Solution to problem 82-4 : A volume problem

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The same method can be used for proving the Hardy integral,

$$\int_{-\infty}^{\infty} \frac{\sin a(y - x)}{(y - x)} \frac{\sin b(y - z)}{(y - z)} \, dy = \frac{\pi \sin b(z - x)}{(z - x)}, \qquad 0 < b < a,$$

where the Parseval equation for the Fourier transform is employed. It is obvious that this method can be utilized for proving more general integrals for the Hankel and for other transforms.

In a second solution, O. P. Lossers shows that the stated equation holds generally for complex x and z.

A Volume Problem

Problem 82-4, *by* M. K. Lewis (Memorial University of Newfoundland).

An asymmetrically positioned hole of radius b is drilled at right angles to the axis of a solid right circular cylinder of radius a $(a > b)$. If the distance between the axis of the drill and the axis of the cylinder is p, determine the volume of material drilled out.

Solution by J. Boersma, P. J. de Doelder *and* J. K. M. Jansen (Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven, the Netherlands).

Introduce Cartesian coordinates x, y, z, then the cylinder and the drill may be described by

$$(1) \qquad\qquad C_a : x^2 + y^2 = a^2, \qquad C_b : (x - p)^2 + z^2 = b^2.$$

The volume of the intersection of C_a and C_b is denoted by $V(a, b, p)$. Without loss of generality, we may assume $0 \leq b \leq a, p \geq 0$.

From a cross-section of C_a and C_b with the plane $z = 0$, it is readily seen that

$$(2) \qquad\qquad V(a, b, p) = 2 \int \int_G [b^2 - (x - p)^2]^{1/2} \, dx \, dy,$$

where the domain of integration is

$$(3) \qquad\qquad G = \{(x, y) \,|\, x^2 + y^2 \leq a^2, |x - p| \leq b\}.$$

We now distinguish three cases:

I. $0 \leq p \leq a - b$. In this case the integral (2) reduces to

$$(4) \qquad V(a, b, p) = 4 \int_{p-b}^{p+b} [(a - x)(b + p - x)(x + b - p)(x + a)]^{1/2} \, dx.$$

The latter integral can be expressed in terms of elliptic integrals by means of Byrd and Friedman [1, form. 254.38]. Omitting the details of the tedious though straightforward calculation, we present the result

$$
\begin{aligned}
(5) \qquad V(a, b, p) = {}& 4(a + b + p)^{-1/2} (a + b - p)^{-1/2} \\
& \cdot [\tfrac{1}{6} (p^2 - 2ap - 2a^2 + 2b^2)(a + b + p)(p + a - b) K(k) \\
& \quad + \tfrac{1}{6} (p^2 + 2a^2 + 2b^2)(a + b + p)(a + b - p) E(k) \\
& \qquad + (a^2 - b^2) p (p + a - b) \Pi(\alpha^2, k)],
\end{aligned}
$$

where

$$(6) \qquad\qquad k^2 = \frac{4ab}{(a + b + p)(a + b - p)}, \qquad \alpha^2 = \frac{2b}{a + b + p}.$$

In (5), $K(k)$, $E(k)$ and $\Pi(\alpha^2, k)$ denote Legendre's complete elliptic integrals of the first, second and third kinds, respectively, as defined in [1, form. 110.06–08].

II. $a - b \leqq p \leqq a + b$. In this case, the integral (2) reduces to

$$(7) \qquad V(a, b, p) = 4 \int_{p-b}^{a} [(b + p - x)(a - x)(x + b - p)(x + a)]^{1/2} \, dx,$$

which can again be evaluated by means of [1, form. 254.38]. Thus we obtain

$$V(a, b, p) = 2a^{-1/2}b^{-1/2}[-\tfrac{1}{3}\, a(p + a - b)(2ab - 2b^2 + (3a + b)p) K(k)$$
$$(8) \qquad\qquad\qquad + \tfrac{2}{3}\, ab(p^2 + 2a^2 + 2b^2) E(k)$$
$$+ (a^2 - b^2)p(p + a - b) \Pi(\alpha^2, k)]$$

where

$$(9) \qquad\qquad k^2 = \frac{(a + b + p)(a + b - p)}{4ab}, \qquad \alpha^2 = \frac{a + b - p}{2a}.$$

III. $p \geqq a + b$. In this case the drill C_b is outside the cylinder C_a, hence

$$(10) \qquad\qquad\qquad\qquad V(a, b, p) = 0.$$

The results in (5) and (8) may be combined into a single formula, viz.,

$$V(a, b, p) = 2^{3/2}[ab(a + b + p)(a + b - p)]^{-1/4}$$
$$(11) \qquad \cdot \{\tfrac{1}{3}\, a(a + b + p)(2ab + 2b^2 - (3a - b)p) k^{1/2} K(k)$$
$$+ \tfrac{1}{3}\, (p^2 + 2a^2 + 2b^2)[ab(a + b + p)(a + b - p)]^{1/2} k^{-1/2} [E(k) - K(k)]$$
$$+ (a^2 - b^2)p(p + a - b) k^{1/2} \Pi(\alpha^2, k)\}$$

where k^2, α^2 are given by (6) in case I, and by (9) in case II. It has been verified that the results (11) for cases I and II are related by the reciprocal modulus transformation (cf. [1, form. 162.02])

$$\operatorname{Re} k_1^{1/2} K(k_1) = k^{1/2} K(k), \ \operatorname{Re} k_1^{-1/2}[E(k_1) - K(k_1)] = k^{-1/2}[E(k) - K(k)],$$
$$(12)$$
$$\operatorname{Re} k_1^{1/2} \Pi(\alpha_1^2, k_1) = k^{1/2} \Pi(\alpha^2, k)$$

where $k_1 = 1/k$, $\alpha_1^2 = \alpha^2/k^2$, $\alpha^2 \leqq k^2 \leqq 1$.

The previous general results simplify in the following special cases:

1. $p = 0$. Then it is found from (5) that

$$(13) \qquad\qquad V(a, b, 0) = \tfrac{4}{3}\, (a + b)[(a^2 + b^2) E(k) - (a - b)^2 K(k)],$$

where $k^2 = 4ab/(a + b)^2$. By means of Gauss' transformation [1, form. 164.02] the latter result can be reduced to

$$(14) \qquad\qquad V(a, b, 0) = \tfrac{8}{3}\, a \left[(a^2 + b^2) E\left(\frac{b}{a}\right) - (a^2 - b^2) K\left(\frac{b}{a}\right)\right].$$

The same result can also be found directly from (4) with $p = 0$, by use of [1, form. 219.11].

2. $p = a - b$. The expression for $V(a, b, p)$ in this case should follow continuously from (5) and (8). Indeed, by taking the limit for $p \to a - b$ in (5) and (8) we find

$$(15) \ V(a, b, a - b) = \tfrac{4}{3}\, (ab)^{1/2}(3a^2 + 3b^2 - 2ab) - 4(a - b)^2(a + b) \log\left(\frac{a^{1/2} + b^{1/2}}{(a - b)^{1/2}}\right),$$

which has been checked by a direct calculation from (4) with $p = a - b$.

3. $p = a + b$. Then C_b is tangent to C_a on the outside, and it is found from (8) and (10) that

(16) $$V(a, b, a + b) = 0.$$

4. $a = b$. Then the result in (8) simplifies to

(17) $$V(a, a, p) = \tfrac{4}{3} a[(p^2 + 4a^2) E(k) - 2p^2 K(k)],$$

where $k^2 = 1 - p^2/4a^2$.

5. $a = b, p = 0$. Then it follows from (14), (15) or (17) that

(18) $$V(a, a, 0) = \frac{16}{3} a^3,$$

which result can also be found immediately from (4).

By means of (5) and (8) we have computed a table of $V(a, b, p)$ to four decimal places, for values of the argument

$$a = 1, \quad b = 0.1(0.1)1, \quad p = 0(0.1)1 + b.$$

The required elliptic integrals were calculated by standard procedures taken from Bulirsch [2].

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TABLE

$V(1, b, p)$

</div>

p \ b	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0.0628	0.2501	0.5591	0.9848	1.5200	2.1550	2.8763	3.6664	4.4999	5.3333
0.1	0.0624	0.2488	0.5561	0.9794	1.5113	2.1416	2.8566	3.6373	4.4534	5.2557
0.2	0.0615	0.2449	0.5472	0.9630	1.4846	2.1006	2.7955	3.5436	4.3029	5.0786
0.3	0.0598	0.2383	0.5320	0.9350	1.4384	2.0290	2.6842	3.3665	4.0833	4.8341
0.4	0.0575	0.2287	0.5098	0.8938	1.3698	1.9180	2.5042	3.1363	3.8161	4.5399
0.5	0.0543	0.2157	0.4795	0.8368	1.2707	1.7501	2.2828	2.8716	3.5149	4.2089
0.6	0.0501	0.1985	0.4389	0.7571	1.1253	1.5509	2.0373	2.5843	3.1900	3.8515
0.7	0.0447	0.1758	0.3826	0.6411	0.9577	1.3364	1.7785	2.2836	2.8505	3.4767
0.8	0.0373	0.1436	0.2997	0.5111	0.7826	1.1168	1.5151	1.9777	2.5040	3.0922
0.9	0.0262	0.0947	0.2107	0.3809	0.6097	0.9002	1.2546	1.6736	2.1576	2.7055
1.0	0.0085	0.0472	0.1283	0.2596	0.4467	0.6937	1.0036	1.3780	1.8179	2.3230
1.1	0.0000	0.0130	0.0611	0.1546	0.3004	0.5037	0.7684	1.0969	1.4910	1.9511
1.2		0.0000	0.0163	0.0724	0.1769	0.3362	0.5548	0.8363	1.1828	1.5958
1.3			0.0000	0.0190	0.0821	0.1967	0.3685	0.6016	0.8990	1.2628
1.4				0.0000	0.0214	0.0908	0.2147	0.3982	0.6449	0.9576
1.5					0.0000	0.0235	0.0987	0.2313	0.4258	0.6855
1.6						0.0000	0.0255	0.1060	0.2468	0.4517
1.7							0.0000	0.0273	0.1129	0.2613
1.8								0.0000	0.0290	0.1193
1.9									0.0000	0.0306
2.0										0.0000

Boersma and Kamminga [3] calculated the volume of intersection, $V(\rho, \eta)$, of a sphere of unit radius and a cylinder of radius ρ, with η denoting the distance of the center of the sphere and the axis of the cylinder. Their expressions for $V(\rho, \eta)$ in terms of elliptic integrals are quite similar in form to the present expressions (5) and (8). In addition it was shown in [3] that $V(\rho, \eta)$ can be represented by an infinite integral of a product of three Bessel functions, viz.,

(19) $$V(\rho, \eta) = 2\pi\rho \sqrt{2\pi} \int_0^\infty J_1(\rho t) J_0(\eta t) \frac{J_{3/2}(t)}{t^{3/2}} \, dt.$$

We shall now derive a similar integral representation for the volume of intersection $V(a, b, p)$ of two cylinders. Following [3], we introduce the function $f(\rho)$ defined by

(20)
$$f(\rho) = b \int_0^\infty J_0(\rho t) J_1(bt) \, dt = \begin{cases} 0 & \text{for } \rho > b, \\ \tfrac{1}{2} & \text{for } \rho = b, \\ 1 & \text{for } 0 \leq \rho < b \end{cases}$$

(cf. Watson [4, form. 13.42(9)]). Let ρ denote the distance to the axis of the cylinder C_b, then $V(a, b, p)$ may be determined by integrating $f(\rho)$ over the volume of the cylinder C_a. Thus, employing cylindrical coordinates r, ϕ, z, one has

(21)
$$\rho = \sqrt{(r \cos \phi - p)^2 + z^2},$$

(22)
$$V(a, b, p) = 2 \int_0^a r \, dr \int_0^{2\pi} d\phi \int_0^\infty f(\sqrt{(r \cos \phi - p)^2 + z^2}) \, dz$$
$$= 2b \int_0^a r \, dr \int_0^{2\pi} d\phi \int_0^\infty dz \int_0^\infty J_0(t\sqrt{(r \cos \phi - p)^2 + z^2}) \, J_1(bt) \, dt.$$

By interchanging the order of integration, we may successively perform the integrations with respect to z, ϕ and r by means of [4, form. 13.47(5), 2.3(1), 5.1(1)]:

(23)
$$\int_0^\infty J_0(t\sqrt{\sigma^2 + z^2}) \, dt = \left(\frac{\pi\sigma}{2t}\right)^{1/2} J_{-1/2}(\sigma t) = \frac{\cos(\sigma t)}{t},$$

(24)
$$\int_0^{2\pi} \cos(t(r \cos \phi - p)) \, d\phi = 2\pi \cos(pt) J_0(rt),$$

(25)
$$\int_0^a r J_0(rt) \, dr = \frac{a}{t} J_1(at).$$

As a result we obtain the integral representation

(26)
$$V(a, b, p) = 4\pi ab \int_0^\infty \frac{J_1(at) J_1(bt)}{t^2} \cos(pt) \, dt,$$

and implicitly it is found that the latter Bessel function integral is given by (5), (8) and (10). Infinite integrals of products of Bessel functions have been studied by Bailey [5], and listed by Luke [6, §13.4], Okui [7]; however none of these references contains the particular integral (26). According to [5, form. (7.1)] the integral (26) can be expressed in terms of an Appell function of the two variables a^2/p^2, b^2/p^2; furthermore, by [5, form. (8.3)] the integral vanishes if $p \geq a + b$, in accordance with (10). The special case $p = 0$ of (26) is given in [7, form. I.2.7(1)] and the result agrees with (13); see also [4, form. 13.4(2)] for a representation in terms of a hypergeometric function of argument b^2/a^2, which is equivalent to (14). The special case $a = b$ of (26) can be derived from [7, form. I. 2.3(3)] and the result agrees with (17).

Also solved by W. B. JORDAN (Scotia, NY) *and the proposer.* M. RENARDY (University of Minnesota) submitted a partial solution.

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A Maximum, Minimum Problem

Problem 82-5, by T. SEKIGUCHI (University of Arkansas).

In Euclidean space E_3, with origin at O and coordinates (x_1, x_2, x_3), let $\overrightarrow{OX}$ be a ray in the closed first orthant, and α_i be the angles between the positive x_i axes and $\overrightarrow{OX}$, $i = 1, 2, 3$.

Determine the maximum and minimum values of $\alpha_1 + \alpha_2 + \alpha_3$.

Solution by O. P. LOSSERS (Eindhoven University of Technology, Eindhoven, the Netherlands).

Without loss of generality we assume (x_1, x_2, x_3) to be on the unit sphere in the closed first octant, denoted by

$$S: \begin{cases} x_1^2 + x_2^2 + x_3^2 = 1, \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{cases}$$

The function to be considered then reads

$$F(x_1, x_2, x_3) := \arccos x_1 + \arccos x_2 + \arccos x_3.$$

(i) At the boundary points of S, we have $F(x_1, x_2, x_3) = \pi$ since

$$\arccos 0 + \arccos x + \arccos \sqrt{1 - x^2} = \pi.$$

(ii) In the interior of S, we find stationary points for F in those points where grad $F = \lambda$ grad $(x_1^2 + x_2^2 + x_3^2 - 1)$, i.e.,

$$x_1\sqrt{1 - x_1^2} = x_2\sqrt{1 - x_2^2} = x_3\sqrt{1 - x_3^2}.$$

Squaring, we obtain

$$x_1^2 - x_2^2 - x_1^4 + x_2^4 = 0, \quad \text{etc.}$$

$$(x_1 - x_2)(x_1 + x_2)(1 - x_1^2 - x_2^2) = 0, \quad \text{etc.}$$

In this product for the interior points only the first factor can be zero, so $x_1 = x_2 = x_3 = 3^{-1/2}$.

At this point, F attains the value $3 \arccos 3^{-1/2}$.

Hence

$$3 \arccos 3^{-1/2} \leq F(x_1, x_2, x_3) \leq \pi.$$

Also solved by G. A. HEUER (Concordia College), C. KNOSHAUG (Bemidji State University), I. I. KOTLARSKI (Oklahoma State University), U. PURSIHEIMO (University of Turku, Turku, Finland), M. RENARDY (University of Minnesota), G. SCHLUETER (Virginia Polytechnic Institute and State University) *and the proposer.*