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STABILITY PROPERTIES OF EQUILIBRIUM SETS OF CONTROLLED LINEAR MECHANICAL SYSTEMS WITH DRY FRICTION

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Abstract

The dynamics of mechanical systems with dry friction elements, modeled by set-valued force laws, can be described by differential inclusions. The switching and set-valued nature of the friction force law is responsible for the hybrid character of such models. An equilibrium set of such a differential inclusion corresponds to a stationary mode for which the friction elements are sticking. The attractivity properties of the equilibrium set are of major importance for the overall dynamic behavior of this type of systems. Conditions for the attractivity of the equilibrium set of linear MDOF mechanical systems with multiple friction elements are presented. These results are obtained by application of a generalization of LaSalle's principle for differential inclusions of Filippov-type. Besides passive systems, also systems with negative viscous damping are considered. For such systems, only local attractivity of the equilibrium set can be assured under certain conditions. Moreover, an estimate for the region of attraction is given for these cases. The results are illustrated by means of a 2DOF example. Moreover, the value of the attractivity results in the context of the control of mechanical systems with friction is illuminated.

Key words

Discontinuous Systems, Dry Friction, Equilibrium Sets, Attractivity, Control.

1 Introduction

The presence of dry friction can influence the behaviour and performance of mechanical systems as it can induce several phenomena, such as friction-induced limit-cycling, damping of vibrations and stiction. Dry friction in mechanical systems is often modelled using set-valued constitutive models (Glocker, 2001), such as the set-valued Coulomb's law. Set-valued friction models have the advantage to properly model stiction, since the friction force is allowed to be non-zero at zero relative velocity. The dynamics of mechanical systems with set-valued friction laws are described by differential inclusions. We limit ourselves to set-valued friction laws which lead to Filippov-type systems (Filippov, 1988). Filippov systems, describing systems with friction, can exhibit equilibrium sets, which correspond to the stiction behaviour of those systems.

The overall dynamics of mechanical systems is largely affected by the stability and attractivity properties of the equilibrium sets. For example, the loss of stability of the equilibrium set can, in certain applications, cause limit-cycling. Moreover, the stability and attractivity properties of the equilibrium set can also seriously affect the performance of control systems. In (Alvarez *et al.*, 2000; Shevitz and Paden, 1994; Bacciotti and Ceragioli, 1999), stability and attractivity properties of (sets of) equilibria in differential inclusions are studied. More specifically, in (Alvarez *et al.*, 2000; Shevitz and Paden, 1994) the attractivity of the equilibrium set of a passive, one-degree-of-freedom friction oscillator with one switching boundary (i.e. one dry friction element) is discussed. Moreover, in (Shevitz and Paden, 1994; Bacciotti and Ceragioli, 1999) the Lyapunov stability of an equilibrium point in the equilibrium set is shown. However, most papers are limited to either one-degree-of-freedom systems or to systems exhibiting only one switching boundary.

We will provide conditions under which the equilibrium set is attractive for multi-degree-of-freedom mechanical systems with an arbitrary number of Coulomb friction elements using Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems. Moreover, passive as well as non-passive systems will be considered. The non-passive systems that will be studied are linear mechanical systems with a non-positive definite damping matrix with additional dry friction elements. The

non-positive-definiteness of the damping matrix of linearized systems can be caused by fluid, aeroelastic, control and gyroscopical forces, which can cause instabilities. It will be demonstrated in this paper that the presence of dry friction in such an unstable linear system can (conditionally) ensure the local attractivity of the equilibrium set of the resulting system with dry friction. Moreover, an estimate of the region of attraction for the equilibrium set will be given. A rigid multibody approach is used for the description of mechanical systems with friction, which allows for a natural physical interpretation of the conditions for attractivity. Finally, a preliminary study of the application of these results in a control context is provided. It should be noted that the results in this paper build on the work presented in (Van de Wouw and Leine, 2004).

In section 2, the equations of motion for linear mechanical systems with frictional elements are formulated and the equilibrium set is defined. In section 3, the attractivity properties of the equilibrium set are studied by means of a generalisation of LaSalle's invariance principle. In section 4, an example is studied in order to illustrate the theoretical results and to investigate the correspondence between the estimated and actual region of attraction. Moreover, in section 5 we illustrate the way in which controller design (static state feedback) can be used to induce attractivity of equilibrium sets in mechanical systems with friction and the way in which the region of attraction can be influenced by control parameter tuning. Finally, a discussion of the obtained results and concluding remarks are given in section 6.

2 Modeling of Mechanical Systems with Coulomb Friction

In this section, we will formulate the equations of motion for linear mechanical systems with m frictional translational joints. These translational joints restrict the motion of the system to a manifold determined by the bilateral holonomic constraint equations imposed by these joints (sliders). Coulomb's friction law is assumed to hold in the tangential direction of the manifold.

Let us formulate the equations of motions for such systems by:

$$M\ddot{q} + C\dot{q} + Kq - W_T\lambda_T = 0, \qquad (1)$$

in which q is a column of independent generalized coordinates, M, C and K represent the mass-matrix, damping-matrix and stiffness-matrix, respectively, and λ_T is a column of friction forces in the translational joints. These friction forces obey the following set-valued force law:

$$\lambda_T \in -\Lambda\text{Sign}(\dot{g}_T), \qquad (2)$$

with

$$\Lambda = \text{diag}\left(\left[\mu_1|\lambda_{N_1}| \ \ldots \ \mu_m|\lambda_{N_m}|\right]\right).$$

Herein, λ_{N_i} and μ_i, $i = 1,\ldots,m$, are the normal contact force and the friction coefficient in translational joint i. Moreover, $W_T^{\text{T}} = \frac{\partial g_T}{\partial \dot{q}}$ is a matrix reflecting the generalized force directions of the friction forces. Herein, $\dot{g}_T$ is a column of relative sliding velocities in the translational joints. Equation (1) forms, together with a set-valued friction law (2), a differential inclusion. Differential inclusions of this type are called Filippov systems which obey Filippov's solution concept (Filippov's convex method). Consequently, the existence of solutions of system (1) is guaranteed. Moreover, due to the fact that $\mu_i \geq 0$, $i = 1,\ldots,m$, which excludes the possibility of repulsive sliding modes along the switching boundaries, also uniqueness of solutions in forward time is guaranteed (Leine and Nijmeijer, 2004).

Due to the set-valued nature of the friction law (2), the system exhibits an equilibrium set. Since we assume that $\dot{g}_T = W_T^T\dot{q}$, $\dot{q} = 0$ implies $\dot{g}_T = 0$. This means that every equilibrium implies sticking in all contact points and obeys the equilibrium inclusion:

$$Kq + W_T\Lambda\text{Sign}(0) \ni 0. \qquad (3)$$

The equilibrium set is therefore given by

$$\mathcal{E} = \left\{(q,\dot{q}) \in \mathbb{R}^{2n}|\ (\dot{q} = 0) \wedge \atop q \in -K^{-1}W_T\Lambda\text{Sign}(0)\right\} \qquad (4)$$

and is positively invariant due to the uniqueness of the solutions in forward time.

3 Attractivity Analysis of the Equilibrium Set

Let us now study the attractivity properties of this equilibrium set $\mathcal{E}$. Hereto, we will use LaSalle's principle (Khalil, 1996), but applied to Filippov systems with uniqueness of solutions in forward time (Van de Wouw and Leine, 2004).

Let us consider the stability of linear systems with friction and positive definite matrices M, K and a non-positive damping matrix C. Note that this implies that the equilibrium point of the linear system without friction is either stable or unstable (not asymptotically stable). In the following theorem we state the conditions under which the equilibrium set of the system with friction is locally attractive. Before stating the results we introduce the following definitions. In the attractivity result, we consider the following energy-based, positive-definite function V:

$$V = \frac{1}{2}\dot{q}^{\text{T}}M\dot{q} + \frac{1}{2}q^{\text{T}}Kq. \qquad (5)$$

Furthermore, we define an open set $\mathcal{I}_\rho$ by

$$\mathcal{I}_\rho = \{(\boldsymbol{q}, \dot{\boldsymbol{q}}) \mid V(\boldsymbol{q}, \dot{\boldsymbol{q}}) < \rho\} \qquad (6)$$

and a number ρ^* by

$$\rho^* = \min_{i=1,\ldots,n_q} \rho_i,$$

with

$$\rho_i = \frac{\beta^2}{2\lambda_i^2} \frac{1}{\|\boldsymbol{e}_{n+i}^T \boldsymbol{S}^{-1}\|^2}, \qquad (7)$$

where $\boldsymbol{S}$ is the square root of $\boldsymbol{P}$ ($\boldsymbol{P} = \boldsymbol{S}^T \boldsymbol{S}$) and $\boldsymbol{P}$ is given by

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{U}_c^T \boldsymbol{K} \boldsymbol{U}_c & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_c^T \boldsymbol{M} \boldsymbol{U}_c \end{bmatrix}. \qquad (8)$$

Moreover, in (7), n_q is the number of eigenvalues of $\boldsymbol{C}$ which lie in the closed left-half complex plane and λ_i are the corresponding eigenvalues of $\boldsymbol{C}$. Moreover, in (7), (8), $\boldsymbol{U}_c$ is the matrix containing the eigencolumns of $\boldsymbol{C}$ and, finally, $\beta := 1/\alpha$ is chosen such that

$$\sum_{i=1}^{n_q} |\boldsymbol{e}_i^T \dot{\boldsymbol{\eta}}| \le \alpha \boldsymbol{p}^T |\boldsymbol{W}_T^T \boldsymbol{U}_c \dot{\boldsymbol{\eta}}| \quad \forall \dot{\boldsymbol{\eta}}. \qquad (9)$$

Herein, $\boldsymbol{e}_i$ is a unit-column with a non-zero element on the i-th position and $\boldsymbol{p} = \{\Lambda_{ii}\}$, for $i = 1, \ldots, m$. The following result will state the conditions under which such constant β exists.

Theorem 1
Consider system (1) with friction law (2). If the following conditions are satisfied:

1. *the matrices $\boldsymbol{M}$, $\boldsymbol{K}$ are positive definite and the matrix $\boldsymbol{C}$ is symmetric,*
2. *define by λ_i, $i = 1, \ldots, n_q$, the eigenvalues of $\boldsymbol{C}$ for which $\lambda_i \le 0$ and the corresponding eigencolumns $\boldsymbol{U}_{c_i}$ satisfy $\boldsymbol{U}_{c_i} \in \text{span}\{\boldsymbol{W}_T\}$ for $i = 1, \ldots, n_q$,*
3. *$\mathcal{E} \subset \mathcal{I}_{\rho^*}$,*

then the equilibrium set (4) is locally attractive.

We present the proof of this theorem here since it unifies the results presented in (Van de Wouw and Leine, 2004) and it provides an estimate for the region of attraction of the equilibrium set, which will be used in sections 4 and 5.

Proof. We consider the positive definite function V as in (5). Using friction law (2) and the fact that $\dot{\boldsymbol{g}}_T =$

$\boldsymbol{W}_T^T \dot{\boldsymbol{q}}$, the time-derivative of V is

$$\begin{aligned} \dot{V} &= \dot{\boldsymbol{q}}^T \left(-\boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{K}\boldsymbol{q} + \boldsymbol{W}_T \boldsymbol{\lambda}_T\right) + \dot{\boldsymbol{q}}^T \boldsymbol{K} \boldsymbol{q} \\ &= -\dot{\boldsymbol{q}}^T \boldsymbol{C} \dot{\boldsymbol{q}} - \dot{\boldsymbol{g}}_T^T \boldsymbol{\Lambda} \text{Sign}(\dot{\boldsymbol{g}}_T) \\ &= -\dot{\boldsymbol{q}}^T \boldsymbol{C} \dot{\boldsymbol{q}} - \boldsymbol{p}^T |\dot{\boldsymbol{g}}_T| \\ &= -\dot{\boldsymbol{q}}^T \boldsymbol{C} \dot{\boldsymbol{q}} - \boldsymbol{p}^T |\boldsymbol{W}_T^T \dot{\boldsymbol{q}}|, \end{aligned} \qquad (10)$$

where the column $|\dot{\boldsymbol{g}}_T|$ is defined by $|\dot{\boldsymbol{g}}_T| = \{|\dot{g}_{T_i}|\}$, for $i = 1, \ldots, m$. Equation (10) implies that $\dot{V}$ is a continuous single-valued function (of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$). It holds that $\boldsymbol{p} \ge \boldsymbol{0}$ and that if $\dot{\boldsymbol{q}} = \boldsymbol{0}$ then $\dot{\boldsymbol{g}}_T = \boldsymbol{0}$.

We now apply a spectral decomposition of $\boldsymbol{C} = \boldsymbol{U}_c \boldsymbol{\Omega}_c \boldsymbol{U}_c^T$, where $\boldsymbol{U}_c$ is an orthonormal matrix containing all eigencolumns of $\boldsymbol{C}$ and $\boldsymbol{\Omega}_c$ is the diagonal matrix containing all eigenvalues of $\boldsymbol{C}$, which are real. Note that also $\boldsymbol{U}_c$ is a real matrix. Moreover, we introduce coordinates $\boldsymbol{\eta}$ such that $\boldsymbol{q} = \boldsymbol{U}_c \boldsymbol{\eta}$. Consequently, $\dot{V}$ satisfies

$$\begin{aligned} \dot{V} &= -\dot{\boldsymbol{q}}^T \boldsymbol{U}_c \boldsymbol{\Omega}_c \boldsymbol{U}_c^T \dot{\boldsymbol{q}} - \boldsymbol{p}^T |\boldsymbol{W}_T^T \dot{\boldsymbol{q}}| \\ &= -\dot{\boldsymbol{\eta}}^T \boldsymbol{\Omega}_c \dot{\boldsymbol{\eta}} - \boldsymbol{p}^T |\boldsymbol{W}_T^T \boldsymbol{U}_c \dot{\boldsymbol{\eta}}|. \end{aligned} \qquad (11)$$

The matrix $\boldsymbol{C}$ has n_q eigenvalues in the closed left-half complex plane; all other eigenvalues lie in the open right-half complex plane. Consequently, $\dot{V}$ obeys the inequality

$$\dot{V} \le -\sum_{i=1}^{n_q} \lambda_i \dot{\eta}_i^2 - \boldsymbol{p}^T |\boldsymbol{W}_T^T \boldsymbol{U}_c \dot{\boldsymbol{\eta}}| \quad \forall \dot{\boldsymbol{\eta}}, \qquad (12)$$

where we assumed that the eigenvalues (and eigencolumns) of $\boldsymbol{C}$ are ordered in such a manner that λ_i, $i = 1, \ldots, n_q$, correspond to the eigenvalues of $\boldsymbol{C}$ in the closed left-half complex plane. Assume that $\exists \alpha > 0$ such that (9) is satisfied. Assuming that such an α can be found, (12) results in

$$\begin{aligned} \dot{V} &\le -\sum_{i=1}^{n_q} \lambda_i \dot{\eta}_i^2 - \beta \sum_{i=1}^{n_q} |\dot{\eta}_i| \le 0, \\ &\forall \dot{\boldsymbol{\eta}} \in \left\{ \dot{\boldsymbol{\eta}} \mid \frac{\beta}{\lambda_i} \le \dot{\eta}_i \le -\frac{\beta}{\lambda_i} \right\} \text{ for } \lambda_i < 0, \\ &\forall \dot{\eta}_i \in \mathbb{R} \text{ for } \lambda_i = 0, \end{aligned} \qquad (13)$$

for $i = 1, \ldots, n_q$ with $\beta = \frac{1}{\alpha}$ and $\dot{\eta}_i = \boldsymbol{e}_i^T \dot{\boldsymbol{\eta}}$. Let us now investigate when $\exists \alpha > 0$ such that (9) is satisfied. Note, hereto, that if

$$\boldsymbol{e}_i \in \text{span} \left\{ \boldsymbol{U}_c^T \boldsymbol{W}_T \right\}, \quad \forall i \in [1, \ldots, n_q],$$

then $\exists \boldsymbol{\gamma}$ such that $\boldsymbol{e}_i^T = \boldsymbol{\gamma}^T \boldsymbol{W}_T^T \boldsymbol{U}_c$. It therefore holds that $|\boldsymbol{e}_i^T \dot{\boldsymbol{\eta}}| = |\boldsymbol{\gamma}^T \boldsymbol{W}_T^T \boldsymbol{U}_c \dot{\boldsymbol{\eta}}|$ and $|\boldsymbol{e}_i^T \dot{\boldsymbol{\eta}}| \le$

$|\boldsymbol{\gamma}^{\mathrm{T}}||\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|$. Choose the smallest $\tilde{\alpha}_i$ such that $|\boldsymbol{\gamma}^{\mathrm{T}}| \leq \tilde{\alpha}_i\boldsymbol{p}^{\mathrm{T}}$, where the sign $\leq$ has to be understood component-wise. Then it holds that $|\boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}| \leq \tilde{\alpha}_i\boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}| \quad \forall\dot{\boldsymbol{\eta}}, \quad \forall i \in [1,\ldots,n_q]$. Note that α in (9) can be taken as $\alpha = \sum_{i=1}^{n_q} \tilde{\alpha}_i$. Finally, one should realize that if and only if

$$\boldsymbol{U}_c\boldsymbol{e}_i \in \text{span}\{\boldsymbol{W}_T\}, \qquad (14)$$

or, in other words, if the i-th column $\boldsymbol{U}_{c_i}$ of $\boldsymbol{U}_c$ satisfies $\boldsymbol{U}_{c_i} \in \text{span}\{\boldsymbol{W}_T\}$ (note in this respect that $\boldsymbol{U}_c$ is real and symmetric), then it holds that $\boldsymbol{e}_i \in \text{span}\{\boldsymbol{U}_c^{\mathrm{T}}\boldsymbol{W}_T\}$. Therefore, a sufficient condition for the validity of (13) can be given by

$$\boldsymbol{U}_{c_i} \in \text{span}\{\boldsymbol{W}_T\}, \quad \forall i \in [1,\ldots,n_q]. \qquad (15)$$

Now, we apply LaSalle's Invariance Principle. Let us, hereto, define a set $\mathcal{C}$ by

$$\mathcal{C} = \left\{(\boldsymbol{q},\dot{\boldsymbol{q}}) \mid \left|\left(\boldsymbol{U}_c^{\mathrm{T}}\dot{\boldsymbol{q}}\right)_i\right| \leq -\frac{\beta}{\lambda_i}, \; i = 1,\ldots,n_q\right\}, \qquad (16)$$

where $\left(\boldsymbol{U}_c^{\mathrm{T}}\dot{\boldsymbol{q}}\right)_i$ denotes the i-th element of the column $\boldsymbol{U}_c^{\mathrm{T}}\dot{\boldsymbol{q}}$. Moreover, let us use the definition of a set $\mathcal{I}_\rho$ as in (6) and choose the maximal constant ρ such that $\mathcal{I}_\rho \subset \mathcal{C}$:

$$\rho_{\max} = \max_{\{\rho:\mathcal{I}_\rho\subset\mathcal{C}\}}\rho. \qquad (17)$$

Let us now prove that $\rho_{\max} = \rho^*$, with ρ^* defined in (7). Note that V can be written as $V = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$, with $\boldsymbol{x}^{\mathrm{T}} = [\boldsymbol{\eta}^{\mathrm{T}} \; \dot{\boldsymbol{\eta}}^{\mathrm{T}}]$ and $\boldsymbol{P}$ defined in (8). The value $\rho_{\max}$ is the lowest value of ρ for which the set $\mathcal{I}_\rho$ touches one of the hyperplanes of $\partial\mathcal{C}$. We define $\rho_i, i = 1,\ldots,n_q$, to be that value of ρ for which the set $\mathcal{I}_\rho$ touches the hyperplane $|\dot{\eta}_i| = -\frac{\beta}{\lambda_i}$. Accordingly, $\rho_{\max}$ is defined by

$$\rho_{\max} = \min_{i=1,\ldots,n_q} \rho_i. \qquad (18)$$

Equating the hyperplane $|\dot{\eta}_i| = -\frac{\beta}{\lambda_i}$ with $\partial\mathcal{I}_{\rho_i}$ gives the relation

$$\sup_{\frac{1}{2}\|\boldsymbol{x}\|_{\boldsymbol{P}}^2=\rho_i} |\dot{\eta}_i| = -\frac{\beta}{\lambda_i}, \qquad (19)$$

where $\|\boldsymbol{x}\|_{\boldsymbol{P}}^2 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$. A decomposition of $\boldsymbol{P}$

$$\boldsymbol{P} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{S}, \qquad \boldsymbol{P} = \boldsymbol{U}_p^{\mathrm{T}}\boldsymbol{\Omega}_p\boldsymbol{U}_p, \qquad \boldsymbol{S} = \boldsymbol{U}_p^{\mathrm{T}}\boldsymbol{\Omega}_p^{\frac{1}{2}}\boldsymbol{U}_p, \qquad (20)$$

where $\boldsymbol{S}$ is the square root of $\boldsymbol{P}$ and a transformation $\boldsymbol{y} = \boldsymbol{S}\boldsymbol{x}$ gives the relationship

$$\sup_{\|\boldsymbol{y}\|=\sqrt{2\rho_i}} |\boldsymbol{e}_{n+i}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{y}| = -\frac{\beta}{\lambda_i}, \qquad (21)$$

with $\|\boldsymbol{y}\| = \|\boldsymbol{x}\|_{\boldsymbol{P}}$ and $\dot{\eta}_i = x_{n+i} = \boldsymbol{e}_{n+i}^{\mathrm{T}}\boldsymbol{x}$. With a transformation $\boldsymbol{z} = \boldsymbol{y}/\sqrt{2\rho_i}$, (21) transforms to

$$\sqrt{2\rho_i}\sup_{\|\boldsymbol{z}\|=1} |\boldsymbol{e}_{n+i}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{z}| = -\frac{\beta}{\lambda_i}. \qquad (22)$$

Using the definition of the norm of a matrix $\boldsymbol{A}$ as $\|\boldsymbol{A}\| = \sup_{\|\boldsymbol{x}\|=1} \|\boldsymbol{A}\boldsymbol{x}\|$, (22) yields

$$\sqrt{2\rho_i}\|\boldsymbol{e}_{n+i}^{\mathrm{T}}\boldsymbol{S}^{-1}\| = -\frac{\beta}{\lambda_i}. \qquad (23)$$

Consequently, ρ_i is given by (7). This concludes the proof that $\rho_{\max} = \rho^*$.

Moreover, we define a set $\mathcal{S} \subset \mathcal{I}_{\rho^*}$ by $\mathcal{S} = \{(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathcal{I}_{\rho^*} : \dot{\boldsymbol{q}} = \boldsymbol{0}\}$. Furthermore, the largest invariant set in $\mathcal{S}$ is the equilibrium set $\mathcal{E}$, since according to the third condition in the theorem $\mathcal{E} \subset \mathcal{I}_{\rho^*}$. Note that $\dot{V} = 0$ if and only if $(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathcal{S}$ and $\dot{V} < 0$ otherwise. Application of LaSalle's invariance principle concludes the proof of the local attractivity of $\mathcal{E}$ under the conditions stated in the theorem. $\qquad\square$

At this point several remarks should be made:

1. It should be noted that the proof of Theorem 1 provides us with a conservative estimate of the region of attraction $\mathcal{A}$ of the locally attractive equilibrium set $\mathcal{E}$. The estimate $\mathcal{B}$ can be formulated in terms of the generalized displacements and velocities: $\mathcal{B} = \mathcal{I}_{\rho^*}$, where ρ^* satisfies (7);

2. The proof of Theorem 1 also shows that boundedness of solutions (starting in $\mathcal{B}$) is ensured and that the equilibrium *point* $(\boldsymbol{q},\dot{\boldsymbol{q}}) = (\boldsymbol{0},\boldsymbol{0})$ is Lyapunov stable.

3. It can be shown that if $\mathbf{1}^{\mathrm{T}}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{K}^{-\mathrm{T}}\boldsymbol{W}_T\boldsymbol{\Lambda}\mathbf{1} < 2\rho^*$, with $\mathbf{1}$ a $m \times 1$-column with ones, then $\mathcal{E} \subset \mathcal{I}_{\rho^*}$. In that case the entire equilibrium set $\mathcal{E}$ is locally attractive.

4. An important consequence of Theorem 1 is that when the damping-matrix $\boldsymbol{C}$ is positive definite, global attractivity of the equilibrium set is assured. Note, hereto, that in the proof of Theorem 1, (15) is automatically satisfied and ρ^* can be taken arbitrarily large in that case.

4 Illustrating example

In this section, we will illustrate the results of the previous section by means of an example concerning a 2DOF mass-spring-damper system, see Figure 1. The equation of motion of this system can be written in the
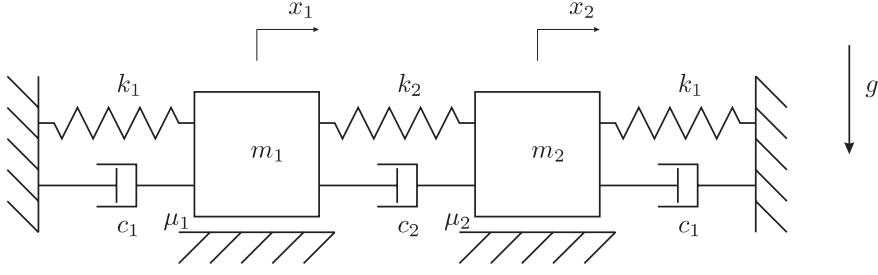
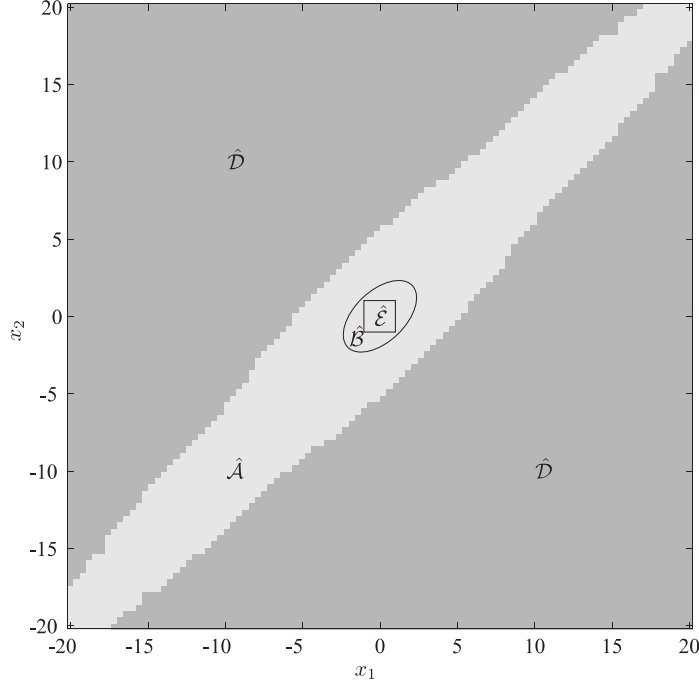Figure 1. 2DOF mass-spring-damper system with Coulomb friction.

Figure 2. Cross-section of the region of attraction $\mathcal{A}$ with the plane defined by $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$.

form (1), with $\boldsymbol{q}^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ and the generalized friction forces $\boldsymbol{\lambda}_T$ given by the Coulomb friction law (2). Herein the matrices $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$, $\boldsymbol{W}_T$ and $\boldsymbol{\Lambda}$ are given by

$$\boldsymbol{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_1 + c_2 \end{bmatrix},$$

$$\boldsymbol{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}, \qquad (24)$$

$$\boldsymbol{W}_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \mu_1 m_1 g & 0 \\ 0 & \mu_2 m_2 g \end{bmatrix},$$

with $m_1, m_2, k_1, k_2 > 0$ and $\mu_1, \mu_2 \geq 0$. Moreover, the tangential velocity $\dot{\boldsymbol{g}}_T$ in the frictional contacts is given by $\dot{\boldsymbol{g}}_T = \begin{bmatrix} \dot{x}_1 & \dot{x}_2 \end{bmatrix}^{\mathrm{T}}$. Let us first compute the spectral decomposition of the damping-matrix, $\boldsymbol{C} = \boldsymbol{U}_c^{-\mathrm{T}} \boldsymbol{\Omega}_c \boldsymbol{U}_c^{-1}$, with (for non-singular $\boldsymbol{C}$):

$$\boldsymbol{U}_c = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{\Omega}_c = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 + 2c_2 \end{bmatrix}. \qquad (25)$$

The equilibrium set $\mathcal{E}$, as defined by (4), is given by

$$\mathcal{E} = \{(x_1, x_2, \dot{x}_1, \dot{x}_2) \mid \dot{x}_1 = 0 \wedge \dot{x}_2 = 0 \wedge$$
$$|x_1| \leq \frac{(k_1 + k_2)\mu_1 m_1 g + k_2 \mu_2 m_2 g}{k_1^2 + 2k_1 k_2} \wedge$$
$$|x_2| \leq \frac{(k_1 + k_2)\mu_2 m_2 g + k_2 \mu_1 m_1 g}{k_1^2 + 2k_1 k_2} \}.$$
$$(26)$$

Let us now consider two different cases for the damping parameters c_1 and c_2:

Firstly, we consider the case that $\mathbf{c_1 > 0}$ **and** $\mathbf{c_2 > -c_1/2}$. Note that $\boldsymbol{C} > 0$ if and only if $c_1 > 0$ and $c_2 > -c_1/2$. Consequently, the global attractivity of the equilibrium set $\mathcal{E}$ is assured. It should be noted that this is also the case when one or both of the friction coefficients μ_1 and μ_2 vanish.

Secondly, we consider the case that $\mathbf{c_1 > 0}$ **and** $\mathbf{c_2 < -c_1/2}$. Clearly, the damping matrix is not positive definite in this case. As a consequence, the equi-

librium point of the system without friction is unstable. Still the equilibrium set of the system with friction can be locally attractive. Therefore, Theorem 1 can be used to investigate the attractivity properties of (a subset of) the equilibrium set. For the friction situation depicted in Figure 1, condition (15) is satisfied if $\mu_1 > 0$ and $\mu_2 > 0$. Namely, $\boldsymbol{W}_T$ spans the two-dimensional space and, consequently, the eigencolumn of the damping matrix corresponding to the unstable eigenvalue $c_1 + 2c_2$, namely $\begin{bmatrix} -1 & 1 \end{bmatrix}^{\mathrm{T}}$, lies in the space spanned by the columns of $\boldsymbol{W}_T$.

Since the attractivity is only local, it is desirable to provide an estimate $\mathcal{B}$ of the region of attraction $\mathcal{A}$ of (a subset of) the equilibrium set. Here, we present a comparison between the actual region of attraction (obtained by numerical simulation) and the estimate $\mathcal{B}$ for the following parameter set: $m_1 = m_2 = 1\,\text{kg}$, $k_1 = k_2 = 1\,\text{N/m}$, $c_1 = 0.5\,\text{Ns/m}$, $c_2 = -0.375\,\text{Ns/m}$, $\mu_1 = \mu_2 = 0.1$ and $g = 10\text{m/s}^2$. An estimate for the region of attraction $\mathcal{B}$ can be provided analytically as $\mathcal{B} = \mathcal{I}_{\rho^*}$, with ρ^* according to (7), which yields for this example:

$$\rho^* = \frac{1}{2}\frac{m_1 m_2 \gamma^2}{m_1 + m_2}, \quad \text{where } \gamma = -\frac{\min(\mu_1 m_1 g, \mu_2 m_2 g)}{c_1 + 2c_2}.$$
(27)

The numerical simulations are performed using an event-driven integration method as described in (Pfeiffer and Glocker, 1996). The event-driven integration method is a hybrid integration technique that uses a standard ODE solver for the integration of smooth phases of the system dynamics and a LCP (Linear Complementarity Problem) formulation to determine the next hybrid mode at the switching boundaries. For these parameter settings, $\mathcal{E} \subset \text{int}(\mathcal{I}_{\rho^*})$ and the local attractivity of the entire equilibrium set $\mathcal{E}$ is ensured. In Figure 2, we show a cross-section of $\mathcal{A}$ with the plane $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, denoted by $\hat{\mathcal{A}}$, which was obtained numerically. Hereto, a grid of initial conditions in the plane $\dot{x}_1 = \dot{x}_2 = 0$ was defined, for which the solutions were obtained by numerically integrating the system over a given time span T. Subsequently, a check was performed to inspect whether the state of the system at time T was in the equilibrium set $\mathcal{E}$. Initial conditions corresponding to attractive solutions are depicted with a light color (set $\hat{\mathcal{A}}$) and initial conditions corresponding to non-attractive solutions are depicted with a dark grey color (set $\hat{\mathcal{D}}$). Moreover, $\hat{\mathcal{E}}$ and $\hat{\mathcal{B}}$ are also shown in the figure, where the ^ indicates that we are referring to cross-sections of the sets. It should be noted that $\hat{\mathcal{E}} \subset \hat{\mathcal{B}}$. As expected the set $\mathcal{B}$ is a conservative estimate for the region of attraction $\mathcal{A}$. In (Van de Wouw and Leine, 2004), more examples are discussed in which the crucial condition for local attractivity (15) is not satisfied.

5 Controlling Attractivity

So far we have concentrated on the asymptotic properties of mechanical systems with friction. In particular, we presented results on the attractivity of equilibrium sets in such systems. Let us now present a preliminary study of the application of these results in a control context. Hereto, we consider a class of controlled mechanical systems with Coulomb friction:

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} - \boldsymbol{W}_T\boldsymbol{\lambda}_T = \boldsymbol{S}\boldsymbol{u}, \qquad (28)$$

where $\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}} > 0$ and $\boldsymbol{u}$ is a column with control inputs and $\boldsymbol{S}$ is a matrix concerning the generalized force directions in which the actuators can influence the system dynamics. This open-loop system may not exhibit positive definite damping- and stiffness matrices; think of linearizations of systems (systems without Coulomb friction) of which the equilibrium undergoes a Hopf or pitchfork bifurcation. Consequently, such systems do not satisfy the conditions of Theorem 1. Now, we propose a PD-type of feedback law $\boldsymbol{u} = -\boldsymbol{F}_1\boldsymbol{q} - \boldsymbol{F}_2\dot{\boldsymbol{q}}$ in order to ensure that the closed-loop dynamics, described by

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{C}_{cl}\dot{\boldsymbol{q}} + \boldsymbol{K}_{cl}\boldsymbol{q} - \boldsymbol{W}_T\boldsymbol{\lambda}_T = \boldsymbol{0}, \qquad (29)$$

with $\boldsymbol{C}_{cl} = \boldsymbol{C} + \boldsymbol{S}\boldsymbol{F}_2$ and $\boldsymbol{K}_{cl} = \boldsymbol{K} + \boldsymbol{S}\boldsymbol{F}_1$, satisfies the conditions of Theorem 1. Consequently, the conditions on the control design can be stated as follows:

1. $(\boldsymbol{K} + \boldsymbol{S}\boldsymbol{F}_1) = (\boldsymbol{K} + \boldsymbol{S}\boldsymbol{F}_1)^{\mathrm{T}} > 0$,
2. $\bar{\boldsymbol{U}}_{c_i} \in \text{span}\{\boldsymbol{W}_T\}$,
3. $\boldsymbol{C}_{cl}$ is symmetric,

where $\bar{\boldsymbol{U}}_{c_i}$, $i = 1, \ldots, \bar{n}_q$, represent the eigencolumns of $\boldsymbol{C}_{cl}$ corresponding to the remaining $\bar{n}_q$ closed left-half plane eigenvalues of $\boldsymbol{C}_{cl}$. The second condition can be formulated as follows: $\exists \boldsymbol{F_2}$ such that the pair $[-(\boldsymbol{C} + \boldsymbol{S}\boldsymbol{F}_2), \boldsymbol{W}_T]$ is stabilizable. The latter condition, in turn, is equivalent to the stabilizability of the pair $[-\boldsymbol{C}, [\boldsymbol{W}_T, \boldsymbol{S}]]$. This stabilizability condition is intuitive since it expresses the fact that one should be able to counteract instabilities due to negative viscous damping either by dry friction effects or by means of feedback in order to attain attractivity. Moreover, we can reformulate the first condition using the following result from (Yakubovich *et al.*, 2004):

Theorem 2
The following two statements are equivalent:

1. $\exists \boldsymbol{P} = \boldsymbol{P}^{\mathrm{T}} > 0$
2. $\exists \boldsymbol{A}$, with $\boldsymbol{A}$ Hurwitz s.t. $\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} < 0$.

Now we use this result to reformulate condition 1; the following two statements are equivalent:

1. $\exists \boldsymbol{F_1}$ s.t. $(\boldsymbol{K} + \boldsymbol{S}\boldsymbol{F}_1) = (\boldsymbol{K} + \boldsymbol{S}\boldsymbol{F}_1)^{\mathrm{T}} > 0$,
2. $\exists \boldsymbol{A}, \boldsymbol{Y}$ with $\boldsymbol{A}$ Hurwitz s.t.
   $\boldsymbol{K}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}} + \boldsymbol{S}\boldsymbol{Y} + \boldsymbol{Y}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}} < 0$ with $\boldsymbol{Y} = \boldsymbol{F}_1\boldsymbol{A}$.
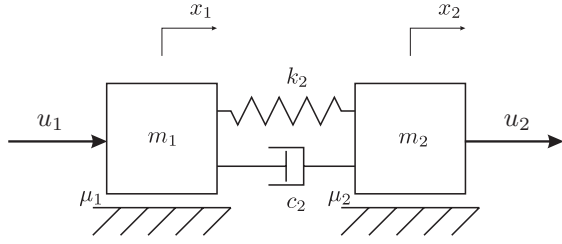
Figure 3. Controlled 2DOF mass-spring-damper system with Coulomb friction.

The LMI in the previous statement can be used to construct the feedback gain-matrix $\boldsymbol{F}_1$ through $\boldsymbol{F}_1 = \boldsymbol{Y}\boldsymbol{A}^{-1}$. In such a way the equilibrium set can be rendered attractive and the following example will illustrate that the region of attraction can also be influenced by tuning the control parameters.

Let us now consider the 2DOF mass-spring-damper system as depicted in figure 3, in which masses m_1 and m_2 can be controlled by actuators u_1 and u_2, respectively. Here PD-control of the two masses will be considered: $u_1 = -p_1 x_1 - d_1 \dot{x}_1, \quad u_2 = -p_2 x_2 - d_2 \dot{x}_2$. The control problem to be addressed can be formulated as follows: how can we influence the attractivity properties (and the corresponding region of attraction) of the equilibrium set by tuning of the control parameters p_1, d_1, p_2 and d_2? For the sake of simplicity we limit ourselves to the case that the control laws for the two actuators are identical, i.e. $p_2 = p_1$ and $d_2 = d_1$. Note that the condition on $\boldsymbol{K}_{cl}$ stated above is guaranteed if $p_1 > 0$. Moreover, the condition $\bar{\boldsymbol{U}}_{c_i} \in \text{span}\{\boldsymbol{W}_T\}$ is satisfied for all values of d_1. However, p_1 and d_1 should be chosen such that condition 3 in Theorem 1 is satisfied. The influence of the control design on the attractivity of the equilibrium set will be investigated along two routes: firstly, we will investigate the influence of the gain d_1 of the derivative action and, secondly, we will investigate the influence of the proportional action p_1. In accordance with (27) the region of attraction for the controlled system is given by $\mathcal{B} = \mathcal{I}_{\rho^*}$, with ρ^* given by (27) with $c_1 = d_1$. This clearly indicates that the region of attraction can be effectively influenced by the control design. In figure 4, the effect of the derivative gain on the region of attraction is depicted. Once more, a similar cross-section of state-space as in figure 2 is considered. Herein, the following system parameter values are used: $m_1 = m_2 = 1\,\text{kg}$, $k_2 = 1\,\text{N/m}$, $c_2 = -0.375\,\text{Ns/m}$, $\mu_1 = \mu_2 = 0.1$, $g = 10\text{m/s}^2$. Moreover, the proportional gain is $p_1 = 1\,\text{N/m}$. Figure 4 clearly displays the increase of the estimated region of attraction for increasing derivative control gain. As this gain approaches the value $d_1 = 0.75$, (i.e. $d_1 = -2c_2$) the estimated region increases progressively until at $d_1 = -2c_2$ the attractivity becomes global. Moreover, when the derivative gain drops below a certain level the attractivity of the equilibrium set

as a whole can not be guaranteed anymore.

The influence of the proportional gain is of an entirely different nature. Namely, (26) (with p_1 substituted for k_1) expresses the fact that the equilibrium set decreases when p_1 increases. However, the estimated region of attraction also decreases. In this respect, it should be noted that for increasing p_1 the set $\mathcal{C}$, defined in (16), and consequently ρ^* remain unchanged; however, since the estimated region of attraction is now given by

$$\mathcal{B} = \{\boldsymbol{x} \in \mathbb{R}^4 \mid \frac{1}{2}\big(m_1\dot{x}_1^2 + m_2\dot{x}_2^2$$
$$+ (p_1 + k_2)(x_1^2 + x_2^2) - 2k_2 x_1 x_2\big) < \rho^*\}, \tag{30}$$

the size of $\mathcal{B}$ decreases for increasing proportional gain.

Obviously, a similar analysis can be performed for the case that different PD-control designs are used for the actuators u_1 and u_2.

6 Conclusions

Conditions for the (local) attractivity of equilibrium sets of mechanical systems with friction are derived. The systems are allowed to have multiple degrees-of-freedom and multiple switching boundaries (friction elements). It is shown that the equilibrium set $\mathcal{E}$ of a linear mechanical system, which without friction exhibits a stable equilibrium point E, will always be attractive when Coulomb friction elements are added. Moreover, it has been shown that even if the system without friction has an unstable equilibrium point E, then the equilibrium set $\mathcal{E}$ of the system with friction can under certain conditions be locally attractive and the equilibrium point $E \subset \mathcal{E}$ is stable. The crucial condition can be interpreted as follows: the space spanned by the eigendirections of the damping matrix, related to negative eigenvalues, lies in the space spanned by the generalized force directions of the dry friction elements.

Lyapunov stability of the equilibrium set of non-passive systems is not addressed, however, the combination of the attractivity property of the equilibrium set and the boundedness of solutions within $\mathcal{B}$ can be a valuable characteristic when the equilibrium set is a desired steady state of the system. Moreover, an estimate of the region of attraction of the equilibrium set is provided.

Finally, the attractivity results are applied to design a linear state-feedback controller to guarantee attractivity of an equilibrium set of a controlled system with friction. Moreover, it is shown that the region of attraction can effectively be shaped by tuning of the control parameters.

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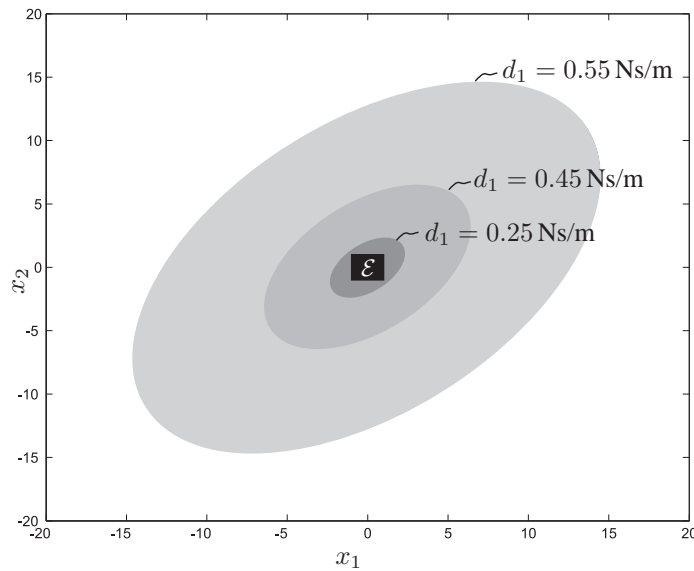
Figure 4. Estimated region of attraction $\mathcal{B}$ for an increasing derivative control gain d_1.

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