbb{F}})$ over an algebraically closed field $\overline{\mathbb{F}}$ is the topology defined by the condition that the closed sets be the solution sets of systems of polynomial equations in the matrix entries and the inverse of the determinant polynomial. An **$\overline{\mathbb{F}}$-algebraic group** is a closed subgroup $\overline{G}$ of $GL_n(\overline{\mathbb{F}})$ for some n. The Zariski topology on $\overline{G}$ is inherited from that of $GL_n(\overline{\mathbb{F}})$.

The **radical** of $\overline{G}$ is the largest normal subgroup of $\overline{G}$ which is closed, connected, and solvable. $\overline{G}$ is **semisimple** if the radical of $\overline{G}$ is trivial.

C.2 Finite groups of Lie type

Definition C.2.1 Let $\overline{G}$ be an algebraic group. Then a **Steinberg endomorphism** of $\overline{G}$ is a surjective algebraic endomorphism σ of $\overline{G}$ whose fixed point subgroup $C_{\overline{G}}(\sigma)$ is finite.

A **σ-setup** (over the algebraic closure $\overline{\mathbb{F}}_p$ of the finite field $\mathbb{F}_p$) is a pair $(\overline{G}, \sigma)$ such that $\overline{G}$ is a semisimple $\overline{F}_p$-algebraic group and σ is a Steinberg endomorphism of $\overline{G}$. If G is a finite group, then a σ-setup for G is a σ-setup $(\overline{G}, \sigma)$ over $\overline{\mathbb{F}}_p$ for some prime p such that G is isomorphic to the subgroup of $C_{\overline{G}}(\sigma)$ generated by all its p-elements.

$Lie(p)$ is the set of finite groups possessing a σ-setup $(\overline{G}, \sigma)$ over $\overline{\mathbb{F}}_p$ such that $\overline{G}$ is simple. Define Lie to be the union of the $Lie(p)$ over all primes p (where for $p = 2$

and $p = 3$ some changes have to be made as given in Definition 2.2.8 of [GLS98]; e.g., one has to remove $A_1(2)$ and $A_1(3)$ and replace $G_2(2)$ by its commutator group). A group contained in *Lie* is called a **finite group of Lie type**.

Chev(p) is the set of all quasisimple groups G (i.e., $[G, G] = 1$ and $G/Z(G)$ is simple) such that $G/Z(G)$ is a finite simple group of Lie type in characteristic p. The set *Chev* is the union of the *Chev*(p) over all primes p. A group contained in *Chev* is called a **finite Chevalley group**.

C.3 Chevalley groups and Steinberg relations

Definition C.3.1 (Steinberg [Ste68], Theorem 8) Let Σ be an indecomposable root system of rank at least two, and let $\mathbb{F}$ be a field. We consider the group G generated by the collection of elements $\{x_r(t) \mid r \in \Sigma, t \in \mathbb{F}\}$ subject to the following relations:

(i) $x_r(t)$ is additive in t.

(ii) If r and s are roots and $r + s \neq 0$, then

$$[x_r(t), x_s(u)] = \prod x_{hr+ks}(C_{hkrs} t^h u^k)$$

with $h, k > 0$, $hr + ks \in \Sigma$ (if there are no such numbers, then $[x_r(t), x_s(u)] = 1$), and structure constants $C_{hkrs} \in \{\pm 1, \pm 2, \pm 3\}$.

(iii) $h_r(t)$ is multiplicative in t, where $h_r(t)$ equals $w_r(t)w_r(-1)$ and $w_r(t)$ equals $x_r(t)x_{-r}(-t^{-1})x_r(t)$ for $t \in \mathbb{F}^*$.

For a certain choice of the structure constants C_{hkrs} (see, e.g., Theorem 1.12.1 of [GLS98]) the group G is called the **universal Chevalley group constructed from Σ and $\mathbb{F}$**. For $r \in \Sigma$ the group $x_r = \{x_r(t) \mid t \in \mathbb{F}\} = (\mathbb{F}, +)$, and any conjugate of x_r in G, is called a **root (sub)group**.

Theorem C.3.2 (Steinberg [Ste68], Theorem 9)
Let Σ be an indecomposable root system of rank at least two, and let $\mathbb{F}$ be an algebraic extension of a finite field. Then the relations (i) *and* (ii) *of Definition C.3.1 suffice to define the corresponding universal Chevalley group, i.e., they imply the relations* (iii).

Section 3.1 provides some more results on defining relations of universal Chevalley groups. See Aschbacher's work [Asc77] for a characterization of finite Chevalley groups over fields of odd order.

C.4 Root subgroups and abstract root subgroups

Theorem C.4.1 (Theorem 12.1 of [AS76])
Let G be a finite Chevalley group of rank at least two, other than ${}^2F_4(q)$. Let X and Y be centers of distinct long root subgroups, of order q. Then one of the following holds:

(i) *$\langle X, Y\rangle$ is elementary Abelian and is the union of $q+1$ long root subgroups which pairwise intersect trivially.*

(ii) *$\langle X, Y\rangle$ is elementary Abelian and the elements of $X \cup Y$ are the only root elements contained in $\langle X, Y\rangle$.*

(iii) *$\langle X, Y\rangle$ is isomorphic to a Sylow subgroup of order q^3 in $SL_3(q)$, the center $Z = Z(\langle X, Y\rangle)$ is a conjugate long root subgroup, and each of XZ, YZ are a union of $q+1$ long root subgroups as in (i).*

(iv) *$\langle X, Y\rangle \cong SL_2(q)$ (or $PSL_2(q)$ in $PSO_4^+(q)$).*

Definition C.4.2 A set Σ of Abelian non-identity subgroups of the group G is called a set of **abstract root subgroups** of G, if it satisfies the following.

(i) $G = \langle \Sigma \rangle$ and $\Sigma^g \subseteq \Sigma$ for each $g \in G$.

(ii) For each pair $A, B \in \Sigma$ one of the following holds:

  (a) $[A, B] = 1$.

  (b) $X = \langle A, B\rangle$ is a rank one group with unipotent (in the sense of Definition 1.1 of Chapter I of [Tim01]) subgroups A and B.

  (c) $Z(\langle A, B\rangle) \geq [A, B] = [a, B] = [A, b] \in \Sigma$ for each $a \in A^*$ and $b \in B^*$.

If, for some field $\mathbb{F}$, in (ii)(b) we always have $X \cong (P)SL_2(\mathbb{F})$, then Σ is called a set of **$\mathbb{F}$-root subgroups** of G. If case (ii)(c) never occurs, the set Σ is a set of **abstract transvection groups**. If both hold, then the set Σ is a set of **$\mathbb{F}$-transvection subgroups**.

If, moreover, in any of the above notions Σ is a conjugacy class of subgroups in G, then Σ is called a **class of abstract root subgroups of** G, respectively a **class of $\mathbb{F}$-root subgroups**, etc.

In case of Σ a set of $\mathbb{F}$-root subgroups, the group $X \cong (P)SL_2(\mathbb{F})$ in (ii)(b), and any conjugate of X in G, is called a **fundamental** SL_2. For example the group $\langle X, Y\rangle \cong SL_2(q)$ of Theorem C.4.1(iv) is a fundamental SL_2. An alternative name for a fundamental SL_2 is **hyperbolic line**.

Franz Georg Timmesfeld proved that the classes of $\mathbb{F}$-root subgroups essentially are the classes of root subgroups of Chevalley groups (of sufficient rank), see the Main Theorem of [Tim91]. Confer to Theorem 4 and Theorem 5 of [Tim99] on results on abstract transvection groups and abstract root subgroups. See also Timmesfeld's book [Tim01].

Following Timmesfeld, for $A \in \Sigma$ define

$$\begin{aligned}
\Sigma_A &:= C_\Sigma(A) \setminus \{A\}\,, \\
\Lambda_A &:= \{B \in \Sigma_A \mid AB \text{ can be partitioned into elements of } \Sigma\}\,, \\
\Psi_A &:= \{B \in \Sigma \mid [A,B] \in \Sigma\}\,, \\
\Omega_A &:= \{B \in \Sigma \mid \langle A,B \rangle \text{ is a rank one group}\}\,.
\end{aligned}$$

The set Σ is the disjoint union of $\{A\}$, Σ_A, Ψ_A, and Ω_A. The pair $A, B \in \Sigma$ is called **strongly commuting** if $B \in \Lambda_A$, **polar** if $B \in \Sigma_A \backslash \Lambda_A$, **special** if $B \in \Psi_A$, and **hyperbolic** if $B \in \Omega_A$. Notice, first, that the relations defined here are symmetric and, second, that, in case of $\mathbb{F}$-root subgroups, the group $\langle A, B\rangle$ for a hyperbolic pair A, B is a fundamental SL_2 of G. The latter observation justifies the alternative notion of a hyperbolic line instead of a fundamental SL_2.

C.5 Centralizers of fundamental SL_2's

Proposition C.5.1
Let G be a Chevalley group of one of the following types, let θ be a hyperbolic line of G, and let $n \geq 2$. Then the following hold for the centralizers $C_G(\theta)$:

(i) $C_G(\theta)$ *is of type* A_{n-2} *for* G *of type* A_n,

(ii) $C_G(\theta)$ *is of type* $A_1 \oplus B_{n-2}$ *for* G *of type* B_n,

(iii) $C_G(\theta)$ *is of type* C_{n-1} *for* G *of type* C_n,

(iv) $C_G(\theta)$ *is of type* $A_1 \oplus D_{n-2}$ *for* G *of type* D_n, *if* $n \geq 4$,

(v) $C_G(\theta)$ *is of type* A_5 *for* G *of type* E_6,

(vi) $C_G(\theta)$ *is of type* D_6 *for* G *of type* E_7,

(vii) $C_G(\theta)$ *is of type* E_7 *for* G *of type* E_8,

(viii) $C_G(\theta)$ *is of type* C_3 *for* G *of type* F_4.

Proof. This is well known. For the statements about the exceptional groups, we refer to [Coo79]. Proofs of the other statements are contained in this thesis, cf. Section 4.1. □

Notice that we can heuristically read off the type of the centralizer from the extended diagram of G. Indeed, the 'local' type is just the extended diagram of G where the vertex corresponding to the root of maximal height and all adjacent vertices have been deleted.

Proposition C.5.2
Let D be a spherical Coxeter diagram of rank at least three, and let G be a Chevalley group of type D. Moreover, let A, B be distinct long root subgroups of G. Then (A, B) is

- *a strongly commuting pair if and only if $[A, B] = 1$ and there does not exist a hyperbolic line l on A with $[l, B] = 1$,*
- *a polar pair if and only if $[A, B] = 1$ and there does exist a hyperbolic line l on A with $[l, B] = 1$,*
- *a special pair if and only if $[A, B] \neq 1$ and there does not exist a hyperbolic line l through A and B, and*
- *a hyperbolic pair if and only if there does exist a hyperbolic line l through A and B.*

Proof. This is easily shown for the classical groups using the description of their long root group geometries given in Section 4.1. For the finite exceptional groups in odd characteristic we cite [Coo79] to see that the permutation rank of a exceptional group of rank at least three on the set of its root subgroups is five. The proposition follows from the fact that there exist strongly commuting, polar, special, and hyperbolic pairs, and Corollary 2.3 of Chapter II of [Tim01], that states that a long root subgroup cannot commute with a whole hyperbolic line if it strongly commutes with a long root subgroup on that hyperbolic line. □

Appendix D

Some Open Problems

Chapter 1

- Recognize all connected graphs that are locally $\mathcal{N}\mathcal{O}_6^+(q)$, $\mathcal{N}\mathcal{O}_6^-(q)$, $\mathcal{N}\mathcal{U}_5(q)$ (see the definition on page 25).
  (*Conjecture:* The local structure uniquely determines the isomorphism type of the graph.)

- Establish the finiteness of the diameter of a connected, locally $\mathbf{H}_2(2)$ graph (see Definition 1.1.1).

Chapter 2

- Recognize all connected graphs that are locally $\mathbf{L}_6(\mathbb{F})$ (see Definition 2.1.1).
  (*Conjecture:* The local structure uniquely determines the isomorphism type of the graph.)

- Try to build global points (see Definition 2.4.12) and global lines (see Definition 2.4.7) on the graph on the fundamental SL_2's of the group $E_6(\mathbb{F})$ with commuting as adjacency, and determine the isomorphism type of that geometry.

Chapter 3

- Prove a Phan-type theorem for the group $F_4(\mathbb{F})$ (see Section 3.2).

- Classify all flips of (spherical) twin buildings (see Section 3.3).

Chapter 4

- Provide a systematic investigation of perp spaces (see Definition 4.2.1).

- Characterize the hyperbolic long root group geometry of orthogonal groups using centralizers of fundamental SL_2's (cf. Corollary 4.3.7, Corollary 4.4.23). (*Conjecture:* The local structure uniquely determines the isomorphism type of the geometry given that the local dimension is sufficiently large.)

- Do the same for the exceptional groups of type E_6, E_7, E_8, and F_4.

- Let Σ be a reduced, crystallographic root system and consider the graph $\Gamma(\Sigma)$ on the unordered pairs $\pm\alpha$, $\alpha \in \Sigma$, as vertices with the orthogonality relation as adjacency relation. Classify all graphs that are connected and locally $\Gamma(\Sigma)$ for some reduced, crystallographic root system Σ.

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Glossary

Index

Dutch Summary

In het grootste deel van dit proefschrift wordt aandacht besteed aan het onderzoek van samenhangs- en lokale eigenschappen van meetkundes en grafen afkomstig van meetkundes. Deze lokale eigenschappen worden vervolgens gebruikt om de meetkundes en grafen te karakteriseren. Omdat er altijd een natuurlijke actie van een groep te vinden is op deze meetkundes, zijn er groepentheoretische gevolgen van de meetkundige resultaten. Het is belangrijk om op te merken, dat wij altijd alleen op het laatste moment overstappen op de taal van groepentheorie en zo veel mogelijk de stellingen bewijzen met meetkundige middelen. De centrale methoden van dit proefschrift zijn dus allemaal meetkundig en combinatorisch.

In het eerste hoofdstuk houden we ons bezig met de studie van grafen op paren van niet-incidente punt-hypervlakken van een projectieve ruimte, waarbij twee paren buren zijn in de graaf dan en slechts dan als het punt van het ene paar in het hypervlak van het andere paar bevat is en andersom. De punt-hypervlak graaf afkomstig van de projektieve ruimte $\mathbb{P}_n(\mathbb{F})$ wordt genoteerd als $\mathbf{H}_n(\mathbb{F})$. Het centrale resultaat van hoofdstuk 1 is als volgt. *Zij $n \geq 3$, zij $\mathbb{F}$ een divisiealgebra en zij Γ een samenhangende graaf die lokaal $\mathbf{H}_n(\mathbb{F})$ is, dan is Γ isomorf met $\mathbf{H}_{n+1}(\mathbb{F})$.* Daarnaast bestuderen we nog een voorbeeld van een graaf die wel lokaal $\mathbf{H}_2(\mathbb{F}_2)$ is, maar niet isomorf met $\mathbf{H}_3(\mathbb{F}_2)$ is. De correspondentie tussen niet-incidente punt-hypervlak paren van een projectieve ruimte en reflectietori van zijn automorfismengroep heeft groepentheoretische gevolgen. Het hoofdstuk eindigt met een korte studie van deelgrafen van de punt-hyperlijn grafen die geïnduceerd worden door een polariteit op de projectieve ruimte.

Hoofdstuk 2 is bijna hetzelfde als hoofdstuk 1. Hier bestuderen we grafen op elkaar niet snijdende paren van lijnen en hyperlijnen van een projectieve ruimte. In principe zijn alle resultaten dezelfde behalve dat de lokale dimensies verdubbeld moeten worden. Toch zijn er een paar verschillen. Bijvoorbeeld is er een natuurlijk voorbeeld voor een graaf die lokaal een lijn-hyperlijn graaf is, maar niet zelf een lijn-hyperlijn graaf is. De elkaar niet snijdende lijn-hyperlijn paren van een projectieve ruimte corresponderen namelijk met de fundamentele SL_2's van zijn automorfismengroep en de buurrelatie in de graaf correspondeert met de commutatierelatie op de SL_2's. Het is ook bekend dat, voor een lichaam $\mathbb{F}$, de centralisator van een fundamentele SL_2 binnen de groep $E_6(\mathbb{F})$ isomorf is met $SL_6(\mathbb{F})$. Op deze manier krijgen we een voorbeeld van een graaf die lokaal een lijn-hyperlijn graaf is, maar

dat zelf niet is. De beschrijving van de lijn-hyperlijn graaf door middel van fundamentele SL_2's heeft groepentheoretische gevolgen en levert ook een soort lokale karakterisering op van de meetkunde op de wortelgroepen en fundamentele SL_2's van een lineare groep van voldoende dimensie. Maar dit tweede gevolg wordt pas in hoofdstuk 4 echt uitgewerkt.

Hoofdstuk 3 beschrijft onderzoek dat de auteur samen met zijn tweede promotor Sergey Shpectorov en met Corneliu Hoffman heeft gedaan. Bestudeert wordt een zekere deelmeetkunde van de symplectische polaire meetkunde. In plaats van een lokale karakterisering wordt hier alleen maar de enkelvoudige samenhang van deze meetkunde bewezen. Het groepentheoretische gevolg is dus geen echte lokale herkenning van de groep maar alleen maar een resultaat over definiërende relaties binnen de groep. Verder houdt dit hoofdstuk zich bezig met de Curtis-Tits stelling en Phans stellingen en met de samenhang tussen deze stellingen en de net beschreven stelling.

Hoofdstuk 4 gaat, zoals al gezegd, over meetkundes op de (lange) wortelgroepen van zekere Chevalleygroepen als punten en de fundamentele SL_2's als lijnen. Deze meetkundes, afkomstig van de lineaire, de unitaire en de symplectische groepen, worden lokaal gekarakteriseerd. Ook wordt er een stel axioma's gegeven waarvoor een meetkunde isomorf is met een hyperbolische lange-wortelgroepmeetkunde van een Chevalleygroep van (getwist) type F_4. De karakteriseringen van de wortelgroepmeetkundes van symplectische en unitaire groepen leveren ook lokale herkenningsstellingen van de groepen zelf op.

Curriculum Vitae

- 13 July 1976: born in Miltenberg, West Germany.
- October 1995 – April 1996: regular student in mathematics at the FernUniversität Hagen, Germany.
- May 1996 – September 1998: regular student in mathematics at the Bayerische Julius-Maximilians-Universität Würzburg, Germany.
- 9 October 1997: awarded the title *Kandidat der Mathematik* by the Bayerische Julius-Maximilians-Universität Würzburg.
- 2 September 1998: awarded the title *Diplom-Mathematiker univ.* by the Bayerische Julius-Maximilians-Universität Würzburg with Prof. Dr. Theo Grundhöfer as supervisor of thesis.
- since October 1998: *Assistent in Opleiding* at the Technische Universiteit Eindhoven, The Netherlands, with Prof. Dr. Arjeh Cohen as supervisor and Dr. Hans Cuypers as co-supervisor.
- March 1999: research stay for one week at the Universiteit Gent, Belgium, on invitation by Prof. Dr. Hendrik Van Maldeghem.
- March 2001 – April 2001: research stay at Bowling Green State University, OH, USA, as a visitor of Prof. Dr. Sergey Shpectorov.
- May 2001 – July 2001: research stay at Michigan State University, MI, USA, as a visitor of Prof. Dr. Jonathan I. Hall and Prof. Dr. Ulrich Meierfrankenfeld.
- March 2002 – April 2002: research stay at Rutgers University, NJ, USA, on invitation by Prof. Dr. Richard Lyons and Prof. Dr. Ronald Solomon.